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A Continuous Family of Equilibria in Ferromagnetic Media are Ground States

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Abstract: We show that a foliation of equilibria (a continuous family of equilibria whose graph covers all the configuration space) in ferromagnetic transitive models are ground states. The result we prove is very general, and it applies to models with long range and many-body interactions. As an application, we consider several models of networks of interacting particles including models of Frenkel–Kontorova type on $\mathbb{Z}^d$ and one-dimensional quasi-periodic media. The result above is an analogue of several results in the calculus of variations (fields of extremals) and in PDE's. Since the models we consider are discrete and long range, new proofs need to be given. We also note that the main hypothesis of our result (the existence of foliations of equilibria) is the conclusion (using KAM theory) of several recent papers. Hence, we obtain that the KAM solutions recently established are minimizers when the interaction is ferromagnetic and transitive (these concepts are defined later).

1. Introduction

Many physical problems lead to variational problems for functions described in discrete sets.

A model to keep in mind as motivation is the Frenkel–Kontorova model [FK39], which considers configurations $u = \{u_i\}_{i\in\mathbb{Z}}$ and tries to find those that minimize the energy given by the formal sum

$$\mathscr{S}(u) = \sum_{i\in\mathbb{Z}} \left[\frac{1}{2}(u_i - u_{i+1})^2 - V(u_i) \right]. \tag{1}$$

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The result we present, however, is significantly more general and allows higher dimensional lattices, many-body interactions as well as long range interactions.

There are several physical interpretations of the FK model [BK04,Sel92], and the original one is the interaction of planar dislocations in a 3-D crystal, but it has appeared as a model of other situations. We can think of u_i as describing the position of the ith atom deposited over a 1-D medium. The first part of the sum describes the interaction between the nearest particles. The function V models the interaction of the atoms with the medium, which is assumed to be periodic or quasi-periodic in models of crystals and quasi-crystals (in this paper, periodicity or quasi-periodicity is not assumed). Note that, with many of these interpretations, it is natural to consider also more general models which involve longer range interactions, multi site interactions, or higher dimensional crystals. For example, if the interpretation of the model is ferromagnetic materials, the physical origin of the interaction is the exchange terms in electrical interaction that many body and long range are natural. The reference [Suh65] includes several such models. Hence, in this paper we will include these generalizations, which are standard in the mathematical treatments of statistical mechanics [Rue69]. Indeed, the limit in which the ranges of the interaction are very long range (sometimes called Kac limit) have special interest [Pre09]. We also note that in numerical analysis, it is standard to use long range operators as higher order approximation of derivatives. Of course, in higher dimensions, the so called *stencils* used in the finite difference study of Laplacians can be considered as many-body and non-nearest neighbor interactions.

In this paper, we will consider the problems of studying configurations which are critical points or ground states of the model (see the precise definitions later). These configurations are important for statistical mechanics in the low temperature limit. They are also discrete analogues of the classical problems of multiple integrals or minimal surfaces in the calculus of variations.

The results we present are more general than the model (1) and include higher dimensional lattices, many body and long range interactions.

For the model (1) when V is a periodic function, the problem of showing existence of plane-like minimizers (i.e., minimizers that differ from a linear function by a bounded function) of (1) with a well defined frequency was solved independently by Mather [Mat82] and Aubry [ALD83], giving rise to what is now referred to as Aubry–Mather theory. Several authors (see [Bla89,Bla90,KdlLR97,CdlL98,dlLV07b,dlLV07a,dlLV10] and references therein) generalized the setting of Aubry–Mather theory to higher dimensional crystals, more general media and for many-body interactions. Related models appear in PDE's [Mos89,RS11], minimal surfaces [CdlL01,Val04,Tor04], fractional Laplacian operators [dlLV09,Dav13].

In the case that V in (1) is a quasi-periodic function, the problem to establish all the results of Aubry–Mather theory as for periodic systems is not completed. Even if some results have been established—e.g. homogenization [LS03,KV08]—others remain open. Notably, the existence of plane-like minimizers is still not settled. In [LS03] there are examples of quasi-periodic potentials in FK models for which no plane-like minimizer exists. When the potential V is quasi-periodic and small enough, the papers [SdlL12b,SdlL12a,dlLSZ16,dlLSZ17] use a rather unusual KAM theory to construct families of equilibria which are plane-like. The results of this paper show that the families constructed by KAM method are minimizers when the problem is ferromagnetic. See also [AS16,AGK16].

Hence, the above papers give that for the quasi-periodic FK model, there are parameters with plane-like minimizers and parameters without them. Therefore, an interesting problem is to study the transition from models with plane-like minimizers to models without them. The papers [SdlL12b, SdlL12a] lead to efficient numerical algorithms, which were implemented in [BdlL13], and lead to several conjectures about the transition between plane-like and non plane-like minimizers. Notably [BdlL13] discovered numerically scaling relations similar to those in phase transitions in the breakdown of analyticity of plane-like solutions in quasi-periodic media. The papers [dlLSZ16, dlLSZ17] also present efficient algorithms for the computation of other solutions, but they have not been implemented yet.

The goal of this paper is to show that for ferromagnetic models when there are continuous families of equilibria whose graphs cover the whole phase space they are actually ground states (also called class-A minimizers). In particular, the solutions produced by KAM theory in [SdlL12b, SdlL12a, dlLSZ16, dlLSZ17] are ground states.

The interest of these results for calculus of variations is that minimizers often have more global properties [Mor24] or better regularity. In statistical mechanics, the ground states play a more important role in the zero temperature limit. From the numerical point of view, as remarked in [NP83, Aub84] it is easy to compute numerically regions in phase space without minimizers (it suffices to compute orbits of finite length and verify that the second variation of action has negative directions). Once we know that rotational KAM tori are minimizers, we can exclude that region. At the time of [NP83] this was known as the *Percival conjecture*, but it was shortly after proved for twist maps.

Results establishing that families of equilibria are minimizers are very common place in the standard calculus of variations. They are often proved by either the methods of *fields of extremals* or *Hilbert integral* [Car99]. In our case, since we are considering discrete space and long range models, these methods do not seem to apply directly and we have to use a different method, sometimes called *sliding* method. For some simple models, in Sect. 4.2, we present arguments that are different from the general one.

We note that, as it is customary, the non-degeneracy equations of KAM theory are weaker than those in the variational theory. Roughly speaking, the KAM theory just requires that certain operators are invertible. The variational theory requires that these quantities are positive definite. On the other hand, the KAM theory is more sensitive to quantitative features. For example, in (1) and periodic V, the KAM only applies for V which are small in a smooth norm, whereas the variational methods apply for any differentiable V.

In Sect. 2.1, we present the results in a very general set up, patterned after the general set up of statistical mechanics [Rue69] allowing multi-body and long range interactions, but we do not assume translation invariance. Indeed in Sect. 4.3, we consider models which are not translation invariant. In Sect. 4, we present again the results for some concrete models, which have appeared in the literature. Even if this could have been avoided logically since the models in Sect. 4 are particular cases of those in Sect. 2.1 we hope that this will add to the readability of the paper and as motivation for those interested in the concrete models and in numerical implementations. Also, the methods of proof used in Sect. 4 are different from the methods used in the proof of the general theorem and are closer to the arguments in the classical calculus of variations.

2. Formulation of the Main Result

2.1. A very general set up. We consider a very general set up motivated by the formulation in [Rue69] of statistical mechanics. Later, in Sect. 4, we will present more details for less general set ups, which may be more familiar.

2.1.1. General assumptions on the systems and its configurations. We consider a discrete countable set Λ. Its elements will be called sites. The set Λ may be imagined as a network of particles. Many cases in statistical mechanics consider that Λ is an integer lattice, corresponding physically to a crystal.

We assume that the state of each site is given by a real number. Hence, the state of the system is given by a function $u : \Lambda \to \mathbb{R}$ which assigns to each site $i \in \Lambda$ the value u_i. For our purposes, it is crucial that the order parameter at each site is a one-dimensional number. We do not know how to deal with two-dimensional phase spaces. Indeed [Bla90] presents counterexamples to several crucial statements in our setting when the order parameters are 2-dimensional. The papers [Mat91,Man97] contain rather satisfactory analogues of several other results of Aubry–Mather theory for higher dimensional order parameters but they do not consider higher dimensional independent variables.

We associate to the finite subsets B of Λ an energy function $H_B : \mathbb{R}^B \to \mathbb{R}$, which models the (possibly many-body and long range) interaction. In physical terms, the interaction may be even among the different sites or among the sites and a substratum. The total energy associated to a configuration u is given by the formal sum:

$$\mathscr{S}(u) = \sum_{\substack{B\subseteq\Lambda \\ \#B<\infty}} H_B(u) \qquad \forall\, u \in \mathbb{R}^\Lambda, \tag{2}$$

where $H_B(u)$ depends only on $u|_B$.

Remark 1. In this paper, we will not assume any periodicity properties of the set Λ and of the interaction, since this will not play any role in our arguments. On the other hand, we note that the main hypothesis of this paper (the existence of a foliation of equilibria) is the conclusion of several other papers which use periodicity assumptions. In [CdlL98,dlLV10], there is a very general set up for periodicity assumptions. In this paper, we do not even need that the models are translation invariant as in [Rue69]. Some interesting models which are not translation invariant are models of structured materials, the Hopfield model, the time dependent Lagrangians, which are discussed in Sect. 4.3. In Sect. 4, we will present the results for some finite range models which are concrete examples of the set up and for which our main hypotheses are verified.

2.1.2. Critical points and ground states. The following definitions are very standard in the calculus of variations.

Definition 2. When the H_B are differentiable, we say that a configuration u is an equilibrium for an energy (2) when

$$\frac{\partial}{\partial u_i}\mathscr{S}(u) \equiv \sum_{B\ni i} \partial_{u_i} H_B(u) = 0 \qquad \forall\, i \in \Lambda. \tag{3}$$

For simplicity, we denote $\frac{\partial}{\partial u_i}\mathscr{S}(u)$ by $E_i(u)$ and $E(u) = \{E_i(u)\}_{i\in\Lambda}$.

We note that, even if the sum defining $\mathscr{S}$ is formal, the equilibrium equations (3) are meant to be well defined equations. The (3) can be taken as defining the derivative of the formal sum $\mathscr{S}$. It is obtained by taking derivatives term by term and retaining only those which are not obviously zero.

Notice that we use standard typographical conventions and we use $\frac{\partial}{\partial u_i}$ ∂_{u_i}, ∂_i indifferently and the choice is dictated to make the typography less cramped.

Hence, we will formulate conditions on the interactions and the configurations that imply that the sums in (3) converge uniformly for all the configurations considered. This can happen for example if $H_B \equiv 0$ whenever $\mathrm{diam}(B) \geq L$. (These are called finite range interactions and Frenkel–Kontorova models are an example.) In Sect. 2.1.7, we will formulate a condition, more general than finite range, which is enough for our purposes.

We are interested in the existence of the following special class of equilibria.

Definition 3 (*Ground states*). We say that a configuration u is a ground state (or a class-A minimizer in the terminology of Morse [Mor24]) if for any configuration φ whose support is a finite subset of Λ we have

$$\mathscr{S}(u) - \mathscr{S}(u + \varphi) \leq 0. \tag{4}$$

Note that (4) should be understood cancelling all the terms that are obviously identical. That is

$$\sum_{\substack{\#B<\infty \\ B\cap \mathrm{supp}(\varphi)\neq\emptyset}} [H_B(u) - H_B(u+\varphi)] \leq 0. \tag{5}$$

We note that the conditions (5) make sense when the interactions are finite range since the sum in (5) involves only finitely many terms. In Sect. 2.1.7, we will make assumptions more general than finite range that ensure that the sum in (5) makes sense. It is clear that the main idea is that we will assume the terms in the sum (5) as well as their derivatives decay fast enough as $\mathrm{diam}(B)$ grows for all u in a class of functions. We will postpone the precise formulation until we have specified which classes of functions we will consider.

Since expressions similar to (5) will appear often in our calculations, we introduce the notation

$$\Gamma(\varphi; u, \tilde{B}) \equiv \sum_{\substack{\#B<\infty \\ B\cap\tilde{B}\neq\emptyset}} [H_B(u+\varphi) - H_B(u)]\,. \tag{6}$$

We remark that if $\mathrm{supp}(\varphi) \subset \tilde{B}$, we have

$$\Gamma(\varphi; u, \tilde{B}) = \Gamma(\varphi; u, \mathrm{supp}(\varphi)).$$

The reason is that, the sums defining the two Γ differ only in sets B which do not intersect the support of φ. Hence, the corresponding term in the sum is zero.

It is easy to check that a ground state is an equilibrium.

2.1.3. Foliations by equilibria. We say that a collection of configurations $\{u^\beta\}_{\beta\in\mathbb{R}}$ is a foliation by equilibria when:

(A1) $E(u^\beta) = 0$, i.e. $E_i(u^\beta) = 0$ for any $\beta \in \mathbb{R}$, $i \in \Lambda$;
(A2) u^β is increasing with respect to β, i.e. if $\beta_1 \leq \beta_2$, $u_i^{\beta_1} \leq u_i^{\beta_2}$ for any $i \in \Lambda$;
(A3) $u_i^\beta \to \pm\infty$ as β goes to $\pm\infty$ for any $i \in \Lambda$;
(A4) for any $i \in \Lambda$, $\left\{u_i^\beta \mid \beta \in \mathbb{R}\right\} = \mathbb{R}$.

The most crucial assumption for us is (A4). This means that, as we move the parameters β, the graphs of the functions u^β sweep out all the space $\Lambda \times \mathbb{R}$.

We say a foliation is strict if

(A2)' $\beta_1 < \beta_2 \Longrightarrow u_i^{\beta_1} < u_i^{\beta_2}$ for all $i \in \Lambda$.

Having a family of critical points satisfying (A1)–(A4) is extremely analogous to the assumption on the *fields of extremals* in the calculus of variations [Car99].[1]

Remark 4. Topologists would prefer to use the name foliation only for what we call strict foliation. Note that in (A2) we allow non-strict inequalities and hence the graphs of u^{β_1} and u^{β_2} can have intersections. This generality is useful for us.

Remark 5. One of the motivation for our study was the Aubry–Mather theory. Nevertheless, we point out that the models in Aubry–Mather theory are assumed to satisfy more properties (e.g. periodicity) than the models considered in this paper.

The usual Aubry–Mather theory for a fixed frequency ω produces a family satisfying (A1)–(A3)—but in general not (A4). On the other hand, for Diophantine frequency ω (and some models) we can use KAM theory to produce families satisfying (A1)–(A4). The calculus of variations methods do not assume that the system is close to integrable, but they require positive definite assumptions on the interaction (ferromagnetism). On the other hand, KAM methods do not require that the system is ferromagnetic but they require that the system admits an approximate solution to the invariance equation (in particular, the approximate solutions for systems close to integrable can be taken as the solutions of the integrable system).

The difference between the quasi-periodic solutions satisfying (A4) and those which do not satisfy (A4) is crucial in Aubry–Mather theory. It is known that for families of models depending on parameters, there are regions of parameter for which the solutions satisfy (A4) and others for which they do not. The boundary between these two regions in parameter space is the well-known *"analyticity breaking"* transition. There is a deep mathematical theory relating to this difference and the consequences of the failure of (A4). See [RS11] for an account of this theory for elliptic PDE.

Remark 6. In the applications to Aubry–Mather theory which we will discuss later in Sect. 4, the set Λ will be $\mathbb{Z}^d$ and the functions u^β will be roughly linear.

We point out, however, that in the case of no interactions, in dimensions bigger than or equal to 2 one could have also harmonic polynomials, which are minimizers. It is pointed out in [Mos86,Mos89] that developing a variational theory starting from the harmonic polynomials of higher degree would be interesting. See [MS92].

Note that the subsequent properties we will assume depend on the class of functions u^β.

2.1.4. Ferromagnetic properties.

Definition 7 (*Ferromagnetic condition*). We say that the C^2 interaction potential H satisfies the ferromagnetic condition if

$$\frac{\partial^2 H_B}{\partial u_p \partial u_q}(u) \le 0 \qquad \forall p, q \in \Lambda,\, p \ne q, \tag{7}$$

for any configuration u on Λ and any finite subset B of Λ.

[1] Note that the fields of extremals in [Car99] are formulated for functions of a one dimensional variable taking values into any dimensional space. Here we are in the opposite situation: we are considering functions of many variables, but taking values in a one dimensional space.

Definition 8 (*Ferromagnetic transitive*). We say that a ferromagnetic interaction in Λ is transitive for a class of configurations $\{u^\beta\}_{\beta\in\mathbb{R}}$ when, given any $p, q \in \Lambda$ there exist an integer $k \geq 1$, a sequence $p_0, \ldots, p_k$ in Λ with $p_0 = p$, $p_k = q$ and sets B_i containing a pair p_i, p_{i+1} for $i = 0, \ldots, k-1$ such that, for any β and any φ with compact support,

$$\partial_{p_i}\partial_{p_{i+1}} H_{B_i}(u^\beta + \varphi) < 0.$$

In the main cases of interest, such as the Frenkel–Kontorova models, we will see that the $\partial_{p_i}\partial_{p_{i+1}} H_{B_i}$ are independent of the configuration, so that this assumption will be very easy to verify in several models of practical importance.

The assumption in Definition 8 appeared in [dlLV07a] where it was shown that it implies that the gradient flow of the formal energy $\mathscr{S}$ in (2) satisfies a strong comparison principle. In the PDE case, a comparison principle for the gradient flow would give a very quick proof of our results, but the conclusion for the long range of the interactions requires an extra argument. See Remark 11.

Remark 9. The name ferromagnetic comes from the physical interpretation of interaction of spins. The convexity condition leads to smaller energy when neighboring spins are aligned. These conditions have appeared in other physical interpretations of the problem. The ferromagnetism assumptions, when $\Lambda = \mathbb{Z}$ and the interactions are nearest neighbor, become the twist conditions in Aubry–Mather theory. We also note that they can be thought of as analogues of ellipticity conditions for continuous variational problems. See [CdlL98, dlLV07b] for some more explanations of these analogies.

2.1.5. Graph theoretic language to describe the ferromagnetic assumptions. We can reformulate some of the assumptions of Sect. 2.1.4 in the language of graph theory. The introduction of a new language is purely cosmetic, but allows us to express future arguments concisely and it may be illuminating.

The key observation is that we can interpret Definition 8 as the existence of a graph structure on Λ.

Whenever there exists B such that for all u^β and all φ of compact support

$$\partial_p \partial_q H_B(u^\beta + \varphi) < 0 \tag{8}$$

then the sites p, q are linked in the graph.

The physical meaning of (8) is that the configuration at p affects the forces experienced at the site q (and vice versa, in agreement with the action-reaction principle). The Definition 8 can be interpreted as saying that any site can influence any other site, if not directly, through influencing intermediate sites that in turn influence some others.

It is natural to endow Λ with a graph structure by considering the points of Λ as vertices and drawing an edge among two linked sites in the sense of (8). Graph theory is a deep subject, but we will only use basic names that can be found in the introductions of any book, for example [BM08].

The assumption in Definition 8 can be interpreted as saying that, starting from any site, we can reach any other jumping only through linked sites or that the graph is connected.

The graph structure allows us to introduce two notions that are standard in graph theory: distance and connectedness.

In graphs, [Bol98, p. 4] we can define a path as a sequence (finite) of edges which connect a sequence of points. (Note that some books require that they do not intersect, this will not make a difference for us). Given a path γ in the graph, we define the $|\gamma|$ the length of a path γ as the number of edges it contains and we say that a path connects two points $i, j \in \Lambda$ when the first edge of γ joins i and the last edge joins j.

We define the *distance between two sites* $i, j \in \Lambda$ as

$$d(i, j) = \inf\{\, |\gamma| \;\big|\; \gamma \text{ joins } i, j\}. \tag{9}$$

When the graph is connected, i.e. all pairs of points have a path joining them, then d can be defined for all pairs satisfying the usual assumptions of distance.

We also define the *distance of a point* i *to a set* $S \subset \Lambda$ as

$$d(i, S) = \inf\{\, d(i, j) \;\big|\; j \in S\}. \tag{10}$$

(Since the $d(i, j)$ takes values in integers, it is clear that the infimum in (10) is a minimum.)

We can also define that a set S is *connected* when any pair of points can be joined by paths all whose edges have end points in the set S.

Given any finite non-empty set S and $k \in \mathbb{N}$, we define the set

$$S_k = \{\, i \;\big|\; d(i, S) \le k\} = \{\, i \;\big|\; j \in S, d(i, j) \le k\}. \tag{11}$$

Note that $S_{k+\ell} = (S_k)_\ell$.

In the Frenkel–Kontorova model, we have that if S is finite, S_k is finite. On the other hand, it is possible that the system involves long range interactions in such a way that S_1 could be infinite even for finite S.

Also the transitivity assumption can be formulated as saying that, for any non-empty set S,

$$\Lambda = \cup_k S_k. \tag{12}$$

2.1.6. Coerciveness assumption. Given a family u^β as before, we will assume that for any compactly supported φ and any $i \in \operatorname{supp}(\varphi)$ we have

$$\lim_{|t|\to\infty} \sum_{B\cap \operatorname{supp}(\varphi)\neq\emptyset} \left[H_B(u^\beta + \varphi + \delta_i t) - H_B(u^\beta + \varphi)\right] = +\infty, \tag{13}$$

where δ_i denotes the Kronecker function which takes the value 1 at i and 0 at any other point.

Note that (13) says that if we make a test function grow at just one point, then the relative energy grows.

2.1.7. A regularity assumption. We will be performing some calculations with the equilibrium equations. In order to justify them, we will need some assumptions on the convergence of the E_i and their derivatives.

The following assumption is sufficient for the methods used in this paper. We note that the finite range of the interaction easily implies our assumption. Many models of interest (e.g. the Frenkel–Kontorova models) are finite range, but there are also models of physical interest which are not. See [SdlL12b] for a discussion of the case when hierarchical models satisfy the assumptions.

Given a class $\{u^\beta\}_{\beta\in\mathbb{R}}$ satisfying (A1)-(A4) we say that the interaction $\{H_B\}$ is u^β summable when: for any φ with compact support, we have for all $\beta \in \mathbb{R}$,

$$\begin{aligned}\lim_{L\to\infty} \sum_{\substack{\mathrm{diam}(B)\geq L\\ B\cap \mathrm{supp}(\varphi)\neq\emptyset}} |\partial_{u_i} H_B(u^\beta + t\varphi)| = 0\\ \lim_{L\to\infty} \sum_{\substack{\mathrm{diam}(B)\geq L\\ B\cap \mathrm{supp}(\varphi)\neq\emptyset}} |\partial_{u_i}\partial_{u_j} H_B(u^\beta + t\varphi)| = 0\end{aligned} \tag{14}$$

and the limits in (14) are uniform in $t \in [0, 1]$.

The following result is a very simple corollary of the coercivity and regularity assumption.

Proposition 10. *Let $\{u^\beta\}_{\beta\in\mathbb{R}}$ be a family of configurations and $\{H_B\}_{\substack{B\subset\Lambda\\ \#B<\infty}}$ be a family of interactions that satisfy the coerciveness and the regularity assumptions with respect to them.*

Fix any function u^β in the family and a finite set $\tilde{B}$.

Then, there is a function φ^* such that

- *supp*$(\varphi^*) \subset \tilde{B}$;
- $$\Gamma(\varphi^*; u^\beta, \tilde{B}) = \inf\{\, \Gamma(\varphi; u^\beta, \tilde{B}) \;\big|\; \mathit{supp}(\varphi) \subset \tilde{B} \,\}$$

  where we use the notation introduced in (6)*;*
- $$E_i(u^\beta + \varphi^*) = 0 \quad \forall\, i \in \tilde{B}. \tag{15}$$

The proof of Proposition 10 is very easy. We note that we are considering a function of finitely many real variables (the values of φ^* at the sites of $\tilde{B}$). By the assumption of regularity this function is differentiable and tends to infinity as any of its arguments goes to infinity. Hence, this function reaches its minimum and the minimum has zero derivative.

Of course, the support of the minimizing function could be smaller than $\tilde{B}$ if some of the values of the minimizing function happen to be zero. □

Remark 11. Note that in Proposition 10 we do not obtain that $u^\beta + \varphi^*$ is an equilibrium. In (15), we only obtain that the equilibrium equations hold in the finite set $\tilde{B}$.

Even if u^β satisfies the equilibrium equations in Λ, when the interaction is long range, modifying the configuration in $\tilde{B}$ can affect the equilibrium equations everywhere.

This is an important difference with the PDE models in the classical calculus of variations and this is the reason why our arguments need to be different.

2.1.8. Statement of the main general result.

Theorem 12. *Let H be a C^2 ferromagnetic interaction potential. Assume that there exists a collection of configurations $\{u^\beta\}_{\beta\in\mathbb{R}}$ such that (A1)-(A4) hold.*

Moreover, assume that, with respect to u^β the interaction satisfies the ferromagnetic transitivity, coercivity and regularity assumptions stated in Sects. 2.1.4, 2.1.6 and 2.1.7.

Then, all the equilibria u^β are ground states.

Remark 13. A referee kindly observed that in some models, it may be possible to exclude that minimizers go through some specific regions of phase space. In such a case, it suffices to verify the hypotheses of the theorem only for the configurations which take values in the complement of the excluded regions.

3. Proof of Theorem 12

Suppose by contradiction that there exist a number β_0 and a configuration φ whose support is nonempty and finite such that

$$\mathscr{S}(u^{\beta_0} + \varphi) - \mathscr{S}(u^{\beta_0}) < 0. \tag{16}$$

That is, using the notation (6)

$$\Gamma(\varphi; u^{\beta_0}, \mathrm{supp}(\varphi)) < 0. \tag{17}$$

We choose any finite set $\tilde{S} \subset \Lambda$ such that $\mathrm{supp}(\varphi) \subsetneq \tilde{S} \subset (\mathrm{supp}(\varphi))_1$ with the notation introduced in (11).

Using Proposition 10 there is a configuration φ^* with support in $\tilde{S}$ such that

$$\Gamma(\varphi^*; u^{\beta_0}, \tilde{S}) = \min_{\mathrm{supp}(\varphi_1) \subseteq \tilde{S}} \Gamma(\varphi_1; u^{\beta_0}, \tilde{S}). \tag{18}$$

We note that, since we can take φ as a test function φ_1 we have

$$\Gamma(\varphi^*; u^{\beta_0}, \tilde{S}) \le \Gamma(\varphi; u^{\beta_0}, \tilde{S}) = \Gamma(\varphi; u^{\beta_0}, \mathrm{supp}(\varphi)) < 0. \tag{19}$$

Hence, if the function u^{β_0} was not a ground state, we could find a non-trivial φ^*. We will show that it is impossible to find a non-trivial φ^*. Therefore, it is impossible that φ exists and, hence u^{β_0} is a ground state.

We denote

$$\begin{aligned} \beta_+ &= \inf\{\beta \in \mathbb{R} \mid u^\beta > u^{\beta_0} + \varphi^*\}, \\ \beta_- &= \sup\{\beta \in \mathbb{R} \mid u^\beta < u^{\beta_0} + \varphi^*\}, \end{aligned} \tag{20}$$

where the partial ordering $u < v$ is defined by $u_i < v_i$ for any $i \in \Lambda$. Analogous definitions hold for "$>$", "$\ge$" and "$\le$".

Consequently, we have that assumption (A2) can be formulated just as $u^{\beta_+} \ge u^{\beta_0} \ge u^{\beta_-}$. Notice that we have $\beta_- \le \beta_+$ such that

$$u^{\beta_-} \le u^{\beta_0} + \varphi^* \le u^{\beta_+}. \tag{21}$$

Moreover, for some $i_-, i_+ \in \tilde{S}$ one can have

$$(u^{\beta_0} + \varphi^*)_{i_+} = u^{\beta_+}_{i_+}, \quad (u^{\beta_0} + \varphi^*)_{i_-} = u^{\beta_-}_{i_-}. \tag{22}$$

By the choice of φ^*, β_- and β_+, we have from Proposition 10 and (A1) that

$$\begin{aligned} &E_i(u^{\beta_0} + \varphi^*) = 0, \quad \forall i \in \tilde{S} \\ &E(u^{\beta_+}) = E(u^{\beta_-}) = 0. \end{aligned} \tag{23}$$

The following is an elementary calculation using the fundamental theorem of calculus which holds for any configuration u^* and any η such that the regularity assumptions hold.

$$\begin{aligned}
&E_{i^*}(u^* + \eta) - E_{i^*}(u^*) \\
&= \int_0^1 dt \left[\sum_{j\in\Lambda} \sum_{B\ni i^*} \frac{\partial^2 H_B}{\partial u_j \partial u_{i^*}}(u^* + t\eta)\eta_j \right] \\
&= \eta_{i^*} \int_0^1 dt \sum_{B\ni i^*} \frac{\partial^2 H_B}{\partial u_{i^*} \partial u_{i^*}}(u^* + t\eta) \\
&\quad + \sum_{\substack{j\in\Lambda \\ j\neq i^*}} \eta_j \int_0^1 dt \sum_{B\ni i^*} \frac{\partial^2 H_B}{\partial u_j \partial u_{i^*}}(u^* + t\eta). \qquad (24)
\end{aligned}$$

Of course, in (24), since there are factors η_j in all the terms, the sum in j can be restricted to the support of η. In the first application of the argument, we use it for a situation when the support of η is $\tilde{S}$. Later on, we will use it for different supports.

The identity (24) leads immediately to the following proposition.

Proposition 14. *Assume that, in the conditions of* (24) *we have*

$$\begin{aligned}
&E_{i^*}(u^*) = E_{i^*}(u^* + \eta) \text{ and } \eta_{i^*} = 0 \text{ for some } i^* \in \tilde{S} \\
&\eta \geq 0 \text{ (or } \eta \leq 0 \text{).}
\end{aligned}$$

Then, we have that $\eta_j = 0$ *for all* j *in* $\tilde{S}$.

We will do the proof by showing that it applies to increasingly general $\tilde{S}$.

Step1. The proof of Proposition 14 is just observing that since $\eta_{i^*} = 0$, and all the other terms in (24) have the same sign, we should have that all of the terms in the sum in (24) should be zero. Hence, either $\eta_j = 0$ or $\int_0^1 \frac{\partial^2 H_B}{\partial u_{i^*} \partial u_j}(u^* + t\eta) = 0$, but for the points j at distance 1, this integral is not zero (because this is the definition of points being linked).

Hence, we obtain the result for any $\tilde{S}$ which is finite and which is contained in $(\{i^*\})_1$.

We also obtain that $\eta_j = 0$ for all $j \in (\{i^*\})_1$. (Of course, if $j \notin \text{supp}(\varphi^*)$ the result is trivial).

Step 2. We can now apply the result starting at any point in $(\{i^*\})_1$ and obtain the result for any $\tilde{S} \subset (\{i^*\})_2$ and, again repeating the argument, for any $j \in (\{i^*\})_2$.

Step 3. Since $\tilde{S}$ is a finite set, due to (12) there exists a $k \geq 1$ such that $\tilde{S} \subset (\{i^*\})_k$. After using the argument in Step 1 at most k times, one can conclude that $\eta_j = 0$ for all j in $\tilde{S}$. Notice that this step uses essentially that the interaction is transitive. □

Now, we come back to the proof of Theorem 12. Notice that Theorem 12 will be established when we conclude that φ^* in (15) could not exist and, hence no φ satisfying (16) could exist.

Taking

$$u^* = u^{\beta_+}, \quad \eta = u^{\beta_0} + \varphi^* - u^{\beta_+},$$

and due to (21) and (22), we see that $\eta \leq 0$, and moreover for $i_+ \in \tilde{S}$ we have

$$E_{i_+}(u^*) = E_{i_+}(u^* + \eta) \text{ and } \eta_{i_+} = 0.$$

Hence, applying Proposition 14, we have $\eta = 0$ in $\tilde{S}$. That is, $\varphi^* = u^{\beta_+} - u^{\beta_0}$ in $\tilde{S}$ and $\varphi^* = u^{\beta_+} - u^{\beta_0} \geq 0$.

We can now proceed in the opposite direction using u^{β_-}. Similarly, we take

$$u^* = u^{\beta_-}, \quad \eta = u^{\beta_0} + \varphi^* - u^{\beta_-},$$

and due to (21) and (22), we see that $\eta \geq 0$, and moreover for $i_- \in \tilde{S}$ we have

$$E_{i_-}(u^*) = E_{i_-}(u^* + \eta) \text{ and } \eta_{i_-} = 0.$$

Applying Proposition 14, we have $\eta = 0$ in $\tilde{S}$. That is, $\varphi^* = u^{\beta_-} - u^{\beta_0}$ in $\tilde{S}$ and $\varphi^* = u^{\beta_+} - u^{\beta_0} \leq 0$, which concludes that $\varphi^* = 0$. That is, we have shown that it is impossible to lower the energy by changing the state in $\tilde{S}$.

4. Some Concrete Examples of the Models Considered

In this section, we will show how several different models in the literature fit in the general framework we have developed. The fact that we can obtain results for different models at the same time is due to the generality of the methods we present here. In this section, we will also present proofs of the results of Theorem 12 for the concrete models. The proofs we will present are different from the general proof and take advantage of the special features of the models.

4.1. General one-dimensional periodic models. The papers [dlL08] considers one dimensional models given by energies of the form:

$$\mathscr{L}(u) = \sum_{k\in\mathbb{Z}} \sum_{L\in\mathbb{N}} H_L(u_k, u_{k+1}, \ldots, u_{k+L}) \tag{25}$$

where $u : \mathbb{Z} \to \mathbb{R}$ and $H_L : \mathbb{R}^{L+1} \to \mathbb{R}$.

Remark 15. Note that the models in (25) enjoy a translation invariance, (the energy of the configuration v defined by $v_i = u_{i+m}$ satisfies that formally, $\mathscr{L}(u) = \mathscr{L}(v)$) which is not present in our general set up, but which is physically justified. One can remove the periodicity by changing the H_L in (25) to $H_{L,k}$. This will be done in models in quasi-periodic media or discounted systems discussed in the next sections.

The corresponding equilibrium equation for the models (25) is:

$$\mathscr{E}_i(u) = \sum_k \sum_L \sum_j \partial_{j+1} H_{L-k}(u_{i-j}, \ldots, u_i, \ldots, u_{L-k+i-j}). \tag{26}$$

The paper [dlL08] includes coercivity and regularity assumptions similar to ours, it includes an extra periodicity property

$$H_L(u_k, u_{k+1}, \ldots, u_{k+L}) = H_L(u_k + 1, u_{k+1} + 1, \ldots, u_{k+L} + 1)$$

as well as higher regularity assumption. On the other hand, the paper [dlL08] does not use the full strength of the ferromagnetic property and indeed they allow some antiferromagnetic terms since the KAM theory allows that. If one changes parameters, there could be critical values when the tori studied go from minimizers to not being minimizers. Note that, since the systems in [dlL08] are translation invariant, the ferromagnetic transitivity is implied by the ferromagnetic property of nearest neighbors (there are other assumptions such as the strict ferromagnetism for other sets of interactions).

The paper [dlL08] considers only equilibrium configurations given by a hull function

$$u_k = \omega k + h(\omega k)$$

where h is a periodic function called the *"hull function"*. The hull function is such that $t + h(t)$ is an increasing function.

It is easy to see that—it is shown with many details in [dlL08] that if h is the hull function for a critical point so is u^β given by

$$h^\beta(\theta) = \beta + h(\theta + \beta).$$

We observe that, when h is a smooth function and $|h|_{L^\infty} < 1$, the configurations obtained for all these hull functions produce a foliation in our sense.

Hence, applying Theorem 12, we obtain the following result:

Theorem 16. *Assume the set up of [dlL08] and assume furthermore, that, for some hull function, the system satisfies the ferromagnetic property*

$$\partial_i \partial_j H_L \leq 0$$

and that, for any given $\eta > 0$,

$$\partial_1 \partial_2 H_1(x, y) \leq -\eta < 0 \qquad \forall x, y \in \mathbb{R}.$$

Then, the quasi-periodic solutions produced in [dlL08] are ground states.

4.2. Application to the Frenkel–Kontorova models on quasi-periodic media. This class of models was considered in [SdlL12b] with nearest neighbor interactions. In [SdlL12a] for more general interactions, many body interactions. The papers [SdlL12b, SdlL12a] consider quasi-periodic solutions which are non-resonant (indeed Diophantine) with the frequency of the medium. The papers [dlLSZ16, dlLSZ17] study quasi-periodic solutions which are resonant with the frequency of the medium in models in which the interactions are only nearest neighbor. Using the results of this paper, we can conclude that the solutions are ground states provided that we assume transitive ferromagnetic conditions.

In this section, we will consider only the problem in [SdlL12b], which will allow us to give a more direct proof of the results. We note that in the models based on the Frenkel–Kontorova models with next neighbor interaction, the transitive ferromagnetic hypothesis is automatic.

We consider the following formal energy

$$\mathscr{S}(\{u_i\}_{i\in\mathbb{Z}}) = \sum_{n\in\mathbb{Z}} \left[\frac{1}{2}(u_n - u_{n+1})^2 - V(u_n\alpha) \right], \tag{27}$$

where $V : \mathbb{T}^d \to \mathbb{R}$ and $\alpha \in \mathbb{R}^d$ satisfying $k \cdot \alpha \neq 0$ when $k \in \mathbb{Z}^d - \{0\}$ where $d \geq 2$.

For simplicity, we denote $H(x, y) = \frac{1}{2}(x-y)^2 - V(x\alpha)$. Consequently, $\partial_{xy} H(x, y) = \partial_{yx} H(x, y) = -1$.

Under the assumption of [SdlL12b, dlLSZ17], using KAM method, we prove the existence of quasi-periodic solutions of the equilibrium equation

$$u_{n+1} + u_{n-1} - 2u_n + \partial_\alpha V(u_n \alpha) = 0, \tag{28}$$

where $\partial_\alpha V \equiv (\alpha \cdot \nabla)V$.

Indeed, the solutions of (28) we found are given by a hull function

$$u_n = n\omega + h(n\omega\alpha)$$

for some given $\omega \in \mathbb{R}$. Therefore, the equilibrium equation we solve in terms of h is

$$h(\sigma + \omega\alpha) + h(\sigma - \omega\alpha) - 2h(\sigma) + \partial_\alpha V(\sigma + \alpha \cdot h(\sigma)) = 0. \tag{29}$$

The papers [dlLSZ16, dlLSZ17] have very different non-resonance assumptions from the assumptions in [SdlL12b, SdlL12a] and require very different methods. Nevertheless, from the point of view of the arguments of this paper, to show that the quasi-periodic solutions produced in both papers are ground states, we can use the same argument.

It is easy to see that if $h(\sigma)$ is a solution (29), for any $\beta \in \mathbb{R}$, $h(\sigma + \beta\alpha) + \beta$ is a solution. We denote $h_\beta(\sigma) = h(\sigma + \beta\alpha) + \beta$.

Hence, let us denote $u_n^\beta = n\omega + h_\beta(n\omega\alpha)$ which is a continuum of equilibria of (28) with respect to the parameter $\beta \in \mathbb{R}$. It is easy to see that, for every fixed $n \in \mathbb{Z}$, u_n^β is monotone with respect to β, i.e.,

$$\frac{\partial u_n^\beta}{\partial \beta} = 1 + \partial_\alpha h(n\omega\alpha + \beta\alpha) \neq 0. \tag{30}$$

Without loss of generality, we assume u_n^β is monotone increasing with respect to β.

The following result is a particular case of Theorem 12, but in this section, we will present a different proof.

Theorem 17. *For every $\beta \in \mathbb{R}$, the configurations $u^\beta \equiv \{u_i^\beta\}_{i\in\mathbb{Z}}$ are ground states of* (27).

Proof. Suppose by contradiction that there exists β_0 such that $\{u_i^{\beta_0}\}_{i\in\mathbb{Z}}$ is not a ground state of (27). That is, there exists two integers $m < n$ and a configuration $\{v_i\}_{i\in\mathbb{Z}}$ satisfying $v_i = u_i^{\beta_0}$ for any $i \le m$ or $i > n$ such that

$$\mathscr{S}_m^n(\{v_i\}_{i\in\mathbb{Z}}) \equiv \sum_{i=m}^{n} \left[\frac{1}{2}(v_i - v_{i+1})^2 - V(v_i\alpha) \right] < \sum_{i=m}^{n} \left[\frac{1}{2}(u_i^{\beta_0} - u_{i+1}^{\beta_0})^2 - V(u_i^{\beta_0}\alpha) \right]. \tag{31}$$

Since $\min_{v\in\mathbb{R}^\mathbb{Z}} \mathscr{S}_m^n(v)$ is a minimizing problem of finite variables and $\mathscr{S}_m^n(v)$ is bounded from below, there exists a minimizing segment $\{w_i\}_{i=m}^{n+1}$ of $\mathscr{S}_m^n$ with the boundary condition $w_i = u_i^{\beta_0}$ for $i = m$ or $i = n+1$. One can suppose, without loss of generality, that there exists $m < i_0 \le n$ such that $w_{i_0} > u_{i_0}^{\beta_0}$. Since u^β is a foliation, there exist $\beta_1 > \beta_0$ and $m < i_1 \le n$ such that

$$w_{i_1} = u_{i_1}^{\beta_1} \text{ and } w_i \le u_i^{\beta_1}, \quad \forall\, m \le i \le n+1. \tag{32}$$

Indeed, one can choose $m < i_2 \le n$ such that $w_{i_2} = u_{i_2}^{\beta_1}$ and $w_{i_2-1} < u_{i_2-1}^{\beta_1}$.

We use the adaptation of the standard technique of the Hilbert integral in calculus of variations (see also [CdlL98]). For every $m \le i \le n$ we calculate

$$
\begin{aligned}
0 &= \partial_x H(w_i, w_{i+1}) + \partial_y H(w_{i-1}, w_i) - \partial_x H(u_i^\beta, u_{i+1}^\beta) - \partial_y H(u_{i-1}^\beta, u_i^\beta) \\
&= \int_0^1 \frac{d}{dt}\Big[\partial_x H(tw_i + (1-t)u_i^\beta, tw_{i+1} + (1-t)u_{i+1}^\beta) \\
&\qquad + \partial_y H(tw_{i-1} + (1-t)u_{i-1}^\beta, tw_i + (1-t)u_i^\beta)\Big]\, dt \\
&= \int_0^1 \Big[(\partial_{xx} H)(w_i - u_i^\beta) + (\partial_{xy} H)(w_{i+1} - u_{i+1}^\beta) \\
&\qquad + (\partial_{yx} H)(w_{i-1} - u_{i-1}^\beta) + (\partial_{yy} H)(w_i - u_i^\beta)\Big]\, dt.
\end{aligned} \tag{33}
$$

Let $i = i_2,\ \beta = \beta_1$ in the above calculation, we obtain

$$0 = w_{i_2+1} - u_{i_2+1}^{\beta_1} + w_{i_2-1} - u_{i_2-1}^{\beta_1}.$$

Hence, due to the choice of i_2, we have $w_{i_2+1} - u_{i_2+1}^{\beta_1} = u_{i_2-1}^{\beta_1} - w_{i_2-1} > 0$, which contradicts (32). □

4.3. Conformally symplectic systems. The so-called conformally symplectic systems have attracted recent mathematical attention but they have been in the applied literature for a very long time.

These models are described by actions that include exponentially decreasing factors. These exponentially decreasing factors are very natural in finance since they account for inflation. In recent times, the KAM theory for these systems has been developed in [CCdlL13] and an Aubry–Mather theory in [MS16].

An example to keep in mind is a variation of (1). In the same set up as (1), we consider the formal action for $\lambda > 0$, $\eta \in \mathbb{R}$

$$\mathscr{S}_{\lambda,\eta}(u) = \sum_{i\in\mathbb{Z}} e^{i\lambda}\left[\frac{1}{2}(u_i - u_{i+1})^2 - V(u_i) + \eta u_i\right]. \tag{34}$$

The equilibrium equations for (34) are equivalent to orbits of the so-called dissipative standard map. Note that these models involve two parameters (besides the parameters in the periodic potential). The λ plays the role of a dissipation and the η is called the drift. In the financial applications, each term of the action is the cost of a transaction at time i and the exponential factor reduces it to constant money.

In [CCdlL13] one can find a rather general KAM theory, which in particular applies to (34). One fixes the rotation number, and determines the quasi-periodic orbit with this rotation and the parameter η. Applying Theorem 12, we obtain that the KAM tori are minimizers, so that one could extend the algorithms for conservative systems in [NP83,Aub84,Mac89]. Other computations of invariant tori and their breakdown appear in [CM07,CC10,CF12].

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