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Rigidity of inversive distance circle packings revisited

Xu Xu [a,b]

[a] School of Mathematics and Statistics, Wuhan University, Wuhan 430072, PR China

[b] Hubei Key Laboratory of Computational Science (Wuhan University), Wuhan 430072, PR China

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A B S T R A C T

Inversive distance circle packing metric was introduced by P Bowers and K Stephenson [7] as a generalization of Thurston's circle packing metric [34]. They conjectured that the inversive distance circle packings are rigid. For nonnegative inversive distance, Guo [22] proved the infinitesimal rigidity and then Luo [27] proved the global rigidity. In this paper, based on an observation of Zhou [37], we prove this conjecture for inversive distance in $(-1, +\infty)$ by variational principles. We also study the global rigidity of a combinatorial curvature introduced in [14,16,19] with respect to the inversive distance circle packing metrics where the inversive distance is in $(-1, +\infty)$.

1. Introduction

1.1. Background

In his work on constructing hyperbolic structure on 3-manifolds, Thurston ([34], Chapter 13) introduced the notion of circle packing metric on triangulated surfaces with

E-mail address: xuxu2@whu.edu.cn.

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non-obtuse intersection angles. The requirement of prescribed intersection angles corresponds to the fact that the intersection angle of two circles is invariant under the Möbius transformations. For triangulated surfaces with Thurston's circle packing metrics, there are singularities at the vertices. The classical combinatorial curvature K_i is introduced to describe the singularity at the vertex v_i, which is defined as the angle deficit at v_i. Thurston's work generalized Andreev's work on circle packing metrics on a sphere [1,2] and gave a complete characterization of the space of the classical combinatorial curvature. As a corollary, he obtained the combinatorial-topological obstacle for the existence of a constant curvature circle packing with non-obtuse intersection angles, which could be written as combinatorial-topological inequalities. Zhou [37] recently generalized Andreev–Thurston Theorem to the case that the intersection angles are in $[0, \pi)$. Chow and Luo [9] introduced a combinatorial Ricci flow, a combinatorial analogue of the smooth surface Ricci flow, for triangulated surfaces with Thurston's circle packing metrics and established the equivalence between the existence of a constant curvature circle packing metric and the convergence of the combinatorial Ricci flow.

Inversive distance circle packing on triangulated surfaces was introduced by Bowers and Stephenson [7] as a generalization of Thurston's circle packing. Different from Thurston's circle packing, adjacent circles in inversive distance circle packing are allowed to be disjoint and the relative distance of the adjacent circles is measured by the inversive distance, which is a generalization of intersection angle. See Bowers–Hurdal [6], Stephenson [33] and Guo [22] for more information. The inversive distance circle packings have practical applications in medical imaging and computer graphics, see [24,35,36] for example. Bowers and Stephenson [7] conjectured that the inversive distance circle packings are rigid. Guo [22] proved the infinitesimal rigidity and then Luo [27] solved affirmably the conjecture for nonnegative inversive distance with Euclidean and hyperbolic background geometry. For the spherical background geometry, Ma and Schlenker [29] had a counterexample showing that there is in general no rigidity and John C. Bowers and Philip L. Bowers [4] obtained a new construction of their counterexample using the inversive geometry of the 2-sphere. John Bowers, Philip Bowers and Kevin Pratt [5] recently proved the global rigidity of convex inversive distance circle packings in the Riemann sphere. Ge and Jiang [12,13] recently studied the deformation of combinatorial curvature and found a way to search for inversive distance circle packing metrics with constant cone angles. They also obtained some results on the image of curvature map for inversive distance circle packings. Ge and Jiang [14] and Ge and the author [19] further extended a combinatorial curvature introduced by Ge and the author in [16–18] to inversive distance circle packings and studied the rigidity and deformation of the curvature.

In this paper, based on an obversion of Zhou [37], we prove Bowers and Stephenson's rigidity conjecture for inversive distance in $(-1, +\infty)$. The main tools are the variational principle established by Guo [22] for inversive distance circle packings and the extension of locally convex function introduced by Bobenko, Pinkall and Springborn [3] and systematically developed by Luo [27]. We refer to Glickenstein [20] for a nice geometric interpretation of the variational principle in [22]. There are many other works on variational principles on circle packings. See Brägger [8], Rivin [31], Leibon [25], Chow–Luo [9], Bobenko–Springborn [7], Marden–Rodin [30], Spingborn [32], Stephenson [33], Luo [28], Guo–Luo [23], Dai–Gu–Luo [10], Guo [21] and others.

1.2. Inversive distance circle packings

In this subsection, we briefly recall the inversive distance circle packing introduced by Bowers and Stephenson [7] in Euclidean and hyperbolic background geometry. For more information on inversive distance circle packing metrics, the readers can refer to Stephenson [33], Bowers and Hurdal [6] and Guo [22].

Suppose M is a closed surface with a triangulation $\mathcal{T} = \{V, E, F\}$, where V, E, F represent the sets of vertices, edges and faces respectively. Let $I : E \to (-1, +\infty)$ be a function assigning each edge $\{ij\}$ an inversive distance $I_{ij} \in (-1, +\infty)$, which is denoted as $I > -1$ in the paper. The triple $(M, \mathcal{T}, I)$ will be referred to as a weighted triangulation of M below. All the vertices are ordered one by one, marked by $v_1, \cdots, v_N$, where $N = |V|$ is the number of vertices, and we often use i to denote the vertex v_i for simplicity below. We use $i \sim j$ to denote that the vertices i and j are adjacent, i.e., there is an edge $\{ij\} \in E$ with i, j as end points. All functions $f : V \to \mathbb{R}$ will be regarded as column vectors in $\mathbb{R}^N$ and $f_i = f(v_i)$ is the value of f at v_i. And we use $C(V)$ to denote the set of functions defined on V. $\mathbb{R}_{>0}$ denotes the set of positive numbers in the paper.

Each map $r : V \to (0, +\infty)$ is a circle packing, which could be taken as the radius r_i of a circle attached to the vertex i. Given $(M, \mathcal{T}, I)$, we assign each edge $\{ij\}$ the length

$$l_{ij} = \sqrt{r_i^2 + r_j^2 + 2r_i r_j I_{ij}} \tag{1.1}$$

for Euclidean background geometry and

$$l_{ij} = \cosh^{-1}(\cosh(r_i)\cosh(r_j) + I_{ij}\sinh(r_i)\sinh(r_j)) \tag{1.2}$$

for hyperbolic background geometry, where I_{ij} is the Euclidean and hyperbolic inversive distance of the two circles centered at v_i and v_j with radii r_i and r_j respectively. Note that the length l_{ij} in (1.1) and (1.2) is well-defined for all $r_i > 0, r_j > 0$ under the condition $I_{ij} > -1$. If $I_{ij} \in (-1, 0)$, the two circles attached to the vertices i and j intersect with an obtuse angle. If $I_{ij} \in [0, 1]$, the two circles intersect with a non-obtuse angle. We can take $I_{ij} = \cos\Phi_{ij}$ with $\Phi_{ij} \in [0, \frac{\pi}{2}]$ and then the inversive distance circle packing is reduced to Thurston's circle packing. If $I_{ij} \in (1, +\infty)$, the two circles attached to the vertices i and j are disjoint. See Fig. 1 for possible arrangements of the circles. Guo [22] and Luo [27] systematically studied the rigidity of inversive distance circle packing metrics for nonnegative inversive distance $I \geq 0$, i.e. $I_{ij} \geq 0$ for every edge $\{ij\} \in E$. In this paper, we focus on the case that $I > -1$.

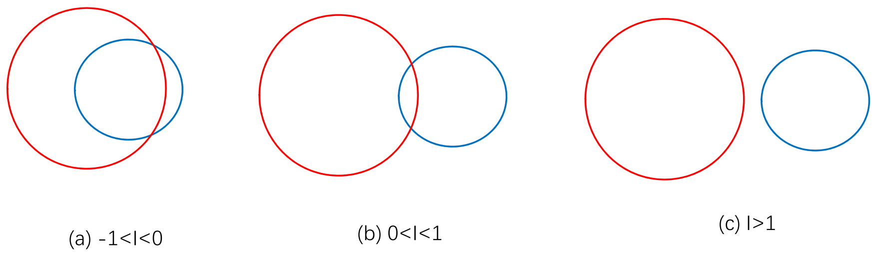

Fig. 1. Inversive distance circle packings.

The following is our main result, which solves Bowers and Stephenson's rigidity conjecture for inversive distance in $(-1, +\infty)$.

Theorem 1.1. *Given a closed triangulated surface $(M, \mathcal{T}, I)$ with inversive distance $I : E \to (-1, +\infty)$ satisfying*

$$I_{ij} + I_{ik}I_{jk} \geq 0, I_{ik} + I_{ij}I_{jk} \geq 0, I_{jk} + I_{ij}I_{ik} \geq 0 \tag{1.3}$$

for any topological triangle $\triangle ijk \in F$.

(1) *A Euclidean inversive distance circle packing on $(M, \mathcal{T})$ is determined by its combinatorial curvature $K : V \to \mathbb{R}$ up to scaling.*
(2) *A hyperbolic inversive distance circle packing on $(M, \mathcal{T})$ is determined by its combinatorial curvature $K : V \to \mathbb{R}$.*

Remark 1. For $I \in [0, 1]$, the above result was Andreev and Thurston's rigidity for circle packing with intersection angles in $[0, \frac{\pi}{2}]$. For $I \in (-1, 1]$, the above result was the rigidity for circle packing with intersection angles in $[0, \pi)$ recently obtained by Zhou [37]. For $I \geq 0$, the above result was the rigidity for inversive distance circle packing obtained by Guo [22] and Luo [27]. Our result unifies these results and allows the inversive distances to take values in a larger domain.

Remark 2. It is interesting to note that in Theorem 1.1, for a topological triangle $\triangle ijk \in F$, if one of I_{ij}, I_{ik}, I_{jk} is negative, the other two must be nonnegative. So at most one of I_{ij}, I_{ik}, I_{jk} is negative.

We further extend the rigidity to combinatorial α-curvature introduced in [14–19], which is defined as

$$R_{\alpha,i} = \frac{K_i}{s_i^\alpha}$$

for $\alpha \in \mathbb{R}$, where $s_i = r_i$ for the Euclidean background geometry and $s_i = \tanh \frac{r_i}{2}$ for the hyperbolic background geometry.

Theorem 1.2. *Given a closed triangulated surface $(M, \mathcal{T}, I)$ with inversive distance $I : E \to (-1, +\infty)$ satisfying*

$$I_{ij} + I_{ik}I_{jk} \geq 0, I_{ik} + I_{ij}I_{jk} \geq 0, I_{jk} + I_{ij}I_{ik} \geq 0$$

for any topological triangle $\triangle ijk \in F$. $\overline{R}$ is a given function defined on the vertices of $(M, \mathcal{T})$.

(1) *If $\alpha\overline{R} \equiv 0$, there exists at most one Euclidean inversive distance circle packing metric with combinatorial α-curvature $\overline{R}$ up to scaling. If $\alpha\overline{R} \leq 0$ and $\alpha\overline{R} \not\equiv 0$, there exists at most one Euclidean inversive distance circle packing metric with combinatorial α-curvature $\overline{R}$.*
(2) *If $\alpha\overline{R} \leq 0$, there exists at most one hyperbolic inversive distance packing metric with combinatorial α-curvature $\overline{R}$.*

1.3. Plan of paper

The paper is organized as follows. In Section 2, we study the Euclidean inversive distance circle packing metrics and prove Theorem 1.1 and 1.2 for the Euclidean background geometry. In Section 3, we study the hyperbolic inversive distance circle packing metrics and prove Theorem 1.1, 1.2 for the hyperbolic background geometry.

2. Euclidean inversive distance circle packings

2.1. Admissible space of Euclidean inversive distance circle packing metrics for a single triangle

Given a weighted triangulated surface $(M, \mathcal{T}, I)$ with weight $I > -1$. Suppose $\triangle ijk$ is a topological triangle in F. Here and in the following, to simplify notations, when we are discussing a triangle $\triangle ijk$, we use l_i to denote the length of the edge $\{jk\}$ and use I_i to denote the inversive distance of the two circles at the vertices j and k. In the Euclidean background geometry, the length l_i of the edge $\{jk\}$ is then defined by

$$l_i = \sqrt{r_j^2 + r_k^2 + 2r_jr_kI_i}. \tag{2.1}$$

For $I > -1$, in order that the lengths l_i, l_j, l_k for $\Delta ijk \in F$ satisfy the triangle inequalities, there are some restrictions on the radii. Denote the admissible space of the radius vectors for a face $\Delta ijk \in F$ as

$$\Omega_{ijk}^E := \{(r_i, r_j, r_k) \in \mathbb{R}_{>0}^3 | l_i < l_j + l_k, l_j < l_i + l_k, l_k < l_i + l_j\}. \tag{2.2}$$

In the case of $I \in [0,1]$, as noted by Thurston [34], $\Omega^E_{ijk} = \mathbb{R}^3_{>0}$. However, in general, $\Omega^E_{ijk} \neq \mathbb{R}^3_{>0}$ for $I \in (-1,+\infty)$. It is proved [22] that the admissible space Ω^E_{ijk} for $I \geq 0$ is a simply connected open subset of $\mathbb{R}^3_{>0}$ and Ω^E_{ijk} may not be convex. Set

$$\Omega^E = \cap_{\Delta ijk \in F} \Omega^E_{ijk} \tag{2.3}$$

to be the space of admissible radius function on the surface. Ω^E is obviously an open subset of $\mathbb{R}^N_{>0}$. Every $r \in \Omega$ is called an inversive distance circle packing metric.

As noted in [22], in order that the edge lengths l_i, l_j, l_k satisfy the triangle inequalities, we just need

$$\begin{aligned} 0 &< (l_i + l_j + l_k)(l_i + l_j - l_k)(l_i + l_k - l_j)(l_j + l_k - l_i) \\ &= 4l_i^2 l_k^2 - (l_i^2 + l_k^2 - l_j^2)^2 \\ &= 2l_i^2 l_j^2 + 2l_i^2 l_k^2 + 2l_j^2 l_k^2 - l_i^4 - l_j^4 - l_k^4. \end{aligned} \tag{2.4}$$

Substituting the definition of edge length (2.1) in the Euclidean background geometry into (2.4), by direct calculations, we have

$$\begin{aligned} &\frac{1}{4}(l_i + l_j + l_k)(l_i + l_j - l_k)(l_i + l_k - l_j)(l_j + l_k - l_i) \\ &\quad = r_i^2 r_j^2 (1 - I_k^2) + r_i^2 r_k^2 (1 - I_j^2) + r_j^2 r_k^2 (1 - I_i^2) \\ &\qquad + 2r_i^2 r_j r_k (I_i + I_j I_k) + 2r_i r_j^2 r_k (I_j + I_i I_k) + 2r_i r_j r_k^2 (I_k + I_i I_j) > 0. \end{aligned}$$

Denote

$$\gamma_{ijk} := I_i + I_j I_k, \gamma_{jik} := I_j + I_i I_k, \gamma_{kij} := I_k + I_i I_j, \tag{2.5}$$

then we have the following result on Euclidean triangle inequalities.

Lemma 2.1 *([22]). Suppose $(M, \mathcal{T}, I)$ is a weighted triangulated surface with weight $I > -1$ and $\triangle ijk$ is a topological triangle in F. The edge lengths l_i, l_j, l_k defined by (2.1) satisfy the triangle inequalities if and only if*

$$\begin{aligned} &r_i^2 r_j^2 (1 - I_k^2) + r_i^2 r_k^2 (1 - I_j^2) + r_j^2 r_k^2 (1 - I_i^2) + 2r_i^2 r_j r_k \gamma_{ijk} + 2r_i r_j^2 r_k \gamma_{jik} \\ &\quad + 2r_i r_j r_k^2 \gamma_{kij} > 0. \end{aligned} \tag{2.6}$$

We have the following direct corollary obtained in [37] by Lemma 2.1.

Corollary 2.2. *If $I_i, I_j, I_k \in (-1, 1]$ and $\gamma_{ijk} \geq 0$, $\gamma_{jik} \geq 0$, $\gamma_{kij} \geq 0$, then the triangle inequalities are satisfied for any $(r_i, r_j, r_k) \in \mathbb{R}^3_{>0}$.*

Remark 3. Specially, if $I_i = \cos\Phi_i, I_j = \cos\Phi_j, I_k = \cos\Phi_k$ with $\Phi_i, \Phi_j, \Phi_k \in [0, \frac{\pi}{2}]$, then we have $I_i, I_j, I_k \in (-1, 1]$ and $\gamma_{ijk} \geq 0, \gamma_{jik} \geq 0, \gamma_{kij} \geq 0$. So the triangle inequalities are satisfied for all radius vectors in $\mathbb{R}^3_{>0}$, which was obtained by Thurston in [34]. However, if we only require $\Phi_i, \Phi_j, \Phi_k \in [0, \pi)$, then (2.6) is equivalent to

$$\begin{aligned} r_i^2 r_j^2 \sin^2\Phi_k + r_i^2 r_k^2 \sin^2\Phi_j + r_j^2 r_k^2 \sin^2\Phi_i + 2r_i^2 r_j r_k(\cos\Phi_i + \cos\Phi_j\cos\Phi_k) \\ + 2r_i r_j^2 r_k(\cos\Phi_j + \cos\Phi_i\cos\Phi_k) + 2r_i r_j r_k^2(\cos\Phi_k + \cos\Phi_i\cos\Phi_j) > 0. \end{aligned}$$

Specially, if $\Phi_i + \Phi_j \leq \pi, \Phi_i + \Phi_k \leq \pi, \Phi_j + \Phi_k \leq \pi$ [37], or $\Phi_i = \Phi_j \in [0, \frac{\pi}{2}]$ [37], or $\Phi_i = \Phi_j = \Phi_k \in [0, \pi)$, the triangle inequalities are satisfied.

By Lemma 2.1, the admissible space Ω^E_{ijk} for the topological triangle $\triangle ijk \in F$ may not be the whole space $\mathbb{R}^3_{>0}$. Furthermore, it is not always convex for all $I_i, I_j, I_k \in (-1, +\infty)$. However, we have the following useful lemma on the structure of the admissible space Ω^E_{ijk}.

Lemma 2.3. *Given a weighted triangulated surface $(M, \mathcal{T}, I)$ with $I > -1$. For a topological triangle $\triangle ijk \in F$, if*

$$\gamma_{ijk} \geq 0, \gamma_{jik} \geq 0, \gamma_{kij} \geq 0, \tag{2.7}$$

then the admissible space Ω^E_{ijk} is a simply connected open subset of $\mathbb{R}^3_{>0}$. Furthermore, for each connected component V of $\mathbb{R}^3_{>0} \setminus \Omega^E_{ijk}$, the intersection $V \cap \overline{\Omega}^E_{ijk}$ is a connected component of $\overline{\Omega}^E_{ijk} \setminus \Omega^E_{ijk}$, on which θ_i is a constant function.

Proof. Define

$$\begin{aligned} F : \mathbb{R}^3_{>0} &\to \mathbb{R}^3_{>0} \\ (r_i, r_j, r_k) &\mapsto (r_j^2 + r_k^2 + 2r_j r_k I_i, r_i^2 + r_k^2 + 2r_i r_k I_j, r_i^2 + r_j^2 + 2r_i r_j I_k) \end{aligned}$$

and

$$\begin{aligned} G : \mathbb{R}^3_{>0} &\to \mathbb{R}^3_{>0} \\ (l_i, l_j, l_k) &\mapsto (l_i^2, l_j^2, l_k^2), \end{aligned}$$

then G is a diffeomorphism of $\mathbb{R}^3_{>0}$ and $H = G^{-1} \circ F$ is the map sending (r_i, r_j, r_k) to (l_i, l_j, l_k).

We first prove that H is injective. To prove this, we just need to prove that F is injective. Note that

$$\frac{\partial(F_i, F_j, F_k)}{\partial(r_i, r_j, r_k)} = 2\begin{pmatrix} 0 & r_j + r_k I_i & r_k + r_j I_i \\ r_i + r_k I_j & 0 & r_k + r_i I_j \\ r_i + r_j I_k & r_j + r_i I_k & 0 \end{pmatrix},$$

which implies that

$$\begin{aligned}
&\left|\frac{\partial(F_i,F_j,F_k)}{\partial(r_i,r_j,r_k)}\right| \\
&\quad = 8(r_j+r_kI_i)(r_k+r_iI_j)(r_k+r_iI_j)+8(r_k+r_jI_i)(r_i+r_kI_j)(r_j+r_iI_k) \\
&\quad = 16r_ir_jr_k(1+I_iI_jI_k)+8r_i\gamma_{ijk}(r_j^2+r_k^2)+8r_j\gamma_{jik}(r_i^2+r_k^2)+8r_k\gamma_{kij}(r_i^2+r_j^2).
\end{aligned}$$

By the condition (2.7) and the Cauchy inequality, we have

$$\begin{aligned}
\left|\frac{\partial(F_i,F_j,F_k)}{\partial(r_i,r_j,r_k)}\right| &\geq 16r_ir_jr_k(1+I_iI_jI_k+\gamma_{ijk}+\gamma_{jik}+\gamma_{kij}) \\
&= 16r_ir_jr_k(1+I_i)(1+I_j)(1+I_k).
\end{aligned}$$

By the condition that $I_i,I_j,I_k\in(-1,+\infty)$, we have $\left|\frac{\partial(F_i,F_j,F_k)}{\partial(r_i,r_j,r_k)}\right|>0$ for any $r\in\mathbb{R}^3_{>0}$. If there are $r=(r_i,r_j,r_k)\in\mathbb{R}^3_{>0}$ and $r'=(r_i',r_j',r_k')\in\mathbb{R}^3_{>0}$ satisfying $F(r)=F(r')$, then we have

$$0=F(r)-F(r')=\frac{\partial(F_i,F_j,F_k)}{\partial(r_i,r_j,r_k)}\Big|_{r+\theta(r-r')}\cdot(r-r')^T,\quad \theta\in(0,1),$$

which implies $r=r'$ by the nondegeneracy of $\frac{\partial(F_i,F_j,F_k)}{\partial(r_i,r_j,r_k)}$ on $\mathbb{R}^3_{>0}$. So the map F is injective on $\mathbb{R}^3_{>0}$, which implies that H is injective on $\mathbb{R}^3_{>0}$.

Note that

$$\begin{aligned}
F_i &= r_j^2+r_k^2+2r_jr_kI_i\geq 2r_jr_k(1+I_i), \\
F_j &= r_i^2+r_k^2+2r_ir_jI_k\geq 2r_ir_k(1+I_j), \\
F_k &= r_i^2+r_j^2+2r_ir_jI_k\geq 2r_ir_j(1+I_k).
\end{aligned}$$

By the condition that $I_i,I_j,I_k\in(-1,+\infty)$, if F is bounded, we have r_ir_j, r_ir_k, r_jr_k are bounded, which implies that $r_i^2+r_j^2, r_i^2+r_k^2, r_j^2+r_k^2$ are bounded. Similarly, we have $F_i\leq(1+|I_i|)(r_j^2+r_j^2)$. This implies that F is a proper map from $\mathbb{R}^3_{>0}$ to $\mathbb{R}^3_{>0}$. By the invariance of domain theorem, we have F is a diffeomorphism between $\mathbb{R}^3_{>0}$ and $F(\mathbb{R}^3_{>0})$. And then H is a diffeomorphism between $\mathbb{R}^3_{>0}$ and $H(\mathbb{R}^3_{>0})$.

Set

$$\mathcal{L}=\{(l_i,l_j,l_k)|l_i+l_j>l_k, l_i+l_k>l_j, l_j+l_k>l_i\},$$

then $\Omega_{ijk}^E=H^{-1}(H(\mathbb{R}^3_{>0})\cap\mathcal{L})$. To prove that Ω_{ijk}^E is simply connected, we just need to prove that $H(\mathbb{R}^3_{>0})\cap\mathcal{L}$ is a cone. Note that $\mathcal{L}$ is a cone in $\mathbb{R}^3_{>0}$ bounded by three planes

$$L_i = \{(l_i, l_j, l_k) \in \mathbb{R}^3_{>0} | l_i = l_j + l_k\},$$
$$L_j = \{(l_i, l_j, l_k) \in \mathbb{R}^3_{>0} | l_j = l_i + l_k\},$$
$$L_k = \{(l_i, l_j, l_k) \in \mathbb{R}^3_{>0} | l_k = l_i + l_j\}.$$

Note that H is a diffeomorphism between $\mathbb{R}^3_{>0}$ and $H(\mathbb{R}^3_{>0})$, $H(\mathbb{R}^3_{>0})$ is a cone bounded by three quadratic surfaces

$$\Sigma_i = \{(l_i, l_j, l_k) \in \mathbb{R}^3_{>0} | l_i^2 = l_j^2 + l_k^2 + 2l_jl_kI_i\},$$
$$\Sigma_i = \{(l_i, l_j, l_k) \in \mathbb{R}^3_{>0} | l_j^2 = l_i^2 + l_k^2 + 2l_il_kI_j\},$$
$$\Sigma_i = \{(l_i, l_j, l_k) \in \mathbb{R}^3_{>0} | l_k^2 = l_i^2 + l_j^2 + 2l_il_jI_k\}.$$

In fact, if $r_i = 0$, then $l_j = r_k, l_k = r_j$ and $l_i^2 = r_j^2 + r_k^2 + 2r_jr_kI_i = l_j^2 + l_k^2 + 2l_jl_kI_i$. Σ_i is in fact the image of $r_i = 0$ under H. By the diffeomorphism of H, Σ_i, Σ_j, Σ_k are mutually disjoint. Furthermore, if $I_i \in (-1, 1]$, we have $(l_j - l_k)^2 < l_i^2 \leq (l_j + l_k)^2$ on Σ_i. And if $I_i \in (1, +\infty)$, we have $l_i^2 > (l_j + l_k)^2$ on Σ_i. This implies that $\Sigma_i \subset \overline{\mathcal{L}}$ if $I_i \in (-1, 1]$ and $\Sigma_i \cap \mathcal{L} = \emptyset$ if $I_i \in (1, +\infty)$. Similar results hold for Σ_j and Σ_k.

To prove that $H(\mathbb{R}^3_{>0}) \cap \mathcal{L}$ is a cone, we just need to consider the following cases by the symmetry between i, j, k.

If $I_i, I_j, I_k \in (-1, 1]$, $H(\mathbb{R}^3_{>0}) \cap \mathcal{L}$ is a cone bounded by $\Sigma_i, \Sigma_j, \Sigma_k$ and $H(\mathbb{R}^3_{>0}) \cap \mathcal{L} = H(\mathbb{R}^3_{>0})$.

If $I_i, I_j \in (-1, 1]$ and $I_k \in (1, +\infty)$, $H(\mathbb{R}^3_{>0}) \cap \mathcal{L}$ is a cone bounded by Σ_i, Σ_j and L_k.

If $I_i \in (-1, 1]$ and $I_j, I_k \in (1, +\infty)$, $H(\mathbb{R}^3_{>0}) \cap \mathcal{L}$ is a cone bounded by Σ_i, L_j and L_k.

If $I_i, I_j, I_k \in (1, +\infty)$, $H(\mathbb{R}^3_{>0}) \cap \mathcal{L}$ is a cone bounded by L_i, L_j and L_k. In this case, $H(\mathbb{R}^3_{>0}) \cap \mathcal{L} = \mathcal{L}$.

For any case, $H(\mathbb{R}^3_{>0}) \cap \mathcal{L}$ is a cone in $\mathbb{R}^3_{>0}$. By the fact that H is a diffeomorphism between $\mathbb{R}^3_{>0}$ and $H(\mathbb{R}^3_{>0})$, we have the admissible space $\Omega^E_{ijk} = H^{-1}(H(\mathbb{R}^3_{>0}) \cap \mathcal{L})$ is simply connected.

By the analysis above, if $H(\mathbb{R}^3_{>0}) \subset \mathcal{L}$, then $\Omega^E_{ijk} = H^{-1}(H(\mathbb{R}^3_{>0}) \cap \mathcal{L}) = \mathbb{R}^3_{>0}$. If $H(\mathbb{R}^3_{>0}) \setminus \mathcal{L} \neq \emptyset$, then Ω^E_{ijk} is a proper subset of $\mathbb{R}^3_{>0}$. If $I_i > 1$, the boundary component $\Sigma_i = \{l_i^2 = l_j^2 + l_k^2 + 2l_jl_kI_i\}$ is out of the set $\mathcal{L}$. By the fact that $\Omega^E_{ijk} = H^{-1}(H(\mathbb{R}^3_{>0}) \cap \mathcal{L})$ and $H : \mathbb{R}^3_{>0} \to H(\mathbb{R}^3_{>0})$ is a diffeomorphism, we have $H^{-1}(L_i)$ is a connected boundary component of Ω^E_{ijk}, on which $\theta_i = \pi, \theta_j = \theta_k = 0$. This completes the proof of the lemma. □

Corollary 2.4. *For a topological triangle $\triangle ijk \in F$ with inversive distance $I > -1$ and $\gamma_{ijk} \geq 0, \gamma_{jik} \geq 0, \gamma_{kij} \geq 0$, the functions $\theta_i, \theta_j, \theta_k$ defined on Ω^E_{ijk} could be continuously extended by constant to $\widetilde{\theta}_i, \widetilde{\theta}_j, \widetilde{\theta}_k$ defined on $\mathbb{R}^3_{>0}$.*

Remark 4.

(1) If $I_i, I_j, I_k \in [0, +\infty)$, obviously we have $\gamma_{ijk} \geq 0$, $\gamma_{jik} \geq 0$, $\gamma_{kij} \geq 0$. So Lemma 2.3 generalizes Lemma 3 in [22] obtained by Guo.
(2) If $I_i, I_j, I_k \in (-1, 1]$ and $\gamma_{ijk} \geq 0$, $\gamma_{jik} \geq 0$, $\gamma_{kij} \geq 0$, by the proof of Lemma 2.3, $\Omega^E_{ijk} = \mathbb{R}^3_{>0}$, which is obtained by Zhou [37].
(3) The condition $I_i, I_j, I_k \in (-1, +\infty)$ and $\gamma_{ijk} \geq 0$, $\gamma_{jik} \geq 0$, $\gamma_{kij} \geq 0$ contains more cases, for example, $I_i = -\frac{1}{2}$, $I_j = 1$ and $I_k = 2$, in which case the admissible space Ω^E_{ijk} is still simply connected.

2.2. Infinitesimal rigidity of Euclidean inversive distance circle packings

Set $u_i = \ln r_i$, then we have $\mathcal{U}^E_{ijk} := \ln(\Omega^E_{ijk})$ is a simply connected subset of $\mathbb{R}^3$ by Lemma 2.3. If $(r_i, r_j, r_k) \in \Omega^E_{ijk}$, l_i, l_j, l_k satisfy the triangle inequalities and forms a Euclidean triangle. Denote the inner angle at the vertex i as θ_i. We have the following useful lemma.

Lemma 2.5. *For any topological triangle $\triangle ijk \in F$, we have*

$$\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i} = \frac{1}{Al_k^2}\left[r_i^2 r_j^2 (1 - I_k^2) + r_i^2 r_j r_k \gamma_{ijk} + r_i r_j^2 r_k \gamma_{jik}\right] \tag{2.8}$$

on $\mathcal{U}^E_{ijk}$, where $A = l_j l_k \sin\theta_i$.

Proof. By the cosine law, we have $l_i^2 = l_j^2 + l_k^2 - 2l_j l_k \cos\theta_i$. Taking the derivative with respect to l_i, we have $\frac{\partial\theta_i}{\partial l_i} = \frac{l_i}{A}$, where $A = l_j l_k \sin\theta_i$ is two times of the area of $\triangle ijk$. Similarly, we have $\frac{\partial\theta_i}{\partial l_j} = \frac{-l_i\cos\theta_k}{A}$, $\frac{\partial\theta_i}{\partial l_k} = \frac{-l_i\cos\theta_j}{A}$. By the definition of l_i, l_j, l_k, we have

$$\frac{\partial l_i}{\partial r_j} = \frac{r_j + r_k I_i}{l_i}, \frac{\partial l_j}{\partial r_j} = 0, \frac{\partial l_k}{\partial r_j} = \frac{r_j + r_i I_k}{l_k}.$$

Then

$$\begin{aligned}
\frac{\partial\theta_i}{\partial u_j} &= r_j \frac{\partial\theta_i}{\partial r_j} \\
&= r_j\left(\frac{\partial\theta_i}{\partial l_i}\frac{\partial l_i}{\partial r_j} + \frac{\partial\theta_i}{\partial l_k}\frac{\partial l_k}{\partial r_j}\right) \\
&= r_j\left[\frac{r_j + r_k I_i}{A} - \frac{l_i\cos\theta_j(r_j + r_i I_k)}{Al_k}\right] \\
&= \frac{1}{Al_k}\left[l_k(r_j^2 - r_j r_k I_i) - \frac{l_i^2 + l_k^2 - l_j^2}{2l_k}(r_j^2 + r_i r_j I_k)\right] \\
&= \frac{1}{Al_k^2}\left[r_i^2 r_j^2(1 - I_k^2) + r_i^2 r_j r_k\gamma_{ijk} + r_i r_j^2 r_k \gamma_{jik}\right],
\end{aligned}$$

where the cosine law is used in the third line and the definition of the length (2.1) is used in the fourth line. This also implies $\frac{\partial\theta_i}{\partial u_j} = \frac{\partial\theta_j}{\partial u_i}$. $\square$

Remark 5. The equation $\frac{\partial\theta_i}{\partial u_j} = \frac{\partial\theta_j}{\partial u_i}$ has been obtained under different conditions in [9, 11,22] and the formulas for $\frac{\partial\theta_i}{\partial l_j}$ and $\frac{\partial\theta_i}{\partial l_i}$ was obtained by Chow and Luo [9]. In general, for $I_i, I_j, I_k \in (-1, +\infty)$, $\frac{\partial\theta_i}{\partial u_j}$ have no sign. However, if $I_i, I_j, I_k \in (-1, 1]$ and $\gamma_{ijk} \geq 0$, $\gamma_{jik} \geq 0$, $\gamma_{kij} \geq 0$, by (2.8), we have $\frac{\partial\theta_i}{\partial u_j} \geq 0$. Furthermore, $\frac{\partial\theta_i}{\partial u_j} = 0$ if and only if $I_k = 1$ and $I_i + I_j = 0$. Especially, if $I_i = \cos\Phi_i, I_j = \cos\Phi_j, I_k = \cos\Phi_k$ with $\Phi_i, \Phi_j, \Phi_k \in [0, \frac{\pi}{2}]$, we have $\frac{\partial\theta_i}{\partial u_j} \geq 0$, and $\frac{\partial\theta_i}{\partial u_j} = 0$ if and only if $\Phi_k = 0$ and $\Phi_i = \Phi_j = \frac{\pi}{2}$.

Remark 6. Geometrically, the three circles at the vertices have a power center O. It is known [35,36] that $\frac{\partial\theta_i}{\partial u_j} = \frac{h_k}{l_k}$, where h_k is the signed distance of the power center O to the edge $\{ij\}$, which is positive if O is in the interior of the triangle $\triangle ijk$ and negative if the power center O is out of the triangle $\triangle ijk$. So under the condition $I_i, I_j, I_k \in (-1, 1]$ and $\gamma_{ijk} \geq 0$, $\gamma_{jik} \geq 0$, $\gamma_{kij} \geq 0$, the power center O is in the triangle $\triangle ijk$.

Lemma 2.5 shows that the matrix

$$\Lambda^E_{ijk} := \frac{\partial(\theta_i, \theta_j, \theta_k)}{\partial(u_i, u_j, u_k)} = \begin{pmatrix} \frac{\partial\theta_i}{\partial u_i} & \frac{\partial\theta_i}{\partial u_j} & \frac{\partial\theta_i}{\partial u_k} \\ \frac{\partial\theta_j}{\partial u_i} & \frac{\partial\theta_j}{\partial u_j} & \frac{\partial\theta_j}{\partial u_k} \\ \frac{\partial\theta_k}{\partial u_i} & \frac{\partial\theta_k}{\partial u_j} & \frac{\partial\theta_k}{\partial u_k} \end{pmatrix}$$

is symmetric on $\mathcal{U}^E_{ijk}$. For the matrix Λ^E_{ijk}, we have the following useful property.

Lemma 2.6. *For any topological triangle $\triangle ijk \in F$ with inversive distance $I_i, I_j, I_k \in (-1, +\infty)$ and $\gamma_{ijk} \geq 0$, $\gamma_{jik} \geq 0$, $\gamma_{kij} \geq 0$, the matrix Λ^E_{ijk} is negative semi-definite with rank 2 and kernel $\{t(1,1,1)^T | t \in \mathbb{R}\}$ on $\mathcal{U}^E_{ijk}$.*

Proof. The proof is parallel to that of Lemma 6 in [22] with some modifications. By the calculations in Lemma 2.5, for a triangle $\triangle ijk \in F$, we have

$$\begin{pmatrix} d\theta_i \\ d\theta_j \\ d\theta_k \end{pmatrix} = -\frac{1}{A} \begin{pmatrix} l_i & 0 & 0 \\ 0 & l_j & 0 \\ 0 & 0 & l_k \end{pmatrix} \begin{pmatrix} -1 & \cos\theta_k & \cos\theta_j \\ \cos\theta_k & -1 & \cos\theta_i \\ \cos\theta_j & \cos\theta_i & -1 \end{pmatrix}$$

$$\times \begin{pmatrix} 0 & \frac{l_i^2 + r_j^2 - r_k^2}{2l_i r_j} & \frac{l_i^2 + r_k^2 - r_j^2}{2l_i r_k} \\ \frac{l_j^2 + r_i^2 - r_k^2}{2l_j r_i} & 0 & \frac{l_j^2 + r_k^2 - r_i^2}{2l_j r_k} \\ \frac{l_k^2 + r_i^2 - r_j^2}{2l_k r_i} & \frac{l_k^2 + r_j^2 - r_i^2}{2l_k r_i} & 0 \end{pmatrix} \begin{pmatrix} r_i & 0 & 0 \\ 0 & r_j & 0 \\ 0 & 0 & r_k \end{pmatrix} \begin{pmatrix} du_i \\ du_j \\ du_k \end{pmatrix}.$$

Write the above formula as

$$\begin{pmatrix} d\theta_i \\ d\theta_j \\ d\theta_k \end{pmatrix} = -\frac{1}{A}N \begin{pmatrix} du_i \\ du_j \\ du_k \end{pmatrix}.$$

By the cosine law, we have

$$4N = \begin{pmatrix} -2l_i^2 & l_i^2+l_j^2-l_k^2 & l_k^2+l_i^2-l_j^2 \\ l_i^2+l_j^2-l_k^2 & -2l_j^2 & l_j^2+l_k^2-l_i^2 \\ l_k^2+l_i^2-l_j^2 & l_j^2+l_k^2-l_i^2 & -2l_k^2 \end{pmatrix} \begin{pmatrix} \frac{1}{l_i^2} & 0 & 0 \\ 0 & \frac{1}{l_j^2} & 0 \\ 0 & 0 & \frac{1}{l_k^2} \end{pmatrix}$$
$$\times \begin{pmatrix} 0 & l_i^2+r_j^2-r_k^2 & l_i^2+r_k^2-r_j^2 \\ l_j^2+r_i^2-r_k^2 & 0 & l_j^2+r_k^2-r_i^2 \\ l_k^2+r_i^2-r_j^2 & l_k^2+r_j^2-r_i^2 & 0 \end{pmatrix}$$

By Lemma 2.5, we have $4N$ is symmetric. Furthermore, note that $\theta_i+\theta_j+\theta_k=\pi$, we have $0=\frac{\partial\theta_i}{\partial u_i}+\frac{\partial\theta_j}{\partial u_i}+\frac{\partial\theta_k}{\partial u_i}=\frac{\partial\theta_i}{\partial u_i}+\frac{\partial\theta_i}{\partial u_j}+\frac{\partial\theta_i}{\partial u_k}$. Then we can write $4N$ as

$$4N = \begin{pmatrix} -A-B & A & B \\ A & -A-C & C \\ B & C & -B-C \end{pmatrix}.$$

To prove Λ_{ijk}^E is negative semi-definite, we just need to prove that $4N$ is positive semi-definite. By direct calculations, we have

$$\begin{aligned} |\lambda I - 4N| &= \begin{vmatrix} \lambda+A+B & -A & -B \\ -A & \lambda+A+C & -C \\ -B & -C & \lambda+B+C \end{vmatrix} \\ &= \lambda[\lambda^2+2(A+B+C)\lambda+3(AB+AC+BC)]. \end{aligned}$$

We want to prove that the equation

$$\lambda^2+2(A+B+C)\lambda+3(AB+AC+BC)=0$$

has two positive roots. Note that for this quadratic equation, we have

$$\Delta = 4(A+B+C)^2-12(AB+AC+BC) = 4(A^2+B^2+C^2-AB-AC-BC) \geq 0,$$

so we just need to prove that $A+B+C<0$ and $AB+AC+BC>0$.

By direct calculations, we have

$$-2(A+B+C) = l_i^2+l_j^2+l_k^2+(l_j^2-l_k^2)\frac{r_j^2-r_k^2}{l_i^2}+(l_k^2-l_i^2)\frac{r_j^2-r_i^2}{l_j^2}+(l_i^2-l_j^2)\frac{r_i^2-r_j^2}{l_k^2}.$$

So $A+B+C<0$ is equivalent to

$$l_i^2+l_j^2+l_k^2+(l_j^2-l_k^2)\frac{r_j^2-r_k^2}{l_i^2}+(l_k^2-l_i^2)\frac{r_j^2-r_i^2}{l_j^2}+(l_i^2-l_j^2)\frac{r_i^2-r_j^2}{l_k^2}>0,$$

which is equivalent to

$$\begin{aligned}&l_i^2l_j^2l_k^2(l_i^2+l_j^2+l_k^2)+l_i^2r_i^2(l_i^2l_j^2+l_i^2l_k^2-l_j^4-l_k^4)+l_j^2r_j^2(l_i^2l_j^2+l_j^2l_k^2-l_i^4-l_k^4)\\&\quad+l_k^2r_k^2(l_i^2l_k^2+l_j^2l_k^2-l_i^4-l_j^4)>0.\end{aligned}$$

Note that

$$\begin{aligned}&2[l_i^2l_j^2l_k^2(l_i^2+l_j^2+l_k^2)+l_i^2r_i^2(l_i^2l_j^2+l_i^2l_k^2-l_j^4-l_k^4)\\&\qquad+l_j^2r_j^2(l_i^2l_j^2+l_j^2l_k^2-l_i^4-l_k^4)+l_k^2r_k^2(l_i^2l_k^2+l_j^2l_k^2-l_i^4-l_j^4)]\\&\quad=2l_i^2l_j^2l_k^2(l_i^2+l_j^2+l_k^2)+l_i^2r_i^2(l_i^4-l_j^4-l_k^4-2l_j^2l_k^2)+l_j^2r_j^2(l_j^4-l_i^4-l_k^4-2l_i^2l_k^2)\\&\qquad+l_k^2r_k^2(l_k^4-l_i^4-l_j^4-2l_i^2l_j^2)+(l_i^2r_i^2+l_j^2r_j^2+l_k^2r_k^2)(2l_i^2l_j^2\\&\qquad+2l_i^2l_k^2+2l_j^2l_k^2-l_i^4-l_j^4-l_k^4).\end{aligned}$$

By the triangle inequalities, we have

$$2l_i^2l_j^2+2l_i^2l_k^2+2l_j^2l_k^2-l_i^4-l_j^4-l_k^4>0$$

on Ω_{ijk}^E. So to prove $A+B+C<0$, we just need to prove

$$\begin{aligned}&2l_i^2l_j^2l_k^2(l_i^2+l_j^2+l_k^2)+l_i^2r_i^2(l_i^4-l_j^4-l_k^4-2l_j^2l_k^2)+l_j^2r_j^2(l_j^4-l_i^4-l_k^4-2l_i^2l_k^2)\\&\quad+l_k^2r_k^2(l_k^4-l_i^4-l_j^4-2l_i^2l_j^2)>0.\end{aligned}$$

By direct calculations, we have

$$\begin{aligned}&2l_i^2l_j^2l_k^2(l_i^2+l_j^2+l_k^2)+l_i^2r_i^2(l_i^4-l_j^4-l_k^4-2l_j^2l_k^2)\\&\qquad+l_j^2r_j^2(l_j^4-l_i^4-l_k^4-2l_i^2l_k^2)+l_k^2r_k^2(l_k^4-l_i^4-l_j^4-2l_i^2l_j^2)\\&\quad=4[r_i^2r_j^2r_k^2(1+I_i^2+I_j^2+I_k^2+4I_iI_jI_k)+r_i^2r_jr_k(I_i+I_jI_k)(r_j^2+r_k^2)\\&\qquad+r_ir_j^2r_k(I_j+I_iI_k)(r_i^2+r_k^2)+r_ir_jr_k^2(I_k+I_iI_j)(r_i^2+r_j^2)]\\&\quad\geq 4r_i^2r_j^2r_k^2(1+I_i^2+I_j^2+I_k^2+4I_iI_jI_k+2I_i+2I_jI_k+2I_j+2I_iI_k+2I_k+2I_iI_j)\\&\quad=4r_i^2r_j^2r_k^2[(1+I_i)(1+I_j)(1+I_k)+(1+I_i)\gamma_{ijk}+(1+I_j)\gamma_{jik}+(1+I_k)\gamma_{kij}]\\&\quad>0,\end{aligned}$$

where the condition $I_i,I_j,I_k\in(-1,+\infty)$ and $\gamma_{ijk}=I_i+I_jI_k\geq 0,\gamma_{jik}=I_j+I_iI_k\geq 0,\gamma_{kij}=I_k+I_iI_j\geq 0$ is used. So we have $A+B+C<0$.

For the term $AB+AC+BC$, by direct calculations, we have

$$\begin{aligned}
&AB+AC+BC\\
&=\frac{1}{l_i^2l_j^2l_k^2}(2l_i^2l_j^2+2l_i^2l_k^2+2l_j^2l_k^2-l_i^4-l_j^4-l_k^4)\\
&\quad\times\left[(r_i^2-r_j^2)(r_k^2-r_i^2)l_i^2+(r_i^2-r_j^2)(r_j^2-r_k^2)l_j^2+(r_k^2-r_i^2)(r_j^2-r_k^2)l_k^2+l_i^2l_j^2l_k^2\right].
\end{aligned}$$

So by the triangle inequalities, $AB+AC+BC>0$ is equivalent to

$$(r_i^2-r_j^2)(r_k^2-r_i^2)l_i^2+(r_i^2-r_j^2)(r_j^2-r_k^2)l_j^2+(r_k^2-r_i^2)(r_j^2-r_k^2)l_k^2+l_i^2l_j^2l_k^2>0.$$

By direct calculations, combining with the condition $I_i, I_j, I_k\in(-1,+\infty)$ and $\gamma_{ijk}\geq 0, \gamma_{jik}\geq 0, \gamma_{kij}\geq 0$, we have

$$\begin{aligned}
&(r_i^2-r_j^2)(r_k^2-r_i^2)l_i^2+(r_i^2-r_j^2)(r_j^2-r_k^2)l_j^2+(r_k^2-r_i^2)(r_j^2-r_k^2)l_k^2+l_i^2l_j^2l_k^2\\
&=8r_i^2r_j^2r_k^2(1+I_iI_jI_k)+4r_i^2r_jr_k(I_i+I_jI_k)(r_j^2+r_k^2)\\
&\quad+4r_ir_j^2r_k(I_j+I_iI_k)(r_i^2+r_k^2)+4r_ir_jr_k^2(I_k+I_iI_j)(r_i^2+r_j^2)\\
&\geq 8r_i^2r_j^2r_k^2(1+I_iI_jI_k+I_i+I_jI_k+I_j+I_iI_k+I_k+I_iI_j)\\
&=8r_i^2r_j^2r_k^2(1+I_i)(1+I_j)(1+I_k)\\
&>0.
\end{aligned}$$

So we have $AB+AC+BC>0$. Then the matrix Λ^E_{ijk} has a zero eigenvalue with eigenvector $(1,1,1)^T$ and two negative eigenvalues on $\mathcal{U}^E_{ijk}$. $\square$

Now suppose that for each topological face $\Delta ijk\in F$, the triangle inequalities are satisfied, i.e. $r\in\Omega^E$, then the weighted triangulated surface $(M,\mathcal{T},I)$ could be taken as gluing many triangles along the edges coherently, which produces a cone metric on the triangulated surface with singularities at V. To describe the singularity at the vertex i, the classical discrete curvature is introduced, which is defined as

$$K_i=2\pi-\sum_{\triangle ijk\in F}\theta_i^{jk}, \tag{2.9}$$

where the sum is taken over all the triangles with i as one of its vertices and θ_i^{jk} is the inner angle of the triangle $\triangle ijk$ at the vertex i. Lemma 2.6 has the following corollary.

Corollary 2.7. *Given a triangulated surface $(M,\mathcal{T})$ with inversive distance $I>-1$ and $\gamma_{ijk}\geq 0, \gamma_{jik}\geq 0, \gamma_{kij}\geq 0$ for any topological triangle $\triangle ijk\in F$. Then the matrix $\Lambda^E=\frac{\partial(K_1,\cdots,K_N)}{\partial(u_1,\cdots,u_N)}$ is symmetric and positive semi-definite with rank $N-1$ and kernel $\{t\mathbf{1}|t\in\mathbb{R}\}$ on $\mathcal{U}^E$ for the Euclidean background geometry.*

Proof. This follows from the fact that $\Lambda^E = -\sum_{\triangle ijk\in F}\Lambda^E_{ijk}$, Lemma 2.5 and Lemma 2.6, where Λ^E_{ijk} is extended by zeros to a $N\times N$ matrix so that Λ^E_{ijk} acts on a vector $(v_1,\cdots,v_N)$ only on the coordinates corresponding to vertices v_i, v_j and v_k in the triangle $\triangle ijk$. $\square$

Remark 7. Guo [22] obtained a result paralleling to Corollary 2.7 for nonnegative inversive distance.

By Lemma 2.3 and Lemma 2.5, we can define an energy function

$$\mathcal{E}_{ijk}(u) = \int_{u_0}^{u} \theta_i du_i + \theta_j du_j + \theta_k du_k$$

on $\mathcal{U}^E_{ijk}$. Lemma 2.6 ensures that $\mathcal{E}_{ijk}$ is locally concave on $\mathcal{U}^E_{ijk}$. Define the Ricci energy function as

$$\mathcal{E}(u) = -\sum_{\triangle ijk\in F}\mathcal{E}_{ijk}(u) + \int_{u_0}^{u}\sum_{i=1}^{N}(2\pi - \overline{K}_i)du_i, \tag{2.10}$$

then $\nabla_u\mathcal{E} = K - \overline{K}$ and $\mathcal{E}(u)$ is locally convex on $\mathcal{U}^E = \cap_{\triangle ijk\in F}\mathcal{U}^E_{ijk}$. The local convexity of $\mathcal{E}$ implies the infinitesimal rigidity of K with respect to u, which is the infinitesimal rigidity of inversive distance circle packings.

2.3. Global rigidity of Euclidean inversive distance circle packings

In this subsection, we shall prove the global rigidity of inversive distance circle packings under the condition $I > -1$ and $\gamma_{ijk}\geq 0, \gamma_{jik}\geq 0, \gamma_{kij}\geq 0$ for any triangle $\triangle ijk\in F$. We need to extend the energy function defined on $\mathcal{U}^E$ to be a convex function defined on $\mathbb{R}^3$. Before going on, we recall the following definition and theorem of Luo in [27].

Definition 2.8. A differential 1-form $w = \sum_{i=1}^n a_i(x)dx^i$ in an open set $U\subset\mathbb{R}^n$ is said to be continuous if each $a_i(x)$ is continuous on U. A differential 1-form w is called closed if $\int_{\partial\tau} w = 0$ for each triangle $\tau\subset U$.

Theorem 2.9 *([27] Corollary 2.6). Suppose $X\subset\mathbb{R}^n$ is an open convex set and $A\subset X$ is an open subset of X bounded by a C^1 smooth codimension-1 submanifold in X. If $w = \sum_{i=1}^n a_i(x)dx_i$ is a continuous closed 1-form on A so that $F(x) = \int_a^x w$ is locally convex on A and each a_i can be extended continuous to X by constant functions to a function $\widetilde{a}_i$ on X, then $\widetilde{F}(x) = \int_a^x \sum_{i=1}^n \widetilde{a}_i(x)dx_i$ is a C^1-smooth convex function on X extending F.*

Combining Lemma 2.3, Corollary 2.4 and Theorem 2.9, we have the following useful lemma.

Lemma 2.10. *For any triangle $\triangle ijk \in F$ with inversive distance $I > -1$ and*

$$\gamma_{ijk} \geq 0, \gamma_{jik} \geq 0, \gamma_{kij} \geq 0,$$

the energy function $\mathcal{E}_{ijk}(u)$ defined on $\mathcal{U}^E_{ijk}$ by (2.10) could be extended to the following function

$$\widetilde{\mathcal{E}}_{ijk}(u) = \int_{u_0}^{u} \widetilde{\theta}_i du_i + \widetilde{\theta}_j du_j + \widetilde{\theta}_k du_k, \tag{2.11}$$

which is a C^1-smooth concave function defined on $\mathbb{R}^3$ with

$$\nabla_u \widetilde{\mathcal{E}}_{ijk} = (\widetilde{\theta}_i, \widetilde{\theta}_j, \widetilde{\theta}_k)^T.$$

Using Lemma 2.10, we can prove the following global rigidity of Euclidean inversive distance circle packings, which is the Euclidean part of Theorem 1.1.

Theorem 2.11. *Given a triangulated surface $(M, \mathcal{T})$ with inversive distance $I > -1$ and $\gamma_{ijk} \geq 0, \gamma_{jik} \geq 0, \gamma_{kij} \geq 0$ for any topological triangle $\triangle ijk \in F$. Then for any $\overline{K} \in C(V)$ with $\sum_{i=1}^N \overline{K}_i = 2\pi\chi(M)$, there exists at most one Euclidean inversive distance circle packing metric r up to scaling with $K(r) = \overline{K}$.*

Proof. By Lemma 2.10, the Ricci potential function $\mathcal{E}(u)$ in (2.10) could be extended from $\mathcal{U}^E$ to the whole space $\mathbb{R}^N$ as follows

$$\widetilde{\mathcal{E}}(u) = -\sum_{\triangle ijk \in F} \widetilde{\mathcal{E}}_{ijk}(u) + \int_{u_0}^{u} \sum_{i=1}^{N} (2\pi - \overline{K}_i) du_i.$$

As $\widetilde{\mathcal{E}}_{ijk}(u)$ is C^1-smooth concave by Lemma 2.10 and $\int_{u_0}^{u} \sum_{i=1}^N (2\pi - \overline{K}_i) du_i$ is a well-defined convex function on $\mathbb{R}^N$, we have $\widetilde{\mathcal{E}}(u)$ is a C^1-smooth convex function on $\mathbb{R}^N$. By Corollary 2.7, we have $\widetilde{\mathcal{E}}(u)$ is locally strictly convex on $\mathcal{U}^E \cap \{\sum_{i=1}^N u_i = 0\}$. Furthermore,

$$\nabla_{u_i} \widetilde{\mathcal{E}} = -\sum_{\triangle ijk \in F} \widetilde{\theta}_i + 2\pi - \overline{K}_i = \widetilde{K}_i - \overline{K}_i,$$

where $\widetilde{K}_i = 2\pi - \sum_{\triangle ijk \in F} \widetilde{\theta}_i$, which implies that $r \in \Omega^E$ is a metric with curvature $\overline{K}$ if and only if the corresponding $u \in \mathcal{U}^E$ is a critical point of $\widetilde{\mathcal{E}}$.

If there are two different inversive distance circle packing metrics $\overline{r}_A, \overline{r}_B \in \Omega^E$ with the same combinatorial Curvature $\overline{K}$, then $\overline{u}_A = \ln \overline{r}_A \in \mathcal{U}^E$, $\overline{u}_B = \ln \overline{r}_B \in \mathcal{U}^E$ are both critical points of the extended Ricci potential $\widetilde{\mathcal{E}}(u)$. It follows that

$$\nabla \widetilde{\mathcal{E}}(\overline{u}_A) = \nabla \widetilde{\mathcal{E}}(\overline{u}_B) = 0.$$

Set

$$\begin{aligned} f(t) &= \widetilde{\mathcal{E}}((1-t)\overline{u}_A + t\overline{u}_B) \\ &= \sum_{\triangle ijk \in F} f_{ijk}(t) + \int_{u_0}^{(1-t)\overline{u}_A + t\overline{u}_B} \sum_{i=1}^{N} (2\pi - \overline{K}_i) du_i, \end{aligned}$$

where

$$f_{ijk}(t) = -\widetilde{\mathcal{E}}_{ijk}((1-t)\overline{u}_A + t\overline{u}_B).$$

Then $f(t)$ is a C^1 convex function on $[0,1]$ and $f'(0) = f'(1) = 0$, which implies that $f'(t) \equiv 0$ on $[0,1]$. Note that $\overline{u}_A$ belongs to the open set $\mathcal{U}^E$, so there exists $\epsilon > 0$ such that $(1-t)\overline{u}_A + t\overline{u}_B \in \mathcal{U}^E$ for $t \in [0,\epsilon]$ and $f(t)$ is smooth on $[0,\epsilon]$.

Note that $f(t)$ is C^1 convex on $[0,1]$ and smooth on $[0,\epsilon]$. $f'(t) \equiv 0$ on $[0,1]$ implies that $f''(t) \equiv 0$ on $[0,\epsilon]$. Note that, for $t \in [0,\epsilon]$,

$$f''(t) = (\overline{u}_A - \overline{u}_B)\Lambda^E(\overline{u}_A - \overline{u}_B)^T,$$

where $\Lambda^E = -\sum_{\triangle ijk \in F} \Lambda^E_{ijk}$. By Corollary 2.7, we have $\overline{u}_A - \overline{u}_B = c(1, \cdots, 1)$ for some constant $c \in \mathbb{R}$, which implies that $\overline{r}_A = e^{c/2}\overline{r}_B$. So there exists at most one Euclidean inversive distance circle packing metric with combinatorial curvature $\overline{K}$ up to scaling. □

Remark 8. The proof of Theorem 2.11 is based on a variational principle, which was introduce by Colin de Verdiere [11]. Guo [22] used the variational principle to study the infinitesimal rigidity of inversive distance circle packing metrics for nonnegative inversive distances. Bobenko, Pinkall and Springborn [3] introduced a method to extend a local convex function on a nonconvex domain to a convex function and solved affirmably a conjecture of Luo [26] on the global rigidity of piecewise linear metrics. Based on the extension method, Luo [27] proved the global rigidity of inversive distance circle packing metrics for nonnegative inversive distance using the variational principle.

2.4. Rigidity of combinatorial α-curvature in Euclidean background geometry

As noted in [16], the classical definition of combinatorial curvature K_i with Euclidean background geometry in (2.9) has two disadvantages. The first is that the classical combinatorial curvature is scaling invariant, i.e. $K_i(\lambda r) = K_i(r)$ for any $\lambda > 0$; The second is that, as the triangulated surfaces approximate a smooth surface, the classical combinatorial curvature K_i could not approximate the smooth Gauss curvature, as we obviously have K_i tends zero. Motivated by the observations, Ge and the author introduced a new combinatorial curvature for triangulated surfaces with Thurston's circle packing metrics in [16–18]. Ge and Jiang [14] and Ge and the author [19] further generalized the curvature to inversive distance circle packing metrics. Set

$$s_i(r) = \begin{cases} r_i, & \text{Euclidean background geometry} \\ \tanh\frac{r_i}{2}, & \text{hyperbolic background geometry} \end{cases}. \tag{2.12}$$

We have the following definition of combinatorial α-curvature on triangulated surfaces with inversive distance circle packing metrics.

Definition 2.12. Given a triangulated surface $(M, \mathcal{T})$ with inversive distance $I > -1$ and an inversive distance circle packing metric $r \in \Omega$, the combinatorial α-curvature at the vertex i is defined to be

$$R_{\alpha,i} = \frac{K_i}{s_i^\alpha}, \tag{2.13}$$

where $\alpha \in \mathbb{R}$ is a constant, K_i is the classical combinatorial curvature at i given by (2.9) and s_i is given by (2.12).

Specially, if $\alpha = 0$, then $R_{\alpha,i} = K_i$. As the inversive distance generalizes Thurston's intersection angle, the Definition 2.12 of combinatorial α-curvature naturally generalizes the definition of combinatorial curvature in [16–18].

For the α-curvature $R_{\alpha,i}$, we have the following global rigidity of Euclidean inversive distance circle packing metrics for inversive distance in $(-1, +\infty)$, which is the Euclidean part of Theorem 1.2.

Theorem 2.13. *Given a closed triangulated surface $(M, \mathcal{T})$ with inversive distance $I > -1$ and $\gamma_{ijk} \geq 0, \gamma_{jik} \geq 0, \gamma_{kij} \geq 0$ for any topological triangle $\triangle ijk \in F$. $\overline{R}$ is a given function defined on the vertices of $(M, \mathcal{T})$. If $\alpha\overline{R} \equiv 0$, there exists at most one Euclidean inversive distance circle packing metric $\overline{r} \in \Omega^E$ with α-curvature $\overline{R}$ up to scaling. If $\alpha\overline{R} \leq 0$ and $\alpha\overline{R} \not\equiv 0$, there exists at most one Euclidean inversive distance circle packing metric $\overline{r} \in \Omega^E$ with α-curvature $\overline{R}$.*

As the proof of Theorem 2.13 is almost parallel to that of Theorem 2.11 using the energy function

$$\widetilde{\mathcal{E}}_\alpha(u) = -\sum_{\triangle ijk \in F} \widetilde{\mathcal{E}}_{ijk}(u) + \int_{u_0}^{u} \sum_{i=1}^{N} (2\pi - \overline{R}_i r_i^\alpha) du_i,$$

we omit the details of the proof.

3. Hyperbolic inversive distance circle packing metrics

3.1. Admissible space of hyperbolic inversive distance circle packing metrics for a single triangle

In this subsection, we investigate the admissible space of hyperbolic inversive distance circle packings for a single topological triangle $\triangle ijk \in F$ with inversive distance $I_i, I_j, I_k \in (-1, +\infty)$ and

$$\gamma_{ijk} \geq 0, \gamma_{jik} \geq 0, \gamma_{kij} \geq 0. \tag{3.1}$$

Suppose $\triangle ijk$ is a topological triangle in F. In the hyperbolic background geometry, the length l_i of the edge $\{jk\}$ is defined by

$$l_i = \cosh^{-1}(\cosh r_j \cosh r_k + I_i \sinh r_j \sinh r_k), \tag{3.2}$$

where I_i is the hyperbolic inversive distance between the two circles attached to the vertices j and k. In order that the edge lengths l_i, l_j, l_k satisfy the triangle inequalities, there are some restrictions on the radius vectors. So we first study the triangle inequalities for the hyperbolic background geometry. To simplify the notations, we use the following simplification

$$C_i = \cosh r_i, S_i = \sinh r_i,$$

when there is no confusion. We have the following lemma on the hyperbolic triangle inequalities.

Lemma 3.1. *Suppose $(M, \mathcal{T}, I)$ is a weighted triangulated surface with hyperbolic inversive distance $I > -1$ and $\triangle ijk$ is a topological triangle in F. Suppose l_i, l_j, l_k are the edge lengths defined by the hyperbolic inversive distance I_i, I_j, I_k using the radius r_i, r_j, r_k by (3.2), then the triangle inequalities are satisfied if and only if*

$$\begin{aligned} &2S_i^2S_j^2S_k^2(1 + I_iI_jI_k) + S_i^2S_j^2(1 - I_k^2) + S_i^2S_k^2(1 - I_j^2) + S_j^2S_k^2(1 - I_i^2) \\ &+ 2C_jC_kS_i^2S_jS_k\gamma_{ijk} + 2C_iC_kS_iS_j^2S_k\gamma_{jik} + 2C_iC_jS_iS_jS_k^2\gamma_{kij} > 0. \end{aligned} \tag{3.3}$$

Proof. In order that $l_i + l_j > l_k, l_i + l_k > l_j, l_j + l_k > l_i$, we just need

$$\sinh \frac{l_i + l_j - l_k}{2} > 0, \sinh \frac{l_i + l_k - l_j}{2} > 0, \sinh \frac{l_j + l_k - l_i}{2} > 0.$$

Note that $l_i > 0, l_j > 0, l_k > 0$, this is equivalent to

$$\sinh \frac{l_i + l_j + l_k}{2} \sinh \frac{l_i + l_j - l_k}{2} \sinh \frac{l_i + l_k - l_j}{2} \sinh \frac{l_j + l_k - l_i}{2} > 0.$$

By direct calculations, we have

$$\begin{aligned}
&4\sinh\frac{l_i+l_j+l_k}{2}\sinh\frac{l_i+l_j-l_k}{2}\sinh\frac{l_i+l_k-l_j}{2}\sinh\frac{l_j+l_k-l_i}{2}\\
&=(\cosh(l_i+l_j)-\cosh l_k)(\cosh l_k-\cosh(l_i-l_j))\\
&=(\cosh^2 l_i-1)(\cosh l_j^2-1)-(\cosh l_i\cosh l_j-\cosh l_k)^2\\
&=(2C_i^2C_j^2C_k^2-C_i^2C_j^2-C_i^2C_k^2-C_j^2C_k^2+1)-(S_i^2S_j^2I_k^2+S_i^2S_k^2I_j^2+S_j^2S_k^2I_i^2)\\
&\quad+2C_jC_kS_i^2S_jS_kI_i+2C_iC_kS_iS_j^2S_kI_j+2C_iC_jS_iS_jS_k^2I_k\\
&\quad+2C_iC_jS_iS_jS_k^2I_iI_j+2C_iC_kS_iS_j^2S_kI_iI_k+2C_jC_kS_i^2S_jS_kI_jI_k+2S_i^2S_j^2S_k^2I_iI_jI_k,
\end{aligned}$$

where the definition of edge length (3.2) is used in the last line. Note that

$$C_i^2=\cosh^2 r_i=\sinh^2 r_i+1=S_i^2+1,$$

we have

$$\begin{aligned}
&4\sinh\frac{l_i+l_j+l_k}{2}\sinh\frac{l_i+l_j-l_k}{2}\sinh\frac{l_i+l_k-l_j}{2}\sinh\frac{l_j+l_k-l_i}{2}\\
&=2S_i^2S_j^2S_k^2(1+I_iI_jI_k)+S_i^2S_j^2(1-I_k^2)+S_i^2S_k^2(1-I_j^2)+S_j^2S_k^2(1-I_i^2)\\
&\quad+2C_jC_kS_i^2S_jS_k(I_i+I_jI_k)+2C_iC_kS_iS_j^2S_k(I_j+I_iI_k)+2C_iC_jS_iS_jS_k^2(I_k+I_iI_j).
\end{aligned}$$

This completes the proof of the lemma. □

Denote the admissible space of hyperbolic inversive distance circle packing metrics for a triangle $\triangle ijk\in F$ as Ω_{ijk}^H, i.e.

$$\Omega_{ijk}^H:=\{(r_i,r_j,r_k)\in\mathbb{R}^3_{>0}|l_i+l_j>l_k,l_i+l_k>l_j,l_j+l_k>l_i\}.$$

By Lemma 3.1, we have the following direct corollary, which was obtained by Zhou [37].

Corollary 3.2. *Suppose $\triangle ijk$ is a topological triangle in F with hyperbolic inversive distance $I_i,I_j,I_k\in(-1,1]$ and $\gamma_{ijk}\geq 0,\gamma_{jik}\geq 0,\gamma_{kij}\geq 0$, then $\Omega_{ijk}^H=\mathbb{R}^3_{>0}$, i.e. the triangle inequalities are satisfied for all radius vectors in $\mathbb{R}^3_{>0}$.*

Specially, if $I_i=\cos\Phi_i,I_j=\cos\Phi_j,I_k=\cos\Phi_k$ with $\Phi_i,\Phi_j,\Phi_k\in[0,\frac{\pi}{2}]$, the triangle inequalities are satisfied for all radius vectors, which was obtained by Thurston in [34].

By Lemma 3.1, we can also get the following useful result.

Corollary 3.3. *Suppose $\triangle ijk$ is a topological triangle in F with hyperbolic inversive distance $I>-1$ and $\gamma_{ijk}\geq 0,\gamma_{jik}\geq 0,\gamma_{kij}\geq 0$. Suppose the edge lengths l_i,l_j,l_k are generated by the radius vector (s,s,s) with $s\in\mathbb{R}_{>0}$. If $s\in\mathbb{R}_{>0}$ satisfies*

$$\sinh^2 s > \frac{I_i^2 + I_j^2 + I_k^2 - 3}{2(1+I_i)(1+I_j)(1+I_j)}, \tag{3.4}$$

we have $(s,s,s) \in \Omega_{ijk}^H$.

Proof. By Lemma 3.1, for $s > 0$, $(s,s,s) \in \Omega_{ijk}^H$ if and only if

$$2\cosh^2 s(\gamma_{ijk} + \gamma_{jik} + \gamma_{kij}) + 2\sinh^2 s(1 + I_iI_jI_k) + 3 - I_i^2 - I_j^2 - I_k^2 > 0.$$

By $\gamma_{ijk} \geq 0$, $\gamma_{jik} \geq 0$, $\gamma_{kij} \geq 0$, we have $\gamma_{ijk} + \gamma_{jik} + \gamma_{kij} \geq 0$. Then

$$\begin{aligned}
&2\cosh^2 s(\gamma_{ijk} + \gamma_{jik} + \gamma_{kij}) + 2\sinh^2 s(1 + I_iI_jI_k) + 3 - I_i^2 - I_j^2 - I_k^2 \\
&\quad \geq 2\sinh^2 s(1 + I_iI_jI_k + \gamma_{ijk} + \gamma_{jik} + \gamma_{kij}) + 3 - I_i^2 - I_j^2 - I_k^2 \\
&\quad = 2\sinh^2 s(1+I_i)(1+I_j)(1+I_j) + 3 - I_i^2 - I_j^2 - I_k^2.
\end{aligned}$$

Note that $I_i, I_j, I_k \in (-1, +\infty)$, to ensure the triangle inequalities, we just need

$$\sinh^2 s > \frac{I_i^2 + I_j^2 + I_k^2 - 3}{2(1+I_i)(1+I_j)(1+I_j)}. \qquad \square$$

Guo [22] obtained a result similar to Corollary 3.3 for $I \geq 0$.

By Lemma 3.1, $\Omega_{ijk}^H \neq \mathbb{R}_{>0}^3$ for general $I_i, I_j, I_k \in (-1, +\infty)$. Furthermore, Ω_{ijk}^H is not convex. Similar to the case of Euclidean background geometry, we have the following lemma on the structure of Ω_{ijk}^H.

Lemma 3.4. *Suppose $\triangle ijk$ is a topological triangle in F with hyperbolic inversive distance $I > -1$ and $\gamma_{ijk} \geq 0, \gamma_{jik} \geq 0, \gamma_{kij} \geq 0$, then the admissible space Ω_{ijk}^H is simply connected. Furthermore, for each connected component V of $\mathbb{R}_{>0}^3 \setminus \Omega_{ijk}^H$, the intersection $V \cap \overline{\Omega}_{ijk}^H$ is a connected component of $\overline{\Omega}_{ijk}^H \setminus \Omega_{ijk}^H$, on which θ_i is a constant function.*

Proof. Define the map

$$\begin{aligned}
F : \mathbb{R}_{>0}^3 &\to \mathbb{R}_{>0}^3 \\
(r_i, r_j, r_k) &\mapsto (F_i, F_j, F_k)
\end{aligned}$$

where

$$\begin{aligned}
F_i &= \cosh r_j \cosh r_k + I_i \sinh r_j \sinh r_k, \\
F_j &= \cosh r_i \cosh r_k + I_j \sinh r_i \sinh r_k, \\
F_k &= \cosh r_i \cosh r_j + I_k \sinh r_i \sinh r_j.
\end{aligned}$$

By direct calculations, we have

$$\frac{\partial(F_i,F_j,F_k)}{\partial(r_i,r_j,r_k)} = \begin{pmatrix} 0 & S_jC_k + I_iC_jS_k & C_jS_k + I_iS_jC_k \\ S_iC_k + I_jC_iS_k & 0 & C_iS_k + I_jS_iC_k \\ S_iC_j + I_kC_iS_j & C_iS_j + I_kS_iC_j & 0 \end{pmatrix}$$

and

$$\begin{aligned} \left|\frac{\partial(F_i,F_j,F_k)}{\partial(r_i,r_j,r_k)}\right| &= 2C_iC_jC_kS_iS_jS_k(1+I_iI_jI_k) + \gamma_{kij}C_kS_k(C_i^2S_j^2 + C_j^2S_i^2) \\ &\quad + \gamma_{jik}C_jS_j(C_k^2S_i^2 + C_i^2S_k^2) + \gamma_{ijk}C_iS_i(C_k^2S_j^2 + C_j^2S_k^2). \end{aligned}$$

By $I > -1$ and $\gamma_{ijk} \geq 0, \gamma_{jik} \geq 0, \gamma_{kij} \geq 0$, we have

$$\begin{aligned} \left|\frac{\partial(F_i,F_j,F_k)}{\partial(r_i,r_j,r_k)}\right| &\geq 2C_iC_jC_kS_iS_jS_k(1+I_iI_jI_k + \gamma_{ijk} + \gamma_{jik} + \gamma_{kij}) \\ &= 2C_iC_jC_kS_iS_jS_k(1+I_i)(1+I_j)(1+I_k) > 0, \end{aligned}$$

which implies that F is globally injective. In fact, if there are two different $r = (r_i, r_j, r_k)$ and $r' = (r_i', r_j', r_k')$ satisfying $F(r) = F(r')$, then we have

$$0 = F(r) - F(r') = \frac{\partial(F_i,F_j,F_k)}{\partial(r_i,r_j,r_k)}|_{r+\theta(r-r')} \cdot (r-r')^T, 0 < \theta < 1,$$

which implies $r = r'$ by the nondegeneracy of $\frac{\partial(F_i,F_j,F_k)}{\partial(r_i,r_j,r_k)}$ on $\mathbb{R}^3_{>0}$. So the map F is injective on $\mathbb{R}^3_{>0}$.

Note that F has the following property

$$0 < (1+I_i)\sinh r_j \sinh r_k \leq F_i \leq (1+|I_i|)\cosh(r_i + r_j),$$

which implies that F is a proper map. By the invariance of domain theorem, we have $F : \mathbb{R}^3_{>0} \to F(\mathbb{R}^3_{>0})$ is a diffeomorphism.

Define

$$\begin{aligned} G : \mathbb{R}^3_{>0} &\to \mathbb{R}^3_{>0} \\ (l_i, l_j, l_k) &\mapsto (\cosh l_i, \cosh l_j, \cosh l_k), \end{aligned}$$

then $G : \mathbb{R}^3_{>0} \to G(\mathbb{R}^3_{>0})$ is a diffeomorphis and $H = G^{-1} \circ F$ is the map defining the edge length by the inversive distance which maps (r_i, r_j, r_k) to (l_i, l_j, l_k).

Set

$$\mathcal{L} = \{(l_i, l_j, l_k) | l_i + l_j > l_k, l_i + l_k > l_j, l_j + l_k > l_i\},$$

then $\Omega^H_{ijk} = H^{-1}(H(\mathbb{R}^3_{>0}) \cap \mathcal{L})$. To prove that Ω^H_{ijk} is simply connected, we just need to prove that $H(\mathbb{R}^3_{>0}) \cap \mathcal{L}$ is simply connected.

Note that $\mathcal{L}$ is a cone in $\mathbb{R}^3_{>0}$ bounded by three planes

$$L_i = \{(l_i, l_j, l_k) \in \mathbb{R}^3_{>0} | l_i = l_j + l_k\},$$
$$L_j = \{(l_i, l_j, l_k) \in \mathbb{R}^3_{>0} | l_j = l_i + l_k\},$$
$$L_k = \{(l_i, l_j, l_k) \in \mathbb{R}^3_{>0} | l_k = l_i + l_j\}.$$

By the fact that H is a diffeomorphism between $\mathbb{R}^3_{>0}$ and $H(\mathbb{R}^3_{>0})$, $H(\mathbb{R}^3_{>0})$ is the set bounded by three surfaces

$$\Sigma_i = \{(l_i, l_j, l_k) \in \mathbb{R}^3_{>0} | \cosh l_i = \cosh l_j \cosh l_k + I_i \sinh l_j \sinh l_k\},$$
$$\Sigma_j = \{(l_i, l_j, l_k) \in \mathbb{R}^3_{>0} | \cosh l_j = \cosh l_i \cosh l_k + I_j \sinh l_i \sinh l_k\},$$
$$\Sigma_k = \{(l_i, l_j, l_k) \in \mathbb{R}^3_{>0} | \cosh l_k = \cosh l_i \cosh l_j + I_k \sinh l_i \sinh l_j\}.$$

In fact, if $r_i = 0$, then $l_j = r_k, l_k = r_j$ and $\cosh l_i = \cosh r_j \cosh r_k + I_i \sinh r_j \sinh r_k = \cosh l_j \cosh l_k + I_i \sinh l_j \sinh l_k$. Σ_i is in fact the image of $r_i = 0$ under H. By the diffeomorphism of H, Σ_i, Σ_j, Σ_k are mutually disjoint. Furthermore, if $I_i \in (-1, 1]$, we have $\cosh(l_j - l_k) < \cosh l_i \le \cosh(l_j + l_k)$ on Σ_i. And if $I_i \in (1, +\infty)$, we have $\cosh l_i > \cosh(l_j + l_k)$ on Σ_i. This implies that $\Sigma_i \subset \overline{\mathcal{L}}$ if $I_i \in (-1, 1]$ and $\Sigma_i \cap \mathcal{L} = \emptyset$ if $I_i \in (1, +\infty)$. Similar results hold for Σ_j and Σ_k. To prove that $H(\mathbb{R}^3_{>0}) \cap \mathcal{L}$ is simply connected, we just need to consider the following cases by the symmetry between i, j, k.

If $I_i, I_j, I_k \in (-1, 1]$, $H(\mathbb{R}^3_{>0}) \cap \mathcal{L}$ is bounded by $\Sigma_i, \Sigma_j, \Sigma_k$ and $H(\mathbb{R}^3_{>0}) \cap \mathcal{L} = H(\mathbb{R}^3_{>0})$.

If $I_i, I_j \in (-1, 1]$ and $I_k \in (1, +\infty)$, $H(\mathbb{R}^3_{>0}) \cap \mathcal{L}$ is bounded by Σ_i, Σ_j and L_k.

If $I_i \in (-1, 1]$ and $I_j, I_k \in (1, +\infty)$, $H(\mathbb{R}^3_{>0}) \cap \mathcal{L}$ is bounded by Σ_i, L_j and L_k.

If $I_i, I_j, I_k \in (1, +\infty)$, $H(\mathbb{R}^3_{>0}) \cap \mathcal{L}$ is bounded by L_i, L_j and L_k. In this case, $H(\mathbb{R}^3_{>0}) \cap \mathcal{L} = \mathcal{L}$.

For any case, $H(\mathbb{R}^3_{>0}) \cap \mathcal{L}$ is a simply connected subset of $\mathbb{R}^3_{>0}$. By the fact that H is a diffeomorphism between $\mathbb{R}^3_{>0}$ and $H(\mathbb{R}^3_{>0})$, we have the admissible space $\Omega^H_{ijk} = H^{-1}(H(\mathbb{R}^3_{>0}) \cap \mathcal{L})$ is simply connected.

By the analysis above, if $H(\mathbb{R}^3_{>0}) \subset \mathcal{L}$, then $\Omega^H_{ijk} = H^{-1}(H(\mathbb{R}^3_{>0}) \cap \mathcal{L}) = \mathbb{R}^3_{>0}$. If $H(\mathbb{R}^3_{>0}) \setminus \mathcal{L} \neq \emptyset$, then Ω^H_{ijk} is a proper subset of $\mathbb{R}^3_{>0}$. If $I_i > 1$, the boundary component $\Sigma_i = \{(l_i, l_j, l_k) \in \mathbb{R}^3_{>0} | \cosh l_i = \cosh l_j \cosh l_k + I_i \sinh l_j \sinh l_k\}$ is out of the set $\mathcal{L}$. By the fact that $\Omega^H_{ijk} = H^{-1}(H(\mathbb{R}^3_{>0}) \cap \mathcal{L})$ and $H : \mathbb{R}^3_{>0} \to H(\mathbb{R}^3_{>0})$ is a diffeomorphism, we have $H^{-1}(L_i)$ is a connected boundary component of Ω^H_{ijk}, on which $\theta_i = \pi, \theta_j = \theta_k = 0$. This completes the proof of the lemma. □

Corollary 3.5. *For a topological triangle $\triangle ijk \in F$ with inversive distance $I > -1$ and $\gamma_{ijk} \ge 0, \gamma_{jik} \ge 0, \gamma_{kij} \ge 0$, the functions $\theta_i, \theta_j, \theta_k$ defined on Ω^H_{ijk} could be continuously extended by constant to $\widetilde{\theta}_i, \widetilde{\theta}_j, \widetilde{\theta}_k$ defined on $\mathbb{R}^3_{>0}$.*

3.2. Infinitesimal rigidity of hyperbolic inversive distance circle packings

Set $u_i = \ln\tanh\frac{r_i}{2}$, then we have $\mathcal{U}^H_{ijk} := u(\Omega^H_{ijk})$ is a simply connected subset of $\mathbb{R}^3_{>0}$. If $(r_i, r_j, r_k) \in \Omega^H_{ijk}$, l_i, l_j, l_k form a hyperbolic triangle. Denote the inner angle at the vertex i as θ_i. We have the following lemma.

Lemma 3.6. *For any triangle $\triangle ijk \in F$, we have*

$$\frac{\partial\theta_i}{\partial u_j} = \frac{\partial\theta_j}{\partial u_i} = \frac{1}{A\sinh^2 l_k}[C_k S_i^2 S_j^2(1-I_k^2) + C_i S_i S_j^2 S_k \gamma_{jik} + C_j S_i^2 S_j S_k \gamma_{ijk}] \tag{3.5}$$

on $\mathcal{U}^H_{ijk}$, where $A = \sinh l_j \sinh l_k \sin\theta_i$.

Proof. By cosine law, we have $\cosh l_i = \cosh l_j \cosh l_k - \sinh l_j \sinh l_k \cos\theta_i$. Taking the derivative with respect to l_i gives

$$\frac{\partial\theta_i}{\partial l_i} = \frac{\sinh l_i}{A},$$

where $A = \sinh l_j \sinh l_k \sin\theta_i$. Similarly, taking the derivative with respect to l_j and l_k and using the cosine law again, we have

$$\frac{\partial\theta_i}{\partial l_j} = \frac{-\sinh l_i \cos\theta_k}{A}, \frac{\partial\theta_i}{\partial l_k} = \frac{-\sinh l_i \cos\theta_j}{A}.$$

By the definition of edge length l_i, l_j and l_k, we have

$$\begin{aligned}
\frac{\partial l_i}{\partial r_j} &= \frac{\sinh r_j \cosh r_k + I_i \cosh r_j \sinh r_k}{\sinh l_i}, \frac{\partial l_j}{\partial r_j} = 0,\\
\frac{\partial l_k}{\partial r_j} &= \frac{\sinh r_j \cosh r_i + I_k \cosh r_j \sinh r_i}{\sinh l_k}.
\end{aligned}$$

Then

$$\begin{aligned}
A\frac{\partial\theta_i}{\partial u_j} &= A\sinh r_j \frac{\partial\theta_i}{\partial r_j}\\
&= A\sinh r_j\left(\frac{\partial\theta_i}{\partial l_i}\frac{\partial l_i}{\partial r_j} + \frac{\partial\theta_i}{\partial l_k}\frac{\partial l_k}{\partial r_j}\right)\\
&= \sinh r_j(\sinh r_j \cosh r_k + I_i \cosh r_j \sinh r_k)\\
&\quad - \frac{1}{\sinh l_k}\sinh r_j \sinh l_i \cos\theta_j(\sinh r_j \cosh r_i + I_k \cosh r_j \sinh r_i),
\end{aligned}$$

which implies that

$$
\begin{aligned}
\sinh^2 l_k A\frac{\partial \theta_i}{\partial u_j} &= (\cosh^2 l_k - 1)\sinh r_j(\sinh r_j \cosh r_k + I_i \cosh r_j \sinh r_k) \\
&\quad + (\cosh l_j - \cosh l_i \cosh l_k)\sinh r_j(\sinh r_j \cosh r_i + I_k \cosh r_j \sinh r_i)
\end{aligned}
$$

Note that

$$
\begin{aligned}
\sinh r_j(\sinh r_j \cosh r_k + I_i \cosh r_j \sinh r_k) &= \cosh r_j \cosh l_i - \cosh r_k, \\
\sinh r_j(\sinh r_j \cosh r_i + I_k \cosh r_j \sinh r_i) &= \cosh r_j \cosh l_k - \cosh r_i.
\end{aligned}
$$

Using the definition of edge lengths l_i, l_j and l_k, by direct calculations, we have

$$
\frac{\partial \theta_i}{\partial u_j} = \frac{1}{A\sinh^2 l_k}[C_k S_i^2 S_j^2(1-I_k^2) + C_i S_i S_j^2 S_k \gamma_{jik} + C_j S_i^2 S_j S_k \gamma_{ijk}],
$$

which implies also $\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i}$. $\square$

Remark 9. For $I_i, I_j, I_k \in (-1, 1]$ and $\gamma_{ijk} \geq 0$, $\gamma_{jik} \geq 0$, $\gamma_{kij} \geq 0$, by Lemma 3.6, we have $\frac{\partial \theta_i}{\partial u_j} \geq 0$, and $\frac{\partial \theta_i}{\partial u_j} = 0$ if and only if $I_k = 1$ and $I_i + I_j = 0$. Especially, if $I_i = \cos\Phi_i, I_j = \cos\Phi_j, I_k = \cos\Phi_k$ with $\Phi_i, \Phi_j, \Phi_k \in [0, \frac{\pi}{2}]$, we have $\frac{\partial \theta_i}{\partial u_j} \geq 0$, and $\frac{\partial \theta_i}{\partial u_j} = 0$ if and only if $\Phi_k = 0$ and $\Phi_i = \Phi_j = \frac{\pi}{2}$.

Lemma 3.6 shows that the matrix

$$
\Lambda^H_{ijk} = \frac{\partial(\theta_i, \theta_j, \theta_k)}{\partial(u_i, u_j, u_k)} = \begin{pmatrix} \frac{\partial \theta_i}{\partial u_i} & \frac{\partial \theta_i}{\partial u_j} & \frac{\partial \theta_i}{\partial u_k} \\ \frac{\partial \theta_j}{\partial u_i} & \frac{\partial \theta_j}{\partial u_j} & \frac{\partial \theta_j}{\partial u_k} \\ \frac{\partial \theta_k}{\partial u_i} & \frac{\partial \theta_k}{\partial u_j} & \frac{\partial \theta_k}{\partial u_k} \end{pmatrix}
$$

is symmetric on $\mathcal{U}^H_{ijk}$. Similar to the case of Euclidean background geometry, we have the following lemma for the matrix Λ^H_{ijk}.

Lemma 3.7. *In the hyperbolic background geometry, for any triangle $\triangle ijk \in F$ with $I_i, I_j, I_k > -1$ and $\gamma_{ijk} \geq 0$, $\gamma_{jik} \geq 0$, $\gamma_{kij} \geq 0$, the matrix Λ^H_{ijk} is negative definite on $\mathcal{U}^H_{ijk}$.*

Proof. The proof is parallel to that of Lemma 12 in [22] with some modifications. By the proof of Lemma 3.6, we have

$$\begin{pmatrix} d\theta_i \\ d\theta_j \\ d\theta_k \end{pmatrix} = -\frac{1}{A} \begin{pmatrix} \sinh l_i & 0 & 0 \\ 0 & \sinh l_j & 0 \\ 0 & 0 & \sinh l_k \end{pmatrix} \begin{pmatrix} -1 & \cos\theta_k & \cos\theta_j \\ \cos\theta_k & -1 & \cos\theta_i \\ \cos\theta_j & \cos\theta_i & -1 \end{pmatrix}$$
$$\times \begin{pmatrix} \frac{1}{\sinh l_i} & 0 & 0 \\ 0 & \frac{1}{\sinh l_j} & 0 \\ 0 & 0 & \frac{1}{\sinh l_k} \end{pmatrix} \begin{pmatrix} 0 & R_{ijk} & R_{ikj} \\ R_{jik} & 0 & R_{jki} \\ R_{kij} & R_{kji} & 0 \end{pmatrix} \tag{3.6}$$
$$\times \begin{pmatrix} \sinh r_i & 0 & 0 \\ 0 & \sinh r_j & 0 \\ 0 & 0 & \sinh r_k \end{pmatrix} \begin{pmatrix} du_i \\ du_j \\ du_k \end{pmatrix},$$

where

$$A = \sinh l_i \sinh l_j \sin\theta_k, R_{ijk} = \sinh r_j \cosh r_k + I_i \cosh r_j \sinh r_k.$$

Write the equation (3.6) as

$$\begin{pmatrix} d\theta_i \\ d\theta_j \\ d\theta_k \end{pmatrix} = -\frac{1}{A}\mathcal{J} \begin{pmatrix} du_i \\ du_j \\ du_k \end{pmatrix} \tag{3.7}$$

and denote the second and fourth matrix in the product of the right hand side of (3.6) as Θ and $\mathcal{R}$ respectively. Then Λ^H_{ijk} is negative definite is equivalent to $\mathcal{J}$ is positive definite.

We first prove that $\det\mathcal{J}$ is positive. To prove this, we just need to prove that $\det(\Theta)$ and $\det\mathcal{R}$ are positive. By direct calculations, we have

$$\begin{aligned}\det\Theta &= -1 + \cos\theta_i^2 + \cos\theta_j^2 + \cos\theta_k^2 + 2\cos\theta_i\cos\theta_j\cos\theta_k \\ &= 4\cos\frac{\theta_i+\theta_j-\theta_k}{2}\cos\frac{\theta_i-\theta_j+\theta_k}{2}\cos\frac{\theta_i+\theta_j+\theta_k}{2}\cos\frac{\theta_i-\theta_j-\theta_k}{2}.\end{aligned}$$

By the Gauss–Bonnet formula for hyperbolic triangles, we have

$$\theta_i + \theta_j + \theta_k = \pi - Area(\triangle ijk),$$

which implies $\frac{\theta_i+\theta_j+\theta_k}{2}, \frac{\theta_i+\theta_j-\theta_k}{2}, \frac{\theta_i-\theta_j+\theta_k}{2}, \frac{\theta_i-\theta_j-\theta_k}{2} \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then we have $\det\Theta > 0$.

By direct calculations, we have

$$\begin{aligned}\det\mathcal{R} &= R_{ijk}R_{jki}R_{kij} + R_{ikj}R_{jik}R_{kji} \\ &= 2C_iC_jC_kS_iS_jS_k(1 + I_iI_jI_k) + C_kS_k(I_k + I_iI_j)(C_i^2S_j^2 + C_j^2S_i^2) \\ &\quad + C_jS_j(I_j + I_iI_k)(C_k^2S_i^2 + C_i^2S_k^2) + C_iS_i(I_i + I_jI_k)(C_k^2S_j^2 + C_j^2S_k^2) \\ &\geq 2C_iC_jC_kS_iS_jS_k(1 + I_iI_jI_k + I_k + I_iI_j + I_j + I_iI_k + I_i + I_jI_k) \\ &= 2C_iC_jC_kS_iS_jS_k(1 + I_i)(1 + I_j)(1 + I_k) > 0,\end{aligned}$$

where the conditions $I_i, I_j, I_k \in (-1, +\infty)$ and $\gamma_{ijk} \geq 0$, $\gamma_{jik} \geq 0$, $\gamma_{kij} \geq 0$ are used. Then we have $\det \mathcal{J} > 0$ on $\mathcal{U}^H_{ijk}$.

By the connectivity of Ω^H_{ijk} and the continuity of the eigenvalues of Λ^H_{ijk}, we just need to prove $\mathcal{J}$ is positive definite for some radius vector in Ω^H_{ijk}. By Corollary 3.3, for sufficient large s, the radius vector $(s, s, s) \in \Omega^H_{ijk}$. We shall prove $\mathcal{J}$ is positive definite for some s large enough. At (s, s, s), we have

$$\begin{aligned}\mathcal{J} = \sinh^2 s \cosh s &\begin{pmatrix} \sinh l_i & 0 & 0 \\ 0 & \sinh l_j & 0 \\ 0 & 0 & \sinh l_k \end{pmatrix} \begin{pmatrix} -1 & \cos\theta_k & \cos\theta_j \\ \cos\theta_k & -1 & \cos\theta_i \\ \cos\theta_j & \cos\theta_i & -1 \end{pmatrix} \\ &\times \begin{pmatrix} \frac{1}{\sinh l_i} & 0 & 0 \\ 0 & \frac{1}{\sinh l_j} & 0 \\ 0 & 0 & \frac{1}{\sinh l_k} \end{pmatrix} \begin{pmatrix} 0 & 1+I_i & 1+I_i \\ 1+I_j & 0 & 1+I_j \\ 1+I_k & 1+I_k & 0 \end{pmatrix}.\end{aligned}$$

Write the above equation as $\mathcal{J} = \sinh^2 s \cosh s N$. Then we just need to prove that the leading 1×1 and 2×2 minor of N is positive for some s large enough.

For the leading 1×1 minor, we have

$$\begin{aligned} N_{11} &= \frac{\sinh l_i \cos\theta_k}{\sinh l_j}(1+I_j) + \frac{\sinh l_i \cos\theta_j}{\sinh l_k}(1+I_k) \\ &= \frac{1}{\sinh^2 l_j \sinh^2 l_k}[(1+I_j)(\cosh l_i \cosh l_j - \cosh l_k)(\cosh^2 l_k - 1) \\ &\quad + (1+I_k)(\cosh l_i \cosh l_k - \cosh l_j)(\cosh^2 l_j - 1)] \\ &= \frac{(1+I_j)(1+I_k)\sinh^4 s}{\sinh^2 l_j \sinh^2 l_k}[2(1+I_i)(1+I_j)(1+I_k)\sinh^4 s \\ &\quad + (6 + 6I_i + 3I_j + 3I_k + 3I_iI_j + 3I_iI_k + 2I_jI_k - I_j^2 - I_k^2)\sinh^2 s + 4(1+I_i)]. \end{aligned} \tag{3.8}$$

Note that, by Corollary 3.3, under the condition

$$2\sinh^2 s(1+I_i)(1+I_j)(1+I_j) > I_i^2 + I_j^2 + I_k^2 - 3,$$

the triangle inequalities are satisfied, which implies

$$\begin{aligned} &\frac{\sinh l_i \cos\theta_k}{\sinh l_j}(1+I_j) + \frac{\sinh l_i \cos\theta_j}{\sinh l_k}(1+I_k) \\ &\quad \geq \frac{(1+I_j)(1+I_k)\sinh^4 s}{\sinh^2 l_j \sinh^2 l_k} \\ &\qquad \times [(3 + 6I_i + 3I_j + 3I_k + 3I_iI_j + 3I_iI_k + 2I_jI_k + I_i^2)\sinh^2 s + 4(1+I_i)] \end{aligned}$$

$$
\begin{aligned}
&= \frac{(1+I_j)(1+I_k)\sinh^4 s}{\sinh^2 l_j \sinh^2 l_k} \\
&\quad \times [((1+I_i)(3+I_i)+2\gamma_{ijk}+3\gamma_{jik}+3\gamma_{kij})\sinh^2 s + 4(1+I_i)].
\end{aligned}
$$

Therefor the leading 1×1 minor of N is positive by the condition $I_i, I_j, I_k \in (-1, +\infty)$ and $\gamma_{ijk}\geq 0$, $\gamma_{jik}\geq 0$, $\gamma_{kij}\geq 0$.

Similar to (3.8), we have

$$
\begin{aligned}
N_{22} &= \frac{\sinh l_j \cos\theta_k}{\sinh l_i}(1+I_i) + \frac{\sinh l_j \cos\theta_i}{\sinh l_k}(1+I_k) \\
&= \frac{(1+I_i)(1+I_k)\sinh^4 s}{\sinh^2 l_i \sinh^2 l_k}[2(1+I_i)(1+I_j)(1+I_k)\sinh^4 s \\
&\quad + (6+3I_i+6I_j+3I_k+3I_iI_j+2I_iI_k+3I_jI_k-I_i^2-I_k^2)\sinh^2 s + 4(1+I_j)].
\end{aligned} \tag{3.9}
$$

Note that

$$
\begin{aligned}
N_{12}N_{21} &= [-(1+I_i) + \frac{\sinh l_i \cos\theta_j}{\sinh l_k}(1+I_k)][-(1+I_j) + \frac{\sinh l_j \cos\theta_i}{\sinh l_k}(1+I_k)] \\
&= \frac{1}{\sinh^4 l_k}[(1+I_k)\sinh l_k \sinh l_i \cos\theta_j - (1+I_i)\sinh^2 l_k] \\
&\quad \times [(1+I_k)\sinh l_k \sinh l_j \cos\theta_i - (1+I_j)\sinh^2 l_k] \\
&= \frac{1}{\sinh^4 l_k}[(1+I_k)(\cosh l_i \cosh l_k - \cosh l_j) - (1+I_i)\sinh^2 l_k] \\
&\quad \times [(1+I_k)(\cosh l_j \cosh l_k - \cosh l_i) - (1+I_j)\sinh^2 l_k] \\
&= \frac{(1+I_k)^4 \sinh^4 s}{\sinh^4 l_k}(1+I_i+I_j-I_k)^2,
\end{aligned} \tag{3.10}
$$

where $\cosh l_i = \cosh^2 s + I_i \sinh^2 s = 1 + (1+I_i)\sinh^2 s$ is used in the last line.

Combining (3.8), (3.9), (3.10), we have the leading 2×2 minor of N is

$$
\begin{aligned}
&\frac{(1+I_i)(1+I_j)(1+I_k)^2\sinh^8 s}{\sinh^2 l_i \sinh^2 l_j \sinh^4 l_k} \\
&\quad \times \big[2(1+I_i)(1+I_j)(1+I_k)\sinh^4 s \\
&\quad + (6+6I_i+3I_j+3I_k+3I_iI_j+3I_iI_k+2I_jI_k-I_j^2-I_k^2)\sinh^2 s + 4(1+I_i)\big] \\
&\quad \times [2(1+I_i)(1+I_j)(1+I_k)\sinh^4 s \\
&\quad + (6+3I_i+6I_j+3I_k+3I_iI_j+2I_iI_k+3I_jI_k-I_i^2-I_k^2)\sinh^2 s + 4(1+I_j)] \\
&\quad - \frac{(1+I_k)^4\sinh^4 s}{\sinh^4 l_k}(1+I_i+I_j-I_k)^2
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(1+I_k)^2 \sinh^4 s}{\sinh^2 l_i \sinh^2 l_j \sinh^4 l_k} \\
&\quad \times \{(1+I_i)(1+I_j)\sinh^4 s[4(1+I_i)^2(1+I_j)^2(1+I_k)^2 \sinh^8 s \\
&\quad + A\sinh^6 s + B\sinh^4 s + C\sinh^2 s + D] \\
&\quad - (1+I_k)^2(1+I_i+I_j-I_k)^2 \sinh^2 l_i \sinh^2 l_j\},
\end{aligned}
$$

where A, B, C, D are polynomials of I_i, I_j, I_k. Note that $\sinh^2 l_i = \cosh^2 l_i - 1 = (1+I_i)\sinh^2 s[2+(1+I_i)\sinh^2 s]$, we have the leading 2×2 minor of N is

$$
\begin{aligned}
&\frac{(1+I_i)(1+I_j)(1+I_k)^2 \sinh^8 s}{\sinh^2 l_i \sinh^2 l_j \sinh^4 l_k} \\
&\quad \times \{4(1+I_i)^2(1+I_j)^2(1+I_k)^2 \sinh^8 s + A\sinh^6 s + B\sinh^4 s + C\sinh^2 s + D \\
&\quad - (1+I_k)^2(1+I_i+I_j-I_k)^2[2+(1+I_i)\sinh^2 s][2+(1+I_j)\sinh^2 s]\}.
\end{aligned}
$$

The term in the last two lines is a polynomial in $\sinh s$ with positive leading coefficient $4(1+I_i)^2(1+I_j)^2(1+I_k)^2$, so for s large enough, the leading 2×2 minor of N is positive.

Combining with the fact that the determinant of $\mathcal{J}$ is positive, we have the matrix Λ^H_{ijk} is negative definite. This completes the proof. $\square$

Remark 10. The matrix $\mathcal{J}$ in (3.7) is the same matrix M in the proof of Lemma 12 of Guo [22], where M was proved to be positive definite for nonnegative inversive distance. Here we produces another proof of the fact.

Remark 11. If $I_i, I_j, I_k \in (-1, 1]$ and $\gamma_{ijk} \geq 0$, $\gamma_{jik} \geq 0$, $\gamma_{kij} \geq 0$, the negative definiteness of Λ^H_{ijk} was proved by Zhou [37] using the same method as that of Lemma 3.7. In this case, the negative definiteness of Λ^H_{ijk} could be proved alternatively. In fact, by direct but tedious calculations, we have

$$
\begin{aligned}
&\frac{\partial\theta_i}{\partial u_i} + \frac{\partial\theta_j}{\partial u_i} + \frac{\partial\theta_k}{\partial u_i} \\
&\quad = \frac{1}{A(\cosh l_j + 1)(\cosh l_k + 1)} \cdot \\
&\qquad \{C_i S_i^2 S_j^2 (I_k^2 - 1) + S_i^2 S_j^2 C_k (I_k^2 - 1) - S_k(C_j S_i^2 S_j \gamma_{ijk} + C_i S_i S_j^2 \gamma_{jik}) \\
&\qquad + C_k S_k[-S_i^2 S_j(2C_iC_j + 1)\gamma_{ijk} - (C_i^2 + S_i^2) S_i S_j^2 \gamma_{jik}] \\
&\qquad + S_k^2[-2C_i S_i^2 S_j^2 (I_i I_j I_k + 1) - S_i S_j \gamma_{kij}(C_j S_i^2 + C_i) + C_i S_i^2 (I_j^2 - 1) \\
&\qquad + C_j S_i^2 (I_j^2 - 1)]\}.
\end{aligned}
\tag{3.11}
$$

In general, $\frac{\partial\theta_i}{\partial u_i} + \frac{\partial\theta_j}{\partial u_i} + \frac{\partial\theta_k}{\partial u_i}$ have no sign. However, if $I_i, I_j, I_k \in (-1, 1]$ and $\gamma_{ijk} \geq 0$, $\gamma_{jik} \geq 0$, $\gamma_{kij} \geq 0$, we have $\frac{\partial\theta_i}{\partial u_i} + \frac{\partial\theta_j}{\partial u_i} + \frac{\partial\theta_k}{\partial u_i} < 0$ by (3.11). Combining with Remark 9, this implies $-\Lambda^H_{ijk}$ is diagonal dominant and then Λ^H_{ijk} is negative definite.

Set

$$\Lambda^H = \frac{\partial(K_1, \cdots, K_N)}{\partial(u_1, \cdots, u_N)} = -\sum_{\triangle ijk \in F} \Lambda^H_{ijk},$$

where Λ^H_{ijk} is extended by zeros to a $N \times N$ matrix so that Λ^H_{ijk} acts on a vector $(v_1, \cdots, v_N)$ only on the coordinates corresponding to vertices v_i, v_j and v_k in the triangle $\triangle ijk$. Lemma 3.6 and Lemma 3.7 have the following direct corollary.

Corollary 3.8. *Given a triangulated surface $(M, \mathcal{T}, I)$ with inversive distance $I > -1$ and $\gamma_{ijk} \geq 0, \gamma_{jik} \geq 0, \gamma_{kij} \geq 0$ for any topological triangle $\triangle ijk \in F$. Then the matrix $\Lambda^H = \frac{\partial(K_1,\cdots,K_N)}{\partial(u_1,\cdots,u_N)}$ is symmetric and positive definite on $\mathcal{U}^H := \cap_{\triangle ijk \in T} \mathcal{U}^H_{ijk}$ for the hyperbolic background geometry.*

Guo [22] once obtained a result paralleling to Corollary 3.8 for $I \geq 0$.

By Lemma 3.4 and Lemma 3.6, we can define an energy function

$$\mathcal{E}_{ijk}(u) = \int_{u_0}^{u} \theta_i du_i + \theta_j du_j + \theta_k du_k$$

on $\mathcal{U}^H_{ijk} = \ln(\Omega^H_{ijk})$. Lemma 3.7 ensures that $\mathcal{E}_{ijk}$ is locally concave on $\mathcal{U}^H_{ijk}$. Define the Ricci potential as

$$\mathcal{E}(u) = -\sum_{\triangle ijk \in T} \mathcal{E}_{ijk}(u) + \int_{u_0}^{u} \sum_{i=1}^{N} (2\pi - \overline{K}_i) du_i, \tag{3.12}$$

then $\nabla_u \mathcal{E} = K - \overline{K}$ and $\mathcal{E}(u)$ is locally convex on $\mathcal{U}^H = \cap_{\triangle ijk \in T} \mathcal{U}^H_{ijk}$. The local convexity of $\mathcal{E}$ implies the infinitesimal rigidity of K with respect to u, which is the infinitesimal rigidity of hyperbolic inversive distance circle packings.

3.3. Global rigidity of hyperbolic inversive distance circle packings

In this subsection, we shall prove the global rigidity of hyperbolic inversive distance circle packings under the condition $I \in (-1, +\infty)$ and $\gamma_{ijk} \geq 0, \gamma_{jik} \geq 0, \gamma_{kij} \geq 0$ for any triangle $\triangle ijk \in F$.

By Corollary 3.5, the functions $\theta_i, \theta_j, \theta_k$ defined on $\mathcal{U}^H_{ijk}$ could be continuously extended by constants to $\widetilde{\theta}_i, \widetilde{\theta}_j, \widetilde{\theta}_k$ defined on $\mathbb{R}^3$. Using Theorem 2.9, we have the following extension.

Lemma 3.9. *In the hyperbolic background geometry, for any triangle $\triangle ijk \in F$ with $I_i, I_j, I_k > -1$ and $\gamma_{ijk} \geq 0$, $\gamma_{jik} \geq 0$, $\gamma_{kij} \geq 0$, the function $\mathcal{E}_{ijk}(u)$ defined on $\mathcal{U}^H_{ijk}$ could be extended to the following function*

$$\widetilde{\mathcal{E}}_{ijk}(u)=\int_{u_0}^{u}\widetilde{\theta}_i du_i+\widetilde{\theta}_j du_j+\widetilde{\theta}_k du_k, \tag{3.13}$$

which is a C^1-smooth concave function defined on $\mathbb{R}^3$ with

$$\nabla_u\widetilde{\mathcal{E}}_{ijk}=(\widetilde{\theta}_i,\widetilde{\theta}_j,\widetilde{\theta}_k)^T.$$

Using Lemma 3.9, we can prove the following global rigidity of hyperbolic inversive distance circle packing metrics, which is the hyperbolic part of Theorem 1.1.

Theorem 3.10. *Given a triangulated surface $(M,\mathcal{T})$ with inversive distance $I\in(-1,+\infty)$ and $\gamma_{ijk}\geq 0,\gamma_{jik}\geq 0,\gamma_{kij}\geq 0$ for any topological triangle $\triangle ijk\in F$. Then for any $\overline{K}\in C(V)$, there is at most one hyperbolic inversive distance circle packing metric r with $K(r)=\overline{K}$.*

Proof. The Ricci energy function $\mathcal{E}(u)$ in (3.12) could be extended from $\mathcal{U}^H$ to the whole space $\mathbb{R}^N$, where $\mathcal{U}^H$ is the image of Ω^H under the map $u_i=\ln\tanh\frac{r_i}{2}$. In fact, the function $\mathcal{E}_{ijk}(u)$ defined on $\mathcal{U}^H_{ijk}$ could be extended to $\widetilde{\mathcal{E}}_{ijk}(u)$ defined by (3.13) on $\mathbb{R}^N$ by Lemma 3.9 and the second term $\int_{u_0}^{u}\sum_{i=1}^N(2\pi-\overline{K}_i)du_i$ in (3.12) can be naturally defined on $\mathbb{R}^N$, then we have the following extension $\widetilde{\mathcal{E}}(u)$ defined on $\mathbb{R}^N$ of the Ricci potential function $\mathcal{E}(u)$

$$\widetilde{\mathcal{E}}(u)=-\sum_{\triangle ijk\in F}\widetilde{\mathcal{E}}_{ijk}(u)+\int_{u_0}^{u}\sum_{i=1}^{N}(2\pi-\overline{K}_i)du_i.$$

As $\widetilde{\mathcal{E}}_{ijk}(u)$ is C^1-smooth concave by Lemma 3.9 and $\int_{u_0}^{u}\sum_{i=1}^N(2\pi-\overline{K}_i)du_i$ is a well-defined convex function on $\mathbb{R}^N$, we have $\widetilde{\mathcal{E}}(u)$ is a C^1-smooth convex function on $\mathbb{R}^N$. Furthermore,

$$\nabla_{u_i}\widetilde{F}=-\sum_{\triangle ijk\in F}\widetilde{\theta}_i+2\pi-\overline{K}_i=\widetilde{K}_i-\overline{K}_i,$$

where $\widetilde{K}_i=2\pi-\sum_{\triangle ijk\in F}\widetilde{\theta}_i$.

If there are two different inversive distance circle packing metrics $\overline{r}_A,\overline{r}_B\in\Omega^H$ with the same combinatorial Curvature $\overline{K}$, then $\overline{u}_A=\ln\tanh\frac{\overline{r}_A}{2}\in\mathcal{U}^H$, $\overline{u}_B=\ln\tanh\frac{\overline{r}_B}{2}\in\mathcal{U}^H$ are both critical points of the extended Ricci potential $\widetilde{\mathcal{E}}(u)$. It follows that

$$\nabla\widetilde{\mathcal{E}}(\overline{u}_A)=\nabla\widetilde{\mathcal{E}}(\overline{u}_B)=0.$$

Set

$$
\begin{aligned}
f(t) &= \widetilde{\mathcal{E}}((1-t)\overline{u}_A + t\overline{u}_B) \\
&= \sum_{\triangle ijk \in F} f_{ijk}(t) + \int_{u_0}^{(1-t)\overline{u}_A + t\overline{u}_B} \sum_{i=1}^{N} (2\pi - \overline{K}_i) du_i,
\end{aligned}
$$

where

$$f_{ijk}(t) = -\widetilde{\mathcal{E}}_{ijk}((1-t)\overline{u}_A + t\overline{u}_B).$$

Then $f(t)$ is a C^1 convex function on $[0,1]$ and $f'(0) = f'(1) = 0$, which implies $f'(t) \equiv 0$ on $[0,1]$. Note that $\overline{u}_A$ belongs to the open set $\mathcal{U}^H$, there exists $\epsilon > 0$ such that $(1-t)\overline{u}_A + t\overline{u}_B \in \mathcal{U}^H$ for $t \in [0, \epsilon]$. So $f(t)$ is smooth on $[0, \epsilon]$.

Note that $f(t)$ is C^1 convex on $[0,1]$ and smooth on $[0, \epsilon]$. $f'(t) \equiv 0$ on $[0,1]$ implies that $f''(t) \equiv 0$ on $[0, \epsilon]$. Note that, for $t \in [0, \epsilon]$,

$$f''(t) = (\overline{u}_A - \overline{u}_B)\Lambda^H(\overline{u}_A - \overline{u}_B)^T,$$

where $\Lambda^H = -\sum_{\triangle ijk \in F} \Lambda^H_{ijk}$. By Corollary 3.8, we have Λ^H is positive definite and then $\overline{u}_A - \overline{u}_B = 0$, which implies that $\overline{r}_A = \overline{r}_B$. So there exists at most one hyperbolic inversive distance circle packing metric with combinatorial curvature $\overline{K}$. $\square$

3.4. Rigidity of combinatorial α-curvature in hyperbolic background geometry

We have the following global rigidity for α-curvature with respect to hyperbolic inversive distance circle packing metrics for inversive distance in $(-1, +\infty)$, which is the hyperbolic part of Theorem 1.2.

Theorem 3.11. *Given a closed triangulated surface $(M, \mathcal{T})$ with inversive distance $I > -1$ and $\gamma_{ijk} \geq 0, \gamma_{jik} \geq 0, \gamma_{kij} \geq 0$ for any topological triangle $\triangle ijk \in F$, $\overline{R}$ is a given function defined on the vertices of $(M, \mathcal{T})$. If $\alpha\overline{R} \leq 0$, there exists at most one hyperbolic inversive distance circle packing metric $\overline{r} \in \Omega^H$ with combinatorial α-curvature $\overline{R}$.*

As the proof of Theorem 3.11 is almost parallel to that of Theorem 3.10 using the energy function

$$\widetilde{\mathcal{E}}_\alpha(u) = -\sum_{\triangle ijk \in F} \widetilde{\mathcal{E}}_{ijk}(u) + \int_{u_0}^{u} \sum_{i=1}^{N} (2\pi - \overline{R}_i \tanh^\alpha \frac{r_i}{2}) du_i,$$

we omit the details of the proof here. Theorem 3.11 is an generalization of Theorem 3.10. Specially, if $\alpha = 0$, Theorem 3.11 is reduced to Theorem 3.10.

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