Wigner rotation and its SO(3) model: an active-frame approach

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ABSTRACT

As an important issue in special relativity, Wigner rotation is notoriously difficult for beginners for two major reasons: this physical phenomenon is highly unintuitive, and the mathematics behind it can be extremely challenging. To remove the first obstacle, we introduce a clear and easy toy model under the guidance of group theory. To overcome the second, a concise mathematical method is developed by the integration of geometric algebra and active-frame formalism.

I. INTRODUCTION

First discovered by L. Silberstein then rediscovered by L. Thomas [1,2], the phenomenon that two successive non-parallel boosts (i.e. pure Lorentz transformations) lead to a boost and a rotation is generally called Wigner rotation [3]. It has been studied by many authors for near a century [4-7], the mysterious aura still persists nevertheless. "The spatial rotation resulting from the successive application of two non-parallel Lorentz transformations have been declared every bit as paradoxical as the more frequently discussed apparent violations of common sense, such as the so-called 'twin paradox.' But the present apparent paradox has important applications..." said H. Goldstein in his classic work *Classical Mechanics* [8].

Intuitively, Wigner rotation seems to violate the transitive relation of parallelism, which is certainly unreasonable. As a matter of fact, as long as there is a relative velocity, the apparent parallelism between the spatial parts of two reference frames is just a three-dimensional illusion. The situation will be quite different if we switch to the four-dimensional perspective. Consider two reference frames with no relative velocity at first, we may assume their individual spatial axes are parallel to each other. However, when one of the frames begins to move with a constant velocity, the parallelism vanishes at least partially. This is because boost is essentially a "pseudo-rotation" in four-dimensional spacetime.

When the problem involves two non-parallel boosts and three reference frames, say K_0, K_1 and K_2, it is justifiable not all of the spatial axes of K_0

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(K_1) are parallel to that of K_1 (K_2), therefore it makes sense some of the spatial axes of K_0 and K_2 might not be parallel. What contradicts to one's intuition is another frame K_3 can be constructed from K_2 by a boost transformation so that K_3 and K_0 are relatively at rest, some spatial axes of K_3 and K_0 still differ by a rotation—the Wigner rotation—even though there is no temporal dimension involved.

To comprehend this fact, it is better to use geometric picture to replace the physical one, i.e., consider a series of frame pseudo-rotations ($K_0 \rightarrow K_1 \rightarrow K_2 \rightarrow K_3$) instead of a snap shot of these four frames. Even then, since the pseudo-rotation is quite different to the ordinary one, it is hard to build a clear picture in one's mind. The best policy is to find a toy model for this process which contains only ordinary rotations. This model may be considered as the first achievement of this paper.

Although Wigner rotation emerges whenever the two boost velocities are not parallel, people usually let these velocities be perpendicular to each other to simplify the calculations. It will be called the simple Wigner rotation in this paper. Interpreting this simple case as a series of pseudo-rotations, we are able to build an $SO(3)$ toy model to mimic this process (Section III). Being a model, it contains the essence of the original problem nonetheless. Thus we can use what we learn to study the simple Wigner rotation (Section IV) then generalize it to the general case (Section V).

As for the second achievement, by working directly on the active frame we show there is no need to consider the passive coordinate transformation at all. (Fig. 1) Allying with geometric algebra, the active-frame formalism enables us to derive all the important results of Wigner rotation and condense them into three neat geometric theorems.

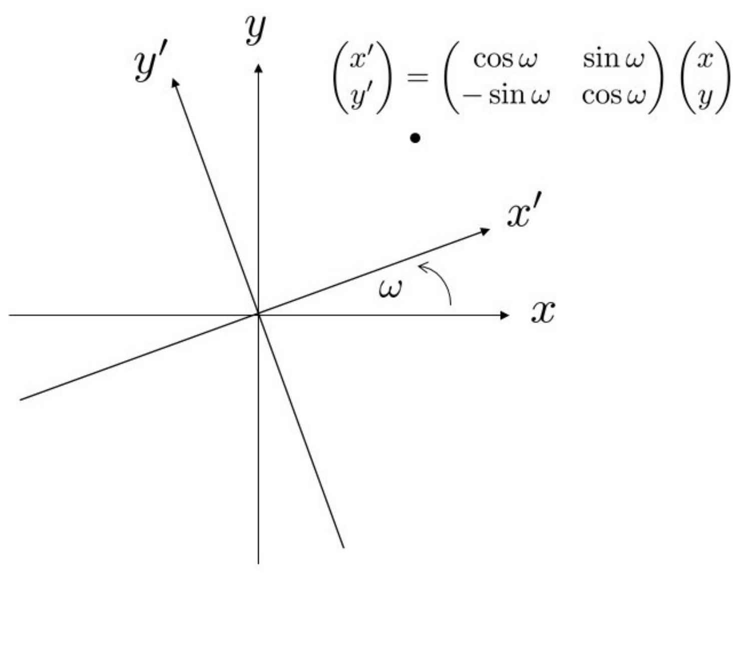

Fig. 1. Passive transformation of coordinates is generated by active transformation of frame, using the former to describe the latter is indirect and unintuitive. When we need to study the transformation of a frame, it is better to use the active-frame formalism.

II. PRELIMINARIES

A. Active frame

To make the mathematics amiable, the two-dimensional Euclidean space $\mathbb{R}^2$ is served as our first example. For a point with coordinates (x, y), the position vector of this point is $x\hat{x} + y\hat{y} = (\hat{x}\,\hat{y})(x\,y)^\top$, where $\top$ is the notation for matrix transpose and $(\hat{x}, \hat{y})$ is the frame which may be taken as a set of bases at the origin. From now on, we will always use this kind of bases instead of coordinate axes, the only difference is the former have unit length while the length of the latter is infinite.

Now consider a passive linear transformation in this space, if the transformation law for the coordinates is $(x'\,y')^\top = [T](x\,y)^\top$, where $[T]$ is the matrix representation of the transformation T, then the frame must obey $(\hat{x}'\,\hat{y}') = (\hat{x}\,\hat{y})[T]^{-1}$ to balance the change made by $[T]$ and render the position vector intact. Conversely, in order to generate the passive coordinate transformation, the frame actively transforms from its original orientation to the new one.

When the transformation is a two-dimensional rotation, i.e., an element of the orthogonal group $SO(2)$, we have $[T]^{-1} = [T]^\top$so that the transformations for the coordinates and frame are formally the same.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = [T] \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$
$$\begin{pmatrix} \hat{x}' \\ \hat{y}' \end{pmatrix} = [T] \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}. \tag{1}$$

For a positive ω, (1) represents a counterclockwise (i.e. positive-sense) rotation of the frame $(\hat{x}, \hat{y})$. We shall think this rotation taking place along the xy-plane instead of around some axis, the reason will be clear soon.

If the coordinates (x, y) is replaced by (x, ct), where c is light speed, the space corresponding to the new coordinates is called the two-dimensional Minkowski space $\mathbb{R}^{1+1}$. It differs from $\mathbb{R}^2$ in the following aspects.

1. Although the second coordinate is ct with the dimension of length, the corresponding basis is $\hat{t}$ which is a dimensionless quantity like $\hat{x}$ and $\hat{y}$.

2. There is no ordinary rotation in $\mathbb{R}^{1+1}$, instead we have the pseudo-rotation (or more precisely the hyperbolic rotation) defined by

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \cosh\Omega & -\sinh\Omega \\ -\sinh\Omega & \cosh\Omega \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}, \tag{2}$$

which leaves $x^2 - c^2t^2$ unchanged. We may think it taking place along the xt-plane just like ordinary rotations performing along an Euclidean plane.

3. The transformation matrix in (2) belongs to the group $SO(1,1)$ instead of $SO(2)$. They are symmetric but not orthogonal unless $\Omega = 0$.

4. With regard to the coordinate transformation (2), the transformation law for the Minkowski frame is

$$\begin{pmatrix} \hat{x}' \\ \hat{t}' \end{pmatrix} = \begin{pmatrix} \cosh\Omega & \sinh\Omega \\ \sinh\Omega & \cosh\Omega \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{t} \end{pmatrix}. \tag{3}$$

It is obvious the transformed frame is not orthonormal as long as $\Omega \neq 0$, i.e., the new bases will no longer have unit length and will no longer be perpendicular to each other. (Fig. 2)

From the physical perspective, there are several more noteworthy points.

5. If the frame $(\hat{x}, \hat{t})$ is interpreted as a physical reference frame, then the space basis $\hat{x}$ is usually thought as a rigid rule and the time basis $\hat{t}$ a set of clocks fixed on the rule. When talking about a moving frame, we mean the rule carries those clocks (not the basis $\hat{t}$!) moving along the x-direction.

6. If we interpret the hyperbolic angle Ω as the rapidity of the relative speed u between those two frames, then $u = c\tanh\mathbf{\Omega}$ and (2) and (3) become the boost transformations of the spacetime coordinates (x, ct) and the frame $(\hat{x}, \hat{t})$ respectively. Note that the rigid rules corresponding to the space bases $\hat{x}$ and $\hat{x}'$ are still parallel to each other from the one-dimensional point of view.

7. We can see from (3) the length of the new bases is larger than the old one, meaning the scale of the frame increases after the transformation. This change of scale is, from geometric point of view, the origin of length contraction and time dilation in special relativity. Hence these two phenomena can be easily demonstrated and understood when we use active frame as the mathematical tool.

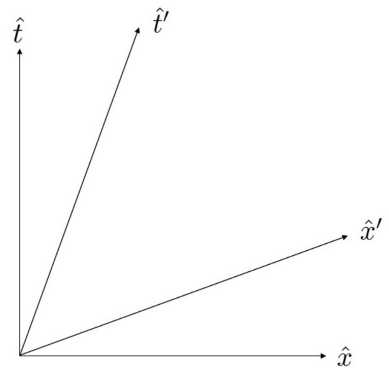

Fig. 2. Hyperbolic rotation of the Minkowski frame $(\hat{x}, \hat{t})$.

B. $SO(2,1)$ and $SO(3)$ groups

Although physical spacetime is the four-dimensional Minkowski space $\mathbb{R}^{3+1}$, we work on its subspace in many cases without losing the generality. For example, when discussing a boost transformation between two frames $(\hat{x}, \hat{y}, \hat{z}, \hat{t})$

and $(\hat{x}', \hat{y}', \hat{z}', \hat{t}')$, we may assume the relative velocity is along the x-direction and consider just the transformation of coordinates (x, ct) which is depicted by (2). Similarly, since the problem of Wigner rotation involves only two relative velocities, it is legitimate to put them in the xy-plane so that none of the z-components shows up in the calculations. Therefore the space $\mathbb{R}^{2+1}$ and the transformation group $SO(2,1)$ are sufficient for us to derive all of the related results.

From the point of view of group theory, there are many similarities between $SO(2,1)$ and $SO(3)$, the rotation group of three-dimensional Euclidean space $\mathbb{R}^3$. For example, the invariants of these two groups are $x^2 + y^2 - c^2t^2$ and $x^2 + y^2 + z^2$ respectively. This allows us to model Wigner rotation problem with a series of rotations in $\mathbb{R}^3$ which provides a concrete and clear picture. Our strategy is to study this model in detail and then apply what we learn to the original problem.

C. Geometric algebra

Wigner rotation, like many other problems in special relativity, is usually studied by using vector and matrix as the mathematical tools. However, there is an alternative choice named geometric algebra (also known as Clifford algebra) which might be more suitable. Putting it simply, geometric algebra is nothing but the traditional vector algebra plus a new operation, the so called geometric product. Since none of the original structure is affected after the new operation is added, we may say geometric algebra is a straightforward extension of vector algebra.

For the four-dimensional Minkowski space $\mathbb{R}^{3+1}$, the geometric product is defined as below.

1. $\hat{x}\hat{x} = \hat{y}\hat{y} = \hat{z}\hat{z} = 1$ and $\hat{t}\hat{t} = -1$. The first three correspond to the unit length of those bases and the fourth one reflects the "Minkowskiness" of $\mathbb{R}^{3+1}$.

2. The geometric products of different bases obey the anti-commutative relation, e.g., $\hat{x}\hat{y} = -\hat{y}\hat{x}$, $\hat{y}\hat{t} = -\hat{t}\hat{y}$, etc.

3. The dagger conjugation changes the ordering in the products, e.g., $(\hat{x}\hat{y})^\dagger = \hat{y}\hat{x}$, $(\hat{x}\hat{y}\hat{z})^\dagger = \hat{z}\hat{y}\hat{x}$, etc.

With the aid of geometric algebra, we can express (1) and its three-dimensional version in the following neat forms.

$$\begin{pmatrix} \hat{x}' \\ \hat{y}' \end{pmatrix} = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = R \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} R^\dagger,$$

$$\begin{pmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{pmatrix} = \begin{pmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = R \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} R^\dagger, \tag{4}$$

where $R = \exp(-\frac{\omega}{2}\hat{x}\hat{y})$ is called rotor. Note that $\hat{z}$ behaves like a constant because $(\hat{x}\hat{y})\hat{z} = \hat{z}(\hat{x}\hat{y})$ and $\exp(-\frac{\omega}{2}\hat{x}\hat{y})\hat{z} = \hat{z}\exp(-\frac{\omega}{2}\hat{x}\hat{y})$. Using the identity $(\hat{x}\hat{y})(\hat{x}\hat{y}) = -1$, R can be expanded as $\cos\frac{\omega}{2} - \hat{x}\hat{y}\sin\frac{\omega}{2}$ and its action to the bases can be calculated easily. For example,

$$\hat{x}' = R\hat{x}R^\dagger = (\cos\frac{\omega}{2} - \hat{x}\hat{y}\sin\frac{\omega}{2})\hat{x}(\cos\frac{\omega}{2} + \hat{x}\hat{y}\sin\frac{\omega}{2}) = \hat{x}\cos\omega + \hat{y}\sin\omega. \quad (5)$$

If we replace ω with $\omega + 2\pi$, $R = \exp(-\frac{\omega}{2}\hat{x}\hat{y})$ becomes $-R$ but the rotation transformation (4) is not affected at all. Therefore we adopt the identification $R \equiv -R$.

The rotor R introduced above can be derived rigorously from the so called Cartan-Dieudonné theorem [6,9]. Here we start with the semi-finished result $R = C(\hat{x}+\hat{x}')\hat{x} = C(\hat{y}+\hat{y}')\hat{y}$, where $\hat{x}'$ and $\hat{y}'$ are given by (1) and C is the real normalization constant in order that $RR^\dagger = 1$. Since $R = C(1+\hat{x}'\hat{x})$ and $\hat{x}'\hat{x}' = \hat{x}\hat{x} = 1$,

$$R\hat{x}R^\dagger = C^2(1+\hat{x}'\hat{x})\hat{x}(1+\hat{x}\hat{x}') = C^2\hat{x}'(1+\hat{x}'\hat{x})(1+\hat{x}\hat{x}') = \hat{x}',$$

which is consistent with (5).

The term $\hat{x}$ in $R = C(\hat{x}+\hat{x}')\hat{x}$ may be interpreted metaphorically as "initial position" and $(\hat{x}+\hat{x}')$ on its left as "halfway between $\hat{x}$ and $\hat{x}'$". This combination may be taken as a general rule.

By using the explicit expression of $\hat{x}'$ and some trigonometric identities, it is straightforward to derive $R = \cos\frac{\omega}{2} - \hat{x}\hat{y}\sin\frac{\omega}{2}$. Obviously we can use $R = C(\hat{y}+\hat{y}')\hat{y}$ to obtain the same result.

The rotors for the hyperbolic rotations in $\mathbb{R}^{2+1}$ are similar to those in $\mathbb{R}^3$. For example, when the rotation is along the xt-plane, the corresponding rotor B takes the form $\exp(-\frac{\Omega}{2}\hat{x}\hat{t})$ and the basis $\hat{x}$ is transformed as

$$\hat{x}^* = B\hat{x}B^\dagger = (\cosh\frac{\Omega}{2} - \hat{x}\hat{t}\sinh\frac{\Omega}{2})\hat{x}(\cosh\frac{\Omega}{2} + \hat{x}\hat{t}\sinh\frac{\Omega}{2}) = \hat{x}\cosh\Omega + \hat{t}\sinh\Omega,$$

where the hyperbolic functions come from the identity $(\hat{x}\hat{t})(\hat{x}\hat{t}) = 1$.

Analogous to (4), the transformation of the frame $(\hat{x}, \hat{y}, \hat{t})$ is

$$\begin{pmatrix}\hat{x}^*\\ \hat{y}^*\\ \hat{t}^*\end{pmatrix} = \begin{pmatrix}\cosh\Omega & 0 & \sinh\Omega\\ 0 & 1 & 0\\ \sinh\Omega & 0 & \cosh\Omega\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{t}\end{pmatrix} = B\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{t}\end{pmatrix}B^\dagger.$$

To derive the hyperbolic rotors, we need to use the Minkowski version of Cartan-Dieudonné theorem. The procedure is similar to that of the rotors in $\mathbb{R}^3$. First we have $B = C'(\hat{x}+\hat{x}^*)\hat{x} = -C'(\hat{t}+\hat{t}^*)\hat{t}$ with the condition $BB^\dagger = 1$. Then it can be proved $B = \cosh\frac{\Omega}{2} - \hat{x}\hat{t}\sinh\frac{\Omega}{2}$ as long as the identification $R \equiv -R$ is generalized to include hyperbolic rotors.

D. Composition of velocities

In special relativity, boost transformation takes place between two inertial frames, hence each boost is defined by a constant velocity which is the relative velocity between the frames. Since Wigner rotation involves two successive boosts, it is inherently related to the problem of adding two velocities relativistically. We give a brief review of this problem below.

In the four-dimensional Minkowski space $\mathbb{R}^{3+1}$, the transformation law for a general four-vector Q is $Q' = [L]Q$, where $[L]$ is an element of the matrix group $SO(3,1)$ which contains all the possible passive boosts and rotations as well as their products. For a four-velocity undergoes a boost $B(\vec{V})$, the transformation is $W' = [B(\vec{V})]W$, where W and W' are the four-velocities of the old and the new frames respectively, and $[B(\vec{V})]$ is the matrix representation of $B(\vec{V})$ which can be proved being always symmetric. Conversely, the inverse boost transformation $W = [B(\vec{V})]^{-1}W' = [B(-\vec{V})]W'$ allows us to calculate the four-velocity of the old frame from that of the new one.

Now consider an object resting in the new frame, since its three-velocity relative to the old frame equals to the relative velocity between the two frames, transforming its four-velocity from the new frame back to the old one reveals the information of the boost velocity.

$$W = [B(\vec{V})]^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ c \end{pmatrix} = \gamma(\vec{V}) \begin{pmatrix} V_x \\ V_y \\ V_z \\ c \end{pmatrix},$$

$$\text{where } \gamma(\vec{V}) = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} = \cosh(\tanh^{-1} \frac{V}{c}).$$

The above result has a remarkable geometric representation in the active-frame formalism. Given a boost transformation $B(\vec{V})$, the time basis $\hat{t}'$ of the new frame is proportional to the four-velocity of the boost velocity,

$$\hat{t}' = (\hat{x}'\, \hat{y}'\, \hat{z}'\, \hat{t}') \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = (\hat{x}\, \hat{y}\, \hat{z}\, \hat{t})[B(\vec{V})]^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{\gamma(\vec{V})}{c}(\vec{V} + c\hat{t}). \tag{6}$$

When the problem involves three inertial frames and two successive boosts, first $B(\vec{V}_1)$ then $B(\vec{V}_2)$, the four-velocity of a rest object in the third frame can be transformed to that of the first frame by

$$W = ([B(\vec{V}_2)][B(\vec{V}_1)])^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ c \end{pmatrix} = [B(\vec{V}_1)]^{-1}[B(\vec{V}_2)]^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ c \end{pmatrix}.$$

The three-velocity contained in this four-velocity is the composition of the two boost velocities in that order and is usually denoted by $\vec{V}_1 \oplus \vec{V}_2$. Therefore the above formula is equivalent to

$$[B(\vec{V}_1)]^{-1}[B(\vec{V}_2)]^{-1}\begin{pmatrix}0\\0\\0\\c\end{pmatrix}=[B(\vec{V}_1\oplus\vec{V}_2)]^{-1}\begin{pmatrix}0\\0\\0\\c\end{pmatrix}=\gamma(\vec{V}_1\oplus\vec{V}_2)\begin{pmatrix}(\vec{V}_1\oplus\vec{V}_2)_x\\(\vec{V}_1\oplus\vec{V}_2)_y\\(\vec{V}_1\oplus\vec{V}_2)_z\\c\end{pmatrix},\tag{7}$$

and (6) is accordingly generalized to

$$\hat{t}'=\frac{\gamma(\vec{V}_1\oplus\vec{V}_2)}{c}[\vec{V}_1\oplus\vec{V}_2+c\hat{t}\,]\text{ or }c\hat{t}'=\gamma(\vec{V}_1\oplus\vec{V}_2)[\vec{V}_1\oplus\vec{V}_2+c\hat{t}\,].\tag{8}$$

In short, $c\hat{t}'$ equals to the four-velocity which corresponds to the three-velocity $\vec{V}_1\oplus\vec{V}_2$ that defines the composite boost.

If the order of the two boosts is exchanged, then (7) becomes

$$[B(\vec{V}_2)]^{-1}[B(\vec{V}_1)]^{-1}\begin{pmatrix}0\\0\\0\\c\end{pmatrix}=[B(\vec{V}_2\oplus\vec{V}_1)]^{-1}\begin{pmatrix}0\\0\\0\\c\end{pmatrix}=\gamma(\vec{V}_2\oplus\vec{V}_1)\begin{pmatrix}(\vec{V}_2\oplus\vec{V}_1)_x\\(\vec{V}_2\oplus\vec{V}_1)_y\\(\vec{V}_2\oplus\vec{V}_1)_z\\c\end{pmatrix}.$$

An important identity $\gamma(\vec{V}_1\oplus\vec{V}_2)=\gamma(\vec{V}_2\oplus\vec{V}_1)$ can be proved by the following arguments. $\gamma(\vec{V}_1\oplus\vec{V}_2)$ and $\gamma(\vec{V}_2\oplus\vec{V}_1)$ equal to the $(4,4)$ elements of $[B(\vec{V}_1)]^{-1}[B(\vec{V}_2)]^{-1}$and $[B(\vec{V}_2)]^{-1}[B(\vec{V}_1)]^{-1}$ respectively. Since boost matrices are all symmetric, $[B(\vec{V}_1)]^{-1}[B(\vec{V}_2)]^{-1}$and $[B(\vec{V}_2)]^{-1}[B(\vec{V}_1)]^{-1}$ share the same diagonal elements. Q.E.D. This identity together with the marvelous relation (8) will prove useful in later discussion.

It is worthwhile to elaborate (7) a little further before closing this section. First notice (7) leads to

$$[B(\vec{V}_2)]^{-1}\begin{pmatrix}0\\0\\0\\c\end{pmatrix}=[B(-\vec{V}_1)]^{-1}[B(\vec{V}_1\oplus\vec{V}_2)]^{-1}\begin{pmatrix}0\\0\\0\\c\end{pmatrix}=[B(-\vec{V}_1\oplus(\vec{V}_1\oplus\vec{V}_2))]^{-1}\begin{pmatrix}0\\0\\0\\c\end{pmatrix},$$

which implies $\vec{V}_2=-\vec{V}_1\oplus(\vec{V}_1\oplus\vec{V}_2)$. Comparing with $\vec{V}_{A|B}=\vec{V}_A-\vec{V}_B$, the non-relativistic formula for the relative velocity of object A with respect to object B, we get the formula for relativistic relative velocity,

$$\vec{V}_{A|B}=-\vec{V}_B\oplus\vec{V}_A.$$

Next, if there are three successive boosts, $B(\vec{V}_1)$, $B(\vec{V}_2)$ and $B(\vec{V}_3)$ in this order, then (7) is generalized to

$$[B(\vec{V}_1)]^{-1}[B(\vec{V}_2)]^{-1}[B(\vec{V}_3)]^{-1}\begin{pmatrix}0\\0\\0\\c\end{pmatrix}=[B(\vec{V}_1\oplus(\vec{V}_2\oplus\vec{V}_3))]^{-1}\begin{pmatrix}0\\0\\0\\c\end{pmatrix},\tag{9}$$

which means the composite boost velocity is $\vec{V}_1 \oplus (\vec{V}_2 \oplus \vec{V}_3)$. Accordingly, if more boosts are added, the composite boost velocity becomes

$$\vec{V}_1 \oplus (\vec{V}_2 \oplus (\vec{V}_3 \oplus (\vec{V}_4 \oplus \cdots (\vec{V}_{n-1} \oplus \vec{V}_n) \cdots))).$$

The parentheses are indispensable since velocity composition rule is not associative, which will be proved in the Appendix.

So much for the general review. In fact, since the problem we are dealing with may be restricted to $\mathbb{R}^{2+1}$, there is no chance for us to use the term three-velocity or four-velocity hereafter. In order to avoid confusion, the counterpart of four-velocity in $\mathbb{R}^{2+1}$ will be called (2+1)-velocity.

III. THE TOY MODEL

A. *process a* and Theorem 1

To construct the $SO(3)$ toy model of the simple Wigner rotation, first we define a series of rotations of a Euclidean frame $(\hat{x}, \hat{y}, \hat{z})$ as below and name it *process a*.

Step 1. Let the frame rotate along the zx-plane (equivalent to around the y-axis) by an angle θ, and call the new frame $(\hat{x}'_a, \hat{y}'_a, \hat{z}'_a)$, where $\hat{y}'_a = \hat{y}$.

Step 2. Let the new frame rotate along the new yz-plane (equivalent to around the new x-axis) by an angle ϕ, and call the newer frame $(\hat{x}''_a, \hat{y}''_a, \hat{z}''_a)$, where $\hat{x}''_a = \hat{x}'_a$.

Step 3. Let the newer frame rotate along the plane defined by $\hat{z}$ and $\hat{z}''_a$ to the extent that the final $\hat{z}'''_a$ coincides with the original $\hat{z}$.

Using the formulation of geometric algebra, these three rotations can be expressed as follows:

$$\text{Step 1.} \quad \begin{pmatrix} \hat{x}'_a \\ \hat{y}'_a \\ \hat{z}'_a \end{pmatrix} = \exp(-\frac{\theta}{2}\hat{z}\hat{x}) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \exp(-\frac{\theta}{2}\hat{z}\hat{x})^\dagger,$$

$$\begin{aligned} \text{Step 2.} \quad \begin{pmatrix} \hat{x}''_a \\ \hat{y}''_a \\ \hat{z}''_a \end{pmatrix} &= \exp(-\frac{\phi}{2}\hat{y}'_a\hat{z}'_a) \begin{pmatrix} \hat{x}'_a \\ \hat{y}'_a \\ \hat{z}'_a \end{pmatrix} \exp(-\frac{\phi}{2}\hat{y}'_a\hat{z}'_a)^\dagger \\ &= \exp(-\frac{\theta}{2}\hat{z}\hat{x}) \exp(-\frac{\phi}{2}\hat{y}\hat{z}) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \exp(-\frac{\phi}{2}\hat{y}\hat{z})^\dagger \exp(-\frac{\theta}{2}\hat{z}\hat{x})^\dagger \\ &= XY \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} Y^\dagger X^\dagger, \end{aligned}$$

$$\text{Step 3.} \quad \begin{pmatrix} \hat{x}'''_a \\ \hat{y}'''_a \\ \hat{z}'''_a \end{pmatrix} = M \begin{pmatrix} \hat{x}''_a \\ \hat{y}''_a \\ \hat{z}''_a \end{pmatrix} M^\dagger = MXY \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} Y^\dagger X^\dagger M^\dagger,$$

where $X = \exp(-\frac{\theta}{2}\hat{z}\hat{x}), Y = \exp(-\frac{\phi}{2}\hat{y}\hat{z})$ and M is constructed by employing the Cartan-Dieudonné theorem,

$$\begin{aligned} M &= C(\hat{z}''_a + \hat{z})\hat{z}''_a = C(1 + \hat{z}\hat{z}''_a) \\ &= C(1 + X^\dagger Y^{\dagger 2} X^\dagger) \end{aligned} \tag{10}$$

with the condition $MM^\dagger = 1$. Note that $\hat{z}XY = X^\dagger Y^\dagger \hat{z}$ has been used to obtain (10).

In the light of (4), the expressions of the first two steps can be easily transformed to matrix form. In contrast, because there is no concise formulation for Cartan-Dieudonné theorem in the traditional vector algebra, the elegance of (10) of step 3 will be lost if we use matrix to replace geometric algebra.

Now we are ready to prove that the result of *process a* is generated by a rotor along the original xy-plane.

Theorem 1. $MXY = \exp(\hat{x}\hat{y}\frac{\epsilon}{2}) =: R_{\mathbf{w}}$ generates the toy Wigner rotation, where the toy Wigner angle ϵ is defined by

$$\tan\frac{\epsilon}{2} = \tan\frac{\theta}{2}\tan\frac{\phi}{2} \text{ with } \epsilon \in (-\pi, \pi].$$

The proof is provided by some simple calculations with three notes as follows:

$$\begin{aligned} MXY &= C(1 + X^\dagger Y^{\dagger 2} X^\dagger)XY = C(XY + X^\dagger Y^\dagger) \\ &= C[\exp(-\frac{\theta}{2}\hat{z}\hat{x})\exp(-\frac{\phi}{2}\hat{y}\hat{z}) + \exp(\frac{\theta}{2}\hat{z}\hat{x})\exp(\frac{\phi}{2}\hat{y}\hat{z})] \\ &= 2C(\cos\frac{\theta}{2}\cos\frac{\phi}{2} + \hat{x}\hat{y}\sin\frac{\theta}{2}\sin\frac{\phi}{2}) \\ &= 2C\cos\frac{\theta}{2}\cos\frac{\phi}{2}\sec\frac{\epsilon}{2}(\cos\frac{\epsilon}{2} + \hat{x}\hat{y}\sin\frac{\epsilon}{2}) \\ &= \exp(\hat{x}\hat{y}\frac{\epsilon}{2}). \end{aligned}$$

Note 1. A normalized M implies the product MXY is also normalized. Hence we assert the coefficient of $\exp(\hat{x}\hat{y}\frac{\epsilon}{2})$ equals to unity without doing practical calculation.

Note 2. Since $R_{\mathbf{w}} = \exp(-\hat{x}\hat{y}\frac{-\epsilon}{2})$, a positive ϵ corresponds to a clockwise (i.e. negative-sense) rotation according to (4). If the range of θ and ϕ is taken to be $(-\pi, \pi]$, then $\epsilon > 0$ if and only if $\theta\phi > 0$. The maximum of the toy Wigner angle is π which corresponds to, for example, $\theta = \frac{\pi}{2}$ and $\phi = \pi$.

Note 3. This theorem tells us although the basis $\hat{z}$ comes back to its original orientation at the end of the process, the other two bases deviate from the original $(\hat{x}, \hat{y})$ by a rotation. (Fig. 3)

$$MXY \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} Y^\dagger X^\dagger M^\dagger = R_{\mathbf{w}} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} R^\dagger_{\mathbf{w}} = \begin{pmatrix} R_{\mathbf{w}}\hat{x}R^\dagger_{\mathbf{w}} \\ R_{\mathbf{w}}\hat{y}R^\dagger_{\mathbf{w}} \\ \hat{z} \end{pmatrix}.$$

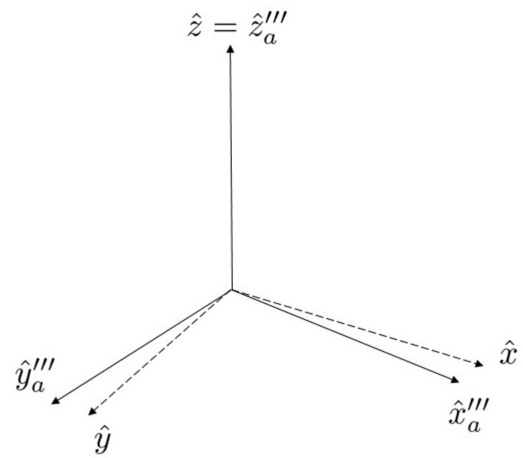

Fig. 3. The frame rotates clockwise along the xy-plane by a toy Wigner angle at the end of *process a*. The result of *process b* is the same except the rotation is counterclockwise.

B. *process b* and Theorem 2 & 3

To fully model the problem of simple Wigner rotation, we have to construct another process which will be named *process b*. The only difference between these two processes is the order of the first two rotations. In *process b*, we let the frame rotate along the yz-plane by an angle ϕ first and then the new frame rotate along the new zx-plane by an angle θ. We can express the result of *process b* as

$$\begin{pmatrix} \hat{x}_b''' \\ \hat{y}_b''' \\ \hat{z}_b''' \end{pmatrix} = N \begin{pmatrix} \hat{x}_b'' \\ \hat{y}_b'' \\ \hat{z}_b'' \end{pmatrix} N^\dagger = NYX \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} X^\dagger Y^\dagger N^\dagger,$$

where X and Y are the same as those in *process a*, while N as the counterpart of M can be constructed in the similar way,

$$\begin{aligned} N &= C'(\hat{z}_b'' + \hat{z})\hat{z}_b'' = C'(1 + \hat{z}\hat{z}_b'') \\ &= C'(1 + Y^\dagger X^{\dagger 2} Y^\dagger) \end{aligned} \tag{11}$$

with the condition $NN^\dagger = 1$.

It is easy to prove NYX is also a rotor along the xy-plane. Moreover, it is the inverse of the rotor MXY.

Theorem 2. $NYX = (MXY)^\dagger = R_{\mathbf{w}}^\dagger = R_{\mathbf{w}}^{-1}$.

The proof is also made of a few simple calculations:

$$\begin{aligned} NYX &= C'(1 + Y^\dagger X^{\dagger 2} Y^\dagger)YX = C'(YX + Y^\dagger X^\dagger) \\ &= C' Y^\dagger X^\dagger (1 + XY^2X) = \frac{C'}{C} Y^\dagger X^\dagger M^\dagger = (MXY)^\dagger, \end{aligned}$$

where $C = C'$ comes from both MXY and NYX are normalized.

This theorem implies that, except for the sense of the rotation, the result of *process b* is the same as that of *process a*.

$$NYX \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} X^\dagger Y^\dagger N^\dagger = R_{\mathbf{w}}^\dagger \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} R_{\mathbf{w}} = \begin{pmatrix} R_{\mathbf{w}}^\dagger \hat{x} R_{\mathbf{w}} \\ R_{\mathbf{w}}^\dagger \hat{y} R_{\mathbf{w}} \\ \hat{z} \end{pmatrix}.$$

In addition to Theorem 2, there is another interesting relation between the rotors N and M.

Theorem 3. $R_{\mathbf{w}} N R_{\mathbf{w}}^\dagger = M$.

The trick of the proof is substituting $X^\dagger Y^\dagger N^\dagger$ for $R_{\mathbf{w}}$ and NYX for $R_{\mathbf{w}}^\dagger$.

An intriguing relation $R_{\mathbf{w}} \hat{z}_b'' R_{\mathbf{w}}^\dagger = \hat{z}_a''$ can be derived from Theorem 3 when M is expressed by $C(1+\hat{z}\hat{z}_a'')$ and N by $C(1+\hat{z}\hat{z}_b'')$. It implies the two rotations generated by M and N performing along two different planes and the N-plane can be transformed to M-plane by the toy Wigner rotation given by Theorem 1. (Fig. 4)

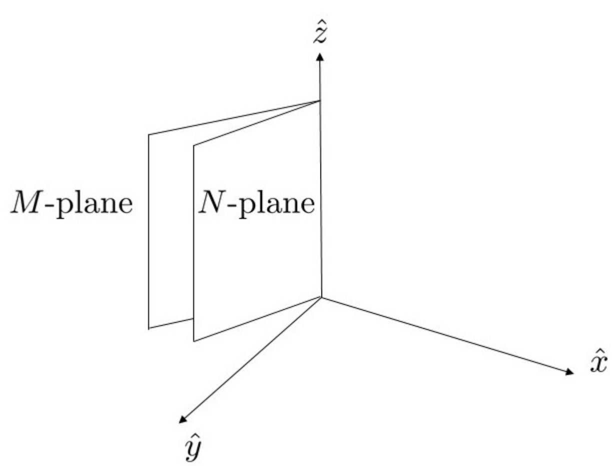

Fig. 4. The third rotation plane of *process b* (N-plane) can be transformed to that of *process a* (M-plane) by the toy Wigner rotation.

In summary, each theorem of this toy model has a precise geometric meaning.

Theorem 1. *process a* brings about a toy Wigner rotation of the frame along the xy-plane.

Theorem 2. The only difference between the results of *process a* and *process b* is the sense of the toy Wigner rotation (clockwise vs. counterclockwise).

Theorem 3. The third rotation planes of the two processes differ each other by a toy Wigner rotation, hence the angle between them equals to the toy Wigner angle.

These theorems may be illustrated in one schematic diagram as shown in Fig. 5.

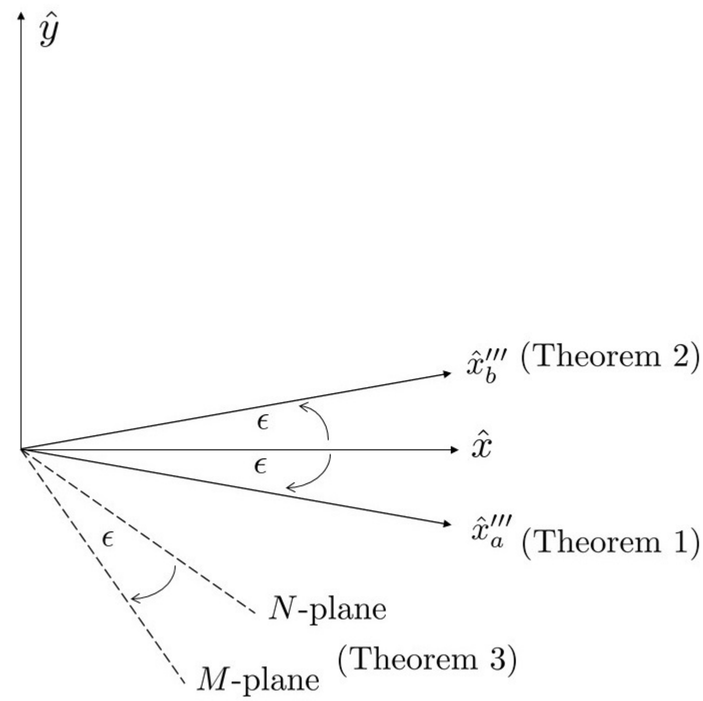

Fig. 5. Schematic diagram for Theorem 1∼3 of the toy Wigner rotation.

IV. SIMPLE WIGNER ROTATION

A. *process c* and Theorem I

After familiarizing ourselves with the $SO(3)$ toy model, we are ready to study the processes that yield the simple Wigner rotations. First we define *process c* which contains three hyperbolic rotations of a Minkowski frame $(\hat{x}, \hat{y}, \hat{t})$.

Step 1. Let the frame rotate along the xt-plane by a hyperbolic angle Θ, and call the new frame $(\hat{x}_c', \hat{y}_c', \hat{t}_c')$, where $\hat{y}_c' = \hat{y}$.

Step 2. Let the new frame rotate along the new yt-plane by a hyperbolic angle Φ, and call the newer frame $(\hat{x}_c'', \hat{y}_c'', \hat{t}_c'')$, where $\hat{x}_c'' = \hat{x}_c'$.

Step 3. Let the newer frame hyperbolically rotate along the plane defined by $\hat{t}$ and $\hat{t}_c''$ to the extent that the final $\hat{t}_c'''$ coincides with the original $\hat{t}$.

The expressions of these three hyperbolic rotations are similar to those of *process a*.

$$\text{Step 1.}\quad \begin{pmatrix}\hat{x}'_c\\ \hat{y}'_c\\ \hat{t}'_c\end{pmatrix} = \exp(-\frac{\Theta}{2}\hat{x}\hat{t})\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{t}\end{pmatrix}\exp(-\frac{\Theta}{2}\hat{x}\hat{t})^\dagger,$$

$$\begin{aligned}\text{Step 2.}\quad \begin{pmatrix}\hat{x}''_c\\ \hat{y}''_c\\ \hat{t}''_c\end{pmatrix} &= \exp(-\frac{\Phi}{2}\hat{y}'_c\hat{t}'_c)\begin{pmatrix}\hat{x}'_c\\ \hat{y}'_c\\ \hat{t}'_c\end{pmatrix}\exp(-\frac{\Phi}{2}\hat{y}'_c\hat{t}'_c)^\dagger\\ &= \exp(-\frac{\Theta}{2}\hat{x}\hat{t})\exp(-\frac{\Phi}{2}\hat{y}\hat{t})\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{t}\end{pmatrix}\exp(-\frac{\Phi}{2}\hat{y}\hat{t})^\dagger\exp(-\frac{\Theta}{2}\hat{x}\hat{t})^\dagger\\ &= \mathcal{X}\mathcal{Y}\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{t}\end{pmatrix}\mathcal{Y}^\dagger\mathcal{X}^\dagger,\end{aligned}$$

$$\text{Step 3.}\quad \begin{pmatrix}\hat{x}'''_c\\ \hat{y}'''_c\\ \hat{t}'''_c\end{pmatrix} = \mathcal{M}\begin{pmatrix}\hat{x}''_c\\ \hat{y}''_c\\ \hat{t}''_c\end{pmatrix}\mathcal{M}^\dagger = \mathcal{M}\mathcal{X}\mathcal{Y}\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{t}\end{pmatrix}\mathcal{Y}^\dagger\mathcal{X}^\dagger\mathcal{M}^\dagger, \tag{12}$$

where $\mathcal{X}, \mathcal{Y}$ and $\mathcal{M}$ are the counterparts of X, Y and M of *process a* respectively. The construction of $\mathcal{M}$ is analogous to (10),

$$\begin{aligned}\mathcal{M} &= -\mathcal{C}(\hat{t}''_c + \hat{t})\hat{t}''_c = \mathcal{C}(1 - \hat{t}\hat{t}''_c)\\ &= \mathcal{C}(1 + \mathcal{X}^\dagger\mathcal{Y}^{\dagger 2}\mathcal{X}^\dagger)\end{aligned} \tag{13}$$

with the condition $\mathcal{M}\mathcal{M}^\dagger = 1$.

With regard to this process, we can deduce a theorem which is analogous to Theorem 1 of the toy model.

Theorem I. $\mathcal{M}\mathcal{X}\mathcal{Y} = \exp(\hat{x}\hat{y}\frac{\varepsilon}{2}) =: \mathcal{R}_{\mathbf{w}}$ is the rotor of the simple Wigner rotation, where the simple Wigner angle ε is defined by

$$\tan\frac{\varepsilon}{2} = \tanh\frac{\Theta}{2}\tanh\frac{\Phi}{2} \text{ with } \varepsilon \in (-\frac{\pi}{2}, \frac{\pi}{2}).$$

The proof is omitted owing to its resemblance to that of Theorem 1.

Similar to the toy Wigner angle ϵ, the simple Wigner angle ε is positive if and only if $\Theta\Phi > 0$. However, the range of ε is half of that of ϵ because the range of $\tanh\frac{\Omega}{2}$ is $(-1, 1)$ while that of $\tan\frac{\omega}{2}$ is $(-\infty, \infty)$.

B. *process d* and Theorem II & III

Imitating the procedure for constructing the toy model, we now exchange the first two rotations in *process c*, i.e., let the frame rotate along the yt-plane by a hyperbolic angle Φ first and then the new frame rotate along the new xt-plane by a hyperbolic angle Θ. This new process will be named *process d* and its result can be expressed as

$$\begin{pmatrix} \hat{x}_d''' \\ \hat{y}_d''' \\ \hat{t}_d''' \end{pmatrix} = \mathcal{N} \begin{pmatrix} \hat{x}_d'' \\ \hat{y}_d'' \\ \hat{t}_d'' \end{pmatrix} \mathcal{N}^\dagger = \mathcal{N}\mathcal{Y}\mathcal{X} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \mathcal{X}^\dagger \mathcal{Y}^\dagger \mathcal{N}^\dagger,$$

where $\mathcal{N}$ is the counterpart of $\mathcal{M}$,

$$\begin{aligned} \mathcal{N} &= -\mathcal{C}'(\hat{t}_d'' + \hat{t})\hat{t}_d'' = \mathcal{C}'(1 - \hat{t}\hat{t}_d'') \\ &= \mathcal{C}'(1 + \mathcal{Y}^\dagger \mathcal{X}^{\dagger 2} \mathcal{Y}^\dagger) \end{aligned} \tag{14}$$

with the condition $\mathcal{N}\mathcal{N}^\dagger = 1$.

Thanks to the isomorphisms between (10) and (13) and between (11) and (14), we acquire the following two theorems by change of notations.

Theorem II. $\mathcal{N}\mathcal{Y}\mathcal{X} = (\mathcal{M}\mathcal{X}\mathcal{Y})^\dagger = \mathcal{R}_{\mathbf{w}}^{-1}$.

Theorem III. $\mathcal{R}_{\mathbf{w}}\mathcal{N}\mathcal{R}^\dagger{}_{\mathbf{w}} = \mathcal{M}$.

C. The physical meanings

To discuss the physical meanings of the processes and theorems introduced in this section, we begin with identifying the rotors $\mathcal{X}$ and $\mathcal{Y}$ with the boosts defined by the velocities $\vec{u} = c \tanh \Theta \hat{x}$ and $\vec{v} = c \tanh \Phi \hat{y}$ respectively. Then the (2+1)-dimensional version of (8) gives us the following results,

$$c\hat{t}_c'' = \gamma(\vec{u} \oplus \vec{v})[\vec{u} \oplus \vec{v} + c\hat{t}]; \; c\hat{t}_d'' = \gamma(\vec{v} \oplus \vec{u})[\vec{v} \oplus \vec{u} + c\hat{t}], \tag{15}$$

where $\gamma(\vec{u} \oplus \vec{v}) = \gamma(\vec{v} \oplus \vec{u})$ as proved at the end of Section II.

Since $\hat{t}_c'' = \mathcal{M}^\dagger \hat{t}_c''' \mathcal{M} = \mathcal{M}^\dagger \hat{t} \mathcal{M}$ according to (12), it implies that $\mathcal{M}^\dagger = \mathcal{M}^{-1}$ as a boost is defined by the velocity $\vec{u} \oplus \vec{v}$. Therefore the boost $\mathcal{M}$ is defined by $-(\vec{u} \oplus \vec{v})$, and for the same reason $\mathcal{N}^\dagger$ is defined by $\vec{v} \oplus \vec{u}$ and $\mathcal{N}$ by $-(\vec{v} \oplus \vec{u})$.

Now we are ready to use physical language to rephrase the two processes in this section (omitting the suffixes c & d).

process c:

Step 1. A frame $(\hat{x}', \hat{y}', \hat{t}')$ is found which moves with velocity $\vec{u}$ relative to the original frame $(\hat{x}, \hat{y}, \hat{t})$.

Step 2. Another frame $(\hat{x}'', \hat{y}'', \hat{t}'')$ is found which moves with velocity $\vec{v}$ relative to $(\hat{x}', \hat{y}', \hat{t}')$.

Step 3. The third frame $(\hat{x}''', \hat{y}''', \hat{t}''')$ is found which moves with velocity $-(\vec{u} \oplus \vec{v})$ relative to $(\hat{x}'', \hat{y}'', \hat{t}'')$.

Since $\hat{t}''' = \hat{t}$, we know from (6) there is no relative velocity between the third and the original unprimed frames. According to Theorem I, the spatial bases of these two frames differ by a simple Wigner rotation.

$$\begin{pmatrix} \hat{x}''' \\ \hat{y}''' \\ \hat{t}''' \end{pmatrix} = \mathcal{M}\mathcal{X}\mathcal{Y} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \mathcal{Y}^\dagger \mathcal{X}^\dagger \mathcal{M}^\dagger = \mathcal{R}_{\mathbf{w}} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \mathcal{R}_{\mathbf{w}}^\dagger = \begin{pmatrix} \mathcal{R}_{\mathbf{w}} \hat{x} \mathcal{R}_{\mathbf{w}}^\dagger \\ \mathcal{R}_{\mathbf{w}} \hat{y} \mathcal{R}_{\mathbf{w}}^\dagger \\ \hat{t} \end{pmatrix}.$$

The above relation can be transformed to the following form,

$$\mathcal{X}\mathcal{Y}\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{t}\end{pmatrix}\mathcal{Y}^\dagger\mathcal{X}^\dagger = \mathcal{M}^\dagger\mathcal{R}_{\mathbf{w}}\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{t}\end{pmatrix}\mathcal{R}_{\mathbf{w}}^\dagger\mathcal{M} = \mathcal{R}_{\mathbf{w}}\mathcal{N}^\dagger\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{t}\end{pmatrix}\mathcal{N}\mathcal{R}_{\mathbf{w}}^\dagger, \tag{16}$$

where Theorem III has been used. It means the action of the first two boosts to the original frame is equivalent to that of a boost of velocity $\vec{u}\oplus\vec{v}$ preceded by a simple Wigner rotation, or a boost of velocity $\vec{v}\oplus\vec{u}$ followed by the same rotation. However, if the term "Wigner rotation" connotes no temporal dimension involved, the latter should be ruled out as a Wigner rotation because $\mathcal{N}^\dagger\hat{t}\mathcal{N}$ is rotated.

process d:

Step 1. A frame $(\hat{x}', \hat{y}', \hat{t}')$ is found which moves with velocity $\hat{v}$ relative to the original frame $(\hat{x}, \hat{y}, \hat{t})$.

Step 2. Another frame $(\hat{x}'', \hat{y}'', \hat{t}'')$ is found which moves with velocity $\vec{u}$ relative to $(\hat{x}', \hat{y}', \hat{t}')$.

Step 3. The third frame $(\hat{x}''', \hat{y}''', \hat{t}''')$ is found which moves with velocity $-(\vec{v}\oplus\vec{u})$ relative to $(\hat{x}'', \hat{y}'', \hat{t}'')$.

Similarly we can conclude that (i) Theorem II asserts the spatial bases of the third and the original frames differ by an inverse simple Wigner rotation,

$$\begin{pmatrix}\hat{x}'''\\ \hat{y}'''\\ \hat{t}'''\end{pmatrix} = \mathcal{N}\mathcal{Y}\mathcal{X}\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{t}\end{pmatrix}\mathcal{X}^\dagger\mathcal{Y}^\dagger\mathcal{N}^\dagger = \mathcal{R}_{\mathbf{w}}^\dagger\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{t}\end{pmatrix}\mathcal{R}_{\mathbf{w}} = \begin{pmatrix}\mathcal{R}_{\mathbf{w}}^\dagger\hat{x}\mathcal{R}_{\mathbf{w}}\\ \mathcal{R}_{\mathbf{w}}^\dagger\hat{y}\mathcal{R}_{\mathbf{w}}\\ \hat{t}\end{pmatrix},$$

(ii) the action of the first two boosts to the original frame is equivalent to a boost of velocity $\vec{v}\oplus\vec{u}$ preceded by an inverse simple Wigner rotation, or a boost of velocity $\vec{u}\oplus\vec{v}$ followed by the same rotation which may be ruled out as a Wigner rotation,

$$\mathcal{Y}\mathcal{X}\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{t}\end{pmatrix}\mathcal{X}^\dagger\mathcal{Y}^\dagger = \mathcal{N}^\dagger\mathcal{R}_{\mathbf{w}}^\dagger\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{t}\end{pmatrix}\mathcal{R}_{\mathbf{w}}\mathcal{N} = \mathcal{R}_{\mathbf{w}}^\dagger\mathcal{M}^\dagger\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{t}\end{pmatrix}\mathcal{M}\mathcal{R}_{\mathbf{w}}. \tag{17}$$

In addition to the above results, Theorem III leads to a relation $\mathcal{R}_{\mathbf{w}}\hat{t}_d''\mathcal{R}_{\mathbf{w}}^\dagger = \hat{t}_c''$ which is the counterpart of $R_{\mathbf{w}}\hat{z}_b''R_{\mathbf{w}}^\dagger = \hat{z}_a''$ of the toy model. We can use (15) to convert it to $\mathcal{R}_{\mathbf{w}}(\vec{v}\oplus\vec{u})\mathcal{R}_{\mathbf{w}}^\dagger = \vec{u}\oplus\vec{v}$ which implies the angle between these two composite velocities equals to the simple Wigner angle.

In summary, each of the three theorems in this section has a precise physical meaning. (Fig. 6)

Theorem I. *process c* brings about a simple Wigner rotation of the frame along the xy-plane.

Theorem II. The only difference between the results of *process c* and *process d* is the sense of the simple Wigner rotation.

Theorem III. The composite velocities of the two processes differ each other by a simple Wigner rotation, implying the angle between them equals to the simple Wigner angle.

Lastly, although the explicit expressions of $\vec{u} \oplus \vec{v}$ and $\vec{v} \oplus \vec{u}$ are not needed here, their derivations are provided as a demonstration.

Comparing (8) with the expression of $\hat{t}''_c$ which can be calculated easily,

$$\begin{aligned}\hat{t}''_c &= \sinh\Theta\cosh\Phi\hat{x} + \sinh\Phi\hat{y} + \cosh\Theta\cosh\Phi\hat{t}\\ &= \cosh\Theta\cosh\Phi(\tanh\Theta\hat{x} + \frac{\tanh\Phi}{\cosh\Theta}\hat{y} + \hat{t})\\ &= \gamma(\vec{u})\gamma(\vec{v})[\frac{u}{c}\hat{x} + \frac{1}{\gamma(\vec{u})}\frac{v}{c}\hat{y} + \hat{t}\,],\end{aligned}$$

we obtain $\gamma(\vec{u} \oplus \vec{v}) = \gamma(\vec{u})\gamma(\vec{v})$ and $\vec{u} \oplus \vec{v} = [u, \frac{v}{\gamma(\vec{u})}]$. For the same reason the expression of $\hat{t}''_d$ leads to $\vec{v} \oplus \vec{u} = [\frac{u}{\gamma(\vec{v})}, v]$.

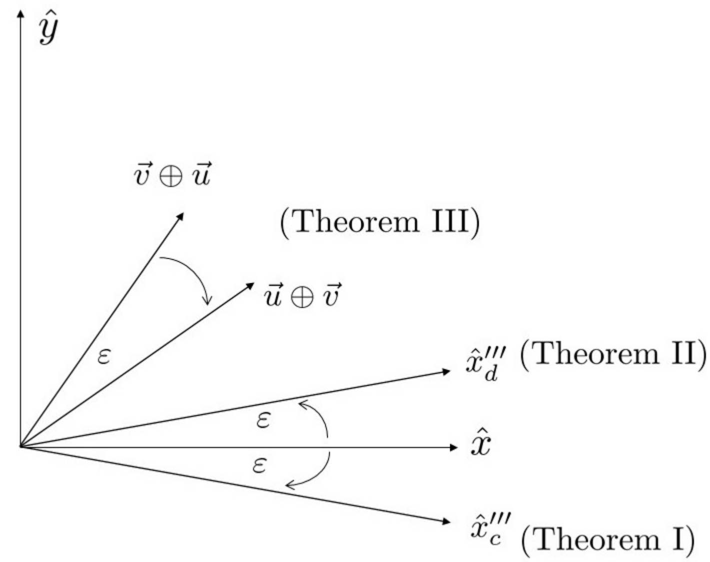

Fig. 6. Schematic diagram for Theorem I∼III of the simple Wigner rotation.

V. GENERAL WIGNER ROTATION

A. *process e* and Theorem I'

Now we release the 90° constraint on the two boost velocities, allowing the angle between them to be arbitrary. Without loss of generality, we still put the velocity $\vec{u}$ along the x-direction, while the velocity $\vec{v}$ deviates from the y-direction clockwise by an angle $\eta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. (Fig. 7) The basis $\hat{y}$ can be rotated to the direction of $\vec{v}$ by a rotor and this new basis will be called $\hat{w}$, i.e., $\hat{w} = \exp(\frac{\eta}{2}\hat{x}\hat{y})\hat{y}\exp(-\frac{\eta}{2}\hat{x}\hat{y}) = \hat{x}\sin\eta + \hat{y}\cos\eta$.

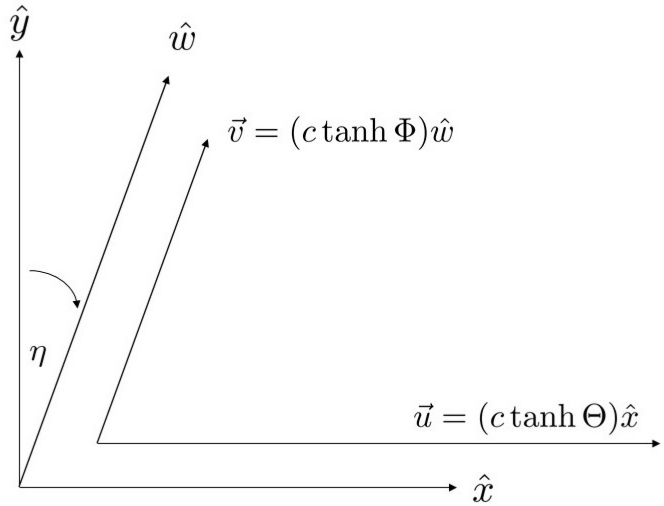

Fig. 7. The orientations of velocities $\vec{u}$ and $\vec{v}$ in the problem of general Wigner rotation. Since Θ and Φ can be any real number, the range $\eta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is adequate for all possible configurations.

Using this new basis $\hat{w}$, we generalize *process c* to the following one which will be named *process e*.

Step 1. Let the frame rotate along the xt-plane by a hyperbolic angle Θ, and call the new frame $(\hat{x}'_e, \hat{y}'_e, \hat{t}'_e)$, where $\hat{y}'_e = \hat{y}$. Accordingly $\hat{w}$ is transformed to $\hat{w}'_e$.

Step 2. Let the new frame rotate along the new wt-plane by a hyperbolic angle Φ, and call the newer frame $(\hat{x}''_e, \hat{y}''_e, \hat{t}''_e)$.

Step 3. Let the newer frame hyperbolically rotate along the plane defined by $\hat{t}$ and $\hat{t}''_e$ to the extent that the final $\hat{t}'''_e$ coincides with the original $\hat{t}$.

The expressions for this process are as follows:

$$\text{Step 1.} \quad \begin{pmatrix} \hat{x}'_e \\ \hat{y}'_e \\ \hat{t}'_e \\ \hat{w}'_e \end{pmatrix} = \exp(-\frac{\Theta}{2}\hat{x}\hat{t}) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \\ \hat{w} \end{pmatrix} \exp(-\frac{\Theta}{2}\hat{x}\hat{t})^{\dagger},$$

$$\text{Step 2.} \quad \begin{pmatrix} \hat{x}_e'' \\ \hat{y}_e'' \\ \hat{t}_e'' \end{pmatrix} = \exp(-\frac{\Phi}{2}\hat{w}_e'\hat{t}_e') \begin{pmatrix} \hat{x}_e' \\ \hat{y}_e' \\ \hat{t}_e' \end{pmatrix} \exp(-\frac{\Phi}{2}\hat{w}_e'\hat{t}_e')^\dagger$$

$$= \exp(-\frac{\Theta}{2}\hat{x}\hat{t}) \exp(-\frac{\Phi}{2}\hat{w}\hat{t}) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \exp(-\frac{\Phi}{2}\hat{w}\hat{t})^\dagger \exp(-\frac{\Theta}{2}\hat{x}\hat{t})^\dagger$$

$$= \mathcal{X}\mathcal{W} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \mathcal{W}^\dagger \mathcal{X}^\dagger,$$

$$\text{Step 3.} \quad \begin{pmatrix} \hat{x}_e''' \\ \hat{y}_e''' \\ \hat{t}_e''' \end{pmatrix} = \boldsymbol{\mathcal{M}} \begin{pmatrix} \hat{x}_e'' \\ \hat{y}_e'' \\ \hat{t}_e'' \end{pmatrix} \boldsymbol{\mathcal{M}}^\dagger = \boldsymbol{\mathcal{M}}\mathcal{X}\mathcal{W} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \mathcal{W}^\dagger \mathcal{X}^\dagger \boldsymbol{\mathcal{M}}^\dagger,$$

where $\mathcal{W} = \exp(-\frac{\Phi}{2}\hat{w}\hat{t})$ and

$$\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{C}}(1 - \hat{t}\hat{t}_e'') = \boldsymbol{\mathcal{C}}(1 + \mathcal{X}^\dagger \mathcal{W}^{\dagger 2} \mathcal{X}^\dagger) \text{ with } \boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{M}}^\dagger = 1.$$

Now we can generalize Theorem I of the simple Wigner rotation to the general case.

Theorem I'.

$$\boldsymbol{\mathcal{M}}\mathcal{X}\mathcal{W} = \exp(\hat{x}\hat{y}\frac{\boldsymbol{\varepsilon}}{2}) =: \boldsymbol{\mathcal{R}}_{\mathbf{w}},$$

$$\text{where } \tan\frac{\boldsymbol{\varepsilon}}{2} = \frac{\cos\eta}{\coth\frac{\Theta}{2}\coth\frac{\Phi}{2} + \sin\eta} \text{ with } \boldsymbol{\varepsilon} \in (-\frac{\pi}{2} - \eta, \frac{\pi}{2} - \eta).$$

The proof is contained in the following calculations,

$$\begin{aligned} \boldsymbol{\mathcal{M}}\mathcal{X}\mathcal{W} &= \boldsymbol{\mathcal{C}}(\mathcal{X}\mathcal{W} + \mathcal{X}^\dagger\mathcal{W}^\dagger) \\ &= \boldsymbol{\mathcal{C}}[\exp(-\frac{\Theta}{2}\hat{x}\hat{t})\exp(-\frac{\Phi}{2}\hat{w}\hat{t}) + \exp(\frac{\Theta}{2}\hat{x}\hat{t})\exp(\frac{\Phi}{2}\hat{w}\hat{t})] \\ &= 2\boldsymbol{\mathcal{C}}[\cosh\frac{\Theta}{2}\cosh\frac{\Phi}{2} + \hat{x}\hat{w}\sinh\frac{\Theta}{2}\sinh\frac{\Phi}{2}] \\ &= 2\boldsymbol{\mathcal{C}}[(\cosh\frac{\Theta}{2}\cosh\frac{\Phi}{2} + \sin\eta\sinh\frac{\Theta}{2}\sinh\frac{\Phi}{2}) + \hat{x}\hat{y}\cos\eta\sinh\frac{\Theta}{2}\sinh\frac{\Phi}{2}] \\ &= \cos\frac{\boldsymbol{\varepsilon}}{2} + \hat{x}\hat{y}\sin\frac{\boldsymbol{\varepsilon}}{2}. \end{aligned}$$

The limits of $\boldsymbol{\varepsilon}$ corresponds to $\coth\frac{\Theta}{2}\coth\frac{\Phi}{2} = \pm 1$.

B. *process f* and Theorem II' & III'

Similar to that *process e* is generalized from *process c*, *process f* is the generalization of *process d.*

$$\begin{pmatrix} \hat{x}_f''' \\ \hat{y}_f''' \\ \hat{t}_f''' \end{pmatrix} = \boldsymbol{\mathcal{N}} \begin{pmatrix} \hat{x}_f'' \\ \hat{y}_f'' \\ \hat{t}_f'' \end{pmatrix} \boldsymbol{\mathcal{N}}^\dagger = \boldsymbol{\mathcal{N}} \mathcal{W} \mathcal{X} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \mathcal{X}^\dagger \mathcal{W}^\dagger \boldsymbol{\mathcal{N}}^\dagger,$$

where $\boldsymbol{\mathcal{N}} = \boldsymbol{\mathcal{C}}'(1 - \hat{t}\hat{t}_f'') = \boldsymbol{\mathcal{C}}'(1 + \mathcal{W}^\dagger \mathcal{X}^{\dagger 2} \mathcal{W}^\dagger)$ with $\boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{N}}^\dagger = 1$.

Because the expressions of the rotors $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{N}}$ are respectively isomorphic to those of $\mathcal{M}$ and $\mathcal{N}$ of the simple Wigner rotation, Theorem II and Theorem III are generalized to the followings.

Theorem II'. $\boldsymbol{\mathcal{N}} \mathcal{W} \mathcal{X} = (\boldsymbol{\mathcal{M}} \mathcal{X} \mathcal{W})^\dagger = \boldsymbol{\mathcal{R}}_{\mathbf{w}}^{-1}$.

Theorem III'. $\boldsymbol{\mathcal{R}}_{\mathbf{w}} \boldsymbol{\mathcal{N}} \boldsymbol{\mathcal{R}}_{\mathbf{w}}^\dagger = \boldsymbol{\mathcal{M}}$.

In summary, except the formula of Wigner angle is more complicated, the related theorems and their physical meanings have no significant change for the general Wigner rotation. For example, the counterpart of (16) is

$$\mathcal{X}\mathcal{W} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \mathcal{W}^\dagger \mathcal{X}^\dagger = \boldsymbol{\mathcal{M}}^\dagger \boldsymbol{\mathcal{R}}_{\mathbf{w}} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \boldsymbol{\mathcal{R}}_{\mathbf{w}}^\dagger \boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{R}}_{\mathbf{w}} \boldsymbol{\mathcal{N}}^\dagger \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \boldsymbol{\mathcal{N}} \boldsymbol{\mathcal{R}}_{\mathbf{w}}^\dagger, \tag{18}$$

and that of (17) is

$$\mathcal{W}\mathcal{X} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \mathcal{X}^\dagger \mathcal{W}^\dagger = \boldsymbol{\mathcal{N}}^\dagger \boldsymbol{\mathcal{R}}_{\mathbf{w}}^\dagger \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \boldsymbol{\mathcal{R}}_{\mathbf{w}} \boldsymbol{\mathcal{N}} = \boldsymbol{\mathcal{R}}_{\mathbf{w}}^\dagger \boldsymbol{\mathcal{M}}^\dagger \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{t} \end{pmatrix} \boldsymbol{\mathcal{M}} \boldsymbol{\mathcal{R}}_{\mathbf{w}}. \tag{19}$$

VI. CONCLUSION

By the joint effort of group theory, geometric algebra and active-frame formalism, the geometry of Wigner rotation problem is clarified and the mathematics is simplified. Among other things, the $SO(3)$ toy model provides an easy way to comprehend the essence of this problem.

As a byproduct, the active-frame formulas (6) and (8) suggest an alternative approach to teaching relativistic kinematics.

APPENDIX: MATRIX-COORDINATE FORMALISM

The results in this paper are mainly obtained and expressed by geometric algebra. In order to make comparisons with those in other literature, there is a need to translate the results into matrix formulation.

First of all, let's practice ourselves with an example from *process c,*

$$
\begin{aligned}
&\exp(-\frac{\Theta}{2}\hat{x}\hat{t})\exp(-\frac{\Phi}{2}\hat{y}\hat{t})\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{t}\end{pmatrix}\exp(-\frac{\Phi}{2}\hat{y}\hat{t})^{\dagger}\exp(-\frac{\Theta}{2}\hat{x}\hat{t})^{\dagger}\\
&=\exp(-\frac{\Theta}{2}\hat{x}\hat{t})\begin{pmatrix}1 & 0 & 0\\ 0 & \cosh\Phi & \sinh\Phi\\ 0 & \sinh\Phi & \cosh\Phi\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{t}\end{pmatrix}\exp(-\frac{\Theta}{2}\hat{x}\hat{t})^{\dagger},\\
&=\begin{pmatrix}1 & 0 & 0\\ 0 & \cosh\Phi & \sinh\Phi\\ 0 & \sinh\Phi & \cosh\Phi\end{pmatrix}\begin{pmatrix}\cosh\Theta & 0 & \sinh\Theta\\ 0 & 1 & 0\\ \sinh\Theta & 0 & \cosh\Theta\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{t}\end{pmatrix}.
\end{aligned}
$$

Using the aforementioned notations $\mathcal{X}$ and $\mathcal{Y}$ for the rotors, and denoting the corresponding matrices by $[\mathcal{X}]$ and $[\mathcal{Y}]$, the above equality becomes

$$
\mathcal{X}\mathcal{Y}\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{t}\end{pmatrix}\mathcal{Y}^{\dagger}\mathcal{X}^{\dagger}=[\mathcal{Y}][\mathcal{X}]\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{t}\end{pmatrix}.
$$

The correspondence between these two formulations may be put formally as $\mathcal{X}\mathcal{Y} \iff [\mathcal{Y}][\mathcal{X}]$. However, in order to make those matrices act on the coordinates (x, y, ct) instead of the frame $(\hat{x}, \hat{y}, \hat{t})$, we need to go further to take the transpose-inverse of $[\mathcal{Y}][\mathcal{X}]$. Hence the correspondence between the rotor-frame formalism and matrix-coordinate formalism should be $\mathcal{X}\mathcal{Y} \iff [\mathcal{Y}]^{-\top}[\mathcal{X}]^{-\top}$ in this example.

Now we are ready to apply this rule to the results of general Wigner rotation. Starting with Theorem I' and bearing in mind the boost matrix is symmetric while the rotation matrix orthogonal, we have

$$
\boldsymbol{\mathcal{M}}\mathcal{X}\mathcal{W}=\boldsymbol{\mathcal{R}}_{\mathbf{w}} \iff [\mathcal{W}]^{-1}[\mathcal{X}]^{-1}[\boldsymbol{\mathcal{M}}]^{-1}=[\boldsymbol{\mathcal{R}}_{\mathbf{w}}].
$$

If we use the notation $B(\vec{V})$ in Section II to unify those hyperbolic rotors, since $B(\vec{V})$ corresponds to passive coordinate transformations, we have $[\mathcal{X}]^{-1} = [B(\vec{u})]$, $[\mathcal{W}]^{-1} = [B(\vec{v})]$ and $[\boldsymbol{\mathcal{M}}]^{-1} = [B(-(\vec{u}\oplus\vec{v}))]$, and the above matrix equality becomes

$$
\begin{aligned}
&[B(\vec{v})][B(\vec{u})][B(-(\vec{u}\oplus\vec{v}))]=[\boldsymbol{\mathcal{R}}_{\mathbf{w}}],\\
&\text{or } [B(\vec{v})][B(\vec{u})]=[\boldsymbol{\mathcal{R}}_{\mathbf{w}}][B(\vec{u}\oplus\vec{v})].
\end{aligned}
\tag{20}
$$

Applying the same rule to Theorem II' gives us a similar result,

$$
[B(\vec{u})][B(\vec{v})]=[\boldsymbol{\mathcal{R}}_{\mathbf{w}}]^{-1}[B(\vec{v}\oplus\vec{u})].
$$

As for Theorem III', the rotor-matrix correspondence is

$$
\begin{aligned}
&\boldsymbol{\mathcal{R}}_{\mathbf{w}}\boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{R}}_{\mathbf{w}}^{\dagger}=\boldsymbol{\mathcal{M}}\\
&\iff [\boldsymbol{\mathcal{R}}_{\mathbf{w}}]^{-1}[\boldsymbol{\mathcal{N}}]^{-1}[\boldsymbol{\mathcal{R}}_{\mathbf{w}}]=[\boldsymbol{\mathcal{M}}]^{-1}\\
&\iff [\boldsymbol{\mathcal{R}}_{\mathbf{w}}]^{-1}[B(-(\vec{v}\oplus\vec{u}))][\boldsymbol{\mathcal{R}}_{\mathbf{w}}]=[B(-(\vec{u}\oplus\vec{v}))]\\
&\iff [\boldsymbol{\mathcal{R}}_{\mathbf{w}}]^{-1}[B(\vec{v}\oplus\vec{u})][\boldsymbol{\mathcal{R}}_{\mathbf{w}}]=[B(\vec{u}\oplus\vec{v})].
\end{aligned}
\tag{21}
$$

Substituting (21) into (20), the result is the matrix form of (18),

$$[B(\vec{v})][B(\vec{u})] = [\boldsymbol{\mathcal{R}}_{\mathbf{w}}][B(\vec{u}\oplus\vec{v})] = [B(\vec{v}\oplus\vec{u})][\boldsymbol{\mathcal{R}}_{\mathbf{w}}]. \tag{22}$$

Taking the transpose of (22) gives us the matrix form of (19),

$$[B(\vec{u})][B(\vec{v})] = [B(\vec{u}\oplus\vec{v})][\boldsymbol{\mathcal{R}}_{\mathbf{w}}]^{-1} = [\boldsymbol{\mathcal{R}}_{\mathbf{w}}]^{-1}[B(\vec{v}\oplus\vec{u})]. \tag{23}$$

Needless to say, (22) and (23) can also be obtained by applying the correspondence rule to the rotor formulas (18) and (19).

Next, the inverse and transpose-inverse of (22) are respectively

$$[B(\vec{u})]^{-1}[B(\vec{v})]^{-1} = [B(\vec{u}\oplus\vec{v})]^{-1}[\boldsymbol{\mathcal{R}}_{\mathbf{w}}]^{-1} = [\boldsymbol{\mathcal{R}}_{\mathbf{w}}]^{-1}[B(\vec{v}\oplus\vec{u})]^{-1}, \tag{24}$$

and

$$[B(\vec{v})]^{-1}[B(\vec{u})]^{-1} = [\boldsymbol{\mathcal{R}}_{\mathbf{w}}][B(\vec{u}\oplus\vec{v})]^{-1} = [B(\vec{v}\oplus\vec{u})]^{-1}[\boldsymbol{\mathcal{R}}_{\mathbf{w}}]. \tag{25}$$

If we apply (24) and (25) to a (2+1)-velocity $\mathsf{0} := (0, 0, c)^\top$, the result is not only compatible with (7) but also provides an elucidation of the velocity composition rule. For example,

$$[B(\vec{u})]^{-1}[B(\vec{v})]^{-1}\mathsf{0} = [B(\vec{u}\oplus\vec{v})]^{-1}[\boldsymbol{\mathcal{R}}_{\mathbf{w}}]^{-1}\mathsf{0} = [\boldsymbol{\mathcal{R}}_{\mathbf{w}}]^{-1}[B(\vec{v}\oplus\vec{u})]^{-1}\mathsf{0}$$

shows the (2+1)-velocity thus derived corresponds to $\vec{u}\oplus\vec{v}$ instead of $\vec{v}\oplus\vec{u}$. This mirrors the fact that the second pair $(\mathcal{R}_{\mathbf{w}}, \mathcal{R}_{\mathbf{w}}^\dagger)$ of (16) and the second pair $(\boldsymbol{\mathcal{R}}_{\mathbf{w}}, \boldsymbol{\mathcal{R}}_{\mathbf{w}}^\dagger)$ of (18) may be ruled out as Wigner rotations.

Similarly,

$$[B(\vec{v})]^{-1}[B(\vec{u})]^{-1}\mathsf{0} = [\boldsymbol{\mathcal{R}}_{\mathbf{w}}][B(\vec{u}\oplus\vec{v})]^{-1}\mathsf{0} = [B(\vec{v}\oplus\vec{u})]^{-1}[\boldsymbol{\mathcal{R}}_{\mathbf{w}}]\mathsf{0}$$

evidently just corresponds to $\vec{v}\oplus\vec{u}$.

Moreover, we can use either (24) or (25) to prove the velocity composition rule is non-associative. Taking $\vec{V}\oplus(\vec{u}\oplus\vec{v}) \neq (\vec{V}\oplus\vec{u})\oplus\vec{v}$ as an example, where $\vec{V}$ is another velocity in the xy-plane. $\vec{V}\oplus(\vec{u}\oplus\vec{v})$ corresponds to the (2+1)-velocity

$$\begin{aligned}
&[B(\vec{V})]^{-1}[B(\vec{u}\oplus\vec{v})]^{-1}\mathsf{0}\\
&= [B(\vec{V})]^{-1}[B(\vec{u})]^{-1}[B(\vec{v})]^{-1}[\boldsymbol{\mathcal{R}}_{\mathbf{w}}]\mathsf{0}\\
&= [B(\vec{V})]^{-1}[B(\vec{u})]^{-1}[B(\vec{v})]^{-1}\mathsf{0},
\end{aligned}$$

which is consistent with (9). On the other hand, $(\vec{V}\oplus\vec{u})\oplus\vec{v}$ corresponds to a different (2+1)-velocity

$$\begin{aligned}
&[B(\vec{V}\oplus\vec{u})]^{-1}[B(\vec{v})]^{-1}\mathsf{0}\\
&= [B(\vec{V})]^{-1}[B(\vec{u})]^{-1}[\boldsymbol{\mathcal{R}}'_{\mathbf{w}}][B(\vec{v})]^{-1}\mathsf{0}\\
&= [B(\vec{V})]^{-1}[B(\vec{u})]^{-1}[\boldsymbol{\mathcal{R}}'_{\mathbf{w}}][B(\vec{v})]^{-1}[\boldsymbol{\mathcal{R}}'_{\mathbf{w}}]^{-1}\mathsf{0}\\
&= [B(\vec{V})]^{-1}[B(\vec{u})]^{-1}[B(\vec{v}')]^{-1}\mathsf{0},
\end{aligned}$$

where $\boldsymbol{\mathcal{R}}'_{\mathbf{w}}$ is the Wigner rotation induced by the boosts $B(\vec{V})$ and $B(\vec{u})$, and $\vec{v}' = \boldsymbol{\mathcal{R}}'^{\dagger}_{\mathbf{w}} \vec{v}\, \boldsymbol{\mathcal{R}}'_{\mathbf{w}}$.

Lastly, the explicit forms of these boost matrices are provided below for reference.

$$[B(\vec{u})] = \begin{pmatrix} \cosh\Theta & 0 & -\sinh\Theta \\ 0 & 1 & 0 \\ -\sinh\Theta & 0 & \cosh\Theta \end{pmatrix},\ [B(\vec{u})]^{-1} = \begin{pmatrix} \cosh\Theta & 0 & \sinh\Theta \\ 0 & 1 & 0 \\ \sinh\Theta & 0 & \cosh\Theta \end{pmatrix}.$$

$$\begin{aligned} [B(\vec{v})] &= \begin{pmatrix} \cos\eta & \sin\eta & 0 \\ -\sin\eta & \cos\eta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\Phi & -\sinh\Phi \\ 0 & -\sinh\Phi & \cosh\Phi \end{pmatrix} \begin{pmatrix} \cos\eta & -\sin\eta & 0 \\ \sin\eta & \cos\eta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\eta^2 + \sin\eta^2\cosh\Phi & \sin\eta\cos\eta(\cosh\Phi - 1) & -\sin\eta\sinh\Phi \\ \sin\eta\cos\eta(\cosh\Phi - 1) & \sin\eta^2 + \cos\eta^2\cosh\Phi & -\cos\eta\sinh\Phi \\ -\sin\eta\sinh\Phi & -\cos\eta\sinh\Phi & \cosh\Phi \end{pmatrix}, \end{aligned}$$

$$[B(\vec{v})]^{-1} = \begin{pmatrix} \cos\eta^2 + \sin\eta^2\cosh\Phi & \sin\eta\cos\eta(\cosh\Phi - 1) & \sin\eta\sinh\Phi \\ \sin\eta\cos\eta(\cosh\Phi - 1) & \sin\eta^2 + \cos\eta^2\cosh\Phi & \cos\eta\sinh\Phi \\ \sin\eta\sinh\Phi & \cos\eta\sinh\Phi & \cosh\Phi \end{pmatrix}.$$

$$[B(\vec{u}\oplus\vec{v})] = \frac{1}{P}\begin{pmatrix} P+Q^2 & QR & -PQ \\ QR & P+R^2 & -PR \\ -PQ & -PR & P^2-P \end{pmatrix},$$

$$[B(\vec{u}\oplus\vec{v})]^{-1} = \frac{1}{P}\begin{pmatrix} P+Q^2 & QR & PQ \\ QR & P+R^2 & PR \\ PQ & PR & P^2-P \end{pmatrix},$$

$$\begin{aligned} \text{where } P &= 1 + \cosh\Theta\cosh\Phi + \sin\eta\sinh\Theta\sinh\Phi, \\ Q &= \sinh\Theta\cosh\Phi + \sin\eta\cosh\Theta\sinh\Phi, \\ R &= \cos\eta\sinh\Phi. \end{aligned}$$

With the following modifications for Q and R, the two matrices above can also be used to represent $[B(\vec{v}\oplus\vec{u})]$ and $[B(\vec{v}\oplus\vec{u})]^{-1}$.

$$\begin{aligned} Q &= \cos^2\eta\sinh\Theta + \sin\eta\cosh\Theta\sinh\Phi + \sin^2\eta\sinh\Theta\cosh\Phi, \\ R &= \cos\eta\cosh\Theta\sinh\Phi + \sin\eta\cos\eta(\sinh\Theta\cosh\Phi - \sinh\Theta). \end{aligned}$$

According to (6), we can read out $\gamma(\vec{u}\oplus\vec{v})$ and $\vec{u}\oplus\vec{v}$ from the third column of $[B(\vec{u}\oplus\vec{v})]^{-1}$, i.e., $\gamma(\vec{u}\oplus\vec{v}) = P - 1$ and $\vec{u}\oplus\vec{v} = \frac{c}{P-1}[Q, R]$. Similarly, $\vec{v}\oplus\vec{u}$ can be extracted according to the same rule.

SUPPLEMENTARY MATERIAL

An animation of the six processes discussed in this paper is available at https://www.youtube.com/watch?v=HyVouwd7X2o

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