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Piecewise Continuous Almost Automorphic Functions and Favard's Theorems for Impulsive Differential Equations in Honor of Russell Johnson

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Abstract
We define piecewise continuous almost automorphic (p.c.a.a.) functions in the manners of Bochner, Bohr and Levitan, respectively, to describe almost automorphic motions in impulsive systems, and prove that with certain prefixed possible discontinuities they are equivalent to quasi-uniformly continuous Stepanov almost automorphic ones. Spatially almost automorphic sets on the line, which serve as suitable objects containing discontinuities of p.c.a.a. functions, are characterized in the manners of Bochner, Bohr and Levitan, respectively, and shown to be equivalent. Two Favard's theorems are established to illuminate the importance and convenience of p.c.a.a. functions in the study of almost periodically forced impulsive systems.

Keywords Piecewise continuous and Stepanov almost automorphic functions · Spatially almost automorphic sets · Favard's theorems · Impulsive differential equations

Mathematics Subject Classification 43A60 · 03E15 · 34A37

1 Introduction

Impulsive differential equations model suitably a class of real world evolutionary processes in which the parameters undergo relatively long periods of smooth variation followed by a short-term rapid change in their values [1]. It is in the study of almost periodic motions in

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systems with impulses at fixed times

$$\begin{cases} x' = f(x,t), \quad t \neq \tau_n, \\ x(\tau_n^+) - x(\tau_n) = g(x(\tau_n), n), \quad n \in \mathbb{Z}, \end{cases}$$

that the almost periodicity of both the impulse times $\{\tau_j\}_{j\in\mathbb{Z}}$ and the piecewise continuous solutions are encountered. The class of piecewise continuous almost periodic functions (p.c.a.p., for short, see Definition 4.4) generalizing Bohr almost periodic ones, first introduced in [15], characterizes successfully almost periodic motions in impulsive systems. As well known in researches on continuous systems, almost automorphic dynamics are studied in depth by R. Johnson [16,17], et. al., which have stimulated later works that almost automorphic phenomenon is a fundamental property occurring in almost periodically forced differential equations [28,29]. From this point of view, we aim at in this paper investigating almost automorphic solutions of impulsive differential equations and thus p.c.a.a. functions naturally occur. To the best of our knowledge, the important notion of almost automorphy, is only studied in one specific piecewise continuous setting [6], in which the discontinuities are contained in $\mathbb{Z}$. So, in the present paper, we shall introduce the notions of general almost automorphy of both impulse times and piecewise continuous functions and then explore relations among various almost automorphy and establish the important Favard's theorems.

The concept of almost automorphy (Definition 2.1) is defined by S. Bochner [4] in relation to aspects of differential geometry. Bochner almost automorphy has a close relation with Bohr almost automorphy (Definition 2.8) and Levitan's N-almost periodicity (Definition 2.11). In the classical work of W. A. Veech [30], the essential equivalence between Bochner and Bohr almost automorphic structures are revealed and corresponding harmonic analysis is established. In a series of papers [2,24,29,31], it is shown that numerical uniformly continuous almost automorphic functions are bounded uniformly continuous N-almost periodic, and vice versa. The Bochner almost automorphic functions on groups discussed in [30] are bounded. The impulse times, however, constitute an unbounded sequence in $\mathbb{R}$. So we plan at the same time to define almost automorphy for unbounded sequences $\{\tau_j\}_{j\in\mathbb{Z}}$ in the manners of Bochner and Bohr and Levitan, respectively, and similarly p.c.a.a. functions are also characterized in three different ways.

As investigating various almost automorphy and impulsive differential equations, many concepts are found equivalent. This makes researches on almost automorphic topics convenient. We first define Bochner and Bohr and Levitan spatial almost automorphy (Definition 3.1, 3.3 and 3.12) for unbounded sequences $\{\tau_j\}_{j\in\mathbb{Z}}$ which serve as suitable objects containing discontinuities of p.c.a.a. functions and prove that they are equivalent (Theorems 3.10 and 3.15). The idea of the proof comes from that in Veech's paper [30] with nontrivial improvements since two different groups are involved in our case. These discrete sets not only make the study of p.c.a.a. functions accessible but also provide new mathematically almost automorphic structure of physical quasicrystals. On the other hand, it is natural to impose the quasi-uniform continuity condition directly on Bochner almost automorphic functions to obtain a generalization. With a little modification, this is indeed the right way of investigation. We intend in this paper to adopt the ideas of our previous work [23] in proving the equivalence of p.c.a.p. and quasi-uniformly continuous Stepanov almost periodic functions. Our first task is to extend the equivalence of almost automorphy and N-almost periodicity to vector-valued functions so that the technique of common translation sets locating positions of variables and discontinuities works well. Then we shall make use of the method of quasi-uniform approximation in [23] to show that Bochner and Bohr and Levitan (Definition 4.2, 4.3 and 4.6) piecewise continuous almost automorphy are equivalent

(Theorem 4.8), which verifies the reasonability of various new piecewise continuous almost automorphy to some extent. In view of [23], we continue to study the connection between Stepanov functions and the others. An important generalization of Bochner almost automorphic functions in the sense of Stepanov is introduced in [5], and subsequently studied by [10,18,22]. Note that Stepanov almost periodic functions are indicated useful in impulsive differential equations [25]. Bochner proves an important theorem that Bohr almost periodic and uniformly continuous Stepanov almost periodic functions are equivalent ([7, p. 174], [19, p. 34]). We shall show in this paper the equivalence of Levitan p.c.a.a. and quasi-uniformly continuous Stepanov almost automorphic functions (Theorem 8.2).

Different characterizations of piecewise continuous almost automorphy are convenient to utilize in different situations. Favard's theorems [11,12] are important contents in the theory of almost periodic differential equations. Many works have been devoted to this direction. Imposing Favard's separation condition on a single almost periodic linear differential equation usually results almost automorphic solutions [29,31]. We shall show that the same condition for impulsive differential equations is sufficient for the existence of Bochner p.c.a.a. solutions (Theorem 9.4). Then Favard's theorem on p.c.a.p. solutions and module containment follows naturally (Theorem 9.5).

This paper is organized as follows. Section 2 introduces basics of Bochner and Bohr almost automorphic and N-almost periodic functions. Other concepts and notations shall be introduced at their first use. In Sect. 3 we characterize for discontinuities of functions the Bochner and Bohr and Levitan spacial almost automorphy, and prove two of our main results about their equivalence. In Sect. 4 we define Bochner and Bohr and Levitan p.c.a.a. functions on the basis of spacial almost automorphy. Section 5 extends the equivalence of almost automorphy and N-almost periodicity to vector-valued functions. In Sect. 6 we prove the third main result on the equivalence of Bochner and Bohr and Levitan piecewise continuous almost automorphy. In Sect. 7 we investigate necessary properties of Stepanov almost automorphic functions. In Sect. 8 we prove the forth main result on the equivalence of Levitan piecewise continuous and quasi-uniformly continuous Stepanov almost automorphy. In Sect. 9 we establish the last two main results on Favard's theorems. Some technical proofs are put in Appendix A to avoid influences on main themes.

2 Bochner and Bohr Almost Automorphy and *N*-Almost Periodicity

Our newly defined notions of various almost automorphy are, of course, based on the classical ones. We first introduce some basic properties. Let $\mathbb{G} = \mathbb{R}$ or $\mathbb{Z}$, and $(X, |\cdot|)$ be a Banach space over $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1 [21, p. 11], [30]. A function $f : \mathbb{G} \to X$ is called (Bochner) almost automorphic if given any sequence $\alpha' \subset \mathbb{G}$, there exists a subsequence $\alpha \subset \alpha'$ and a function $g \in X^{\mathbb{G}}$ such that $\mathcal{T}_\alpha f = g$ and $\mathcal{T}_{-\alpha} g = f$ pointwise on $\mathbb{G}$.

Remark 2.2 The operator $\mathcal{T}_\alpha f = g$ is adopt here to ease the notation for taking limits, which means that $g(t) = \lim_{k\to\infty} f(t+\alpha_k)$, $t \in \mathbb{G}$, $\alpha = \{\alpha_k\}_{k=1}^{\infty} \subset \mathbb{G}$, and is written only when the limit exists [14, p. 3]. The mode of convergence will be specified at each use of the symbol, e.g. uniformly on $\mathbb{R}$ and pointwise for $t \in \mathbb{R}\backslash\mathbb{Z}$. The symbol $\beta \subset \alpha$ means that $\beta = \{\beta_k\}_{k=1}^{\infty}$ is a subsequence of $\alpha = \{\alpha_k\}_{k=1}^{\infty}$, and $-\alpha$ is defined to be the sequence $\{-\alpha_k\}_{k=1}^{\infty}$. g is called a generalized translation of f.

Denote by $A(\mathbb{G}, X)$ the set of all almost automorphic functions from $\mathbb{G}$ to X, and by $AA(\mathbb{G}, X)$ the ones which are continuous on $\mathbb{G}$. Equipped with the uniform convergence norm $\|f\| = \sup_{t\in\mathbb{G}} |f(t)|$, $AA(\mathbb{G}, X)$ is a Banach space.

It is well-known that the continuity of g implies the uniform continuity of $f \in AA(\mathbb{R}, X)$ [30]. We prove the converse by uniform continuity.

Lemma 2.3 *An almost automorphic function on $\mathbb{R}$ is uniformly continuous if and only if all of its generalized translations are uniformly continuous.*

Proof Suppose that $\mathcal{T}_\alpha f = g$ and $\mathcal{T}_{-\alpha} g = f$ pointwise on $\mathbb{R}$. From the equality

$$|f(s) - f(t)| = \lim_{k\to\infty} |g(s-\alpha_k) - g(t-\alpha_k)|, \quad \forall s, t \in \mathbb{R}$$

it follows that the uniform continuity of g yields that of f.

Conversely, use

$$|g(s) - g(t)| = \lim_{k\to\infty} |f(s+\alpha_k) - f(t+\alpha_k)|, \quad \forall s, t \in \mathbb{R}.$$

□

Clearly, $AA_{uc}(\mathbb{R}, X) := AA(\mathbb{R}, X) \cap BUC(\mathbb{R}, X)$ is a Banach subspace, where $BUC(\mathbb{R}, X)$ is the space of bounded and uniformly continuous functions from $\mathbb{R}$ to X.

Bohr almost automorphy has proved to be powerful in studying Bochner almost automorphic functions. The following two definitions are elementary.

Definition 2.4 [19,25]. A set $E \subset \mathbb{G}$ is said to be relatively dense if there is a positive number $l = l(E) \in \mathbb{G}$ such that $[a, a+l] \cap E \neq \emptyset$ for all $a \in \mathbb{G}$. l is called an inclusion length for E.

Definition 2.5 A subset E of a group G is called strongly relatively dense if there exist elements $\{s_i\}_{i=1}^m \cup \{t_j\}_{j=1}^n \subset G$ such that $\cup_{i=1}^m s_i E = G = \cup_{j=1}^n E t_j$.

Definition 2.5 is the same as Definition 2.1.1 in [30] except for the name. We use the term "strongly relatively dense" to distinguish from that in [19,25]. The symbol G denotes a general group, while $\mathbb{G}$ denotes $\mathbb{R}$ or $\mathbb{Z}$. Clearly, $\mathbb{Z}$ is relatively dense, but not strongly relatively dense in $\mathbb{R}$.

Although Definitions 2.4 and 2.5 look different, they are closely related.

Lemma 2.6 *Considered in $\mathbb{Z}$, a set $E \subset \mathbb{Z}$ is relatively dense if and only if it is strongly relatively dense.*

Proof Sufficiency. Suppose that $\mathbb{Z} = \cup_{j=1}^n (s_j + E)$ and $M = \max_{1\le j\le n} |s_j|$. For every interval of length $2M$ and with midpoint $t \in \mathbb{Z}$, there exists $1 \le k \le n$ and $\tau \in E$ so that $t = s_k + \tau$. Therefore, $\tau = t - s_k \in [t-M, t+M] \cap E$. Hence $2M$ is an inclusion length for the relatively dense set E.

Necessity. Suppose that E is relatively dense and has an inclusion length l. For every $t \in \mathbb{Z}$ find a $\tau \in [t-l, t] \cap E$. Thus $t - \tau \in [0, l] \cap \mathbb{Z}$. It follows that $\mathbb{Z} = \cup_{j=0}^l (j + E)$. □

Lemma 2.7 *The following statements are true when considered in $\mathbb{R}$.*

(i) *A strongly relatively dense set $E \subset \mathbb{R}$ is relatively dense.*
(ii) *If $E \subset \mathbb{R}$ is relatively dense and $\delta > 0$, then $E + [0, \delta]$ is strongly relatively dense.*

Proof (i) Replace $\mathbb{Z}$ by $\mathbb{R}$ in the proof of the sufficiency in Lemma 2.6.

(ii) Suppose that E is relatively dense with an inclusion length l, and n is an integer satisfying $0 < l/n < \delta$. It suffices to show the subset $E + [0, l/n]$ to be strongly relatively dense. For any $t \in \mathbb{R}$, there is a $\tau \in [t - l, t] \cap E$. Consequently, there exists an integer $0 \leq k \leq n - 1$ such that $t - \tau \in [kl/n, (k + 1)l/n] \subset [0, l]$. Thus $t = kl/n + \tau + s$, where $0 \leq s \leq l/n < \delta$. Therefore, $\mathbb{R} = \cup_{j=0}^{n-1}(jl/n + E + [0, l/n])$.

□

Definition 2.8 A function $f : G \to X$ with a relatively compact range on a group G is called Bohr almost automorphic if for each finite set $E \subset G$ and prescribed $\epsilon > 0$ there is a set $B_\epsilon = B_\epsilon(E) \subset G$ such that

(i) B_ϵ is strongly relatively dense.
(ii) $B_\epsilon = B_\epsilon^{-1} := \{\tau^{-1}; \tau \in B_\epsilon\}$.
(iii) If $\tau \in B_\epsilon$, then $\max_{s,t\in E} |f(s\tau t) - f(st)| < \epsilon$.
(iv) If $\tau_1, \tau_2 \in B_\epsilon$, then $\max_{s,t\in E} |f(s\tau_1\tau_2^{-1}t) - f(st)| < 2\epsilon$.

Correspondingly, the definition of a Bochner almost automorphic function $f : G \to X$ is given by Definition 2.1 with $\mathbb{G}$ and sequences replaced by G and nets of group elements, respectively. Note that complex almost automorphic functions can be generalized naturally to functions taking values in a Banach space except for the property of being pointwise limit of a jointly almost automorphic net of almost periodic functions, which involves the Tietze extension theorem for real valued functions. The following result essentially due to Veech [30] is important.

Theorem 2.9 *A function $f : G \to X$ is Bochner almost automorphic if and only if it is Bohr almost automorphic.*

Remark 2.10 Definition 2.8 is a mimic of Definition 2.1.2 of [30]. The notion of relative denseness in [30] is replaced by the notion of strongly relative denseness here. We have also replaced the boundedness condition of f in Definition 2.1.2 of [30] by having a relatively compact range. One verifies readily that Theorem 2.9 is true with the same proof as Theorem 2.2.1 in [30] since the boundedness condition is proposed only to guarantee the relative compactness.

Levitan introduces the notion of N-almost periodicity, which is intended to be a generalization of Bohr almost periodic one. Because the translation set has a non-uniform restriction on variables, it is convenient to use N-almost periodicity to study almost automorphy.

Definition 2.11 [19, p. 53]. A function $f \in C(\mathbb{R}, X)$ is called N-almost periodic if it satisfies the following two conditions:

(i) For all $\epsilon, N > 0$ there exists a relatively dense set of ϵ, N-almost periods of f,

$$T(f, \epsilon, N) := \{\tau \in \mathbb{R}; |f(t \pm \tau) - f(t)| \leq \epsilon, |t| \leq N\}.$$

(ii) For all $\epsilon, N > 0$ there exists an $\eta = \eta(\epsilon, N) > 0$ such that

$$T(f, \eta, N) \pm T(f, \eta, N) \subset T(f, \epsilon, N).$$

Remark 2.12 The set $T(f, \eta, N)$ could be replaced by a relatively dense subset. See the footnote in [19, p. 54] and Bogolyubov's theorem in [19, p. 55]. Furthermore, such a subset could be made symmetric with respect to 0. Indeed, if $B(f, \eta, N) \subset T(f, \eta, N)$ and $B(f, \eta, N) \pm B(f, \eta, N) \subset T(f, \epsilon, N)$, then

$$B(f, \eta, N) \cup B(f, \eta, N)^{-1} \subset T(f, \eta, N)$$

$$[B(f,\eta,N)\cup B(f,\eta,N)^{-1}]\pm[B(f,\eta,N)\cup B(f,\eta,N)^{-1}]$$
$$=\pm[B(f,\eta,N)\pm B(f,\eta,N)]\subset T(f,\epsilon,N),$$

where $B(f,\eta,N)^{-1}=\{-\tau;\tau\in B(f,\eta,N)\}$.

Denote by $NAP(\mathbb{R},X)$ the set of all N-almost periodic functions from $\mathbb{R}$ to X. [31] proves the following important equality by a jointly almost automorphic net of almost periodic functions.

Theorem 2.13 [31]. $AA_{uc}(\mathbb{R},\mathbb{C})=NAP(\mathbb{R},\mathbb{C})\cap BUC(\mathbb{R},\mathbb{C})$.

Now the following basic observation is clear and shall be used later. If $f\in AA_{uc}(\mathbb{R},\mathbb{C})$ and $\delta>0$ is chosen for $\epsilon>0$ in the statement of uniform continuity, from

$$|f(t\pm(\tau+s))-f(t)|\le|f(t\pm(\tau+s))-f(t\pm\tau)|+|f(t\pm\tau)-f(t)|$$
$$<2\epsilon,\quad |t|\le N, s\in[0,\delta], \tau\in T(f,\epsilon,N)$$

it follows that $T(f,2\epsilon,N)$ is strongly relatively dense since it contains such a subset $T(f,\epsilon,N)+[0,\delta]$.

3 Equivalence of Bochner and Bohr and Levitan Spatial Almost Automorphy

In this section we characterize three new classes of spatially almost automorphic sets on the line which serve as suitable objects containing possible discontinuities of p.c.a.a. functions. The main results are about their equivalence (Theorems 3.10 and 3.15). There are two different ways in defining the discontinuities of p.c.a.p. functions. One is via equi-potentially almost periodicity [25, p. 195] (Definition 3.11) and the other one is to introduce a distance between almost periodic sets on the line [13,26]. It is easy to check that our spatially almost automorphic sets are Delone sets. So physical quasicrystals may have such a structure. For researches on quasicrystals, see e.g. [9,20] and the references therein.

To make our goal clear, let us briefly introduce the function class that solutions of impulsive differential equations belong to. A sequence $\{\tau_j\}_{j\in\mathbb{Z}}\in\mathbb{R}^{\mathbb{Z}}$ is called admissible if $\lim_{j\to\pm\infty}\tau_j=\pm\infty$ and $\tau_j<\tau_{j+1}$ for all $j\in\mathbb{Z}$. Put $\tau_j^k=\tau_{j+k}-\tau_j$ for $j,k\in\mathbb{Z}$. Let $PC(\mathbb{R},X)$ be the set of all piecewise continuous functions $h:\mathbb{R}\to X$ which have discontinuities of the first kind (both $h(t+0)$ and $h(t-0)$ exist) only at the points of a subset of an admissible sequence $\{\tau_j=\tau_j(h)\}_{j\in\mathbb{Z}}$ and are continuous from the left at $\{\tau_j\}_{j\in\mathbb{Z}}$, i.e. $\lim_{t\to\tau_j-0}h(t)=h(\tau_j)$ for all $j\in\mathbb{Z}$. Since the empty set is a subset of every admissible sequence, $PC(\mathbb{R},X)$ contains all continuous functions.

To the best of our knowledge, there are no results on the almost automorphy of unbounded sequences. In the present section we shall focus on the spatial almost automorphy of the class of admissible sequences.

3.1 Bochner and Bohr Spatial Almost Automorphy

Bochner almost automorphic functions on groups are bounded, but admissible sequences containing possible discontinuities of piecewise continuous functions are unbounded. In this subsection we first introduce the notions of Bochner and Bohr spatial almost automorphy for admissible sequences and then prove their equivalence. Levitan spatial almost automorphy and its equivalent relation with the Bohr one shall be discussed in the next subsection.

The following definition will be fully understood later when the generalized translations of a piecewise continuous function are considered.

Definition 3.1 An admissible sequence $\{\tau_j\}_{j\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ with $\inf_{j\in\mathbb{Z}} \tau_j^1 > 0$ is called Bochner spatially almost automorphic (Bochner s.a.a., for short) if for any $\alpha' \subset \mathbb{R}$, there are sequences $\alpha \subset \alpha'$, $\{m_k\}_{k=1}^{\infty} \subset \mathbb{Z}$ and $\{\tau_j^*\}_{j\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ such that for each $n \in \mathbb{Z}$,

$$\lim_{k\to\infty} |\tau_{n+m_k} + \alpha_k - \tau_n^*| = 0, \quad \lim_{k\to\infty} |\tau_{n-m_k}^* - \alpha_k - \tau_n| = 0.$$

The following lemma gives an example of spatially almost automorphic sequences.

Lemma 3.2 *Suppose that* $\xi > 0$, $\zeta \in AA(\mathbb{Z}, \mathbb{R})$ *and the sequence given by*

$$\tau_n = \xi n + \zeta(n), \quad n \in \mathbb{Z}$$

satisfies $\inf_{j\in\mathbb{Z}} \tau_j^1 > 0$. *Then* $\{\tau_j\}_{j\in\mathbb{Z}}$ *is Bochner s.a.a.*

Proof Given any $\alpha' \subset \mathbb{R}$, there are unique $m_k' \in \mathbb{Z}$ and $\vartheta_k' \in [0, \xi)$ such that

$$-\alpha_k' = \xi m_k' + \vartheta_k', \quad k \in \mathbb{Z}_+.$$

Hence there are subsequences $\alpha \subset \alpha'$, $\{m_k\} \subset \{m_k'\}$, $\{\vartheta_k\} \subset \{\vartheta_k'\}$, a sequence $\zeta^* \in \mathbb{R}^{\mathbb{Z}}$ and a number $\vartheta \in [0, \xi]$ such that

$$\lim_{k\to\infty} \zeta(\cdot + m_k) = \zeta^*(\cdot), \ \lim_{k\to\infty} \zeta^*(\cdot - m_k) = \zeta(\cdot), \ \lim_{k\to\infty} \vartheta_k = \vartheta$$

and

$$-\alpha_k = \xi m_k + \vartheta_k, \quad k \in \mathbb{Z}_+.$$

Define a sequence by

$$\tau_n^* = \xi n + \zeta^*(n) - \vartheta, \quad n \in \mathbb{Z}.$$

A direct calculation shows that for each $n \in \mathbb{Z}$,

$$\begin{aligned}
\lim_{k\to\infty} |\tau_{n+m_k} + \alpha_k - \tau_n^*| &= \lim_{k\to\infty} |\zeta(n + m_k) - \zeta^*(n) - \vartheta_k + \vartheta| = 0,\\
\lim_{k\to\infty} |\tau_{n-m_k}^* - \alpha_k - \tau_n| &= \lim_{k\to\infty} |\zeta^*(n - m_k) - \zeta(n) - \vartheta + \vartheta_k| = 0.
\end{aligned}$$

□

On the basis of Definition 2.8, we propose the following new concept.

Definition 3.3 An admissible sequence $\{\tau_j\}_{j\in\mathbb{Z}}$ satisfying

$$0 < \inf_{j\in\mathbb{Z}} \tau_j^1 \le \sup_{j\in\mathbb{Z}} \tau_j^1 < \infty \tag{1}$$

is called Bohr spatially almost automorphic (Bohr s.a.a., for short) if for any $\epsilon > 0$ and $N \in \mathbb{Z}_+$ there is a set $B_{\epsilon,N} \subset \mathbb{R}$ such that

(i) $B_{\epsilon,N}$ is strongly relatively dense.
(ii) $B_{\epsilon,N} = B_{\epsilon,N}^{-1} := \{-r; r \in B_{\epsilon,N}\}$.

(iii) If $r \in B_{\epsilon,N}$, then there exists $p \in \mathbb{Z}$ such that

$$\max_{|n|\le N} |\tau_{n+p} + r - \tau_n| < \epsilon.$$

(iv) If $r, s \in B_{\epsilon,N}$ and $p, q \in \mathbb{Z}$ satisfy

$$\max_{|n|\le N} |\tau_{n+p} + r - \tau_n| < \epsilon, \ \max_{|n|\le N} |\tau_{n+q} + s - \tau_n| < \epsilon,$$

then

$$\max_{|n|\le N} |\tau_{n+p-q} + r - s - \tau_n| < 2\epsilon.$$

Remark 3.4 Condition (1) is a little like the condition of having a relatively compact range in Definition 2.8 and guarantees the convergence and equivalent relative denseness (see the proof of Theorem 3.10). Obviously, in (iii) the p attached to r is unique if $2\epsilon < \inf_{j\in\mathbb{Z}} \tau_j^1$.

The following result corresponds the property that a Bochner almost automorphic function naturally has a relatively compact range.

Lemma 3.5 *Suppose that $\{\tau_j\}_{j\in\mathbb{Z}}$ is Bochner s.a.a., then it satisfies (1).*

Proof It suffices to show the sequence $\{\tau_j^1\}_{j\in\mathbb{Z}}$ to be relatively compact. For any sequence $\{l_k''\}_{k=1}^\infty \subset \mathbb{Z}$, let $\alpha'' = \{-\tau_{l_k''}\}_{k=1}^\infty \subset \mathbb{R}$. By Definition 3.1, there are sequences $\alpha' = \{-\tau_{l_k'}\}_{k=1}^\infty \subset \alpha''$, $\{m_k'\}_{k=1}^\infty \subset \mathbb{Z}$ and $\{\tau_j^*\}_{j\in\mathbb{Z}} \in \mathbb{R}^\mathbb{Z}$ such that for each $n \in \mathbb{Z}$,

$$\lim_{k\to\infty} |\tau_{n+m_k'} - \tau_{l_k'} - \tau_n^*| = 0.$$

Thus $\{\tau_{m_k'} - \tau_{l_k'}\}_{k=1}^\infty$ converges to τ_0^*. From the assumption $\inf_{j\in\mathbb{Z}} \tau_j^1 > 0$ it follows that the sequence of integers $\{m_k' - l_k'\}_{k=1}^\infty$ is bounded. Consequently, there are $p \in \mathbb{Z}$ and subsequences $\{l_k\}_{k=1}^\infty \subset \{l_k'\}_{k=1}^\infty$, $\{m_k\}_{k=1}^\infty \subset \{m_k'\}_{k=1}^\infty$ such that $m_k = l_k + p$ for all $k \in \mathbb{Z}_+$. Therefore,

$$\lim_{k\to\infty} |\tau_{1-p+l_k+p} - \tau_{l_k} - \tau_{1-p}^*| = \lim_{k\to\infty} |\tau_{l_k+1} - \tau_{l_k} - \tau_{1-p}^*| = 0.$$

Therefore, the sequence $\{\tau_j^1\}_{n\in\mathbb{Z}}$ is relatively compact whence bounded. □

The following four lemmas are elementary in proving the equivalence of Bochner and Bohr spatial almost automorphy.

Lemma 3.6 *Suppose that $\{\tau_j\}_{j\in\mathbb{Z}}$ is Bochner s.a.a., $\alpha \subset \mathbb{R}$, $\{m_k\}_{k=1}^\infty \subset \mathbb{Z}$, $\{\tau_j^*\}_{j\in\mathbb{Z}} \in \mathbb{R}^\mathbb{Z}$ and $n \in \mathbb{Z}$ is fixed. If*

$$\lim_{k\to\infty} |\tau_{n+m_k} + \alpha_k - \tau_n^*| = 0,$$

then already

$$\lim_{k\to\infty} |\tau_{n-m_k}^* - \alpha_k - \tau_n| = 0.$$

Proof Assume the contrary that there are $\epsilon > 0$ and two subsequences $\{\beta_j = \alpha_{k(j)}\}_{j=1}^\infty$ and $\{l_j = m_{k(j)}\}_{j=1}^\infty$ with

$$|\tau_{n-l_j}^* - \beta_j - \tau_n| > \epsilon.$$

If $\{\gamma_i = \beta_{j(i)}\}_{i=1}^{\infty}, \{p_i\}_{i=1}^{\infty} \subset \mathbb{Z}$ and $\{\tau'_j\}_{j\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ satisfy

$$\lim_{i\to\infty} |\tau_{n+p_i} + \gamma_i - \tau'_n| = 0,$$

then the sequence $\{\tau_{n+l_{j(i)}} - \tau_{n+p_i}\}_{i=1}^{\infty}$ is bounded and so will be the one $\{l_{j(i)} - p_i\}_{i=1}^{\infty}$ by (1). There are a subsequence, denoting by $\{p_i\}_{i=1}^{\infty}$ again, and an integer q such that $p_i = l_{j(i)} + q$ for each $i \in \mathbb{Z}_+$. Consequently, $\tau'_n = \tau^*_{n+q}$. However, it can never happen that

$$\lim_{i\to\infty} |\tau'_{n-p_i} - \gamma_i - \tau_n| = \lim_{i\to\infty} |\tau^*_{n-l_{j(i)}} - \gamma_i - \tau_n| = 0.$$

This contradicts Definition 3.1. □

Lemma 3.7 *Suppose that $\{\tau_j\}_{j\in\mathbb{Z}}$ is Bochner s.a.a., then for any $\epsilon > 0$ and $N \in \mathbb{Z}_+$, the set*

$$T(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N) := \left\{r \in \mathbb{R};\ \max_{|n|\le N} |\tau_{n+p} + r - \tau_n| < \epsilon \text{ for some } p \in \mathbb{Z}\right\}$$

is strongly relatively dense in $\mathbb{R}$.

Proof Assume the contrary that there are $\epsilon > 0$ and $N \in \mathbb{Z}_+$ so that the set $T(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N)$ is not strongly relatively dense. Let $r_1 \in \mathbb{R}$ be arbitrary, there would be an $r_2 \in \mathbb{R}$ with $r_2 \notin T(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N) + r_1$ by assumption. Having chosen $\{r_j\}_{j=1}^{l} \subset \mathbb{R}$ with $r_k - r_m \notin T(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N)$ for $1 \le m < k \le l$, there exists $r_{l+1} \in \mathbb{R}$ satisfying

$$r_{l+1} \notin \bigcup_{j=1}^{l} [T(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N) + r_j].$$

This produces a sequence $\{r_j\}_{j=1}^{\infty}$ with $r_k - r_l \notin T(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N)$ for $k > l$. By Definition 3.1, there are sequences $\{\alpha_k = r_{j(k)}\}_{k=1}^{\infty} \subset \{r_j\}_{j=1}^{\infty}$, $\{m_k\}_{k=1}^{\infty} \subset \mathbb{Z}$ and $\{\tau^*_j\}_{j\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ such that for each $n \in \mathbb{Z}$,

$$\lim_{k\to\infty} |\tau_{n+m_k} + \alpha_k - \tau^*_n| = 0,\ \lim_{k\to\infty} |\tau^*_{n-m_k} - \alpha_k - \tau_n| = 0.$$

Let l be large so that

$$\max_{|n|\le N} |\tau^*_{n-m_l} - \alpha_l - \tau_n| < \frac{\epsilon}{2},$$

then find $k > l$ with

$$\max_{|n|\le N+|m_l|} |\tau_{n+m_k} + \alpha_k - \tau^*_n| < \frac{\epsilon}{2}.$$

Therefore,

$$\begin{aligned}\max_{|n|\le N} |\tau_{n+m_k-m_l} + \alpha_k - \alpha_l - \tau_n| &\le \max_{|n|\le N} |\tau_{n+m_k-m_l} + \alpha_k - \tau^*_{n-m_l}| \\ &\quad + \max_{|n|\le N} |\tau^*_{n-m_l} - \alpha_l - \tau_n| < \epsilon,\end{aligned}$$

which contradicts the fact that $\alpha_k - \alpha_l \notin T(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N)$ by construction. □

Lemma 3.8 *Suppose that $\{\tau_j\}_{j\in\mathbb{Z}}$ is Bochner s.a.a., then for any $\epsilon > 0$ and $N \in \mathbb{Z}_+$, there are $\delta > 0$ and $M \ge N$ such that $r, s \in T(\{\tau_j\}_{j\in\mathbb{Z}}, \delta, M)$ with*

$$\max_{|n|\le M} |\tau_{n+p} + r - \tau_n| < \delta,\ \max_{|n|\le M} |\tau_{n+q} + s - \tau_n| < \delta$$

yield $r - s \in T(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N)$ *with*

$$\max_{|n|\le N} |\tau_{n+p-q} + r - s - \tau_n| < \epsilon.$$

Proof Assume the contrary that there are $\epsilon > 0$ and $N \in \mathbb{Z}_+$ so that given any $\delta > 0$ and $M \ge N$ there must exist $r, s \in \mathbb{R}$ and $l, m \in \mathbb{Z}$ with

$$\begin{aligned}
&\max_{|n|\le M} |\tau_{n+l} + r - \tau_n| < \delta,\\
&\max_{|n|\le M} |\tau_{n+m} + s - \tau_n| < \delta,\\
&\max_{|n|\le N} |\tau_{n+l-m} + r - s - \tau_n| > \epsilon.
\end{aligned}$$

Let $\{\delta_j\}_{j=1}^{\infty} \subset \mathbb{R}$ be a sequence decreasing to 0 and $\sum_{j=1}^{\infty} \delta_j < \infty$. Put $M_1 = N$, there would be $r_1, s_1 \in \mathbb{R}$ and $l_1, m_1 \in \mathbb{Z}$ with

$$\begin{aligned}
&\max_{|n|\le 2M_1} |\tau_{n+l_1} + r_1 - \tau_n| < \delta_1,\\
&\max_{|n|\le 2M_1} |\tau_{n+m_1} + s_1 - \tau_n| < \delta_1,\\
&\max_{|n|\le N} |\tau_{n+l_1-m_1} + r_1 - s_1 - \tau_n| > \epsilon.
\end{aligned}$$

Letting $M_2 > M_1 + \max\{|l_1|, |m_1|\}$, there are $r_2, s_2 \in \mathbb{R}$ and $l_2, m_2 \in \mathbb{Z}$ with

$$\begin{aligned}
&\max_{|n|\le 2M_2} |\tau_{n+l_2} + r_2 - \tau_n| < \delta_2,\\
&\max_{|n|\le 2M_2} |\tau_{n+m_2} + s_2 - \tau_n| < \delta_2,\\
&\max_{|n|\le N} |\tau_{n+l_2-m_2} + r_2 - s_2 - \tau_n| > \epsilon.
\end{aligned}$$

Inductively, there are sequences

$$\{M_k\}_{k=1}^{\infty} \subset \mathbb{Z}_+,\ \{r_k, s_k\}_{k=1}^{\infty} \subset \mathbb{R},\ \{l_k, m_k\}_{k=1}^{\infty} \subset \mathbb{Z}$$

such that

$$\begin{aligned}
&M_{k+1} > M_k + \max\{|l_k|, |m_k|\},\ \lim_{k\to\infty} M_k = \infty,\\
&\max_{|n|\le 2M_{k+1}} |\tau_{n+l_{k+1}} + r_{k+1} - \tau_n| < \delta_{k+1},\\
&\max_{|n|\le 2M_{k+1}} |\tau_{n+m_{k+1}} + s_{k+1} - \tau_n| < \delta_{k+1},\\
&\max_{|n|\le N} |\tau_{n+l_{k+1}-m_{k+1}} + r_{k+1} - s_{k+1} - \tau_n| > \epsilon.
\end{aligned}$$

Define two sequences $\alpha \subset \mathbb{R}$ and $\{p_k\}_{k=1}^{\infty} \subset \mathbb{Z}$ by

$$\begin{aligned}
&\alpha_1 = r_1,\ \alpha_2 = s_1,\ p_1 = l_1,\ p_2 = m_1,\\
&\alpha_{2k+1} = \sum_{j=1}^{k+1} r_j,\ \alpha_{2k+2} = \alpha_{2k-1} + s_{k+1},\\
&p_{2k+1} = \sum_{j=1}^{k+1} l_j,\ p_{2k+2} = p_{2k-1} + m_{k+1}, \quad k \in \mathbb{Z}_+.
\end{aligned}$$

A direct calculation shows that

$$\begin{aligned}
&\max_{|n|\le M_{k+1}} |(\tau_{n+p_{2k+1}} + \alpha_{2k+1}) - (\tau_{n+p_{2k+2}} + \alpha_{2k+2})| \\
&= \max_{|n|\le M_{k+1}} |(\tau_{n+l_1+l_2+\cdots+l_k+l_{k+1}} + r_{k+1}) - (\tau_{n+l_1+l_2+\cdots+l_k+m_{k+1}} + s_{k+1})| \\
&\le \max_{|n|\le M_{k+1}} |\tau_{n+l_1+l_2+\cdots+l_k+l_{k+1}} + r_{k+1} - \tau_{n+l_1+l_2+\cdots+l_k}| \\
&\quad + \max_{|n|\le M_{k+1}} |\tau_{n+l_1+l_2+\cdots+l_k+m_{k+1}} + s_{k+1} - \tau_{n+l_1+l_2+\cdots+l_k}| < 2\delta_{k+1}
\end{aligned}$$

for $k \in \mathbb{Z}_+$ and for $k > j$,

$$\begin{aligned}
&\max_{|n|\le M_{j+2}} |(\tau_{n+p_{2k+1}} + \alpha_{2k+1}) - (\tau_{n+p_{2j+1}} + \alpha_{2j+1})| \\
&= \max_{|n|\le M_{j+2}} |\tau_{n+l_1+l_2+\cdots+l_k+l_{k+1}} + r_{k+1} + \cdots + r_{j+2} - \tau_{n+l_1+l_2+\cdots+l_{j+1}}| \\
&\le \max_{|n|\le M_{j+2}} |\tau_{n+l_1+l_2+\cdots+l_k+l_{k+1}} + r_{k+1} - \tau_{n+l_1+l_2+\cdots+l_k}| + \cdots \\
&\quad + \max_{|n|\le M_{j+2}} |\tau_{n+l_1+l_2+\cdots+l_{j+2}} + r_{j+2} - \tau_{n+l_1+l_2+\cdots+l_{j+1}}| \\
&< \sum_{i=1}^{k-j} \delta_{j+1+i},
\end{aligned} \tag{2}$$

which tends to 0 as $j \to \infty$. Consequently, $\{\tau_{n+p_k} + \alpha_k\}_{k=1}^{\infty}$ is a Cauchy sequence for each $n \in \mathbb{Z}$. There is a sequence $\{\tau_j^*\}_{j\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ so that

$$\lim_{k\to\infty} |\tau_{n+p_k} + \alpha_k - \tau_n^*| = 0, \quad \forall n \in \mathbb{Z}.$$

By Lemma 3.6, already

$$\lim_{k\to\infty} |\tau^*_{n-p_k} - \alpha_k - \tau_n| = 0, \quad \forall n \in \mathbb{Z}.$$

Let j be so large that

$$\max_{|n|\le N} |\tau^*_{n-p_{2j}} - \alpha_{2j} - \tau_n| < \frac{\epsilon}{4},$$

then choose large $k > j$ with

$$\max_{|n|\le N} |\tau_{n-p_{2j}+p_{2k+1}} + \alpha_{2k+1} - \tau^*_{n-p_{2j}}| < \frac{\epsilon}{4},$$

it follows that

$$\begin{aligned}
&\max_{|n|\le N} |\tau_{n-p_{2j}+p_{2k+1}} + \alpha_{2k+1} - \alpha_{2j} - \tau_n| \\
&= \max_{|n|\le N} |\tau_{n-p_{2j}+p_{2k+1}} + r_{k+1} + \cdots + r_j - s_j - \tau_n| < \frac{\epsilon}{2}.
\end{aligned}$$

By (2) and $N + |p_{2j}| < M_{j+1}$,

$$\begin{aligned}
&\max_{|n|\le N} |\tau_{n-p_{2j}+p_{2k+1}} + r_{k+1} + \cdots + r_{j+1} - \tau_{n-p_{2j}+p_{2j-1}}| \\
&= \max_{|n|\le N} |\tau_{n-p_{2j}+p_{2k+1}} + r_{k+1} + \cdots + r_{j+1} - \tau_{n+l_j-m_j}|
\end{aligned}$$

$$< \sum_{i=0}^{k-j} \delta_{j+1+i} < \frac{\epsilon}{2}$$

for large j. Therefore,

$$\max_{|n|\le N} |\tau_{n+l_j-m_j} + r_j - s_j - \tau_n| < \epsilon,$$

which contradicts the construction. □

Lemma 3.9 *Suppose that $\{\tau_j\}_{j\in\mathbb{Z}}$ is Bochner s.a.a., then for any $\epsilon > 0$ and $N \in \mathbb{Z}_+$, there are $\delta > 0$ and $M \ge N$ such that $\{r_i\}_{i=1}^{\nu} \subset T(\{\tau_j\}_{j\in\mathbb{Z}}, \delta, M)$ with*

$$\max_{|n|\le M} |\tau_{n+p_i} + r_i - \tau_n| < \delta, \quad i = 1, \dots, \nu$$

yield $\sum_{i=1}^{\nu} \omega_i r_i \in T(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N)$ with

$$\max_{|n|\le N} \left|\tau_{n+\sum_{i=1}^{\nu}\omega_i p_i} + \sum_{i=1}^{\nu} \omega_i r_i - \tau_n\right| < \epsilon$$

for any $\{\omega_i\}_{i=1}^{\nu} \subset \{-1, 0, 1\}$.

Proof We prove it by induction. If $\nu = 1$, use Lemma 3.8 and the fact that $0 \in T(\{\tau_j\}_{j\in\mathbb{Z}}, \delta, M)$ with

$$\max_{|n|\le M} |\tau_{n+0} + 0 - \tau_n| < \delta.$$

Suppose the conclusion holds for some $\nu \in \mathbb{Z}_+$. By Lemma 3.8, there are $0 < \delta_2 < \delta_1 < \epsilon =: \delta_0$ and $M_2 \ge M_1 \ge N =: M_0$ such that $r, s \in T(\{\tau_j\}_{j\in\mathbb{Z}}, \delta_i, M_i)$, $i = 1, 2$, with

$$\max_{|n|\le M_i} |\tau_{n+p} + r - \tau_n| < \delta_i, \ \max_{|n|\le M_i} |\tau_{n+q} + s - \tau_n| < \delta_i$$

yield $r - s \in T(\{\tau_j\}_{j\in\mathbb{Z}}, \delta_{i-1}, M_{i-1})$ with

$$\max_{|n|\le M_{i-1}} |\tau_{n+p-q} + r - s - \tau_n| < \delta_{i-1}.$$

Using induction assumption there are $\delta > 0$ and $M \ge M_2$ such that $\{r_i\}_{i=1}^{\nu} \subset T(\{\tau_j\}_{j\in\mathbb{Z}}, \delta, M)$ with

$$\max_{|n|\le M} |\tau_{n+p_i} + r_i - \tau_n| < \delta, \quad i = 1, \dots, \nu$$

yield $\sum_{i=1}^{\nu} \omega_i r_i \in T(\{\tau_j\}_{j\in\mathbb{Z}}, \delta_2, M_2)$ with

$$\max_{|n|\le M_2} \left|\tau_{n+\sum_{i=1}^{\nu}\omega_i p_i} + \sum_{i=1}^{\nu} \omega_i r_i - \tau_n\right| < \delta_2$$

for any $\{\omega_i\}_{i=1}^{\nu} \subset \{-1, 0, 1\}$. Let $\{r_i\}_{i=1}^{\nu+1} \subset T(\{\tau_j\}_{j\in\mathbb{Z}}, \delta, M)$ with

$$\max_{|n|\le M} |\tau_{n+p_i} + r_i - \tau_n| < \delta, \quad i = 1, \dots, \nu + 1.$$

Then both $\sum_{i=1}^{\nu} \omega_i r_i$ and $\omega_{\nu+1} r_{\nu+1} \in T(\{\tau_j\}_{j\in\mathbb{Z}}, \delta_2, M_2)$ with respectively

$$\max_{|n|\le M_2} \left|\tau_{n+\sum_{i=1}^{\nu}\omega_i p_i} + \sum_{i=1}^{\nu} \omega_i r_i - \tau_n\right| < \delta_2, \ \max_{|n|\le M_2} |\tau_{n+\omega_{\nu+1}p_{\nu+1}} + \omega_{\nu+1} r_{\nu+1} - \tau_n| < \delta_2.$$

From the choice of (δ_2, M_2) and (δ_1, M_1) it follows that

$$-\sum_{i=1}^{\nu} \omega_i r_i \in T(\{\tau_j\}_{j\in\mathbb{Z}}, \delta_1, M_1),\ \omega_{\nu+1} r_{\nu+1} \in T(\{\tau_j\}_{j\in\mathbb{Z}}, \delta_1, M_1)$$

with respectively

$$\max_{|n|\le M_2} \left|\tau_{n-\sum_{i=1}^{\nu}\omega_i p_i} - \sum_{i=1}^{\nu} \omega_i r_i - \tau_n\right| < \delta_1,\ \max_{|n|\le M_2} |\tau_{n+\omega_{\nu+1} p_{\nu+1}} + \omega_{\nu+1} r_{\nu+1} - \tau_n| < \delta_1,$$

and $\omega_{\nu+1} r_{\nu+1} - (-\sum_{i=1}^{\nu}\omega_i r_i) \in T(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N)$ with

$$\max_{|n|\le N} \left|\tau_{n-(-\sum_{i=1}^{\nu}\omega_i p_i)+\omega_{\nu+1} p_{\nu+1}} - \left(-\sum_{i=1}^{\nu}\omega_i r_i\right) + \omega_{\nu+1} r_{\nu+1} - \tau_n\right| < \epsilon.$$

□

The following is our first main result.

Theorem 3.10 *A sequence $\{\tau_j\}_{j\in\mathbb{Z}}$ is Bohr s.a.a. if and only if it is Bochner s.a.a.*

Proof Suppose that $\{\tau_j\}_{j\in\mathbb{Z}}$ is Bohr s.a.a. By (1), the sequence $\{\tau_j^1\}_{j\in\mathbb{Z}}$ is bounded and there are unique $m_k' \in \mathbb{Z}$ and ϑ_k' such that

$$\alpha_k' = -\tau_{m_k'} + \vartheta_k',\ \vartheta_k' \in [-\tau_{m_k'}, -\tau_{m_k'-1}),\quad k \in \mathbb{Z}_+.$$

Consequently, there are subsequences $\alpha \subset \alpha'$, $\{m_k\} \subset \{m_k'\}$, $\{\vartheta_k\} \subset \{\vartheta_k'\}$, a sequence u and a number $\vartheta \in [0, \sup_{j\in\mathbb{Z}} \tau_j^1]$ such that $\lim_{k\to\infty} \vartheta_k = \vartheta$ and

$$\lim_{k\to\infty} \tau^1_{n+m_k} = u(n),\quad \forall n \in \mathbb{Z}$$

and

$$\alpha_k = -\tau_{m_k} + \vartheta_k,\ \vartheta_k \in [-\tau_{m_k}, -\tau_{m_k-1}),\quad k \in \mathbb{Z}_+.$$

Define an admissible sequence by

$$\tau_n^* = \begin{cases} \vartheta + \sum_{j=0}^{n-1} u(j), & n \ge 1; \\ \vartheta, & n = 0; \\ \vartheta - \sum_{j=n}^{-1} u(j), & n \le -1. \end{cases}$$

It is easy to check that

$$\tau_{n+m_k} - \tau_{m_k} + \vartheta_k = \begin{cases} \vartheta_k + \sum_{j=0}^{n-1} \tau^1_{j+m_k}, & n \ge 1; \\ \vartheta_k, & n = 0; \\ \vartheta_k - \sum_{j=n}^{-1} \tau^1_{j+m_k}, & n \le -1. \end{cases}$$

Therefore,

$$\lim_{k\to\infty} |\tau_{n+m_k} + \alpha_k - \tau_n^*| = \lim_{k\to\infty} |\tau_{n+m_k} - \tau_{m_k} + \vartheta_k - \tau_n^*| = 0,\ \forall n \in \mathbb{Z}. \tag{3}$$

Let $N \in \mathbb{Z}_+$ be fixed and $\epsilon > 0$ be arbitrary. There is a set $B_{\epsilon,N} \subset \mathbb{R}^d$ satisfying (i)–(iv) of Definition 3.3. Since $B_{\epsilon,N}$ is strongly relatively dense, the sequence α could be written as

$$\alpha_k = r_k + s_{j(k)},\quad r_k \in B_{\epsilon,N}, s_{j(k)} \in \mathbb{R}, k \in \mathbb{Z}_+,$$

where $j(\cdot)$ maps $\mathbb{Z}_+$ to a finite index set. By passing to subsequences if necessary, we may assume that $s_{j(k)} = s_{j_0}$ is independent of k. Consequently, there is a sequence $\{l_k\}_{k=1}^{\infty} \subset \mathbb{Z}$ such that

$$\max_{|n|\le N} |\tau_{n+l_k} + r_k - \tau_n| < \epsilon, \tag{4}$$

$$\max_{|n|\le N} |\tau_{n+l_k-l_j} + r_k - r_j - \tau_n| < 2\epsilon, \quad j, k \in \mathbb{Z}_+. \tag{5}$$

By (3) and (4), for each $|n| \le N$ the sequence $\{\tau_{n+l_k} - \tau_{n+m_k}\}_{k=1}^{\infty}$ is bounded, so is the sequence of integers $\{l_k - m_k\}_{k=1}^{\infty}$ by (1). Therefore, by passing to subsequences if necessary, we may assume that $l_k - m_k$ is a constant integer $p \in \mathbb{Z}$ for all $k \in \mathbb{Z}_+$. Consequently, using (3) and (5), letting j be fixed then k be large in the refined sequence $\{m_k\}_{k=1}^{\infty}$,

$$\begin{aligned}
\max_{|n|\le N} |\tau^*_{n-m_j} - \alpha_j - \tau_n| &\le \max_{|n|\le N} |\tau_{n-m_j+m_k} + \alpha_k - \alpha_j - \tau_n| \\
&\quad + \max_{|n|\le N} |\tau_{n-m_j+m_k} + \alpha_k - \tau^*_{n-m_j}| \\
&= \max_{|n|\le N} |\tau_{n-l_j+l_k} + r_k - r_j - \tau_n| \\
&\quad + \max_{|n|\le N} |\tau_{n-m_j+m_k} + \alpha_k - \tau^*_{n-m_j}| < 3\epsilon.
\end{aligned}$$

Because ϵ is arbitrarily small, τ_n is a limit point of the sequence $\{\tau^*_{n-m_k} - \alpha_k\}_{k=1}^{\infty}$. Hence letting N be free diagonal process produces a subsequence, denoting by $\{\tau^*_{n-m_k} - \alpha_k\}_{k=1}^{\infty}$ again, converging to τ_n for each $n \in \mathbb{Z}$. Thus

$$\lim_{k\to\infty} |\tau^*_{n-m_k} - \alpha_k - \tau_n| = 0, \quad n \in \mathbb{Z}.$$

Conversely, let $\{\tau_j\}_{j\in\mathbb{Z}}$ be Bochner s.a.a. and $\epsilon > 0$, $N \in \mathbb{Z}_+$ be given. Find $\delta > 0$ and $M \ge N$ so that Lemma 3.9 holds for $\nu = 2$. Put

$$B_{\epsilon,N} = T(\{\tau_j\}_{j\in\mathbb{Z}}, \delta, M) \cup T(\{\tau_j\}_{j\in\mathbb{Z}}, \delta, M)^{-1}. \tag{6}$$

Then $B_{\epsilon,N}$ is strongly relatively dense by Lemma 3.7 and $B_{\epsilon,N} = B_{\epsilon,N}^{-1}$ by definition. Obviously, $0 \in T(\{\tau_j\}_{j\in\mathbb{Z}}, \delta, M)$ with

$$\max_{|n|\le M} |\tau_{n+0} + 0 - \tau_n| < \delta.$$

For any $r \in B_{\epsilon,N}$, either $r \in T(\{\tau_j\}_{j\in\mathbb{Z}}, \delta, M)$ or $-r \in T(\{\tau_j\}_{j\in\mathbb{Z}}, \delta, M)$. If $l \in \mathbb{Z}$ fulfills

$$\max_{|n|\le M} |\tau_{n+l} + r - \tau_n| < \delta \text{ or } \max_{|n|\le M} |\tau_{n+l} - r - \tau_n| < \delta,$$

then $r - 0 \in T(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N)$ or $0 - (-r) \in T(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N)$ with respectively

$$\max_{|n|\le N} |\tau_{n+l} + r - \tau_n| < \epsilon \text{ or } \max_{|n|\le N} |\tau_{n-l} + r - \tau_n| < \epsilon.$$

If $r, s \in B_{\epsilon,N}$ with $l, m \in \mathbb{Z}$ such that

$$\max_{|n|\le M} |\tau_{n+l} + r - \tau_n| < \delta, \ \max_{|n|\le M} |\tau_{n+m} + s - \tau_n| < \delta,$$

then by Lemma 3.9 $r - s \in T(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N)$ with

$$\max_{|n|\le N} |\tau_{n+l-m} + r - s - \tau_n| < \epsilon.$$

In view of Lemma 3.5, $\{\tau_j^1\}_{j\in\mathbb{Z}}$ is bounded. Thus it is Bohr s.a.a. □

3.2 Bohr and Levitan Spatial Almost Automorphy

In this subsection, we introduce the notion of Levitan spatial almost automorphy and reveal its equivalence with the Bohr one.

Our definition is a nontrivial improvement of notions of equi-potentially and N-almost periodicity. In the study of p.c.a.p. solutions to impulsive differential equations, requirements on the discontinuities of functions are as follows.

Definition 3.11 [25, p. 195]. Given an admissible sequence $\{\tau_j\}_{j\in\mathbb{Z}}$, the family of derived sequences

$$\{\{\tau_j^k\}\} := \{\{\tau_j^k\}_{j\in\mathbb{Z}}\}_{k\in\mathbb{Z}}$$

is called equi-potentially almost periodic, if for each $\epsilon > 0$ the common ϵ-translation set of all the sequences $\{\{\tau_j^k\}\}$,

$$T(\{\{\tau_j^k\}\}, \epsilon) = \left\{p \in \mathbb{Z};\, |\tau_{j+p}^k - \tau_j^k| < \epsilon \text{ for all } j, k \in \mathbb{Z}\right\}$$

is relatively dense.

In view of Definitions 2.11 and 3.11 and Theorem 2.13, we propose

Definition 3.12 An admissble sequence $\{\tau_j\}_{j\in\mathbb{Z}}$ satisfying (1) shall be called Levitan spatially almost automorphic (Levitan s.a.a., for short), if the family of derived sequences $\{\{\tau_j^k\}\}$ is equi-potentially almost automorphic (e.p.a.a., for short), that is, it satisfies the following two conditions:

(i) For any ϵ, $N > 0$ the common translation set of a finite number of sequences of the family $\{\{\tau_j^k\}\}$,

$$T(\{\{\tau_j^k\}\}, \epsilon, N) := \left\{p \in \mathbb{Z};\, |\tau_{j\pm p}^k - \tau_j^k| < \epsilon \text{ for all } |j|, |j+k| \le N\right\}$$

is relatively dense.

(ii) For any ϵ, $N > 0$, there are an $\eta > 0$ and a relatively dense subset $B(\{\{\tau_j^k\}\}, \eta, N) \subset T(\{\{\tau_j^k\}\}, \eta, N)$ such that

$$B(\{\{\tau_j^k\}\}, \eta, N) = -B(\{\{\tau_j^k\}\}, \eta, N),$$
$$B(\{\{\tau_j^k\}\}, \eta, N) \pm B(\{\{\tau_j^k\}\}, \eta, N) \subset T(\{\{\tau_j^k\}\}, \epsilon, N)$$

and

$$|\tau_0^p \pm \tau_0^q - \tau_0^{p\pm q}| < \epsilon$$

for all $p, q \in B(\{\{\tau_j^k\}\}, \eta, N)$.

Remark 3.13 Condition (ii) corresponds to (iv) of Definition 3.3 and Lemma 3.8. They are all essentially requirements on the pairs $(r, p) \in \mathbb{R} \times \mathbb{Z}$.

Note that the two sets $T(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N)$ and $T(\{\{\tau_j^k\}\}, \epsilon, N)$ consists respectively of real and integer numbers. The following lemma relates together the strongly relative denseness of a set in $\mathbb{R}$ and the relative denseness of a set in $\mathbb{Z}$.

Lemma 3.14 *Suppose that* $\{\tau_j\}_{j\in\mathbb{Z}}$ *is Bohr s.a.a., then for any* $\epsilon > 0$ *and* $N \in \mathbb{Z}_+$*, the set*

$$P(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N) := \left\{p \in \mathbb{Z};\ \max_{|n|\le N} |\tau_{n+p} + r - \tau_n| < \epsilon \text{ for some } r \in \mathbb{R}\right\}$$

is relatively dense in $\mathbb{Z}$.

Proof By Lemma 3.7 and Lemma 2.6, the set $T(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N)$ is relatively dense in $\mathbb{R}$. The inequalities

$$|\tau_{0+p} + r - \tau_0| < \epsilon, \quad r \in T(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N)$$

yield the relative denseness of the set $\{\tau_p;\ p \in P(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N)\}$ in $\mathbb{R}$. Arrange the integers in $P(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N)$ as an increasing sequence $\{p_k\}_{k\in\mathbb{Z}}$. By (1), $\lim_{k\to\pm\infty} p_k = \pm\infty$ and $\{p_{k+1} - p_k\}_{k\in\mathbb{Z}}$ is bounded since $\{\tau_{p_{k+1}} - \tau_{p_k}\}_{k\in\mathbb{Z}}$ is. Thus $P(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N)$ is relatively dense in $\mathbb{Z}$. □

The following is our second main result.

Theorem 3.15 *A sequence* $\{\tau_j\}_{j\in\mathbb{Z}}$ *is Bohr s.a.a. if and only if it is Levitan s.a.a.*

Proof Firstly, (1) is already fulfilled.

Suppose that $\{\tau_j\}_{j\in\mathbb{Z}}$ is Bohr s.a.a., then it is Bochner s.a.a. by Theorem 3.10. For any small $\epsilon > 0$ and $N \in \mathbb{Z}_+$ there is a set $B_{\epsilon,N} \subset \mathbb{R}$ given by (6) satisfying (i)–(iv) of Definition 3.3. By (i) of Lemma 2.6, $B_{\epsilon,N}$ is relatively dense. Put

$$B^*_{\epsilon,N} = \{p \in \mathbb{Z};\ \max_{|n|\le N} |\tau_{n+p} + r - \tau_n| < \epsilon \text{ for some } r \in B_{\epsilon,N}\}.$$

Because ϵ is small, the integer p attached to r is unique. Furthermore, the inequalities connecting r and p imply the relative denseness of the set $\{\tau_p;\ p \in B^*_{\epsilon,N}\}$. (6) and the proof of Lemma 3.14 imply that $B^*_{\epsilon,N}$ is relatively dense. If $r, s \in B_{\epsilon,N}$ with $p, q \in B^*_{\epsilon,N}$ such that

$$\max_{|n|\le N} |\tau_{n+p} + r - \tau_n| < \epsilon,\ \max_{|n|\le N} |\tau_{n+q} + s - \tau_n| < \epsilon,$$

the constructed (6) implies $-r \in B_{\epsilon,N}$ and already

$$\max_{|n|\le N} |\tau_{n\pm p} \pm r - \tau_n| = \max_{|n|\le N} |\tau_n^{\pm p} \pm r| < \epsilon.$$

Hence $B^*_{\epsilon,N} = -B^*_{\epsilon,N}$ and using (iii) and (iv) of Definition 3.3,

$$|\tau_0^p \pm \tau_0^q - \tau_0^{p\pm q}| < 4\epsilon.$$

A straightforward computation shows that

$$\begin{aligned} |\tau^k_{j\pm p} - \tau^k_j| &= |(\tau_{j+k\pm p} - \tau_{j\pm p}) - (\tau_{j+k} - \tau_j)| = |\tau^{\pm p}_{j+k} - \tau^{\pm p}_j| \\ &\le |\tau^{\pm p}_{j+k} \pm r| + |\tau^{\pm p}_j \pm r| < 2\epsilon, \end{aligned}$$

for all $|j|, |j+k| \le N$. Thus the relatively dense set $B^*_{\epsilon,N}$ is contained in $T(\{\{\tau^k_j\}\}, 2\epsilon, N)$. By (iv) of Definition 3.3 and the same calculation as above,

$$B^*_{\epsilon,N} \pm B^*_{\epsilon,N} \subset T(\{\{\tau^k_j\}\}, 4\epsilon, N).$$

Summing up, $\{\tau_j\}_{j\in\mathbb{Z}}$ is Levitan s.a.a.

Conversely, suppose that $\{\tau_j\}_{j\in\mathbb{Z}}$ is Levitan s.a.a., then for any small $\epsilon > 0$, $N \in \mathbb{Z}_+$, there are an $\eta \in (0, \epsilon)$ and a symmetric relatively dense subset $B(\{\{\tau_j^k\}\}, \eta, N) \subset T(\{\{\tau_j^k\}\}, \eta, N)$ satisfying (ii) of Definition 3.12. Define

$$B_{3\epsilon,N} = \{\pm\tau_0^p;\, p \in B(\{\{\tau_j^k\}\}, \eta, N)\} + (-\eta, \eta).$$

Clearly, $B_{3\epsilon,N} = -B_{3\epsilon,N}$. By $\tau_0^p = \tau_p - \tau_0$, (1) and (ii) of Lemma 2.6, the set $B_{3\epsilon,N}$ is strongly relatively dense. Let $p, q \in B(\{\{\tau_j^k\}\}, \eta, N)$ and $r = -\tau_0^p$, $s = -\tau_0^q$ and δ, $\delta' \in (-\eta, \eta)$. From Definition 3.12 it follows that

$$\begin{aligned}
|\tau_0^p + \tau_0^{-p} - 0| &= |\tau_0^{-p} - r| < \epsilon,\\
|\tau_0^p \pm \tau_0^q - \tau_0^{p\pm q}| &= |r \pm s + \tau_0^{p\pm q}| < \epsilon,\\
|\tau_0^{-p-q} - r - s| &\le |\tau_0^{-p-q} - \tau_0^{-p} - \tau_0^{-q}|\\
&\quad + |\tau_0^{-p} + \tau_0^{-q} - r - s| < 3\epsilon.
\end{aligned}$$

Consequently, using the definition of $T(\{\{\tau_j^k\}\}, \eta, N)$,

$$\begin{aligned}
\max_{|n|\le N} |\tau_{n+p} + r + \delta - \tau_n| &\le \max_{|n|\le N} |(\tau_{n+p} - \tau_n) - (\tau_p - \tau_0)| + \eta\\
&= \max_{|n|\le N} |\tau_{0+p}^n - \tau_0^n| + \eta < 2\eta < 3\epsilon,\\
\max_{|n|\le N} |\tau_{n-p} - r - \delta - \tau_n| &\le \max_{|n|\le N} |(\tau_{n-p} - \tau_n) - (\tau_{-p} - \tau_0)| + |\tau_0^{-p} - r| + \eta\\
&< \max_{|n|\le N} |\tau_{0-p}^n - \tau_0^n| + 2\epsilon < 3\epsilon,
\end{aligned}$$

which implies (iii) of Definition 3.3. Moreover,

$$\begin{aligned}
\max_{|n|\le N} |\tau_{n+p-q} + (r+\delta) - (s+\delta') - \tau_n| &\le \max_{|n|\le N} |(\tau_{n+p-q} - \tau_n) - (\tau_{p-q} - \tau_0)|\\
&\quad + |\tau_0^{p-q} + r - s| + 2\eta\\
&< \max_{|n|\le N} |\tau_{0+p-q}^n - \tau_0^n| + 3\epsilon < 6\epsilon,\\
\max_{|n|\le N} |\tau_{n\pm(p+q)} \pm [(r+\delta) + (s+\delta')] - \tau_n| &\le \max_{|n|\le N} |(\tau_{n\pm(p+q)} - \tau_n) - (\tau_{\pm(p+q)} - \tau_0)|\\
&\quad + |\tau_0^{\pm(p+q)} \pm (r+s)| + 2\eta\\
&< \max_{|n|\le N} |\tau_{0\pm(p+q)}^n - \tau_0^n| + 5\epsilon < 6\epsilon,
\end{aligned}$$

which yields (iv) of Definition 3.3. Summing up, $\{\tau_j\}_{j\in\mathbb{Z}}$ is Bohr s.a.a. □

4 Piecewise Continuous Almost Automorphy

On the basis of spatially almost automorphic sequences in $\mathbb{R}^{\mathbb{Z}}$, we are able to propose the new classes of Bochner and Bohr and Levitan p.c.a.a. functions and to state the third main result on equivalence (Theorem 4.8). These functions are natural generalizations of classical almost automorphic functions in the study of almost periodic impulsive differential equations and shall be shown important via establishing Favard's theorems.

The concept of quasi-uniform continuity plays an important role in approximating piecewise continuous functions and turns out to be a basic property.

Definition 4.1 [23]. A function $f \in PC(\mathbb{R}, X)$ which has discontinuities at the points of a subset of an admissible sequence $\{\tau_j\}_{j\in\mathbb{Z}}$ is said to be quasi-uniformly continuous if for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $|f(s) - f(t)| < \epsilon$ whenever $s, t \in (\tau_j, \tau_{j+1}]$ for some $j \in \mathbb{Z}$ and $|s - t| < \delta$.

Definition 4.2 A function $f \in PC(\mathbb{R}, X)$ is called Bochner piecewise continuous almost automorphic (Bochner p.c.a.a., for short) if the following conditions hold:

(i) f has possible discontinuities at the points of a subset of a Bochner s.a.a. sequence $\{\tau_j\}_{j\in\mathbb{Z}}$.
(ii) f is quasi-uniformly continuous.
(iii) Given any sequence $\alpha' \subset \mathbb{R}$, there are a subsequence $\alpha \subset \alpha'$ and a function $g \in PC(\mathbb{R}, X)$ which has possible discontinuities at the points of an admissible sequence $\{\tau_j^*\}_{j\in\mathbb{Z}}$ given by Definition 3.1 for $-\alpha$, such that $\mathcal{T}_\alpha f = g$ pointwise on $\mathbb{R}\backslash\{\tau_j^*\}_{j\in\mathbb{Z}}$ and $\mathcal{T}_{-\alpha} g = f$ pointwise on $\mathbb{R}\backslash\{\tau_j\}_{j\in\mathbb{Z}}$.

The reasonability of (iii) above shall be verified by Theorem 6.5 later. $\{f(\cdot + \alpha_k)\}_{k\in\mathbb{Z}_+}$ may diverge at the points of $\{\tau_j^*\}_{j\in\mathbb{Z}}$ because of possible discontinuities of f at the points of $\{\tau_j\}_{j\in\mathbb{Z}}$ (Remark 6.8). The class of Bochner p.c.a.a. functions are convenient to establish Favard's theorems.

Definition 4.3 A function $f \in PC(\mathbb{R}, X)$ with a relatively compact range is called Bohr piecewise continuous almost automorphic (Bohr p.c.a.a., for short) if the following conditions hold:

(i) f has possible discontinuities at the points of a subset of a Bohr s.a.a. sequence $\{\tau_j\}_{j\in\mathbb{Z}}$.
(ii) f is quasi-uniformly continuous.

and for any $\epsilon > 0$ and finite set $E \subset \mathbb{R}\backslash\{\tau_j\}_{j\in\mathbb{Z}}$, there is a set $B_\epsilon = B_\epsilon(E) \subset \mathbb{R}$ such that

(iii) B_ϵ is strongly relatively dense.
(iv) $B_\epsilon = B_\epsilon^{-1} := \{-\tau; \tau \in B_\epsilon\}$.
(v) If $r \in B_\epsilon$, then $\max_{t\in E} |f(t + r) - f(t)| < \epsilon$.
(vi) If $r, s \in B_\epsilon$, then $\max_{t\in E} |f(t + r - s) - f(t)| < 2\epsilon$.

Since we do not require the convergence on $\{\tau_j\}_{j\in\mathbb{Z}}$ and have imposed an additional condition on the finite set E, Definitions 4.2 and 4.3 are clearly weaker than Definitions 2.1 and 2.8, respectively.

Our notion of Levitan piecewise continuous almost automorphy arises with improvements from the concepts of p.c.a.p. and N-ρ-a.p.p.c. Levitan functions in impulsive differential equations.

Definition 4.4 [25, p. 201]. A function $f \in PC(\mathbb{R}, X)$ is called piecewise continuous almost periodic (p.c.a.p.) if the following conditions hold:

(i) f has possible discontinuities at the points of a subset of an admissible sequence $\{\tau_j\}_{j\in\mathbb{Z}}$ which has an equi-potentially almost periodic family (Definition 3.11) of derived sequences $\{\{\tau_j^k\}\}$.
(ii) f is quasi-uniformly continuous.
(iii) For each $\epsilon > 0$, the ϵ-translation set of f,

$$\check{T}(f, \epsilon) := \{\tau \in \mathbb{R}; |f(t + \tau) - h(t)| < \epsilon \text{ for all } t \in \mathbb{R} \\ \text{such that } |t - \tau_j| > \epsilon, j \in \mathbb{Z}\}$$

is relatively dense.

Definition 4.5 [27]. A function $f \in PC(\mathbb{R}, \mathbb{R})$ with discontinuities of the first kind on an almost periodic discrete set D [13,25,26] is called an N-ρ-a.p.p.c. Levitan function if the following conditions hold:

(i) $\forall \epsilon, N > 0$ there exists a relatively dense set of ϵ-N-almost periods

$$\Omega_{\epsilon,N} := \{\tau \in \mathbb{R}; |f(t \pm \tau) - f(t)| \leq \epsilon,\ \forall t \in \big(\mathbb{R} \backslash F_\epsilon(s(D))\big) \cap [-N, N]\},$$

where $s(D)$ is the set obtained from arranging members of D in a strictly increasing sequence, and $F_\epsilon(s(D))$ is a closed ϵ-neighbourhood of the set $s(D)$.

(ii) $\forall \epsilon, N > 0, \exists \eta(\epsilon, N) > 0$: $\Omega_{\eta,N} \pm \Omega_{\eta,N} \subset \Omega_{\epsilon,N}$.

Our new concept is formulated as follows.

Definition 4.6 A function $f \in PC(\mathbb{R}, X)$ with a relatively compact range is called Levitan piecewise continuous almost automorphic (Levitan p.c.a.a., for short) if the following conditions hold:

(i) f has possible discontinuities at the points of a subset of a Levitan s.a.a. sequence $\{\tau_j\}_{j\in\mathbb{Z}}$.

(ii) f is quasi-uniformly continuous.

(iii) For any $\epsilon, N > 0$, the ϵ, N-translation set of f,

$$\begin{aligned}\check{T}(f, \epsilon, N) := \{\tau \in \mathbb{R}; |f(t \pm \tau) - f(t)| < \epsilon \text{ for all } |t| \leq N \\ \text{such that } |t - \tau_j| > \epsilon, j \in \mathbb{Z}\}\end{aligned}$$

is relatively dense.

(iv) For any $\epsilon, N > 0$, there are an $\eta > 0$ and a relatively dense subset $B(f, \eta, N) \subset \check{T}(f, \eta, N)$ such that

$$B(f, \eta, N) = -B(f, \eta, N),$$
$$B(f, \eta, N) \pm B(f, \eta, N) \subset \check{T}(f, \epsilon, N).$$

Remark 4.7 Note that the symbol $\check{T}(f, \epsilon, N)$ for the ϵ, N-translation set of a Levitan p.c.a.a. function is different from that of a continuous one. If the symmetry of $B(f, \eta, N)$ is not assumed, it can also be obtained as we do in Remark 2.12. Clearly, our definition generalizes the possible discontinuities on almost periodic sets of N-ρ-a.p.p.c. Levitan functions to the ones with some kind of almost automorphy.

Denote by $PCAA(\mathbb{R}, X)$ and $PCAA_B(\mathbb{R}, X)$ and $PCAA_L(\mathbb{R}, X)$ the sets of Bochner and Bohr and Levitan p.c.a.a. functions, respectively.

The following is the third main result in this paper. Its proof is put in Sect. 6.

Theorem 4.8 $PCAA(\mathbb{R}, X) = PCAA_B(\mathbb{R}, X) = PCAA_L(\mathbb{R}, X)$.

5 Equivalence of Bochner Almost Automorphy and *N*-Almost Periodicity

As mentioned before Definition 2.11, it is sometimes convenient to use N-almost periodicity to study almost automorphy. For later use, our goal in this section is to extend Theorem 2.13 to vector-valued functions.

Bohr almost automorphy and N-almost periodicity look similar. In [30], it is not clear what the relationship between the two classes is. We shall analyze basic definitions and show that they are equivalent under suitable conditions. If G is commutative, Definition 2.8 has a simpler form.

Lemma 5.1 *Suppose that G is an abelian group, then a function $f : G \to X$ with a relatively compact range is Bohr almost automorphic if and only if for any finite set $E \subset G$ and $\epsilon > 0$ there is a set $B_\epsilon = B_\epsilon(E) \subset G$ such that*

(i) *B_ϵ is strongly relatively dense.*
(ii) *$B_\epsilon = B_\epsilon^{-1}$.*
(iii) *If $\tau \in B_\epsilon$, then $\max_{t\in E} |f(t+\tau) - f(t)| < \epsilon$.*
(iv) *If $\tau_1, \tau_2 \in B_\epsilon$, then $\max_{t\in E} |f(t+\tau_1-\tau_2) - f(t)| < 2\epsilon$.*

Proof Let a finite $E \subset G$ and $\epsilon > 0$ be given.

Suppose (i)–(iv) in Definition 2.8 for $E \cup \{0\}$ and ϵ. Then (i)–(iv) in Lemma 5.1 follows by setting $s = 0$.

Suppose (i)–(iv) in Lemma 5.1 for $E' = E + E$ and ϵ. Then (i)–(iv) in Lemma 5.1 follows by setting $t' = s + t$. □

Lemma 5.2 *Suppose that $f \in BUC(\mathbb{R}, X)$ has a relatively compact range, then it is Bohr almost automorphic if and only if for any compact set $K \subset \mathbb{R}$ and $\epsilon > 0$ there is a set $B_\epsilon = B_\epsilon(K) \subset \mathbb{R}$ such that*

(i) *B_ϵ is strongly relatively dense.*
(ii) *$B_\epsilon = B_\epsilon^{-1}$.*
(iii) *If $\tau \in B_\epsilon$, then $\max_{t\in K} |f(t+\tau) - f(t)| < \epsilon$.*
(iv) *If $\tau_1, \tau_2 \in B_\epsilon$, then $\max_{t\in K} |f(t+\tau_1-\tau_2) - f(t)| < 2\epsilon$.*

Proof It suffices to prove the necessity. One verifies readily that results in [30] extends naturally to vector-valued functions if none of particular properties of real valued functions are concerned. By Corollary 2.1.2' in [30, p. 742], if $\epsilon > 0$ is given, then for any integer $n > 0$ there exists a compact set $K' \supset K$ and a $\delta > 0$ such that if $\{\tau_j\}_{j=1}^n \subset C_\delta(K')$ and if $\{\omega_j\}_{j=1}^n \subset \{0, \pm 1\}$, then $\sum_{j=1}^n \omega_j \tau_j \in C_\epsilon(K)$, where

$$C_\epsilon(K) := \left\{\tau \in \mathbb{R};\, \max_{t\in K} |f(t+\tau) - f(t)| < \epsilon\right\}.$$

Define $B_\epsilon(K) = C_\delta(K') \cup C_\delta(K')^{-1}$, which yields directly $B_\epsilon = B_\epsilon^{-1}$. Because $C_\delta(K')$ is strongly relatively dense, so is B_ϵ. (iii) and (iv) follows from the relation between $C_\delta(K')$ and $C_\epsilon(K)$ with $n = 2$. □

The following is a vector-valued version of Theorem 2.13, crucial in characterizing almost automorphy by N-almost periodicity.

Theorem 5.3 *$AA_{uc}(\mathbb{R}, X) = NAP(\mathbb{R}, X) \cap KUC(\mathbb{R}, X)$, where $KUC(\mathbb{R}, X)$ denotes the set of uniformly continuous functions with a relatively compact range.*

Proof Suppose that $f \in AA_{uc}(\mathbb{R}, X)$, then f has a relatively compact range and Theorem 2.9 yields (i)–(iv) of Lemma 5.2 for any $\epsilon > 0$ and compact interval $[-N, N]$. By (i) of Lemma 2.6, B_ϵ is relatively dense. (ii) and (iii) of Lemma 5.2 imply $B_\epsilon \subset T(f, \epsilon, N)$. So $T(f, \epsilon, N)$ is relatively dense. (iv) of Lemma 5.2 yields $B_\epsilon \pm B_\epsilon \subset T(f, 2\epsilon, N)$. In view of Remark 2.12, $f \in NAP(\mathbb{R}, X) \cap KUC(\mathbb{R}, X)$.

Conversely, suppose that $f \in NAP(\mathbb{R}, X) \cap KUC(\mathbb{R}, X)$, then (i) and (ii) of Definition 2.11 hold for any $\epsilon, N > 0$ and a suitable $0 < \eta < \epsilon$. By uniform continuity, find a $\delta > 0$ such that $|f(s) - f(t)| < \eta/2$ for all $|s - t| \le \delta$. Therefore,

$$T(f, \eta/2, N) + [-\delta, \delta] \subset T(f, \eta, N).$$

Put $B_\epsilon([-N, N]) = T(f, \eta, N)$. Then B_ϵ is strongly relatively dense since it contains such a subset by (ii) of Lemma 2.6. By definition of $T(f, \eta, N)$ and $\eta < \epsilon$, $B_\epsilon = B_\epsilon^{-1}$ and it satisfies (iii) of Lemma 5.2. (ii) of Definition 2.11 yields (iv) of Lemma 5.2. Because ϵ and N are arbitrary, $f \in AA_{uc}(\mathbb{R}, X)$ by Lemma 5.2 and Theorem 2.9. □

6 Equivalence of Bochner and Bohr and Levitan Piecewise Continuous Almost Automorphy

In this section, we prove the third main *Theorem* 4.8. We first introduce the method of quasi-uniform approximation in the study of piecewise continuous functions.

Lemma 6.1 *Suppose that $h \in PC(\mathbb{R}, X)$ is quasi-uniformly continuous with possible discontinuities at the points of a subset of an admissible sequence $\{\tau_j\}_{j\in\mathbb{Z}}$ satisfying $\theta = \inf_{j\in\mathbb{Z}} \tau_j^1 > 0$. Then given any $\epsilon > 0$, there is $\delta \in (0, \min\{\epsilon, \theta/2\})$ such that the function defined by*

$$h_\sigma(t) := \frac{1}{\sigma}\int_0^\sigma h(t+s)ds, \quad 0 < \sigma < \delta \tag{7}$$

satisfies

$$|h_\sigma(t) - h(t)| < \epsilon, \quad \forall t \in \mathbb{R}, |t - \tau_j| > \epsilon, j \in \mathbb{Z}.$$

Remark 6.2 [23] proves Lemma 6.1 with $\{\tau_j\}_{j\in\mathbb{Z}}$ being a Wexler sequence, which is admissible, has an equi-potentially almost periodic derived family and satisfies $\inf_{j\in\mathbb{Z}} \tau_j^1 > 0$. If $\{\tau_j\}_{j\in\mathbb{Z}}$ is only admissible and $\inf_{j\in\mathbb{Z}} \tau_j^1 > 0$, the proof is exactly the same. See Lemma 3.5 in [23].

The following theorem provides a basic tool in locating positions of variables and discontinuities. It indicates that different almost automorphic objects have a relatively dense common translation set. Its technical proof, however, is a little deviate from our main topics here and put in Appendix A. Given any $\lambda, \epsilon, N > 0$, $f \in AA_{uc}(\mathbb{R}, X)$ and Bochner s.a.a. sequence $\{\tau_j\}_{j\in\mathbb{Z}}$, define

$$\begin{aligned} T^\lambda(f, \epsilon, N) &:= T(f, \epsilon, N) \cap (\lambda\mathbb{Z}) \\ &= \{m\lambda; m \in \mathbb{Z}, |f(t \pm m\lambda) - f(t)| < \epsilon \text{ for all } |t| \le N\}, \\ T_-^\lambda(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N) &: = -T(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N) \cap (\lambda\mathbb{Z}) \\ &= -\Big\{m\lambda; m \in \mathbb{Z}, \max_{|n|\le N} |\tau_{n+p} + m\lambda - \tau_n| < \epsilon \text{ for some } p \in \mathbb{Z}\Big\}. \end{aligned}$$

Theorem 6.3 *Suppose that $f \in AA_{uc}(\mathbb{R}, X)$, $\{\tau_j\}_{j\in\mathbb{Z}}$ is a Bochner s.a.a. sequence. Then for any $\epsilon_1, \epsilon_2, N_1, N_2 > 0$, there are $\eta \in (0, \epsilon_1)$, $\delta \in (0, \epsilon_2)$ so that for any $\lambda \in (0, \min\{\eta, \delta\})$, both the sets $T^\lambda(f, \epsilon_1, N_1) \cap T_-^\lambda(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon_2, N_2)$ and*

$P^\lambda(f, \{\tau_j\}_{j\in\mathbb{Z}}; \epsilon_1, \epsilon_2, N_1, N_2) := \{p \in \mathbb{Z};$ *there exists $m \in \mathbb{Z}$ such that*

$$- m\lambda \in T^{\lambda}(f, \epsilon_1, N_1) \text{ and } \max_{|n|\le N_2} |\tau_{n+p} + m\lambda - \tau_n| < \epsilon_2\}$$

are relatively dense.

Let $KPUCA(\mathbb{R}, X)$ be the set of all functions $h \in PC(\mathbb{R}, X)$ which have a relatively compact range and are quasi-uniformly continuous with possible discontinuities at the points of a subset of a Levitan s.a.a. sequence. Note that Bochner and Bohr and Levitan s.a.a. sequences are equivalent by Theorems 3.10 and 3.15.

Theorem 6.4 (Quasi-uniform approximation). *Suppose that $h \in KPUCA(\mathbb{R}, X)$ with possible discontinuities at the points of a subset of a Levitan s.a.a. sequence $\{\tau_j\}_{j\in\mathbb{Z}}$. If for each $\epsilon > 0$ there exists an $f_\epsilon \in AA_{uc}(\mathbb{R}, X)$ such that $|f_\epsilon(t) - h(t)| < \epsilon$ for all $t \in \mathbb{R}$, $|t - \tau_j| > \epsilon$, $j \in \mathbb{Z}$, then $h \in PCAA_L(\mathbb{R}, X)$.*

Proof It suffices to prove that h satisfies (iii) and (iv) of Definition 4.6. Let $\theta = \inf_{j\in\mathbb{Z}} \tau_j^1$, $\epsilon \in (0, \theta/6)$, and $N_1, N_2 > 0$ with $N_2 = 1 + \max\{|j|; |\tau_j| \le N_1\}$ be given. Find an $\eta \in (0, \epsilon)$ with

$$T(f_\epsilon, \eta, N_1) \pm T(f_\epsilon, \eta, N_1) \subset T(f_\epsilon, \epsilon, N_1).$$

Choose a pair (δ, M) according to Lemma 3.9 for (ϵ, N_2) and $\nu = 2$ so that

$$\max_{|n|\le M} |\tau_{n+p} - r - \tau_n| < \delta, \ \max_{|n|\le M} |\tau_{n+q} - s - \tau_n| < \delta$$

yield

$$\max_{|n|\le N_2} |\tau_{n\pm p} \mp r - \tau_n| < \epsilon \ \max_{|n|\le N_2} |\tau_{n\pm(p\pm q)} \mp (r \pm s) - \tau_n| < \epsilon. \tag{8}$$

By Theorem 6.3, for sufficiently small λ, both $T^{\lambda}(f_\epsilon, \eta, N_1) \cap T_-^{\lambda}(\{\tau_j\}_{j\in\mathbb{Z}}, \delta, M)$ and $P^{\lambda}(f_\epsilon, \{\tau_j\}_{j\in\mathbb{Z}}; \eta, \delta, N_1, M)$ are relatively dense, so will be the symmetric set

$$\begin{aligned} B(h, \eta, N_1) &:= [T^{\lambda}(f_\epsilon, \eta, N_1) \cap T_-^{\lambda}(\{\tau_j\}_{j\in\mathbb{Z}}, \delta, M)] \\ &\quad \cup [-T^{\lambda}(f_\epsilon, \eta, N_1) \cap T_-^{\lambda}(\{\tau_j\}_{j\in\mathbb{Z}}, \delta, M)]. \end{aligned}$$

If $r, s \in B(h, \eta, N_1)$, then

$$|f_\epsilon(t \pm r) - f_\epsilon(t)| \le \eta, \ |f_\epsilon(t \pm (r \pm s)) - f_\epsilon(t)| \le \epsilon, \quad \forall |t| \le N_1$$

and there are $p, q \in \mathbb{Z}$ satisfying (8). Let $|t| \le N_1$ and $\tau_k + 3\epsilon < t < \tau_{k+1} - 3\epsilon$ for some $k \in \mathbb{Z}$, then $|k|, |k+1| \le N_2$. It follows that

$$|\tau_j^{\pm p} \mp r| < \epsilon, \ |\tau_j^{\pm(p\pm q)} \mp (r \pm s)| < \epsilon$$

for $|j| \le N_2$. Hence

$$\begin{aligned} &\tau_{k\pm p} - \tau_k - \epsilon < \pm r < \tau_{k+1\pm p} - \tau_{k+1} + \epsilon, \\ &\tau_{k\pm p} + 2\epsilon < t \pm r < \tau_{k+1\pm p} - 2\epsilon, \\ &\tau_{k\pm(p\pm q)} - \tau_k - \epsilon < \pm(r \pm s) < \tau_{k+1\pm(p\pm q)} - \tau_{k+1} + \epsilon, \\ &\tau_{k\pm(p\pm q)} + 2\epsilon < t \pm (r \pm s) < \tau_{k+1\pm(p\pm q)} - 2\epsilon. \end{aligned}$$

Therefore, $|t - \tau_j| > 3\epsilon > \eta$, $|t \pm r - \tau_j| > 2\epsilon > \eta$ and $|t \pm (r \pm s) - \tau_j| > 2\epsilon > \eta$ for all $j \in \mathbb{Z}$. Consequently,

$$|h(t \pm r) - h(t)| \le |h(t \pm r) - f_\epsilon(t \pm r)| + |f_\epsilon(t \pm r) - f_\epsilon(t)|$$

$$
\begin{aligned}
&+|f_\epsilon(t)-h(t)| < \epsilon+\eta+\epsilon < 3\epsilon,\\
|h(t\pm(r\pm s))-h(t)| &\le |h(t\pm(r\pm s))-f_\epsilon(t\pm(r\pm s))|+|f_\epsilon(t\pm(r\pm s))-f_\epsilon(t)|\\
&+|f_\epsilon(t)-h(t)| < 3\epsilon
\end{aligned}
$$

for all $|t| \le N_1$. Thus the relatively dense set $B(h, \eta, N_1)$ fufills

$$
\begin{aligned}
B(h,\eta,N_1) &= -B(h,\eta,N_1) \subset \check{T}(h, 3\epsilon, N_1),\\
B(h,\eta,N_1) &\pm B(h,\eta,N_1) \subset \check{T}(h, 3\epsilon, N_1).
\end{aligned}
$$

□

The following result is elementary in understanding Bochner piecewise continuous almost automorphic functions. Moreover, it is also a completeness theorem when combined with the later Lemma 9.2 for functions of which possible discontinuities are contained in a Wexler sequences.

Theorem 6.5 *Suppose that $h \in KPUCA(\mathbb{R}, X)$ with possible discontinuities at the points of a subset of a Bochner s.a.a. sequence $\{\tau_j\}_{j\in\mathbb{Z}}$, then for any $\alpha' \subset \mathbb{R}$, there are a subsequence $\alpha \subset \alpha'$ and a function $h^* \in PC(\mathbb{R}, X)$ such that*

(i) $\mathcal{T}_\alpha h = h^*$ pointwise on $\mathbb{R}\backslash\{\tau_j^*\}_{j\in\mathbb{Z}}$, where $\{\tau_j^*\}_{j\in\mathbb{Z}}$ is an admissible sequence with $\inf_{j\in\mathbb{Z}}(\tau_{j+1}^*-\tau_j^*) > 0$ containing possible discontinuities of h^* and given by Definition 3.1 for $-\alpha$.
(ii) h^* is quasi-uniformly continuous and has a relatively compact range.
(iii) The values of h^* on $\mathbb{R}\backslash\{\tau_j^*\}_{j\in\mathbb{Z}}$ depend only on the values of h on $\mathbb{R}\backslash\{\tau_j\}_{j\in\mathbb{Z}}$.

Proof The proof is divided into four steps.

1. We seek for the limits. Since h has a relatively compact range and is bounded and integrable, Tychnoff product and Lebesgue dominated convergence theorems yield the existence of a subsequence $\alpha \subset \alpha'$ and an integrable function h^* with a relatively compact range such that $\mathcal{T}_\alpha h = h^*$ pointwise on $\mathbb{R}$. From Definition 3.1 for $-\alpha$ and by passing to subsequence if necessary, we may assume that

$$
\lim_{k\to\infty} |\tau_{n+m_k}-\alpha_k-\tau_n^*| = 0, \quad \lim_{k\to\infty} |\tau_{n-m_k}^*+\alpha_k-\tau_n| = 0, \quad \forall n \in \mathbb{Z},
$$

for some sequences $\{m_k\}_{k=1}^\infty \subset \mathbb{Z}$ and $\{\tau_j^*\}_{j\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$. Clearly,

$$
\inf_{j\in\mathbb{Z}}(\tau_{j+1}^*-\tau_j^*) \ge \inf_{j\in\mathbb{Z}}\inf_{n\in\mathbb{Z}} \tau_n^1 > 0.
$$

2. We prove that h^* is uniformly continuous on the set

$$
\{t \in \mathbb{R};\, |t-\tau_j^*| > \eta, \quad \forall j \in \mathbb{Z}\}
$$

for each $\eta > 0$. Given $\eta > 0$, let $\delta > 0$ be chosen for h and $\epsilon > 0$ in the statement of quasi-uniform continuity and $s, t \in (\tau_n^*+\eta, \tau_{n+1}^*-\eta)$ for some $n \in \mathbb{Z}$, $|s-t| < \delta$. It follows that

$$
s+\alpha_k, t+\alpha_k \in (\tau_n^*+\eta+\alpha_k, \tau_{n+1}^*-\eta+\alpha_k) \subset (\tau_{n+m_k}, \tau_{n+1+m_k}) \tag{9}
$$

for large k. Therefore, $|h(s+\alpha_k)-h(t+\alpha_k)| < \epsilon$ and

$$
\begin{aligned}
|h^*(s)-h^*(t)| &\le |h^*(s)-h(s+\alpha_k)|+|h(s+\alpha_k)-h(t+\alpha_k)|\\
&+|h(t+\alpha_k)-h^*(t)| < 2\epsilon
\end{aligned}
$$

using large k.

3. We prove the property that for any $\epsilon > 0$ there exists $\delta > 0$ such that $|h^*(s) - h^*(t)| < \epsilon$ whenever $s, t \in (\tau_j^*, \tau_{j+1}^*)$ for some $j \in \mathbb{Z}$ and $|s - t| < \delta$. Assume the contrary that there is an $\epsilon_0 > 0$ so that for any $\delta > 0$ there are $s, t \in (\tau_j^*, \tau_{j+1}^*)$ for some $j \in \mathbb{Z}$, $|s - t| < \delta$ with $|h^*(s) - h^*(t)| > \epsilon_0$. Let $\delta_0 \in (0, \inf_{j \in \mathbb{Z}} \tau_j^1)$ be chosen for h and $\epsilon_0/3$ in the statement of quasi-uniform continuity. By Step 2, there are a sequence $\{\delta_l\}_{l=1}^{\infty} \subset (0, \delta_0)$ decreasing to 0, three sequences $\{n_l\}_{l=1}^{\infty} \subset \mathbb{Z}$, $\{s_l\}_{l=1}^{\infty}$, $\{t_l\}_{l=1}^{\infty}$ such that either

$$\tau_{n_l}^* < s_l, t_l < \tau_{n_l}^* + \delta_l, \ |h^*(s_l) - h^*(t_l)| \geq \epsilon_0, \quad l \in \mathbb{Z}_+$$

or

$$\tau_{n_l}^* - \delta_l < s_l, t_l < \tau_{n_l}^*, \ |h^*(s_l) - h^*(t_l)| \geq \epsilon_0, \quad l \in \mathbb{Z}_+.$$

We only prove the first case. The proof of the other one is similar. Let $l \in \mathbb{Z}_+$ be fixed, it follows that

$$s_l + \alpha_k, t_l + \alpha_k \in [\tau_{n_l}^* + (s_l - \tau_{n_l}^*) + \alpha_k, \tau_{n_l}^* + \delta_0 + \alpha_k) \subset (\tau_{n_l + m_k}, \tau_{n_l + 1 + m_k})$$

for large k. Therefore, $|h(s_l + \alpha_k) - h(t_l + \alpha_k)| < \epsilon_0/3$ and by using large k,

$$\begin{aligned} |h^*(s_l) - h^*(t_l)| &\leq |h^*(s_l) - h(s_l + \alpha_k)| + |h(s_l + \alpha_k) - h(t_l + \alpha_k)| \\ &\quad + |h(t_l + \alpha_k) - h^*(t_l)| < \epsilon_0, \end{aligned}$$

which is a contradiction.

4. The property in Step 3 implies the existence of lateral limits $\lim_{t \to \tau_j^* \pm} h^*(t)$ for $j \in \mathbb{Z}$. Change the values of h^* at $\{\tau_j^*\}_{j \in \mathbb{Z}}$ so that h^* is continuous from left. There results $h^* \in PC(\mathbb{R}, X)$ satisfying (i)–(iii). □

On the basis of Theorem 6.5, the following result verifies our original idea.

Lemma 6.6 *If $f \in A(\mathbb{R}, X) \cap PC(\mathbb{R}, X)$ is quasi-uniformly continuous with possible discontinuities at the points of a subset of a Bochner s.a.a. sequence $\{\tau_j\}_{j \in \mathbb{Z}}$, then it is Bochner p.c.a.a.*

Proof By Definition 2.1, suppose that $\mathcal{T}_\alpha f = g$ and $\mathcal{T}_{-\alpha} g = f$ pointwise on $\mathbb{R}$ for some $\alpha \subset \mathbb{R}$ and $g \in X^{\mathbb{R}}$. From Theorem 6.5 it follows that g has possible discontinuities at the points of a subset of an admissible sequence $\{\tau_j^*\}_{j \in \mathbb{Z}}$ given by Definition 3.1 for $-\alpha$ by passing to subsequences if necessary. With a modification of the values on $\{\tau_j^*\}_{j \in \mathbb{Z}}$, g could be in $PC(\mathbb{R}, X)$. Moreover, the values of f on $\mathbb{R} \backslash \{\tau_j\}_{j \in \mathbb{Z}}$ will not be influenced by an argument similar to that of (iii) of Theorem 6.5. Thus (iii) of Definition 4.2 is true for $(f, \{\tau_j\}_{j \in \mathbb{Z}}, \alpha, g, \{\tau_j^*\}_{j \in \mathbb{Z}})$. □

The proof of Theorem 4.8 is divided into the following five lemmas.

Lemma 6.7 *Functions in $PCAA(\mathbb{R}, X)$ have a relatively compact range.*

Proof Suppose that $f \in PCAA(\mathbb{R}, X)$ with possible discontinuities at the points of a subset of a Bochner s.a.a. sequence $\{\tau_j\}_{j \in \mathbb{Z}}$. For any $\alpha' \subset \mathbb{R}$, let $\alpha \subset \alpha'$, $g \in PC(\mathbb{R}, X)$ and $\{\tau_j^*\}_{j \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ satisfy (iii) of Definition 4.2. If $0 \notin \{\tau_j^*\}_{j \in \mathbb{Z}}$, then $\lim_{k \to \infty} f(\alpha_k) = g(0)$. Otherwise, without loss of generality we may assume that $\tau_0^* = 0$ and

$$\lim_{k \to \infty} |\tau_{n+m_k} - \alpha_k - \tau_n^*| = 0, \ \lim_{k \to \infty} |\tau_{n-m_k}^* + \alpha_k - \tau_n| = 0, \quad n \in \mathbb{Z}$$

for some sequence $\{m_k\}_{k=1}^{\infty} \subset \mathbb{Z}$. Thus $\lim_{k\to\infty} |\tau_{m_k} - \alpha_k| = 0$ and at least one of the two sets $\{\alpha_k; \alpha_k \le \tau_{m_k}\}$ and $\{\alpha_k; \alpha_k > \tau_{m_k}\}$ has infinitely many elements. By passing to subsequences if necessary we may assume that $\alpha_k \le \tau_{m_k}$ for all $k \in \mathbb{Z}_+$. The proof of the other case is similar and so we omit it. For any $\epsilon > 0$ let $\delta \in (0, \inf_{j\in\mathbb{Z}} \tau_j^1)$ be chosen for f and ϵ in the statement of quasi-uniform continuity and fix a $t \in (-\delta/2, 0)$. It follows that

$$\tau_{m_k} - \delta < t + \alpha_k < \alpha_k \le \tau_{m_k}, \ |f(t+\alpha_k) - f(\alpha_k)| < \epsilon$$

for large k. Since $\lim_{k\to\infty} f(t+\alpha_k) = g(t)$, the sequence $\{f(t+\alpha_k)\}_{k\in\mathbb{Z}_+}$ has a finite ϵ-net. Hence $\{f(\alpha_k)\}_{k\in\mathbb{Z}_+}$ has a finite 2ϵ-net. Because ϵ is arbitrary, $\{f(\alpha_k)\}_{k\in\mathbb{Z}_+}$ is totally bounded and contains a convergent subsequence. □

Remark 6.8 The above proof indicates $\{f(\alpha_k + \tau_n^*)\}_{k\in\mathbb{Z}_+}$ may have two limit points for each $n \in \mathbb{Z}$ due to possible discontinuities of f at $\{\tau_j\}_{j\in\mathbb{Z}}$.

The following lemma extends a basic integration technique in [3, p. 80] to Bochner p.c.a.a. functions.

Lemma 6.9 *Suppose that $f \in PCAA(\mathbb{R}, X)$, then for each $\sigma > 0$, the function f_σ defined by (7) belongs to $AA_{uc}(\mathbb{R}, X)$.*

Proof Suppose that $f \in PCAA(\mathbb{R}, X)$ with possible discontinuities at the points of a subset of a Bochner s.a.a. sequence $\{\tau_j\}_{j\in\mathbb{Z}}, \alpha \subset \mathbb{R}, g \in PC(\mathbb{R}, X)$ and $\{\tau_j^*\}_{j\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ satisfy (iii) of Definition 4.2. By Lemma 6.7 and Definition 4.2, f is bounded and locally integrable. Hence g is locally integrable by Lebesgue's dominated convergence theorem. Define a continuous function by

$$g_\sigma(t) = \frac{1}{\sigma}\int_0^\sigma g(t+s)ds, \quad t \in \mathbb{R}.$$

Again by using Lebesgue's dominated convergence theorem as $k \to \infty$,

$$|f_\sigma(t+\alpha_k) - g_\sigma(t)| \le \frac{1}{\sigma}\int_0^\sigma |f(t+\alpha_k+s) - g(t+s)|ds \to 0,$$
$$|g_\sigma(t-\alpha_k) - f_\sigma(t)| \le \frac{1}{\sigma}\int_0^\sigma |g(t-\alpha_k+s) - f(t+s)|ds \to 0.$$

In view of $g_\sigma \in C(\mathbb{R}, X)$, $f_\sigma \in AA_{uc}(\mathbb{R}, X)$ by Lemma 2.3. □

Lemma 6.10 $PCAA(\mathbb{R}, X) \subset PCAA_L(\mathbb{R}, X)$.

Proof Lemmas 6.1 and 6.9 imply that the quasi-uniform approximation Theorem 6.4 holds for functions in $PCAA(\mathbb{R}, X)$, whence the inclusion holds. □

Lemma 6.11 $PCAA_B(\mathbb{R}, X) \supset PCAA_L(\mathbb{R}, X)$.

Proof Let h be in $PCAA_L(\mathbb{R}, X)$ and $\{\tau_j\}_{j\in\mathbb{Z}}$ be the Levitan s.a.a. sequence containing possible discontinuities of h. It suffices to verify (iii)–(vi) of Definition 4.3 for h. Given any finite set $E \subset \mathbb{R}\backslash\{\tau_j\}_{j\in\mathbb{Z}}$, find $\epsilon, N > 0$ with

$$\epsilon < \min_{j\in\mathbb{Z}, t\in E} |t - \tau_j|, \ E \subset (-N+\epsilon, N-\epsilon). \tag{10}$$

By (iv) of Definition 4.6 there are an $\eta, 0 < \eta < \epsilon/2$, and a relatively dense set $B(h, \eta, N) \subset \check{T}(h, \eta, N)$ such that

$$B(h, \eta, N) = -B(h, \eta, N),$$

$$B(h, \eta, N) \pm B(h, \eta, N) \subset \check{T}(h, \epsilon/2, N).$$

Next we construct a set from $B(h, \eta, N)$ which fulfills all the requirement. Because h is quasi-uniformly continuous, there exists a δ, $0 < \delta < \epsilon/2$ such that $|h(s) - h(t)| < \epsilon/2$ whenever $s, t \in (\tau_j, \tau_{j+1}]$ for some $j \in \mathbb{Z}$ and $|s - t| < \delta$. If $r \in \check{T}(h, \epsilon/2, N)$, $s \in (r - \delta, r + \delta)$ and $|t| \leq N - \delta$, $|t - \tau_j| > \epsilon$, $j \in \mathbb{Z}$, a direct calculation shows that $|s - r| < \delta$, $|t \pm (s - r)| \leq N$, and

$$|t \pm (s - r) - \tau_j| \geq |t - \tau_j| - |s - r| > \epsilon - \delta > \frac{\epsilon}{2}$$

for all $j \in \mathbb{Z}$. Therefore, using $r \in \check{T}(h, \epsilon/2, N)$ and quasi-uniform continuity,

$$\begin{aligned} |h(t \pm s) - h(t)| &\leq |h(t \pm (s - r) \pm r) - h(t \pm (s - r))| \\ &\quad + |h(t \pm (s - r)) - h(t)| < \epsilon. \end{aligned}$$

Consequently, using $\eta < \epsilon/2$ and monotonicity,

$$\check{T}(h, \eta, N) + (-\delta, \delta) \subset \check{T}(h, \epsilon/2, N) + (-\delta, \delta) \subset \check{T}(h, \epsilon, N - \delta).$$

Define

$$B_\epsilon = B_\epsilon(E) := B(h, \eta, N) + \Big(-\frac{\delta}{2}, \frac{\delta}{2}\Big),$$

then

$$B_\epsilon \subset \check{T}(h, \eta, N) + \Big(-\frac{\delta}{2}, \frac{\delta}{2}\Big) \subset \check{T}(h, \epsilon, N - \delta).$$

(ii) of Lemma 2.6 implies that the set B_ϵ is strongly relatively dense and by definition $B_\epsilon = B_\epsilon^{-1}$. Properties of $B(h, \eta, N)$ imply

$$B_\epsilon \pm B_\epsilon \subset \check{T}(h, \epsilon/2, N) + (-\delta, \delta) \subset \check{T}(h, \epsilon, N - \delta).$$

At last, (10) implies

$$\begin{aligned} &\max_{t \in E} |h(t + r) - h(t)| < \epsilon, \\ &\max_{t \in E} |h(t + r - s) - h(t)| < \epsilon \end{aligned}$$

for all $r, s \in B_\epsilon$. □

Lemma 6.12 $PCAA_B(\mathbb{R}, X) \subset PCAA(\mathbb{R}, X)$.

Proof We make use of the method in proving the sufficiency of Theorem 2.2.1 in [30]. Let $f \in PCAA_B(\mathbb{R}, X)$ with possible discontinuities at the points of a subset of a Bohr s.a.a. sequence $\{\tau_j\}_{j \in \mathbb{Z}}$, which is already Bochner s.a.a., and $\alpha' \subset \mathbb{R}$. By definition f has a relatively compact range, which implies that there are a subsequence $\alpha \subset \alpha'$ and two functions g, $h \in X^{\mathbb{R}}$ satisfying $\mathcal{T}_\alpha f = g$ and $\mathcal{T}_{-\alpha} g = h$ pointwise on $\mathbb{R}$. Given arbitrary $t \in \mathbb{R} \backslash \{\tau_j\}_{j \in \mathbb{Z}}$ and $\epsilon > 0$, find a set $B_\epsilon = B_\epsilon(\{t\})$ satisfying (iii)–(vi) of Definition 4.3. Because B_ϵ is strongly relatively dense, there exsits $\{s_j\}_{j=1}^m \subset \mathbb{R}$ such that $\mathbb{R} = \cup_{j=1}^n (s_j + B_\epsilon)$. For each $k \in \mathbb{Z}_+$ we may write $\alpha_k = r_k + s_j$, where $r_k \in B_\epsilon$ and $j = j(k)$. There are but finitely many s_j, so there is a subsequence $\beta \subset \alpha$ such that $\beta_k = r_k' + s_{j_0}$, where $r_k' \in B_\epsilon$ and j_0 is independent of k. Obviously, $\mathcal{T}_{-\beta} \mathcal{T}_\beta f = h$ pointwise on $\mathbb{R}$. Let k then j be chosen so large that

$$|f(t - \beta_k + \beta_j) - h(t)| < \epsilon.$$

By $\beta_j - \beta_k = r'_j - r'_k$ and (vi) of Definition 4.3,

$$|f(t - \beta_k + \beta_j) - f(t)| < 2\epsilon,$$

which yields $|h(t) - f(t)| < 3\epsilon$. Since ϵ then $t \in \mathbb{R}\backslash\{\tau_j\}_{j\in\mathbb{Z}}$ are arbitrary, $f = h$ on $\mathbb{R}\backslash\{\tau_j\}_{j\in\mathbb{Z}}$. At last, by Theorem 6.5, g has possible discontinuities at the points of a subset of an admissible sequence $\{\tau^*_j\}_{j\in\mathbb{Z}}$ given by Definition 3.1 for $-\alpha$ by passing to subsequences if necessary and with a modification of its values on $\{\tau^*_j\}_{j\in\mathbb{Z}}$ it could be in $PC(\mathbb{R}, X)$. Moreover, the values of h on $\mathbb{R}\backslash\{\tau_j\}_{j\in\mathbb{Z}}$ will not be influenced by an argument similar to that of (iii) of Theorem 6.5. Thus (iii) of Definition 4.2 is true for $(f, \{\tau_j\}_{j\in\mathbb{Z}}, \alpha, g, \{\tau^*_j\}_{j\in\mathbb{Z}})$. □

7 Stepanov Almost Automorphy

In this section, we mainly reduce Stepanov almost automorphic functions to vector-valued Bochner almost automorphic ones (Lemma 7.4) so that Theorem 5.3 is applicable for the next section.

For any p, $1 \le p < \infty$, consider function spaces $L^p_{loc}(\mathbb{R}, X)$, $Y = L^p([0, 1], X)$ and $C(\mathbb{R}, Y)$. For every $f \in L^p_{loc}(\mathbb{R}, X)$, put

$$\tilde{f}(t)(s) = f(t + s), \quad a.e.\ s \in [0, 1], \forall t \in \mathbb{R}.$$

$\tilde{f}$ is called the Bochner transform of f. It is easy to see that $\tilde{f} \in C(\mathbb{R}, Y)$ and

$$\tilde{f}(t)(s) = \tilde{f}(\tau)(t - \tau + s) \tag{11}$$

for $a.e.\ s \in [0, 1] \cap [\tau - t, \tau - t + 1]$ and all $t \in \mathbb{R}$. If $\tilde{f} \in AA(\mathbb{R}, Y)$, for any sequence $\alpha' \subset \mathbb{R}$, there would exist a subsequence $\alpha \subset \alpha'$ and a measurable function $h \in Y^{\mathbb{R}}$ such that $\mathcal{T}_\alpha \tilde{f} = h$ and $\mathcal{T}_{-\alpha} h = \tilde{f}$ pointwise, i.e.,

$$\lim_{k\to\infty} \|\tilde{f}(t + \alpha_k) - h(t)\|^p_Y = \lim_{k\to\infty} \int_0^1 |f(t + \alpha_k + s) - h(t)(s)|^p ds = 0,$$

$$\lim_{k\to\infty} \|h(t - \alpha_k) - \tilde{f}(t)\|^p_Y = \lim_{k\to\infty} \int_0^1 |h(t - \alpha_k)(s) - f(t + s)|^p ds = 0$$

for all $t \in \mathbb{R}$. In general, we do not know whether h is the Bochner transform of a function in $L^p_{loc}(\mathbb{R}, X)$ or not. To clarify this basic fact, define independently

Definition 7.1 A function $f \in L^p_{loc}(\mathbb{R}, X)$, $p \ge 1$, is called S^p-uniformly continuous almost automorphic (S^p-u.c.a.a., for short) if given any sequence $\alpha' \subset \mathbb{R}$, there exists a subsequence $\alpha \subset \alpha'$ and a function $g \in L^p_{loc}(\mathbb{R}, X)$ such that $\mathcal{T}_\alpha f = g$ and $\mathcal{T}_{-\alpha} g = f$ pointwise in the sense of Stepanov, that is,

$$\lim_{k\to\infty} \int_0^1 |f(t + \alpha_k + s) - g(t + s)|^p ds = 0,$$

$$\lim_{k\to\infty} \int_0^1 |g(t - \alpha_k + s) - f(t + s)|^p ds = 0$$

for all $t \in \mathbb{R}$.

A function $f \in L^p_{loc}(\mathbb{R}, X)$ with $\tilde{f} \in AA(\mathbb{R}, Y)$ will be called S^p-a.a. [5]. Denote by $S^p AA(\mathbb{R}, X)$ and $S^p AA_{uc}(\mathbb{R}, X)$ the sets of all Stepanov a.a. and u.c.a.a. functions (of order p), respectively. It is obvious that $S^p AA_{uc}(\mathbb{R}, X) \subset S^p AA(\mathbb{R}, X)$. Next we show that equipped with a suitable norm, $S^pAA(\mathbb{R}, X)$ and $S^pAA_{uc}(\mathbb{R}, X)$ are isometrically isomorphic to corresponding complete subspaces of $AA(\mathbb{R}, Y)$ and $AA_{uc}(\mathbb{R}, Y)$, respectively.

The translation invariant property (11) of $\tilde{f}$ is crucial in constructing spaces isometrically isomorphic to $S^pAA(\mathbb{R}, X)$ and $S^pAA_{uc}(\mathbb{R}, X)$, respectively. As in [23], set

$$\widetilde{C}(\mathbb{R}, Y) = \{g \in C(\mathbb{R}, Y);\ g \text{ satisfies } (11)\}$$

and define a linear map by

$$\Phi : L^p_{loc}(\mathbb{R}, X) \to \widetilde{C}(\mathbb{R}, Y), \quad f \mapsto \tilde{f}.$$

Let

$$M^p(\mathbb{R}, X) = \left\{ f \in L^p_{loc}(\mathbb{R}, X);\ \sup_{t\in\mathbb{R}} \int_0^1 |f(t+s)|^p ds < \infty \right\}$$

be the Banach space [8, p. 39] of functions bounded in the mean (of order p) equipped with the norm

$$\|f\|_{M^p} = \sup_{t\in\mathbb{R}} \left[\int_0^1 |f(t+s)|^p ds \right]^{\frac{1}{p}}$$

and

$$\widetilde{BC}(\mathbb{R}, Y) = \widetilde{C}(\mathbb{R}, Y) \cap BC(\mathbb{R}, Y)$$

be a subspace of bounded and continuous functions equipped with the uniform convergence norm $\|\cdot\|$. [23] proves the following

Lemma 7.2 $\Phi : L^p_{loc}(\mathbb{R}, X) \to \widetilde{C}(\mathbb{R}, Y)$ *is an isomorphism and* $\Phi : (M^p(\mathbb{R}, X), \|\cdot\|_{M^p}) \to (\widetilde{BC}(\mathbb{R}, Y), \|\cdot\|)$ *is an isometric isomorphism.*

Define

$$\widetilde{AA}(\mathbb{R}, Y) = \widetilde{C}(\mathbb{R}, Y) \cap AA(\mathbb{R}, Y),$$
$$\widetilde{AA}_{uc}(\mathbb{R}, Y) = \widetilde{C}(\mathbb{R}, Y) \cap AA_{uc}(\mathbb{R}, Y).$$

To show the corresponding relations between the spaces $\widetilde{AA}(\mathbb{R}, Y)$, $\widetilde{AA}_{uc}(\mathbb{R}, Y)$ and $S^pAA(\mathbb{R}, X)$, $S^pAA_{uc}(\mathbb{R}, X)$, respectively, we need the following result.

Lemma 7.3 *Generalized translations of functions in* $\widetilde{AA}(\mathbb{R}, Y)$ *also satisfy (11).*

Proof Suppose that $h \in \widetilde{AA}(\mathbb{R}, Y)$ and $\mathcal{T}_\alpha h = g$, $\mathcal{T}_{-\alpha} g = h$ pointwise on $\mathbb{R}$. Given any t, $\tau \in \mathbb{R}$, put $I_{t,\tau} = [\tau - t, \tau - t + 1]$, $I = [0, 1]$ and $\nu = t - \tau + s$. It is easy to see that $s \in I \cap I_{t,\tau}$ if and only if $\nu \in I \cap I_{\tau,t}$. By (11),

$$h(t)(s) = h(\tau)(\nu), \quad h(t + \alpha_k)(s) = h(\tau + \alpha_k)(\nu)$$

for $a.e.\ s \in I \cap I_{t,\tau}$ and all $t \in \mathbb{R}$, $k \in \mathbb{Z}_+$. A direct calculation shows that

$$\begin{aligned}\int_{I\cap I_{t,\tau}} |h(t+\alpha_k)(s) - g(\tau)(\nu)|^p ds &= \int_{I\cap I_{\tau,t}} |h(\tau+\alpha_k)(\nu) - g(\tau)(\nu)|^p d\nu \\ &\le \int_I |h(\tau+\alpha_k)(\nu) - g(\tau)(\nu)|^p d\nu \\ &= \|h(\tau+\alpha_k) - g(\tau)\|_Y^p\end{aligned}$$

and

$$\int_{I\cap I_{t,\tau}} |h(t+\alpha_k)(s) - g(t)(s)|^p ds \leq \|h(t+\alpha_k) - g(t)\|_Y^p.$$

Since $\mathcal{T}_\alpha h = g$ pointwise on $\mathbb{R}$, both $g(t)(\cdot)$ and $g(\tau)(t-\tau+\cdot)$ are limits of $h(t+\alpha_k)(\cdot)$ in $L^p(I\cap I_{t,\tau}, X)$. Therefore,

$$g(t)(s) = g(\tau)(t-\tau+s), \quad a.e.\ s \in I\cap I_{t,\tau}, \forall t \in \mathbb{R}.$$

□

Lemma 7.4 *Both* $\Phi : (S^p AA(\mathbb{R}, X), \|\cdot\|_{M^p}) \to (\widetilde{AA}(\mathbb{R}, Y), \|\cdot\|)$ *and* $\Phi : (S^p AA_{uc}(\mathbb{R}, X), \|\cdot\|_{M^p}) \to (\widetilde{AA}_{uc}(\mathbb{R}, Y), \|\cdot\|)$ *are isometric isomorphisms.*

Proof We shall show that $f \in S^p AA_{uc}(\mathbb{R}, X)$ if and only if $\tilde{f} \in \widetilde{AA}_{uc}(\mathbb{R}, Y)$. Then $S^p AA_{uc}(\mathbb{R}, X) \subset M^p(\mathbb{R}, X)$ and by Lemma 7.2, $\Phi : S^p AA_{uc}(\mathbb{R}, X) \to \widetilde{AA}_{uc}(\mathbb{R}, Y)$ is injective. If $h \in \widetilde{AA}_{uc}(\mathbb{R}, Y)$, h is the Bochner transform of $\Phi^{-1}(h)$. So $\Phi^{-1}(h) \in S^p AA_{uc}(\mathbb{R}, X)$ and $\Phi : S^p AA_{uc}(\mathbb{R}, X) \to \widetilde{AA}_{uc}(\mathbb{R}, Y)$ is surjective. The final conclusion follows from Lemma 7.2. The proof of the other case of $\Phi : S^p AA(\mathbb{R}, X) \to \widetilde{AA}(\mathbb{R}, Y)$ is similar, since by definition $f \in S^p AA(\mathbb{R}, X)$ if and only if $\tilde{f} \in \widetilde{AA}(\mathbb{R}, Y)$.

Suppose that $f \in S^p AA_{uc}(\mathbb{R}, X)$ and $\mathcal{T}_\alpha f = g$, $\mathcal{T}_{-\alpha} g = f$ pointwise in the sense of Stepanov. By definitions of norms and almost automorphy, $\tilde{f} \in AA(\mathbb{R}, Y)$. Clearly, $\tilde{f}$ satisfies (11) and $\tilde{g} \in C(\mathbb{R}, Y)$. Thus $\tilde{f} \in \widetilde{AA}_{uc}(\mathbb{R}, Y)$.

Conversely, let $\tilde{f} \in \widetilde{AA}_{uc}(\mathbb{R}, Y)$ and $\mathcal{T}_\alpha \tilde{f} = h$, $\mathcal{T}_{-\alpha} h = \tilde{f}$ pointwise on $\mathbb{R}$. By Lemmas 2.3 and 7.3, $h \in \widetilde{C}(\mathbb{R}, Y)$. So, h is the Bochner transform of $\Phi^{-1}(h)$ by Lemma 7.2. Consequently, $\mathcal{T}_\alpha f = \Phi^{-1}(h)$, $\mathcal{T}_{-\alpha}\Phi^{-1}(h) = f$ pointwise in the sense of Stepanov. □

The following lemma is a basic integration technique like Lemma 6.9.

Lemma 7.5 *Suppose that* $f \in S^p AA_{uc}(\mathbb{R}, X)$*, then for each* $\sigma > 0$*,*

$$f_\sigma(t) := \frac{1}{\sigma}\int_0^\sigma f(t+s)ds \in AA_{uc}(\mathbb{R}, X).$$

Proof Suppose that $\mathcal{T}_\alpha f = g$, $\mathcal{T}_{-\alpha} g = f$ pointwise in the sense of Stepanov. Define

$$g_\sigma(t) = \frac{1}{\sigma}\int_0^\sigma g(t+s)ds, \quad t \in \mathbb{R}.$$

Using Hölder's inequality for $0 < \sigma \leq 1$,

$$\begin{aligned}
|f_\sigma(t+\alpha_k) - g_\sigma(t)| &\leq \frac{1}{\sigma}\int_0^\sigma |f(t+\alpha_k+s) - g(t+s)|ds \\
&\leq \frac{1}{\sigma}\Big[\int_0^1 |f(t+\alpha_k+s) - g(t+s)|^p ds\Big]^{1/p} \to 0, \\
|g_\sigma(t-\alpha_k) - f_\sigma(t)| &\leq \frac{1}{\sigma}\Big[\int_0^1 |g(t-\alpha_k+s) - f(t+s)|^p ds\Big]^{1/p} \to 0, \quad k \to \infty.
\end{aligned}$$

If $\sigma > 1$, divide the integrals above into a finite sum. In view of $g_\sigma \in C(\mathbb{R}, X)$, $f_\sigma \in AA_{uc}(\mathbb{R}, X)$ by Lemma 2.3. □

8 Equivalence of Levitan Piecewise Continuous and Stepanov Almost Automorphy

In this section, we prove our fourth main result on equivalence relations.

To get a good understanding, one verifies readily that Levitan p.c.a.a. functions include the uniformly continuous almost automorphic ones.

Lemma 8.1 $AA_{uc}(\mathbb{R}, X) \subset PCAA_L(\mathbb{R}, X)$.

Proof For every $h \in AA_{uc}(\mathbb{R}, X)$, h has discontinuities at the points of the empty set, which is a subset of any Levitan s.a.a. sequence $\{\tau_j\}_{j\in\mathbb{Z}}$. Uniform continuity of h implies quasi-uniform continuity. Assume by Theorem 5.3 that $\epsilon, \eta, N > 0$ satisfy

$$T(h, \eta, N) \pm T(h, \eta, N) \subset T(h, \epsilon, N).$$

Let $B(h, \eta, N) = T(h, \eta, N)$. Then $B(h, \eta, N) = -B(h, \eta, N) \subset \check{T}(h, \eta, N)$ by definitions. Moreover,

$$B(h, \eta, N) \pm B(h, \eta, N) \subset T(h, \epsilon, N) \subset \check{T}(h, \epsilon, N).$$

□

Define for every bounded $h \in PC(\mathbb{R}, X)$ a quantity

$$\|h\| = \sup_{t\in\mathbb{R}} |h(t)| = \sup_{j\in\mathbb{Z}} \sup_{\tau_j<t\leq\tau_{j+1}} |h(t)|,$$

The following is the fourth main result in this paper. It generalizes corresponding theorems of Bochner and [23] (see Remark 8.3) respectively on Bohr and piecewise continuous almost periodicity.

Theorem 8.2 *For any* $p \geq 1$,

$$S^p AA_{uc}(\mathbb{R}, X) \cap KPUCA(\mathbb{R}, X) = PCAA_L(\mathbb{R}, X).$$

Proof Let $h \in PCAA_L(\mathbb{R}, X)$ have discontinuities at the points of a subset of a Levitan s.a.a. sequence $\{\tau_j\}_{j\in\mathbb{Z}}$. Using Theorem 5.3 and Lemma 7.4, we show $h \in S^p AA_{uc}(\mathbb{R}, X)$ by proving that $\tilde{h} \in NAP(\mathbb{R}, Y) \cap KUC(\mathbb{R}, Y)$. Let $L > \sup_{j\in\mathbb{Z}} \tau_j^1$, $\theta = \inf_{j\in\mathbb{Z}} \tau_j^1$ and $m \in \mathbb{Z}_+$ satisfy $m\theta > 1$. A direct calculation shows that

$$\begin{aligned}\tau_{n+m} - \tau_n &= \sum_{j=0}^{m-1} (\tau_{n+j+1} - \tau_{n+j}) \\ &= \sum_{j=0}^{m-1} \tau_{n+j}^1 \geq m\theta > 1\end{aligned}$$

for all $n \in \mathbb{Z}$. For every $|t| \leq N_1$ there exists a unique $k \in \mathbb{Z}$ with $\tau_k < t \leq \tau_{k+1}$, then $t + 1 \leq \tau_{k+m+1}$. Because the number of $k = k(t)$ with $|t| \leq N_1$ is finite, there is a number N_2 independent of $|t| \leq N_1$ satisfying $|\tau_j| \leq N_2$ for $j = k, \ldots, k + m + 1$. Consequently,

$$\|\tilde{h}(t \pm r) - \tilde{h}(t)\|_Y^p = \int_t^{t+1} |h(s \pm r) - h(s)|^p ds \leq \int_{\tau_k}^{\tau_{k+1+m}} |h(s \pm r) - h(s)|^p ds$$

$$
\begin{aligned}
&= \sum_{j=k}^{k+m} \Big[\int_{\tau_j}^{\tau_j+\epsilon} |h(s \pm r) - h(s)|^p ds \\
&\quad + \int_{\tau_j+\epsilon}^{\tau_{j+1}-\epsilon} |h(s \pm r) - h(s)|^p ds + \int_{\tau_{j+1}-\epsilon}^{\tau_{j+1}} |h(s \pm r) - h(s)|^p ds \Big] \\
&\le \sum_{j=k}^{k+m} [2\epsilon(2\|h\|)^p + (\tau_j^1 - 2\epsilon)\epsilon^p] \\
&< (m+1)[2^{p+1}\|h\|^p + (L - 2\epsilon)\epsilon^{p-1}]\epsilon =: \epsilon^*(\epsilon)
\end{aligned}
$$

for all $r \in \check{T}(h, \epsilon, N_2)$ and $|t| \le N_1$, where $0 < \epsilon < \theta/2$ and $\|h\| < \infty$ by assumption. Thus $T(\tilde{h}, \epsilon^*(\epsilon), N_1)$ contains a relatively dense subset $\check{T}(h, \epsilon, N_2)$.

Let η be a number such that $0 < \eta < \theta/2$, $\epsilon^*(\eta) < \epsilon^*(\epsilon)$ and there exists relatively dense set $B(h, \eta, N_2) \subset \check{T}(h, \eta, N_2)$ satisfying

$$B(h, \eta, N_2) \pm B(h, \eta, N_2) \subset \check{T}(h, \epsilon, N_2).$$

By the argument above,

$$B(h, \eta, N_2) \subset \check{T}(h, \eta, N_2) \subset T(\tilde{h}, \epsilon^*(\eta), N_1).$$

Therefore,

$$B(h, \eta, N_2) \pm B(h, \eta, N_2) \subset T(\tilde{h}, \epsilon^*(\epsilon), N_1).$$

Hence $\tilde{h} \in NAP(\mathbb{R}, Y)$. By a similar estimate for $\|\tilde{h}(t \pm r) - \tilde{h}(t)\|_Y^p$ as above,

$$\|\tilde{h}(t+s) - \tilde{h}(t)\|_Y^p \le \epsilon^*(\epsilon)$$

for $|s| < \delta < \epsilon$, where δ is chosen for ϵ in the statement of quasi-uniform continuity. Thus considering $\|h\| < \infty$, $\tilde{h} \in BUC(\mathbb{R}, Y)$.

Given any sequence $\{t_k'\}_{k=1}^\infty \subset \mathbb{R}$, because h has a relatively compact range, there exists a subsequence $\{t_k\}_{k=1}^\infty \subset \{t_k'\}_{k=1}^\infty$ such that $\{h(t_k + s)\}_{k=1}^\infty$ is convergent for all $s \in [0, 1]$ by Tychnoff product theorem. Therefore, $\{\tilde{h}(t_k)\}_{k=1}^\infty$ converges in $Y = L^p([0, 1], X)$ by Lebesgue's dominated convergence theorem.

For the reverse containment, assume that $h \in S^pAA_{uc}(\mathbb{R}, X) \cap KPUCA(\mathbb{R}, X)$ has discontinuities at the points of a subset of a generalized Wexler sequence $\{\tau_j\}_{j\in\mathbb{Z}}$ with $\inf_{j\in\mathbb{Z}} \tau_j^1 = \theta$. From Lemma 6.1 it follows that for every $\epsilon > 0$ there exists a δ, $0 < \delta < \min\{\theta/2, \epsilon\}$ such that $|h_\sigma(t) - h(t)| < \epsilon$ for all $\sigma \in \mathbb{R}$, $0 < \sigma < \delta$ and $t \in \mathbb{R}$, $|t - \tau_j| > \epsilon$, $j \in \mathbb{Z}$. Moreover, Lemma 7.5 implies that $h_\sigma \in AA_{uc}(\mathbb{R}, X)$ for $0 < \sigma < \delta$. Therefore, $h \in PCAA_L(\mathbb{R}, X)$ by Theorem 6.4. □

Remark 8.3 Bochner proves $S^p(\mathbb{R}, X) \cap BUC(\mathbb{R}, X) = AP(\mathbb{R}, X)$ ([7, p. 174], [19, p. 34]) and Theorem 3.2 in [23] proves $S^p(\mathbb{R}, X) \cap PUCW(\mathbb{R}, X) = PCAP(\mathbb{R}, X) \cap PUCW(\mathbb{R}, X)$, where $AP(\mathbb{R}, X)$, $S^p(\mathbb{R}, X)$ and $PCAP(\mathbb{R}, X)$ denote the set of Bohr, Stepanov and piecewise continuous almost periodic functions, respectively, and $PUCW(\mathbb{R}, X)$ is the set of functions $h \in PC(\mathbb{R}, X)$ which are quasi-uniformly continuous with possible discontinuities at the points of a subset of a Wexler sequence (see Remark 6.2 for definition).

9 Favard's Theorems

In this section we study p.c.a.a. solutions of almost periodic impulsive differential equations and establish two Favard's theorems.

Favard's theorem on almost periodic differential equations reads as follows.

Theorem 9.1 *Consider the following linear differential equation*

$$x' = A(t)x + f(t), \tag{12}$$

where $A \in AP(\mathbb{R}, \mathbb{R}^{d\times d})$*,* $f \in AP(\mathbb{R}, \mathbb{R}^d)$*. If for any* B *in the hull of* A*, any nontrivial bounded solution* x *of*

$$x' = B(t)x$$

satisfies $\inf_{t\in\mathbb{R}} |x(t)| > 0$ *and (12) admits a bounded solution, then (12) has at least one almost periodic solution* ϕ *such that* $\mathrm{mod}(\phi) \subset \mathrm{mod}(A, f)$*, where* $\mathrm{mod}\ (\varphi)$ *denotes the frequency module defined as the additive group generated by the spectrum of an almost periodic function* φ*.*

[14] proposes Question A on the truth of Theorem 9.1 if only $x' = A(t)x$ is required $\inf_{t\in\mathbb{R}} |x(t)| > 0$ for nontrivial solutions. [17] construct a scalar differential equation of the form (12) which admits bounded solutions, but no almost periodic solutions. [29] proves the existence of almost automorphic soluitons under the condition of [14]. [31] extends Theorem 4.2 of [29] to differential equations with piecewise constant argument. As for impulsive differential equations, we shall consider the linear differential equation with impulses at fixed times

$$\begin{cases} x' = A(t)x + h(t), & t \neq \tau_n, \\ x(\tau_n^+) - x(\tau_n) = B(n)x(\tau_n) + b(n), & n \in \mathbb{Z}, \end{cases} \tag{13}$$

and its homogeneous system

$$\begin{cases} x' = A(t)x, & t \neq \tau_n, \\ x(\tau_n^+) - x(\tau_n) = B(n)x(\tau_n), & n \in \mathbb{Z}, \end{cases} \tag{14}$$

which satisfy the following conditions:

(H1) $\{\tau_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}$ is a Wexler sequence such that

$$\tau_n = \xi n + \zeta(n), \quad n \in \mathbb{Z},$$

where $\xi > 0$, $\zeta \in AP(\mathbb{Z}, \mathbb{R})$ and $\theta = \inf_{j\in\mathbb{Z}} \tau_j^1$.

(H2) $A \in PCAP(\mathbb{R}, \mathbb{R}^{d\times d})$, $h \in PCAP(\mathbb{R}, \mathbb{R}^d)$ has discontinuities at the points of a subset of $\{\tau_j\}_{j\in\mathbb{Z}}$, $B \in AP(\mathbb{Z}, \mathbb{R}^{d\times d})$, $b \in AP(\mathbb{Z}, \mathbb{R}^d)$, where $d \in \mathbb{Z}_+$. $\det[I + B(n)] \neq 0$ for all $n \in \mathbb{Z}$.

Theorem 6 in [27] proves that if (14) has only trivial bounded solution, then any bounded solution of (13) is an N-ρ-a.p.p.c. Levitan function (Definition 4.5). N-ρ-a.p.p.c. Levitan functions, when considered as bounded solutions of impulsive differential equations, are already quasi-uniformly continuous and hence form a subclass of our Levitan p.c.a.a. solutions. The theorem in [27], although lack of details, has indicated this class of solutions to be natural in almost periodically forced impulsive differential equations. We provide here a completely new and easily accessible approach and further results.

Wexler sequences are Bochner s.a.a. by Lemma 3.2. See [13,25,26] for more about almost periodic sets on the line. The following result is adequate for use.

Lemma 9.2 *Suppose that $\{\tau_j\}_{j\in\mathbb{Z}}$ is a Wexler sequence defined by*

$$\tau_n = \xi n + \zeta(n), \quad n \in \mathbb{Z},$$

where $\xi > 0$, $\zeta \in AP(\mathbb{Z}, \mathbb{R})$, then for any $\alpha' \subset \mathbb{R}$, there are sequences $\alpha \subset \alpha'$, $\{m_k\}_{k=1}^{\infty} \subset \mathbb{Z}$ and a Wexler sequence $\{\tau_j^*\}_{j\in\mathbb{Z}}$ of the form

$$\tau_n^* = \xi n + \zeta^*(n) + \vartheta, \quad n \in \mathbb{Z}$$

with ζ^* in the hull of ζ, $\vartheta \in [0, \xi]$, such that

$$\lim_{k\to\infty} \sup_{n\in\mathbb{Z}} |\tau_{n+m_k} + \alpha_k - \tau_n^*| = 0, \ \lim_{k\to\infty} \sup_{n\in\mathbb{Z}} |\tau_{n-m_k}^* - \alpha_k - \tau_n| = 0.$$

Proof Given any $\alpha' \subset \mathbb{R}$, there are unique $m_k' \in \mathbb{Z}$ and $\vartheta_k' \in [0, \xi)$ such that

$$-\alpha_k' = \xi m_k' + \vartheta_k', \quad k \in \mathbb{Z}_+.$$

Hence there are subsequences $\alpha \subset \alpha'$, $\{m_k\} \subset \{m_k'\}$, $\{\vartheta_k\} \subset \{\vartheta_k'\}$, a sequence ζ^* in the hull of ζ and a number $\vartheta \in [0, \xi]$ such that

$$\lim_{k\to\infty} \zeta(\cdot + m_k) = \zeta^*(\cdot), \ \lim_{k\to\infty} \zeta^*(\cdot - m_k) = \zeta(\cdot), \ \lim_{k\to\infty} \vartheta_k = \vartheta$$

and

$$-\alpha_k = \xi m_k + \vartheta_k, \quad k \in \mathbb{Z}_+.$$

Define a Wexler sequence by

$$\tau_n^* = \xi n + \zeta^*(n) - \vartheta, \quad n \in \mathbb{Z}.$$

A direct calculation shows that

$$\begin{aligned}
\sup_{n\in\mathbb{Z}} |\tau_{n+m_k} + \alpha_k - \tau_n^*| &= \sup_{n\in\mathbb{Z}} |\zeta(n + m_k) - \zeta^*(n) - \vartheta_k + \vartheta| \\
&\le \|\zeta(\cdot + m_k) - \zeta^*(\cdot)\| + |\vartheta_k - \vartheta|, \\
\sup_{n\in\mathbb{Z}} |\tau_{n-m_k}^* - \alpha_k - \tau_n| &= \sup_{n\in\mathbb{Z}} |\zeta^*(n - m_k) - \zeta(n) - \vartheta + \vartheta_k| \\
&\le \|\zeta^*(\cdot - m_k) - \zeta(\cdot)\| + |\vartheta_k - \vartheta|,
\end{aligned}$$

which imply the final conclusion. □

The following theorem obtains nearly the same result as Theorem 6 in [27] on N-ρ-a.p.p.c. Levitan solutions by a new and simpler approach.

Theorem 9.3 *Suppose that (13) satisfies (H1) and (H2), and the homogeneous system (14) has only trivial bounded solutions. Then any bounded solution of (13) is Bochner p.c.a.a.*

Proof Let ϕ be a bounded solution of (13) and $\alpha \subset \mathbb{R}$. For each $k \in \mathbb{Z}_+$ the function $\phi_k(\cdot) := \phi(\cdot + \alpha_k)$ has possible discontinuities at the points of a subset of $\{\tau_j - \alpha_k\}_{j\in\mathbb{Z}}$, and satisfies

$$\begin{cases} \frac{d}{dt}\phi_k(t) = A(t + \alpha_k)\phi_k(t) + h(t + \alpha_k), & t \ne \tau_n - \alpha_k, \\ \phi_k((\tau_n - \alpha_k)^+) - \phi_k(\tau_n - \alpha_k) = B(n)\phi_k(\tau_n - \alpha_k) + b(n), & n \in \mathbb{Z}. \end{cases}$$

From the proof of Lemma 9.2 and by passing to subsequence if necessary, we may assume that

$$-\alpha_k = \xi m_k + \vartheta_k, \quad k \in \mathbb{Z}_+,$$

where $m_k \in \mathbb{Z}$ and $\vartheta_k \in [0, \xi)$, and using the fact that p.c.a.p. functions are Stepanov almost periodic (Theorem 3.2 in [23]),

$$\begin{aligned}
&\lim_{k\to\infty} \sup_{t\in\mathbb{R}} \int_t^{t+1} |A(\alpha_k + s) - A^*(s)|ds = 0, \\
&\lim_{k\to\infty} \sup_{t\in\mathbb{R}} \int_t^{t+1} |h(\alpha_k + s) - h^*(s)|ds = 0, \\
&\lim_{k\to\infty} \|B(\cdot - m_k) - B^*(\cdot)\| = 0, \\
&\lim_{k\to\infty} \|b(\cdot - m_k) - b^*(\cdot)\| = 0, \\
&\lim_{k\to\infty} \|\zeta(\cdot - m_k) - \zeta^*(\cdot)\| = 0, \ \lim_{k\to\infty} \vartheta_k = \vartheta,
\end{aligned} \tag{15}$$

where B^*, b^*, ζ^* are in the hull of B, b, ζ, respectively, $\vartheta \in [0, \xi]$ and A^*, h^* satisfy all the conclusions of Theorem 6.5 with possible discontinuities at the points of a subset of a Wexler sequence defined by

$$\tau_n^* = \xi n + \zeta^*(n) + \vartheta, \quad n \in \mathbb{Z}$$

such that

$$\lim_{k\to\infty} \sup_{n\in\mathbb{Z}} |\tau_{n-m_k} - \alpha_k - \tau_n^*| = 0, \ \lim_{k\to\infty} \sup_{n\in\mathbb{Z}} |\tau_{n+m_k}^* + \alpha_k - \tau_n| = 0.$$

Since ϕ is bounded, so is ϕ' on $\mathbb{R}\backslash\{\tau_j\}_{j\in\mathbb{Z}}$ by (13). Hence ϕ is quasi-uniformly continuous with possible discontinuities at the points of a subset of $\{\tau_j\}_{j\in\mathbb{Z}}$. Given $\eta > 0$ and $s, t \in [\tau_n^* + \eta, \tau_{n+1}^* - \eta]$ for some $n \in \mathbb{Z}$, it follows that

$$s + \alpha_k, t + \alpha_k \in [\tau_n^* + \eta + \alpha_k, \tau_{n+1}^* - \eta + \alpha_k] \subset (\tau_{n-m_k}, \tau_{n+1-m_k})$$

for large k. Thus the family $\{\phi(\cdot + \alpha_k)\}$, where $k >> 1$, are uniformly bounded and equicontinuous on $[\tau_n^* + \eta, \tau_{n+1}^* - \eta]$. Consequently, by the Arzela-Ascoli theorem and passing to subsequences if necessary, $\{\phi(\cdot + \alpha_k)\}_{k>>1}$ converges uniformly on $[\tau_n^* + \eta, \tau_{n+1}^* - \eta]$ to a function ϕ^*. By the equation $\{d\phi(\cdot + \alpha_k)/dt\}_{k>>1}$ also converges uniformly. Hence ϕ^* is differentiable and $d\phi^*/dt = \lim_{k\to\infty} d\phi(\cdot + \alpha_k)/dt$. Therefore,

$$\frac{d}{dt}\phi^*(t) = A^*(t)\phi^*(t) + h^*(t), \quad t \neq \tau_n^*, n \in \mathbb{Z}.$$

By Theorem 6.5, with a modification of the values taking at $\{\tau_j^*\}_{j\in Z}$ if necessary, $\phi^* \in PC(\mathbb{R}, \mathbb{R}^d)$. For any $n \in \mathbb{Z}$ and $\epsilon > 0$, there is a small $\delta > 0$ such that

(i) $|\phi^*(t) - \phi^*(\tau_n^{*+})| < \epsilon$ for $t \in (\tau_n^*, \tau_n^* + 2\delta)$.
(ii) $|\phi^*(s) - \phi^*(\tau_n^*)| < \epsilon$ for $t \in (\tau_n^* - 2\delta, \tau_n^*]$.
(iii) $|\phi(s) - \phi(t)| < \epsilon$ whenever $s, t \in (\tau_j, \tau_{j+1}]$ for some $j \in \mathbb{Z}$ and $|s - t| < 3\delta$, which further yields $|\phi(t) - \phi(\tau_j^+)| \leq \epsilon$ for $t \in (\tau_j, \tau_j + 3\delta)$.

Let $t \in (\tau_n^* + \delta, \tau_n^* + 2\delta)$ and $s \in (\tau_n^* - 2\delta, \tau_n^* - \delta)$, then

$$t + \alpha_k \in (\tau_n^* + \delta + \alpha_k, \tau_n^* + 2\delta + \alpha_k) \subset (\tau_{n-m_k}, \tau_{n-m_k} + 3\delta),$$

$$s + \alpha_k \in (\tau_n^* - 2\delta + \alpha_k, \tau_n^* - \delta + \alpha_k) \subset (\tau_{n-m_k} - 3\delta, \tau_{n-m_k})$$

for large k. Consequently, fixing s and t,

$$\begin{aligned}|\phi(\tau_{n-m_k}^+) - \phi^*(\tau_n^{*+})| &\leq |\phi(\tau_{n-m_k}^+) - \phi(t+\alpha_k)| + |\phi(t+\alpha_k) - \phi^*(t)| \\ &\quad + |\phi^*(t) - \phi^*(\tau_n^{*+})| < 3\epsilon, \\ |\phi(\tau_{n-m_k}) - \phi^*(\tau_n^*)| &\leq |\phi(\tau_{n-m_k}) - \phi(s+\alpha_k)| + |\phi(s+\alpha_k) - \phi^*(s)| \\ &\quad + |\phi^*(s) - \phi^*(\tau_n^*)| < 3\epsilon\end{aligned}$$

for large k. From

$$\phi(\tau_{n-m_k}^+) - \phi(\tau_{n-m_k}) = B(n-m_k)\phi(\tau_{n-m_k}) + b(n-m_k)$$

it follows that

$$\phi^*(\tau_n^{*+}) - \phi^*(\tau_n^*) = B^*(n)\phi^*(\tau_n^*) + b^*(n)$$

for all $n \in \mathbb{Z}$. Therefore, $\phi^* \in PC(\mathbb{R}, \mathbb{R}^d)$ is a bounded solution of

$$\begin{cases} x' = A^*(t)x + h^*(t), & t \neq \tau_n^*, \\ x(\tau_n^{*+}) - x(\tau_n^*) = B^*(n)x(\tau_n) + b^*(n), & n \in \mathbb{Z}. \end{cases}$$

Conversely, since $d\phi^*/dt$ is bounded on $\mathbb{R}\backslash\{\tau_j^*\}_{j\in\mathbb{Z}}$, ϕ^* is quasi-uniformly continuous with possible discontinuities at the points of a subset of $\{\tau_j^*\}_{j\in\mathbb{Z}}$. From the argument above and the almost periodicity of A, h, B, b, $\{\tau_j\}_{j\in\mathbb{Z}}$ it follows that $\{\phi^*(\cdot - \alpha_k)\}_{k=1}^\infty$ converges pointwise on $\mathbb{R}\backslash\{\tau_j\}_{j\in\mathbb{Z}}$ to a bounded solution φ of (13). Since the homogeneous system (14) has only trivial bounded solutions, $\phi = \varphi$. Thus $\mathcal{T}_\alpha\phi = \phi^*$ pointwise on $\mathbb{R}\backslash\{\tau_j^*\}_{j\in\mathbb{Z}}$ and $\mathcal{T}_{-\alpha}\phi^* = \phi$ pointwise on $\mathbb{R}\backslash\{\tau_j\}_{j\in\mathbb{Z}}$. So ϕ is Bochner p.c.a.a. by definition. □

The following is the fifth main result on Favard's theorem concerning almost automorphic solutions as [29,31]. It goes further than Theorem 6 in [27] and shows advantages of our Bochner p.c.a.a. functions.

Theorem 9.4 *Suppose that (13) satisfies (H1) and (H2), and any nontrivial bounded solution x of the homogeneous system (14) satisfies $\inf_{t\in\mathbb{R}}|x(t)| > 0$. If (13) admits a bounded solution, then (13) has a Bochner p.c.a.a. solution.*

Proof We first show that if (13) has a bounded solution x_0, then it admits a bounded solution x^* with minimum norm $\|x^*\| = \sup_{t\in\mathbb{R}}|x^*(t)|$. Let $K \subset \mathbb{R}^n$ be the closed ball centered at 0 with radius $\|x_0\|$. Put

$$\lambda = \inf\{\|x\|; |x(t)| \leq \|x_0\|, \forall t \in \mathbb{R} \text{ and } x \text{ is a solution of (13)}\}$$

and let $\{x_k\}_{k\in\mathbb{Z}_+}$ be a sequence of solutions of (13) such that $\lim_{k\to\infty}\|x_k\| = \lambda$. Since $\{x_k\}_{k\in\mathbb{Z}_+}$ are uniformly bounded, so are their derivatives $\{x_k'\}_{k\in\mathbb{Z}_+}$ on $\mathbb{R}\backslash\{\tau_j\}_{j\in\mathbb{Z}}$ by (13). Consequently, given $\epsilon > 0$ there is a $\delta > 0$ such that for all $k \in \mathbb{Z}_+$, $|x_k(s) - x_k(t)| < \epsilon$ whenever $s, t \in (\tau_j, \tau_{j+1}]$ for some $j \in \mathbb{Z}$ and $|s - t| < \delta$. For each $n \in \mathbb{Z}$, redefining $x_k(\tau_n) = x_k(\tau_n^+)$ makes the family $\{x_k\}_{k\in\mathbb{Z}_+}$ uniformly bounded and equi-continuous on $[\tau_n, \tau_{n+1}]$. By the Arzela-Ascoli theorem and passing to subsequences if necessary, $\{x_k\}_{k\in\mathbb{Z}_+}$ converges uniformly on $[\tau_n, \tau_{n+1}]$ to a function x_n^* for each $n \in \mathbb{Z}$. By the equation $\{x_k'\}_{k\in\mathbb{Z}_+}$ also converges uniformly. Hence x_n^* is differentiable and $x_n^{*\prime} = \lim_{k\to\infty} x_k'$. Define

$$x^*(t) = x_n^*(t), \quad t \in (\tau_n, \tau_{n+1}], n \in \mathbb{Z}.$$

It follows that

$$x^{*\prime}(t) = A(t)x^*(t) + h(t), \quad t \neq \tau_n, n \in \mathbb{Z}$$

and

$$\begin{aligned} x^*(\tau_n^+) = x_n^*(\tau_n) &= \lim_{k\to\infty} x_k(\tau_n^+) = \lim_{k\to\infty} [I + B(n)]x_k(\tau_n) + b(n) \\ &= [I + B(n)]x_{n-1}^*(\tau_n) + b(n) = [I + B(n)]x^*(\tau_n) + b(n), \end{aligned}$$

where the $x_k(\tau_n)$ above is the original value, not the modified one. Therefore, x^* is a solution of (13) and

$$\begin{aligned} \|x^*\| &= \sup_{n\in\mathbb{Z}} \sup_{t\in(\tau_n,\tau_{n+1}]} |x^*(t)| = \sup_{n\in\mathbb{Z}} \sup_{t\in[\tau_n,\tau_{n+1}]} |x_n^*(t)| \\ &= \sup_{n\in\mathbb{Z}} \lim_{k\to\infty} \sup_{t\in(\tau_n,\tau_{n+1}]} |x_k(t)| \leq \sup_{n\in\mathbb{Z}} \lim_{k\to\infty} \|x_k\| = \lambda. \end{aligned}$$

Clearly, $\|x^*\| \leq \|x_0\|$. Thus $\|x^*\| = \lambda$ by definition.

The separation condition $\inf_{t\in\mathbb{R}} |x(t)| > 0$ implies that the bounded solution x^* of (13) with minimum norm is unique. Otherwise, if ϕ_1 and ϕ_2 are two such solutions, then $(\phi_1 + \phi_2)/2$ is a solution of (13) and $(\phi_1 - \phi_2)/2$ is a nontrivial solution of (14). By assumption $|\phi_1(t) - \phi_2(t)|/2 \geq \rho$ for all t and some $\rho > 0$. The parallelogram law implies

$$\left|\frac{\phi_1(t) + \phi_2(t)}{2}\right|^2 + \left|\frac{\phi_1(t) - \phi_2(t)}{2}\right|^2 = \frac{|\phi_1(t)|^2 + |\phi_2(t)|^2}{2} \leq \|x^*\|^2.$$

Thus

$$\left\|\frac{\phi_1 + \phi_2}{2}\right\|^2 \leq \|x^*\|^2 - \rho^2$$

which contradicts the minimum property of x^*.

At last, the proof of Theorem 9.3 shall yield that x^* is Bochner p.c.a.a. $\square$

The following is the last main result on Favard's theorem concerning almost periodic solutions and module containment.

Theorem 9.5 *Suppose that (13) satisfies (H1) and (H2) with* $\det[I + B(n)] \neq 0$, $n \in \mathbb{Z}$, *replaced by* $\inf_{n\in\mathbb{Z}} |\det[I + B(n)]| > 0$, *and consider the families of impulsive systems obtained in the proof of Theorem 9.3,*

$$\begin{cases} x' = A^*(t)x + h^*(t), & t \neq \tau_n^*, \\ x(\tau_n^{*+}) - x(\tau_n^*) = B^*(n)x(\tau_n) + b^*(n), & n \in \mathbb{Z}, \end{cases} \tag{16}$$

and

$$\begin{cases} x' = A^*(t)x, & t \neq \tau_n^*, \\ x(\tau_n^+) - x(\tau_n) = B^*(n)x(\tau_n), & n \in \mathbb{Z}, \end{cases} \tag{17}$$

called a hull of (13) and (14), respectively. If for every (17), any bounded solution x of it satisfies $\inf_{t\in\mathbb{R}} |x(t)| > 0$, *and (13) admits a bounded solution, then (13) has a p.c.a.p. solution ϕ such that*

$$\mathrm{mod}(\phi) \subset \mathrm{span}\left(\mathrm{mod}(A, h) \cup \left[\frac{1}{\xi} \cdot \left\{[\mathrm{mod}(B, b, \zeta)]^{(r)} \cup \{2\pi\}\right\}\right]\right),$$

where $E^{(r)} = \{\beta \in [0, 2\pi); (\beta + 2\pi\mathbb{Z}) \in E\}$ *denotes a representative set of* $E \subset \mathbb{R}/2\pi\mathbb{Z}$ *and* $\mathrm{span}(F)$ *is the additive group generated by* $F \subset \mathbb{R}$.

Proof First note that (16) is an almost periodic impulsive system. From the proof of Theorem 9.3, B^* and b^* are in the hull of B and b, respectively, and $\{\tau_j^*\}_{j\in\mathbb{Z}}$ is a Wexler sequence. Moreover, A^* and h^* are Stepanov almost periodic since they are generalized translation of such functions A and h, respectively. By Theorem 6.5, A^* and h^* are quasi-uniformly continuous with possible discontinuities at the points of a subset of $\{\tau_j^*\}_{j\in\mathbb{Z}}$. Theorem 3.2 in [23] implies $A^* \in PCAP(\mathbb{R}, \mathbb{R}^{d\times d})$ and $h^* \in PCAP(\mathbb{R}, \mathbb{R}^d)$. Thus (16) satisfies (H1) and a modified (H2) with $\inf_{n\in\mathbb{Z}} |\det[I + B^*(n)]| > 0$. Consequently, the Bochner p.c.a.a. solution x^* with minimum norm of (13) has all its generalized translations being solutions with minimum norm of systems of the form (16). Hence they are Bochner p.c.a.a. by Theorem 9.4. From Lemma 6.9 and its proof it follows that for each $\sigma > 0$, the uniformly continuous almost automorphic function

$$x_\sigma^*(t) = \frac{1}{\sigma}\int_0^\sigma x^*(t+s)ds, \quad t \in \mathbb{R},$$

has all its generalized translations uniformly continuous almost automorphic. Thus $x_\sigma^* \in AP(\mathbb{R}, \mathbb{R}^d)$ by Theorem 3.3.1 in [30]. Lemma 3.6 in [23] asserts that if $f \in PC(\mathbb{R}, X)$ is quasi-uniformly continuous with possible discontinuities at the points of a subset of a Wexler sequence $\{\tau_j'\}_{j\in\mathbb{Z}}$ and for each $\epsilon > 0$ there exists an $f_\epsilon \in AP(\mathbb{R}, X)$ such that $|f_\epsilon(t) - f(t)| < \epsilon$ for all $t \in \mathbb{R}$, $|t - \tau_j'| > \epsilon$, $j \in \mathbb{Z}$, then $f \in PCAP(\mathbb{R}, X)$. Combined with Lemma 6.1 it follows that $x^* \in PCAP(\mathbb{R}, \mathbb{R}^d)$.

As for the module containment, by filling in the gaps linearly define Bohr almost periodic functions

$$\begin{aligned}
\bar{B}(t) &= (n+1-t)B(n) + (t-n)B(n+1), \quad n < t \le n+1, n \in \mathbb{Z},\\
\bar{b}(t) &= (n+1-t)b(n) + (t-n)b(n+1), \quad n < t \le n+1, n \in \mathbb{Z},\\
\bar{\zeta}(t) &= (n+1-t)\zeta(n) + (t-n)\zeta(n+1), \quad n < t \le n+1, n \in \mathbb{Z}.
\end{aligned}$$

Let $\alpha' \subset \xi\mathbb{Z}$ be a sequence such that

$$\begin{aligned}
&\int_t^{t+1} |A(s+\alpha_k') - A(s)|ds \to 0, \quad \int_t^{t+1} |h(s+\alpha_k') - h(s)|ds \to 0,\\
&\int_t^{t+1} \left|\bar{B}\Big(\frac{s+\alpha_k'}{\xi}\Big) - \bar{B}\Big(\frac{s}{\xi}\Big)\right|ds \to 0, \quad \int_t^{t+1} \left|\bar{b}\Big(\frac{s+\alpha_k'}{\xi}\Big) - \bar{b}\Big(\frac{s}{\xi}\Big)\right|ds \to 0,\\
&\int_t^{t+1} \left|\bar{\zeta}\Big(\frac{s+\alpha_k'}{\xi}\Big) - \bar{\zeta}\Big(\frac{s}{\xi}\Big)\right|ds \to 0, \quad \forall t \in \mathbb{R},
\end{aligned} \tag{18}$$

as $k \to \infty$. There would be sequences $\alpha \subset \alpha'$, $\{m_k\}_{k\in\mathbb{Z}_+} \subset \mathbb{Z}$ with $-\alpha_k = \xi m_k$, $k \in \mathbb{Z}_+$ such that (15) holds for B^*, b^*, ζ^* in the hull of B, b, ζ, respectively, and functions A^*, h^* satisfying all the conclusions of Theorem 6.5 with possible discontinuities at the points of a subset of a Wexler sequence defined by

$$\tau_n^* = \xi n + \zeta^*(n), \quad n \in \mathbb{Z}.$$

Because $\bar{\zeta}$ is Bohr almost periodic, (18) and Theorem 4.10 in [23] yield

$$\|\zeta(\cdot - m_k) - \zeta(\cdot)\| = \left\|\zeta\Big(\frac{\cdot + \alpha_k}{\xi}\Big) - \zeta\Big(\frac{\cdot}{\xi}\Big)\right\| \le \left\|\bar{\zeta}\Big(\frac{\cdot + \alpha_k}{\xi}\Big) - \bar{\zeta}\Big(\frac{\cdot}{\xi}\Big)\right\| \to 0$$

as $k \to \infty$. Thus $\zeta = \zeta^*$ and similarly $B = B^*$, $b = b^*$. Hence A, h, A^*, h^* have possible discontinuities contained in the same Wexler sequence $\{\tau_j\}_{j\in\mathbb{Z}}$. From (15) and (18) it follows that $A = A^*$ and $h = h^*$. The proof of Theorem 9.3 implies the p.c.a.p. solution x^* with minimum norm of (13) has its translation sequence $\{x^*(\cdot + \alpha_k)\}_{k\in\mathbb{Z}_+}$ converging uniformly to itself on each interval $[\tau_n + \eta, \tau_{n+1} - \eta]$ with $\eta > 0$, $n \in \mathbb{Z}$. Therefore,

$$\lim_{k\to\infty} \int_t^{t+1} |x^*(s + \alpha_k) - x^*(s)|ds = 0, \quad \forall t \in \mathbb{R}.$$

Theorem 2.1 of [31] and Theorem 4.7 and Lemma 5.8 of [23] imply the final module containment relation. □

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Appendix A: Common Translation Sets

We prove in this section the basic tool Theorem 6.3. The following lemma concerns an elementary property of an auxiliary function.

Lemma A.1 *Suppose that* $g(n) = (-1)^n$, $n \in \mathbb{Z}$, *is a 2-periodic sequence and*

$$g(t) = (n + 1 - t)(-1)^n + (t - n)(-1)^{n+1}, \quad n \le t \le n + 1, \ n \in \mathbb{Z}.$$

Then for any $0 < \epsilon < 1$ *and* $N > 0$,

$$T(g, \epsilon, N) = \bigcup_{m\in 2\mathbb{Z}} \Big(m - \frac{\epsilon}{2}, m + \frac{\epsilon}{2}\Big).$$

Proof For any $\tau \in \mathbb{R}$, there is a unique $m \in \mathbb{Z}$ so that $m \le \tau < m + 1$. Thus $n + m \le n + \tau < n + m + 1$ and $n - m - 1 < n - \tau \le n - m$ for all $n \in \mathbb{Z}$. From the definition of the function g it follows that

$$\begin{aligned} g(n + \tau) &= (m + 1 - \tau)(-1)^{n+m} + (\tau - m)(-1)^{n+m+1} \\ &= (-1)^{n+m}(m + 1 - \tau - \tau + m) \\ &= (-1)^{n+m}(2m - 2\tau + 1), \\ g(n - \tau) &= (\tau - m)(-1)^{n-m-1} + (m + 1 - \tau)(-1)^{n-m} \\ &= (-1)^{n-m}(m + 1 - \tau - \tau + m) \\ &= (-1)^{n-m}(2m - 2\tau + 1), \quad n \in \mathbb{Z}. \end{aligned}$$

Therefore,

$$\begin{aligned} |g(n + \tau) - g(n)| &= |(-1)^n[(-1)^m(2m - 2\tau + 1) - 1]| \\ &= |(-1)^m(2m - 2\tau + 1) - 1|, \\ |g(n - \tau) - g(n)| &= |(-1)^n[(-1)^{-m}(2m - 2\tau + 1) - 1]| \\ &= |(-1)^{-m}(2m - 2\tau + 1) - 1|, \quad n \in \mathbb{Z}, \end{aligned}$$

which are independent of n.

The proof of the "$\subset$" part is divided into two cases.

Case 1. If $\tau \in T(g, \epsilon, N)$ and m is odd, from equalities above for $|n| \le N$ it follows that

$$|2m - 2\tau + 1 + 1| < \epsilon, \quad m + 1 - \frac{\epsilon}{2} < \tau < m + 1.$$

Case 2. If $\tau \in T(g, \epsilon, N)$ and m is even, it follows that

$$|\tau - m| < \frac{\epsilon}{2}, \quad m \leq \tau < m + \frac{\epsilon}{2}.$$

Summing up, there is an even integer m' such that $|\tau - m'| < \epsilon/2$.

Conversely, suppose that there is an even integer m' such that $|\tau - m'| < \epsilon/2$. Let $t \in [n, n+1)$, $n \in \mathbb{Z}$ and $|t| \leq N$. It is obvious that

$$n + m \leq t + \tau < n + m + 2, \quad n - m - 1 < t - \tau < n - m + 1.$$

The proof of the "$\supset$" part is divided into six cases.

Case 1. m is odd and $t + \tau < n + m + 1$.

Case 2. m is odd and $t + \tau \geq n + m + 1$, $t - \tau < n - m$.

Case 3. m is odd and $t + \tau \geq n + m + 1$, $t - \tau \geq n - m$.

Case 4. m is even and $t + \tau \geq n + m + 1$.

Case 5. m is even and $t + \tau < n + m + 1$, $t - \tau \geq n - m$.

Case 6. m is even and $t + \tau < n + m + 1$, $t - \tau < n - m$.

We only prove Case 1. The proofs of the other cases are similar, so we omit them. Conditions of Case 1 imply $m' = m + 1$ and

$$\begin{aligned}
&m + 1 - \frac{\epsilon}{2} < \tau < m + 1, \quad n + m < n + m + 1 - \frac{\epsilon}{2} < t + \tau < n + m + 1, \\
&n - \frac{\epsilon}{2} < t < n + \frac{\epsilon}{2}, \quad n \leq t < n + \frac{\epsilon}{2}, \\
&n - m - 1 < t - \tau < n - m - 1 + \epsilon < n - m.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|g(t + \tau) - g(t)| &= |[(n + m + 1 - t - \tau)(-1)^{n+m} + (t + \tau - n - m)(-1)^{n+m+1}] \\
&\quad - [(n + 1 - t)(-1)^n + (t - n)(-1)^{n+1}]| \\
&= |[(t + \tau - n - m) - (n + m + 1 - t - \tau)] \\
&\quad - [(n + 1 - t) - (t - n)]| \\
&= | - 2(n + m + 1 - t - \tau) + 2(t - n)| < \epsilon, \\
|g(t - \tau) - g(t)| &= |[(n - m - t + \tau)(-1)^{n-m-1} + (t - \tau - n + m + 1)(-1)^{n-m}] \\
&\quad - [(n + 1 - t)(-1)^n + (t - n)(-1)^{n+1}]| \\
&= |[(n - m - t + \tau) - (t - \tau - n + m + 1)] \\
&\quad - [(n + 1 - t) - (t - n)]| \\
&= 2|\tau - m - 1| < \epsilon.
\end{aligned}$$

Summing up, $\tau \in T(g, \epsilon, N)$. □

The following two lemmas generalize an almost periodic result in [14, p. 164]: for any almost periodic function f, $T(f, \epsilon) \cap \mathbb{Z}$ is relatively dense.

Lemma A.2 *Suppose that $f \in AA_{uc}(\mathbb{R}, X)$, then for any ϵ, $N > 0$, the set $T(f, \epsilon, N) \cap \mathbb{Z}$ is relatively dense, and there is an $\eta > 0$ such that*

$$[T(f, \eta, N) \cap \mathbb{Z}] \pm [T(f, \eta, N) \cap \mathbb{Z}] \subset T(f, \epsilon, N) \cap \mathbb{Z}.$$

Proof Let $\epsilon > 0$ and the auxiliary function g in Lemma A.1 be given. Since $f \in KUC(\mathbb{R}, X)$, there is a $\delta > 0$ such that $|f(s) - f(t)| < \epsilon/2$ for all $s, t \in \mathbb{R}$ with $|s - t| < \delta$. For any $s \in \mathbb{R}$, $|s - \tau| < \delta$, where $\tau \in T(f, \epsilon/2, N)$, it is easy to see that

$$|f(t \pm s) - f(t)| \leq |f(t \pm s) - f(t \pm \tau)| + |f(t \pm \tau) - f(t)| < \epsilon$$

for all $|t| \leq N$. Thus $T(f, \epsilon/2, N) + (-\delta, \delta) \subset T(f, \epsilon, N)$.

Because the pair (f, g) is almost automorphic, Theorem 5.3 implies that the set $S := T(f, \epsilon/2, N) \cap T(g, 2\delta, N)$ is relatively dense. For any $\tau \in S$, by Lemma A.1 there is an $m \in 2\mathbb{Z}$ such that $|\tau - m| < \delta$. Therefore, $m \in T(f, \epsilon, N)$ and $T(f, \epsilon, N) \cap \mathbb{Z}$ is relatively dense.

Let $\eta > 0$ satisfy $T(f, \eta, N) \pm T(f, \eta, N) \subset T(f, \epsilon, N)$. It is easy to check that

$$\begin{aligned}[T(f, \eta, N) \cap \mathbb{Z}] \pm [T(f, \eta, N) \cap \mathbb{Z}] &\subset [T(f, \eta, N) \pm T(f, \eta, N)] \cap \mathbb{Z} \\ &\subset T(f, \epsilon, N) \cap \mathbb{Z}.\end{aligned}$$

□

Lemma A.3 *Suppose that* $f \in AA_{uc}(\mathbb{R}, X)$*, then for any* $\lambda, \epsilon, N > 0$*, the set*

$$\begin{aligned}T^{\lambda}(f, \epsilon, N) :&= T(f, \epsilon, N) \cap (\lambda\mathbb{Z}) \\ &= \{m\lambda;\, m \in \mathbb{Z}, |f(t \pm m\lambda) - f(t)| < \epsilon \text{ for all } |t| \leq N\}\end{aligned}$$

is relatively dense, and there is an $\eta > 0$ *such that*

$$T^{\lambda}(f, \eta, N) \pm T^{\lambda}(f, \eta, N) \subset T^{\lambda}(f, \epsilon, N). \tag{19}$$

Proof For any $\lambda > 0$, define a function $F(t) = f(\lambda t)$, $t \in \mathbb{R}$. So $F \in AA_{uc}(\mathbb{R}, X)$. From the equality

$$F(t \pm \tau) - F(t) = f(\lambda t \pm \lambda\tau) - f(\lambda t)$$

it follows that $T(f, \epsilon, N) = \lambda T(F, \epsilon, \lambda N)$ for all $\epsilon, N > 0$. By Lemma A.2, the set $T(F, \epsilon, \lambda N) \cap \mathbb{Z}$ is relatively dense. Therefore, the set

$$T(f, \epsilon, N) \cap (\lambda\mathbb{Z}) = \lambda[T(F, \epsilon, \lambda N) \cap \mathbb{Z}]$$

is relatively dense for all $\epsilon, N > 0$.

By Lemma A.2, there is an $\eta > 0$ such that

$$[T(F, \eta, \lambda N) \cap \mathbb{Z}] \pm [T(F, \eta, \lambda N) \cap \mathbb{Z}] \subset T(F, \epsilon, \lambda N) \cap \mathbb{Z},$$

which yields by multiplying both sides a factor λ,

$$T^{\lambda}(f, \eta, N) \pm T^{\lambda}(f, \eta, N) \subset T^{\lambda}(f, \epsilon, N).$$

□

Similar results also hold for Bochner s.a.a. sequences. But we do not state results like (19) since the set concerned is not symmetric with respect to 0. A desired symmetric set could be constructed by using Lemma 3.9.

Lemma A.4 *Suppose that* $\{\tau_j\}_{j\in\mathbb{Z}}$ *is a Bochner s.a.a. sequence. Then for any* $\epsilon > 0$*,* $N \in \mathbb{Z}_+$ *and any* $\lambda \in (0, \epsilon)$*, the set*

$$T_{-}^{\lambda}(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N) := -T(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N) \cap (\lambda\mathbb{Z})$$

$$= -\Big\{m\lambda;\, m \in \mathbb{Z},\, \max_{|n|\le N} |\tau_{n+p} + m\lambda - \tau_n| < \epsilon \textit{ for some } p \in \mathbb{Z}\Big\}$$

is relatively dense.

Proof By Lemma 3.7 and (ii) of Lemma 2.6, the set $T(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon/2, N)$ is relatively dense. Hence there is an $l > 0$ such that for any $a \in \mathbb{R}$, there exists $r(p) \in [a + \epsilon/2, a + \epsilon/2 + l] \cap T(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon/2, N)$ with

$$\max_{|n|\le N} |\tau_{n+p} + r(p) - \tau_n| < \frac{\epsilon}{2}.$$

Put $I = [r(p) - \epsilon/2, r(p) + \epsilon/2]$. Then $I \subset [a, a + l + \epsilon]$, and for all $r \in I$,

$$\max_{|n|\le N} |\tau_{n+p} + r - \tau_n| \le \max_{|n|\le N} |\tau_{n+p} + r(p) - \tau_n| + |r - r(p)| < \epsilon,$$

which implies $I \subset T(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N)$. For any $0 < \lambda < \epsilon$, there is a number $r_0 \in I$ such that $r_0 + \lambda \in I$. If r_0 is not a multiple of λ, then by adding a number small than λ we obtain $r = m\lambda \in [r_0, r_0 + \lambda] \subset I$. Since every interval of length $l + \epsilon$ contains a subinterval $I \subset T(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N)$ of length ϵ, the set $T_-^\lambda(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon, N)$ is relatively dense. □

Now we prove Theorem 6.3 which is indispensable in locating positions of variables and discontinuities.

Proof of Theorem 6.3 First choose a pair (δ, M), $\delta \in (0, \epsilon_2)$, $M \ge N_2$, satisfying Lemma 3.9 for (ϵ_2, N_2) and $\nu = 2$. By Lemmas A.3 and A.4, there is $\eta \in (0, \epsilon_1)$, so that for any $\lambda \in (0, \min\{\eta, \delta\})$, both $T^\lambda(f, \eta, N_1)$ and $T_-^\lambda(\{\tau_j\}_{j\in\mathbb{Z}}, \delta, M)$ are relatively dense and

$$\begin{aligned}
&T^\lambda(f, \eta, N_1) \pm T^\lambda(f, \eta, N_1) \subset T^\lambda(f, \epsilon_1, N_1),\\
&T_-^\lambda(\{\tau_j\}_{j\in\mathbb{Z}}, \delta, M) \pm T_-^\lambda(\{\tau_j\}_{j\in\mathbb{Z}}, \delta, M) \subset T_-^\lambda(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon_2, N_2).
\end{aligned}$$

Let $L_1 = L_1(\eta, , N_1)$, $L_2 = L_2(\delta, M) \in \mathbb{Z}$ be such that $L_1\lambda$ and $L_2\lambda$ are the inclusion lengths for $T^\lambda(f, \eta, N_1)$ and $T_-^\lambda(\{\tau_j\}_{j\in\mathbb{Z}}, \delta, M)$ respectively. Put $L_3 = \max\{L_1, L_2\}$ and

$$\begin{aligned}
S_n = \Big\{&(m_1, m_2) \in \mathbb{Z}^2;\, m_1\lambda \in [n\lambda, (n + L_3)\lambda] \cap T^\lambda(f, \eta, N_1),\\
&m_2\lambda \in [n\lambda, (n + L_3)\lambda] \cap T_-^\lambda(\{\tau_j\}_{j\in\mathbb{Z}}, \delta, M),\ n \in \mathbb{Z}\Big\},\\
S = &\bigcup_{n\in\mathbb{Z}} S_n.
\end{aligned}$$

It follows from the relative denseness property that $S_n \ne \emptyset$ for all $n \in \mathbb{Z}$. We say that two pairs of the numbers (m_1, m_2), $(m_1', m_2') \in S$ are equivalent if $m_1 - m_2 = m_1' - m_2'$. Because $|m_1 - m_2| \le L_3$, the difference $m_1 - m_2$ can take only a finite number of values. Hence the number of equivalence classes of this relation is finite. Choose the representative elements for these classes, $\big(m_1^{(\nu)}, m_2^{(\nu)}\big)$, $\nu = 1, 2, \ldots, s$. Set $L_4 = \max_{1\le\nu\le s} \big|m_1^{(\nu)}\big|$ and $L_5 = L_3 + 2L_4$. For any $n \in \mathbb{Z}$ let $(m_1, m_2) \in S_{n+L_4}$ and $\big(m_1^{(\nu)}, m_2^{(\nu)}\big)$ be the representative of the equivalence class containing the pair (m_1, m_2) such that $m_1 - m_2 = m_1^{(\nu)} - m_2^{(\nu)}$. Put $m = m_1 - m_1^{(\nu)} = m_2 - m_2^{(\nu)}$. From $\big|m_1^{(\nu)}\big| \le L_4$ it follows that

$$\begin{aligned}
&n \le m = m_1 - m_1^{(\nu)} \le n + 2L_4 + L_3,\\
&n\lambda \le m\lambda \le (n + L_5)\lambda.
\end{aligned}$$

The number $m\lambda$ defined in this way will be in $T^{\lambda}(f, \epsilon_1, N_1) \cap T^{\lambda}_{-}(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon_2, N_2)$. Indeed, $m_1\lambda, m_1^{(\nu)}\lambda \in T^{\lambda}(f, \eta, N_1)$ yield $m\lambda = m_1\lambda - m_1^{(\nu)}\lambda \in T^{\lambda}(f, \epsilon_1, N_1)$, and similarly $m_2\lambda$, $m_2^{(\nu)}\lambda \in T^{\lambda}_{-}(\{\tau_j\}_{j\in\mathbb{Z}}, \delta, M)$ yield $m\lambda = m_2\lambda - m_2^{(\nu)}\lambda \in T^{\lambda}_{-}(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon_2, N_2)$. Because the set of numbers m defined above is relatively dense with an inclusion length L_5, the set $T^{\lambda}(f, \epsilon_1, N_1) \cap T^{\lambda}_{-}(\{\tau_j\}_{j\in\mathbb{Z}}, \epsilon_2, N_2)$ is relatively dense. Using (1) and by a similar argument in proving Lemma 3.14, the set $P^{\lambda}(f, \{\tau_j\}_{j\in\mathbb{Z}}; \epsilon_1, \epsilon_2, N_1, N_2)$ is also relatively dense. □

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