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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
`annales@ens.fr`

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ON THE K-THEORY STABLE BASES OF THE SPRINGER RESOLUTION

BY CHANGJIAN SU, GUFANG ZHAO AND CHANGLONG ZHONG

ABSTRACT. – Cohomological and K-theoretic stable bases originated from the study of quantum cohomology and quantum K-theory. Restriction formula for cohomological stable bases played an important role in computing the quantum connection of cotangent bundle of partial flag varieties. In this paper we study the K-theoretic stable bases of cotangent bundles of flag varieties. We describe these bases in terms of the action of the affine Hecke algebra and the twisted group algebra of Kostant-Kumar. Using this algebraic description and the method of root polynomials, we give a restriction formula of the stable bases. We apply it to obtain the restriction formula for partial flag varieties. We also build a relation between the stable basis and the Casselman basis in the principal series representations of the Langlands dual group. As an application, we give a closed formula for the transition matrix between Casselman basis and the characteristic functions.

RÉSUMÉ. – Les bases stables cohomologiques et K-théoriques proviennent de l'étude de la cohomologie quantique et de la K-théorie quantique. La formule de restriction pour les bases stables cohomologiques a joué un rôle important dans le calcul de la connexion quantique du fibré cotangent de variétés de drapeaux partielles. Dans cet article, nous étudions les bases stables K-théoriques de fibré cotangents des variétés de drapeaux. Nous décrivons ces bases en fonction de l'action de l'algèbre de Hecke affine et de l'algèbre de Kostant-Kumar. En utilisant cette description algébrique et la méthode des polynômes de racine, nous donnons une formule de restriction des bases stables. Nous l'appliquons pour obtenir la formule de restriction pour les variétés de drapeaux partielles. Nous construisons également une relation entre la base stable et la base de Casselman dans les représentations de la série principale du groupe dual de Langlands p-adique. Comme une application, nous donnons une formule close pour la matrice de transition entre la base de Casselman et les fonctions caractéristiques.

1. Introduction

In [30], Maulik-Okounkov defined the cohomological stable envelope for symplectic resolutions (see also [10]). The image of certain cohomology classes under the stable envelope map are called the cohomological stable bases. The stable envelope is used to construct a quantum group action on the cohomology of quiver varieties, and to compute the quantum

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connection of quiver varieties. Moreover, Nakajima gave a sheaf theoretic definition of the stable envelope [33]. We refer the readers to [8, 32, 44, 46] for other applications.

The K-theoretic stable envelope is defined in [31] (see also [36, 45, 40]). It is constructed in [31] and used to define a quantum group action on the equivariant K-theory of quiver varieties [45]. Based on that, in [36], difference equations in quantum K-theory of quiver varieties are constructed geometrically, which are further identified algebraically with the quantum Knizhnik-Zamolodchikov equations [17, 36] and quantum Weyl group actions [45]. The monodromy of these difference equations is studied in [1] using the elliptic stable envelope. The K-theoretic stable bases for Hilbert scheme of points on $\mathbb{C}^2$ are studied in [35] and [18].

Stable bases for cotangent bundle of flag varieties and partial flag varieties are also of interest. The cohomological stable bases for $T^*(G/B)$ turn out to be the characteristic cycles of certain D-modules on the flag variety G/B. Pulling it back to G/B, we get the Chern-Schwartz-MacPherson classes for the Schubert cells [3, 42]. Moreover, for cohomological stable bases of the cotangent bundle $T^*(G/P)$, in [48], the first-named author obtained their restriction formula, which played an important role in computing the quantum connection of $T^*(G/P)$ in [47].

The goal of the present paper is to study the K-theory stable bases of cotangent bundle of flag varieties, and to find a restriction formula for the K-theoretic stable bases, formula expressing the stable bases in terms of the torus fixed point basis in $T^*(G/B)$. For each choice of a Weyl chamber, there is a set of stable bases, labeled by Weyl group elements $w \in W$. For the positive/negative Weyl chambers, the stable basis will be denoted by $\{\mathrm{stab}_\pm(w) \mid w \in W\}$. (There are other choices involved in the definition. See § 4.2 for the detail.) In the special cases when $w \in W$ is the identity e or the longest element $w_0 \in W$, $\mathrm{stab}_+(e)$ and $\mathrm{stab}_-(w_0)$ are equal to the structure sheaves of the corresponding fixed points, up to a factor.

Let Z be the Steinberg variety and A be the maximal torus of G. The convolution algebra $K_{G\times\mathbb{C}^*}(Z)$, which is isomorphic to the affine Hecke algebra by a well known theorem of Kazhdan-Lusztig and Ginzburg ([22, 16]), acts on $K_{A\times\mathbb{C}^*}(T^*G/B)$ on the left and on the right. Under these two actions, the Demazure-Lusztig operators corresponding to simple root α are denoted respectively by T_α and T'_α. Our first main result is the following:

THEOREM 1.1 (Theorem 4.5). – *The elements* $\mathrm{stab}_\pm(w)$ *are generated by the action of* $K_{G\times\mathbb{C}^*}(Z)$. *More precisely,*

$$\mathrm{stab}_+(w) = q^{-\ell(w)/2}T'_{w^{-1}}(\mathrm{stab}_+(e)), \quad \mathrm{stab}_-(w) = q^{\ell(w_0w)/2}(T_{w_0w})^{-1}(\mathrm{stab}_-(w_0)).$$

In the proof of this theorem, we use the *rigidity* technique (see § 3) to calculate the affine Hecke algebra actions on the stable bases in Proposition 4.3.

Theorem 1.1 allows us to give a purely algebraic definition of the stable bases (Definition 6.3), involving only the affine Hecke algebra, the twisted group algebra of Kostant-Kumar and its dual. The study of properties of the stable bases boils down to combinatorics of the twisted group algebra.

We use Theorem 1.1 and the root polynomial method to find a restriction formula of stable bases. Such polynomials for cohomology and K-theory of flag varieties were studied by Billey, Graham, and Willems [7, 19, 51], and then generalized by Lenart-Zainoulline [25].

In this method, a formula of the Schubert classes in terms of classes of torus fixed points is determined by the coefficients of root polynomials (see Theorem 7.3). Generalizing the root polynomial method, we obtain our second main result. For the cotangent bundle of partial flag varieties in type A, this is also obtained by Rimányi, Tarasov and Varchenko using weight functions in [40, 41]. In a work in progress of Knutson-Zinn-Justin, K-theory stable basis is also studied from the point of view of integrable systems.

THEOREM 1.2 (Theorem 7.5). – *With* $a^+_{w,v}$ *(resp.* $K^\tau_{w,v}$*) defined in Lemma 5.2 (resp. §7.3), we have*

$$\begin{aligned} \mathrm{stab}_+(w)|_v &= q^{-\ell(w)/2} v(a^+_{w^{-1},v^{-1}}) \prod_{\alpha>0}(1-e^\alpha). \\ \mathrm{stab}_-(w)|_v &= q^{\ell(w)/2} K^\tau_{w,v}\Big[\prod_{\alpha>0, v^{-1}\alpha>0}(1-qe^{-\alpha})\Big]\cdot\Big[\prod_{\alpha>0, v^{-1}\alpha<0}(1-e^\alpha)\Big]. \end{aligned}$$

We also give some applications of the above theorems in § 8. We obtain the restriction formula for stable bases in $K_T(T^*G/P_J)$ in Theorem 8.6. This is done by showing that the stable bases coincide with the image of $\mathrm{stab}_\pm(w) \in K_T(T^*G/B)$ via the Lagrange correspondence from T^*G/B to T^*G/P_J.

As an application, we study the relation between K-theory of the Springer resolution and the principal series representations of p-adic groups.

In Theorem 9.4, we relate the T-equivariant K-theory of the Springer resolution to the bases in the Iwahori invariants of an unramified principle series [14, 37]. Such an isomorphism has been well-known, and has been studied by Lusztig [28] and Braverman-Kazhdan [9] from different points of view. However, the present paper explicitly identifies different bases from K-theory and from p-adic representation theory, which had been previously unknown. In particular, the K-theory stable basis is identified with the characteristic functions on certain semi-infinite orbits; the T-fixed-point basis is identified with the Casselman basis. Consequently, Theorem 1.2 also gives a closed formula for the transition matrix between these characteristic functions and the Casselman basis. A formula for the generating function of the matrix coefficients has been previously achieved by Reeder via a different approach [37, Proposition 5.2].

Under the isomorphism in Theorem 9.4, various structures from the p-adic representations, e.g., the intertwiners, Macdonald's formula for the spherical functions [29, 14], and the Casselman-Shalika formula for Whittaker functions [15], have meanings in terms of K-theory. Although this isomorphism is well-known, the K-theory interpretation of these structures is not well-documented. For the convenience of the readers, we also spell these out in § 9.

The results in the present paper also provide a way to study the transition matrix between stable bases and the Schubert classes of $K_T(G/B)$, as will be explained in a future publication. Such transition matrix is related with [24] which studies the (spherical) Whittaker functions of p-adic groups. It is also shadowed by the two geometric realizations of the affine Hecke algebras [5] and the periodic modules [9, 27, 28]. The cohomological analogue of this transition matrix, i.e., the transition matrix from cohomological Schubert classes to the cohomological stable bases, is of independent interest. It was proved in [3] that cohomological stable bases can be identified with Chern-Schwartz-MacPherson classes. In [2], Aluffi and Mihalcea raised a positivity conjecture concerning this matrix. Recently, the non-equivariant case is proved in [3], in which the cohomological stable basis played an important role.

Another future application is a relation between the K-theory stable basis and the localizations of baby Verma modules in modular representations of Lie algebras [6]. This subject is interesting on its own, and is related to the wall-crossings of stable bases. We will briefly discuss this in § 6.4 and postpone the details in a separate paper.

The structure of this paper is as follows: In Section 2 we recall the definition of stable bases. In Section 3 we recall rigidity in K-theory. In Section 4 we define the two convolution actions by the Hecke algebra, and compute their effects on the stable bases. In Section 5 we recall an algebraic description of affine Hecke algebra in terms of Kostant-Kumar's twisted group algebra. In Section 6 we give an algebraic description of stable bases. In Section 7 we define the root polynomials for Hecke algebra and in Theorem 7.3; we relate some coefficients of Hecke algebra with root polynomials, and obtain the restriction formula in Theorem 7.5. In Section 8 we give an algebraic description of the stable bases for partial flag varieties. In Section 9 we talk about the relation between the K-theory stable basis and the Casselman basis in p-adic representations.

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Notations

Throughout this paper, G is a complex reductive group with maximal torus A, a Borel subgroup B and its opposite Borel subgroup B^-. Let Λ be the group of characters of A. Let Σ be the set of roots of G. Let Σ^+ be the roots in B, which is the set of positive roots, and let Σ^- be the negative roots. For each root α, we use $\alpha > 0$ or $\alpha < 0$ to say that it is positive or negative. Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be the set of simple roots, and $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+}\alpha$. Let $\geq$ denote the Bruhat order in the Weyl group W.

Let G/B be the complete flag variety. The maximal torus A acts on G/B by left multiplication. Hence, it also acts on the cotangent bundle T^*G/B and the tangent bundle $T(G/B)$. Let $T = A\times\mathbb{C}^*$. We denote the standard representation of $\mathbb{C}^*$ by $q^{\frac{1}{2}}$. The factor $\mathbb{C}^* \subseteq T$ acts trivially on G/B, and dilates the fibers of T^*G/B by the character q^{-1}. The T-fixed points of T^*G/B and G/B are both bijective to W, the Weyl group of G.

For any $J \subset \Pi$, let $W_J \subset W$ be the corresponding subgroup, and $P_J \supset B$ be the corresponding parabolic subgroup. Let $\Sigma_J = \{\alpha \in \Sigma | s_\alpha \in W_J\}$, and similarly define $\Sigma_J^{\pm}$.

Let w_0 be the longest element of W, and w_0^J the longest element of W_J. We write as W^J the set of minimal length representatives of elements in W_J. For a reduced decomposition $w = s_{i_1} \cdots s_{i_l}$, define

$$\Sigma_w := w\Sigma^- \cap \Sigma^+ = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \dots, s_{i_1}s_{i_2}\cdots s_{i_{l-1}}(\alpha_{i_l})\}.$$

We will frequently use the identities

$$w_0\Sigma^- = \Sigma^+, \quad s_i\Sigma^- = (\{\alpha_i\} \sqcup \Sigma^-)\backslash\{-\alpha_i\}, \quad v(\Sigma^+\backslash\Sigma_J^+) = \Sigma^+\backslash\Sigma_J^+, \quad \text{for } v \in W_J.$$

Let $R = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$, $S = R[\Lambda]$, then $S \cong K_T(\mathbb{C})$, and let $Q = \mathrm{Frac}(S)$ be its field of fractions.

2. Stable bases of T^*G/B

In this section, we recall Maulik and Okounkov's definition of the K-theoretic stable bases for the Springer resolution.

Recall that $\mathbb{C}^*$ act on the cotangent fiber of T^*G/B by a non-trivial character q^{-1}, where $q^{-\frac{1}{2}}$ corresponds to the standard representation of the torus $\mathbb{C}^*$. Therefore $K_{\mathbb{C}^*}(\mathbb{C}) = R = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$, $K_T(\mathbb{C}) \cong S = R[\Lambda]$. For any T-invariant vector space V, let

$$\bigwedge\nolimits^{\bullet} V := \sum_k (-1)^k \bigwedge\nolimits^k V^\vee = \prod (1 - e^{-\alpha}) \in K_T(\mathbb{C}),$$

where the product is over all Lie(T)-weights in V counted with multiplicities.

The set of T-fixed points of T^*G/B is discrete and is in one-to-one correspondence with W. For an element $w \in W$, the corresponding fixed point is still denoted by w. Let ι_w be the inclusion of the fixed point $w \in W$ into T^*G/B. By Thomason's theorem [50], $K_T(T^*G/B) \otimes_S Q$ is a finite dimensional Q-vector space with basis $\{\iota_{w*}1 \mid w \in W\}$. This basis is referred to as the fixed-point basis. For any $\mathscr{F} \in K_T(T^*G/B)$, let $\mathscr{F}|_w$ denote the restriction of $\mathscr{F}$ to the fixed point $w \in T^*G/B$. Let $(\cdot, \cdot)$ denote the K-theoretic pairing on $K_T(T^*G/B)$, which can be defined using localization as follows:

$$(\mathscr{F}_1, \mathscr{F}_2) = \sum_w \frac{\mathscr{F}_1|_w \otimes \mathscr{F}_2|_w}{\prod_{\alpha>0}(1 - e^{w\alpha})(1 - qe^{-w\alpha})}, \quad \mathscr{F}_1, \mathscr{F}_2 \in K_T(T^*G/B).$$

2.1. The definition of stable bases

Let $\Lambda^\vee$ be the lattice of cocharacters of A. The Lie algebra of the maximal compact subgroup of A is $\mathfrak{a}_\mathbb{R} = \Lambda^\vee \otimes_\mathbb{Z} \mathbb{R}$. The A-weights occurring in the normal bundle to $(T^*G/B)^A$ coincide with the usual roots for G. The root hyperplanes $\alpha_i^\perp$ partition $\mathfrak{a}_\mathbb{R}$ into finitely many chambers

$$\mathfrak{a}_\mathbb{R} \setminus \bigcup \alpha_i^\perp = \coprod \mathfrak{C}_i.$$

Let $\mathfrak{C}$ be a chamber. For any cocharacter $\sigma \in \mathfrak{C}$, the stable leaf of $w \in W$ is defined as

$$\mathrm{Leaf}_\mathfrak{C}(w) = \left\{ x \in T^*G/B \mid \lim_{z\to 0} \sigma(z) \cdot x = w \right\}.$$

Note that the limit, and hence $\mathrm{Leaf}_{\mathfrak{C}}(w)$ per se, is independent of the choice of σ. Define a partial order on W as follows:

$$w \preceq_{\mathfrak{C}} v \quad \text{if} \quad \overline{\mathrm{Leaf}_{\mathfrak{C}}(v)} \cap w \neq \emptyset.$$

Then the order determined by the positive (resp. negative) chamber is the same as the Bruhat order (resp. the opposite Bruhat order). Define the slope of a fixed point v by

$$\mathrm{Slope}_{\mathfrak{C}}(v) = \bigcup_{w \preceq_{\mathfrak{C}} v} \mathrm{Leaf}_{\mathfrak{C}}(w).$$

Let $+$ denote the chamber such that all roots in Σ^+ are positive on it, and $-$ the opposite chamber. In particular, $\mathrm{Leaf}_+(w) = T^*_{BwB/B}G/B$, and $\mathrm{Leaf}_-(w) = T^*_{B^-wB/B}G/B$.

Definition 2.1. – A *polarization* $T^{\frac{1}{2}} \in K_T(T^*G/B)$ is the choice of a Lagrangian subbundle of the tangent bundle $T(T^*G/B) \in K_T(T^*G/B)$, so that

$$T^{\frac{1}{2}} + q^{-1}(T^{\frac{1}{2}})^\vee = T(T^*G/B)$$

as T-equivariant vector bundles.

For any polarization $T^{\frac{1}{2}}$, there is an opposite one defined as

$$T^{\frac{1}{2}}_{\mathrm{opp}} = q^{-1}(T^{\frac{1}{2}})^\vee.$$

There are two natural polarizations: $T(G/B)$ and T^*G/B which are opposite to each other. Let N_w denote the normal bundle of T^*G/B at $w \in W$.

Any chamber $\mathfrak{C}$ determines a decomposition $N_w = N_{w,+} \oplus N_{w,-}$ into A-weight spaces which are positive and negative with respect to $\mathfrak{C}$ respectively. For any polarization $T^{\frac{1}{2}}$, denote $N_w^{\frac{1}{2}}$ by $N_w \cap T^{1/2}|_w$. Similarly, we have $N_{w,+}^{\frac{1}{2}}$ and $N_{w,-}^{\frac{1}{2}}$. In particular, $N_{w,-} = N_{w,-}^{\frac{1}{2}} \oplus q^{-1}(N_{w,+}^{\frac{1}{2}})^\vee$. Consequently, we have

$$N_{w,-} - N_w^{\frac{1}{2}} = q^{-1}(N_{w,+}^{\frac{1}{2}})^\vee - N_{w,+}^{\frac{1}{2}}$$

as virtual vector bundles. The determinantal bundle of the virtual bundle $N_{w,-} - N_w^{\frac{1}{2}}$ is a complete square, hence, its square root will be denoted by

$$\left(\frac{\det N_{w,-}}{\det N_w^{\frac{1}{2}}}\right)^{\frac{1}{2}}.$$

Recall that for any weight λ, let $\mathcal{L}_\lambda$ be the associated line bundle on G/B. Pulling it back to T^*G/B via the projection map, we get the corresponding line bundle on T^*G/B, denoted by $\mathcal{O}(\lambda)$. The assignment associating $\lambda \in \Lambda$ to $\mathcal{O}(\lambda) \in \mathrm{Pic}_A(T^*G/B)$ induced an isomorphsim. For every rational weight $\lambda \in P \otimes_{\mathbb{Z}} \mathbb{Q}$, let $\mathcal{O}(\lambda)$ denote the corresponding element in $\mathrm{Pic}_A(T^*G/B) \otimes_{\mathbb{Z}} \mathbb{Q}$. We say λ, or the corresponding $\mathcal{O}(\lambda)$ is sufficiently general if

$$\lambda - w\lambda \notin \Lambda \text{ for any } w \in W. \tag{1}$$

For a Laurent polynomial $f := \sum_\mu f_\mu e^\mu \in K_T(\mathrm{pt})$, where $e^\mu \in K_A(\mathrm{pt})$ and $f_\mu \in R$, we define its *Newton Polygon*, denoted by $\deg_A f$ to be

$$\deg_A f = \text{Convex hull}(\{\mu | f_\mu \neq 0\}) \subset \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}.$$

We use the following theorem as the definition of K-theoretic stable bases.

THEOREM 2.2. – [36, §9.1] *For any chamber $\mathfrak{C}$, a sufficiently general $\mathscr{L}$, and a polarization $T^{1/2}$, there exists a unique map of S-modules*

$$\operatorname{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathscr{L}}: K_T((T^*G/B)^A) \to K_T(T^*G/B)$$

such that for any $w \in W$, $\Gamma = \operatorname{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathscr{L}}(w)$ satisfies:

1. *(support)* $\operatorname{supp}\Gamma \subset \operatorname{Slope}_{\mathfrak{C}}(w)$;
2. *(normalization)* $\Gamma|_w = (-1)^{\operatorname{rank} N_{w,+}^{\frac{1}{2}}} \left(\frac{\det N_{w,-}}{\det N_w^{\frac{1}{2}}}\right)^{\frac{1}{2}} \mathcal{O}_{\operatorname{Leaf}_{\mathfrak{C}}(w)}|_w$;
3. *(degree)* $\deg_A\left(\Gamma|_v \otimes \mathscr{L}|_w\right) \subseteq \deg_A\left((\operatorname{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathscr{L}}(v) \otimes \mathscr{L})|_v\right)$, *for any $v \prec_{\mathfrak{C}} w$,*

where w in $\operatorname{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathscr{L}}(w)$ *is the unit in $K_T^*(w)$.*

2.2. Comments and examples

REMARK 2.3. – (1). From the characterization, the transition matrix from $\{\operatorname{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathscr{L}}(w) \mid w \in W\}$ to the fixed point basis is a triangular matrix with nonzero diagonal entries. Hence, after localization, $\{\operatorname{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathscr{L}}(w) \mid w \in W\}$ form a basis, which is called the *stable bases*.

(2). It is shown in [45, Proposition 1] that via the K-theory pairing, $\{\operatorname{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathscr{L}}(w) | w \in W\}$ and $\{\operatorname{stab}_{-\mathfrak{C},T_{\mathrm{opp}}^{\frac{1}{2}},\mathscr{L}^{-1}}(w) | w \in W\}$ are dual to each other, i.e.,

$$\left(\operatorname{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathscr{L}}(v),\ \operatorname{stab}_{-\mathfrak{C},T_{\mathrm{opp}}^{\frac{1}{2}},\mathscr{L}^{-1}}(w)\right) = \delta_{v,w}.$$

(3). Let the alcoves of $\mathfrak{g} = \operatorname{Lie} G$ be the connected components of $(\mathfrak{a}_{\mathbb{R}})^* \setminus H_{\alpha,n}$, with $H_{\alpha,n} = \{\lambda \in \mathfrak{a}_{\mathbb{R}}^* = (\operatorname{Lie} A_{\mathbb{R}})^* \mid (\lambda, \alpha^\vee) = n\}$. Then $\operatorname{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathscr{L}}$ stays the same if $\mathscr{L}$ is in the same alcove. In other words, $\operatorname{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathscr{L}}$ depends on $\mathscr{L}$ locally.

(4). By the uniqueness property, we have

$$\operatorname{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathscr{L}\otimes\mathcal{O}(\lambda)}(w) = e^{-w\lambda}[\mathcal{O}(\lambda)] \otimes \operatorname{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathscr{L}}(w),$$

where λ is an integral weight of $\mathfrak{g}$. Combining with part (3), it is sufficient to study stable bases for those alcoves near $0 \in \operatorname{Lie} A_{\mathbb{R}}^*$.

(5). Let us explain why Condition (1) on $\mathscr{L} = \mathcal{O}(\lambda)$ is imposed. Suppose $\mu := v\lambda - u\lambda \in \Lambda$ for some $v \prec_{\mathfrak{C}} u$, and suppose we already have a map $\operatorname{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathscr{L}}$ as in Theorem 2.2. For any Laurent polynomial $f(q)$, we define a new map $\operatorname{stab}'_{\mathfrak{C},T^{\frac{1}{2}},\mathscr{L}}$ as follows:

$$\operatorname{stab}'_{\mathfrak{C},T^{1/2},\mathscr{L}}(y) = \begin{cases} \operatorname{stab}_{\mathfrak{C},T^{1/2},\mathscr{L}}(y), & \text{if } y \neq u; \\ \operatorname{stab}_{\mathfrak{C},T^{1/2},\mathscr{L}}(u) + f(q)e^{\mu} \operatorname{stab}_{\mathfrak{C},T^{1/2},\mathscr{L}}(v), & \text{if } y = u. \end{cases}$$

Then the new map also satisfies the conditions in Theorem 2.2, which contradicts with the uniqueness. We check this as follows: The first two conditions are obvious. For the degree condition, there are four cases: $u \prec_{\mathfrak{C}} y$, $y = u$, $y \prec_{\mathfrak{C}} u$, and y is not comparable with u. The last two cases are easy to check, so we only consider the first two.

— Case $u \prec_{\mathfrak{C}} y$.
In this case, $\mathrm{stab}'_{\mathfrak{C},T^{1/2},\mathscr{L}}(y) = \mathrm{stab}_{\mathfrak{C},T^{1/2},\mathscr{L}}(y)$. If $w \neq u$, then $\mathrm{stab}'_{\mathfrak{C},T^{1/2},\mathscr{L}}(w) = \mathrm{stab}_{\mathfrak{C},T^{1/2},\mathscr{L}}(w)$. Thus the condition is satisfied. If $w = u$, then $\mathrm{stab}'_{\mathfrak{C},T^{1/2},\mathscr{L}}(u)|_u = \mathrm{stab}_{\mathfrak{C},T^{1/2},\mathscr{L}}(u)|_u$ because $v \prec_{\mathfrak{C}} u$. So the condition is also satisfied.

— Case $y = u$.
In this case, $\mathrm{stab}'_{\mathfrak{C},T^{1/2},\mathscr{L}}(w) = \mathrm{stab}_{\mathfrak{C},T^{1/2},\mathscr{L}}(w)$, for any $w \prec_{\mathfrak{C}} u$. By definition,

$$\deg_A\left(\mathrm{stab}'_{\mathfrak{C},T^{1/2},\mathscr{L}}(u)|_w\right) + u\lambda = \deg_A\left(\mathrm{stab}_{\mathfrak{C},T^{1/2},\mathscr{L}}(u)|_w + f(q)e^{\mu}\,\mathrm{stab}_{\mathfrak{C},T^{1/2},\mathscr{L}}(v)|_w\right) + u\lambda.$$

Since

$$\deg_A\left(\mathrm{stab}_{\mathfrak{C},T^{1/2},\mathscr{L}}(u)|_w\right) + u\lambda \subset \deg_A\left(\mathrm{stab}_{\mathfrak{C},T^{1/2},\mathscr{L}}(w)|_w\right) + w\lambda$$

and

$$\deg_A\left(f(q)e^{\mu}\,\mathrm{stab}_{\mathfrak{C},T^{1/2},\mathscr{L}}(v)|_w\right) + u\lambda \subset \deg_A\left(\mathrm{stab}_{\mathfrak{C},T^{1/2},\mathscr{L}}(w)|_w\right) + w\lambda,$$

therefore,

$$\deg_A\left(\mathrm{stab}'_{\mathfrak{C},T^{1/2},\mathscr{L}}(u)|_w\right) + u\lambda \subset \deg_A\left(\mathrm{stab}'_{\mathfrak{C},T^{1/2},\mathscr{L}}(w)|_w\right) + w\lambda.$$

EXAMPLE 2.4. – Let us study the easiest example in which $G = \mathrm{SL}(2,\mathbb{C})$, and hence $G/B = \mathbb{P}^1$. Let α be the unique positive root. Let 0 and ∞ denote the two fixed points, which correspond to 1 and s_α in the Weyl group. Then $S = \mathbb{Z}[q^{\pm\frac{1}{2}}][e^{\pm\frac{\alpha}{2}}]$, $T_0\mathbb{P}^1$ has weight $e^{-\alpha}$, $T_0^*\mathbb{P}^1$ has weight $q^{-1}e^{\alpha}$, $T_\infty\mathbb{P}^1$ has weight e^{α}, and $T_\infty^*\mathbb{P}^1$ has weight $q^{-1}e^{-\alpha}$. The condition (1) on λ is equivalent to $\lambda \notin \frac{\mathbb{Z}}{4}\alpha$. The alcoves are $(\frac{n\alpha}{2}, \frac{(n+1)\alpha}{2})$, where $n \in \mathbb{Z}$.

Let us pick the negative chamber, and fix the polarization to be $T^*\mathbb{P}^1$. Then $\mathrm{Leaf}(\infty) = T_\infty^*\mathbb{P}^1$, and $\mathrm{Leaf}(0) = \mathbb{P}^1 \setminus \{\infty\}$. Thus $0 > \infty$. It is easy to see that for any slope $\mathcal{O}(\lambda)$,

$$\mathrm{stab}(\infty) = -q^{\frac{1}{2}}e^{\alpha}[\mathcal{O}_{T_\infty^*\mathbb{P}^1}].$$

By Remark 2.3.(4), we only need to study the case for a fixed $\lambda \in (0, \frac{\alpha}{2})$. By the support and normalization conditions in Theorem 2.2, we get

$$\mathrm{stab}(0) = [\mathcal{O}_{\mathbb{P}^1}] + a[\mathcal{O}_{T_\infty^*\mathbb{P}^1}], \quad a \in Q.$$

However, by the support condition, $(\mathrm{stab}(0), [\mathcal{O}_{\mathbb{P}^1}]) \in S = \mathbb{Z}[e^{\pm\frac{\alpha}{2}}][q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$. Since $([\mathcal{O}_{\mathbb{P}^1}], [\mathcal{O}_{\mathbb{P}^1}]) \in S$ and $([\mathcal{O}_{T_\infty^*\mathbb{P}^1}], [\mathcal{O}_{\mathbb{P}^1}]) = 1$, we get $a \in S$.

We have

$$\mathrm{stab}(0)|_\infty = 1 - qe^{\alpha} + a(1 - e^{-\alpha}).$$

Since $\deg_A \mathrm{stab}(\infty)|_\infty = [0, \alpha]$, we have

$$\deg_A(1 - qe^{\alpha} + a(1 - e^{-\alpha})) \subset [s_\alpha\lambda - \lambda, s_\alpha\lambda - \lambda + \alpha] = [-(\lambda, \alpha^\vee)\alpha, -(\lambda, \alpha^\vee)\alpha + \alpha].$$

There are two cases.

(1) *Case* $\lambda \in (0, \frac{\alpha}{4})$. – In this case,

$$\deg_A(1 - qe^{\alpha} + a(1 - e^{-\alpha})) \subset [-(\lambda, \alpha^\vee)\alpha, -(\lambda, \alpha^\vee)\alpha + \alpha] \subset (-\frac{\alpha}{2}, \alpha).$$

Since $a \in S$, we get $a = qe^{\alpha}$.

(2) *Case* $\lambda \in (\frac{\alpha}{4}, \frac{\alpha}{2})$. – In this case,

$$\deg_A(1 - qe^{\alpha} + a(1 - e^{-\alpha})) \subset [-(\lambda, \alpha^{\vee})\alpha, -(\lambda, \alpha^{\vee})\alpha + \alpha] \subset (-\alpha, \frac{\alpha}{2}).$$

Then a must be of the form $a_1 e^{\alpha} + a_2 e^{\frac{\alpha}{2}}$ for some $a_i \in R$. Plugging into $1 - qe^{\alpha} + a(1 - e^{-\alpha})$, we get $1 - qe^{\alpha} + a_1 e^{\alpha} + a_2 e^{\frac{\alpha}{2}} + a_1 + a_2 e^{-\frac{\alpha}{2}}$. Since $\deg_A(1 - qe^{\alpha} + a_1 e^{\alpha} + a_2 e^{\frac{\alpha}{2}} + a_1 + a_2 e^{-\frac{\alpha}{2}}) \subset (-\alpha, \frac{\alpha}{2})$, we get $a_1 = q, a_2 = 0$. Thus $a = qe^{\alpha}$.

To conclude, when $\lambda_0 \in (0, \frac{\alpha}{2})$, we have

$$\mathrm{stab}_{-,T^*\mathbb{P}^1,\lambda_0}(0) = [\mathcal{O}_{\mathbb{P}^1}] + qe^{\alpha}[\mathcal{O}_{T^*_{\infty}\mathbb{P}^1}],$$

and

$$\mathrm{stab}_{-,T^*\mathbb{P}^1,\lambda_0}(\infty) = -q^{\frac{1}{2}} e^{\alpha}[\mathcal{O}_{T^*_{\infty}\mathbb{P}^1}].$$

In general, when $\lambda_n \in (\frac{n\alpha}{2}, \frac{(n+1)\alpha}{2})$, then $\lambda - \frac{n}{2}\alpha \in (0, \frac{\alpha}{2})$. Thus for $w = 1, s_{\alpha}$,

$$\mathrm{stab}_{-,T^*\mathbb{P}^1,\lambda_n}(w) = e^{-\frac{n}{2}w\alpha}[\mathcal{O}(\frac{n}{2}\alpha)] \otimes \mathrm{stab}(w).$$

For the positive chamber, the opposite polarization $T\mathbb{P}^1$ and the opposite slope $\lambda_{-1} \in (-\frac{\alpha}{2}, 0)$, we have

$$\mathrm{stab}_{+,T\mathbb{P}^1,\lambda_{-1}}(0) = [\mathcal{O}_{T^*_0\mathbb{P}^1}],$$

and

$$\mathrm{stab}_{+,T\mathbb{P}^1,\lambda_{-1}}(\infty) = -q^{-\frac{1}{2}} e^{-\alpha}[\mathcal{O}_{\mathbb{P}^1}] + \left(-q^{\frac{1}{2}} e^{-2\alpha} + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})e^{-\alpha}\right)[\mathcal{O}_{T^*_0\mathbb{P}^1}].$$

It is easy to check that

$$\left(\mathrm{stab}_{+,T\mathbb{P}^1,\lambda_{-1}}(v), \mathrm{stab}_{-,T^*\mathbb{P}^1,\lambda_0}(w)\right) = \delta_{v,w}.$$

3. Rigidity

In this section, we introduce rigidity, and make the normalization axiom for stable bases more explicit.

In equivariant cohomology, degree counting is a very useful method in computations. In equivariant K-theory, this method is often replaced by a rigidity argument. If a T-equivariant sheaf $\mathcal{F}$ has compact support, then the equivariant holomorphic Euler characteristic $\chi(\mathcal{F}) \in K_T(\mathrm{pt})$ is a Laurent polynomial, which, in general, is difficult to calculate. However, the calculation is simplified if $\chi(\mathcal{F})$ depends on few or even no equivariant variables. This property is known as rigidity. One standard way to prove such a property is to use the following elementary observation: for any $p(z) \in \mathbb{C}[z^{\pm}]$,

$$p(z) \text{ is bounded as } z^{\pm 1} \to \infty \Longleftrightarrow p = \text{constant}. \tag{2}$$

For applications of this observation, see [36, §2.4].

We will need the following lemma.

Lemma 3.1. – *Suppose* $f = \sum_{\mu \in I} a_{\mu} e^{\mu} \in S$ *is a Laurent polynomial, with* $\mu \in (\mathrm{Lie}\, A)^*$ *and* $0 \neq a_{\mu} \in R$.

1. *There exists* $\xi \in \mathrm{Lie}\, A_{\mathbb{R}}$ *in the positive chamber, such that for any* $\mu \in I$, $(\xi, \mu) \in \mathbb{Z}$, *and furthermore,* $(\xi, \mu) \neq (\xi, \mu')$ *for any* $\mu \neq \mu'$ *in* I.
2. *Moreover, if both of the limits* $\lim\limits_{t\to\pm\infty} f(t\xi)$ *are bounded, and one of them equals* $g(q)$ *for some* $g(q) \in R$, *then* $f = g(q)$.

Proof. – The existence of such ξ follows easily from the fact that f has only finitely many terms. The second part follows from the equivalence in Equation (2) in the paragraph before this lemma. □

With Lemma 3.1 we can define the following two scalars

$$max_{\xi} f = \max_{\mu\in I}(\mu, \xi) \quad \text{and} \quad min_{\xi} f = \min_{\mu\in I}(\mu, \xi).$$

For any $v \in W$, we denote $q^{\ell(v)}$ simply by q_v.

By Theorem 2.2, we have the following.

Lemma 3.2. – *For* $v, w \in W$,

1. $\mathrm{stab}_{-,T^*G/B,\mathcal{L}}(v)|_w = 0$, *unless* $w \geq v$;
2. $\mathrm{stab}_{-,T^*G/B,\mathcal{L}}(v)|_v = q_v^{\frac{1}{2}} \prod_{\alpha\in\Sigma^-\cap v\Sigma^-}(1 - qe^{\alpha}) \cdot \prod_{\alpha\in\Sigma^+\cap v\Sigma^-}(1 - e^{\alpha})$;
3. $\mathrm{stab}_{+,T(G/B),\mathcal{L}}(v)|_w = 0$, *unless* $w \leq v$;
4. $\mathrm{stab}_{+,T(G/B),\mathcal{L}}(v)|_v = q_v^{-\frac{1}{2}} \prod_{\alpha\in\Sigma^-\cap v\Sigma^+}(q - e^{\alpha}) \cdot \prod_{\alpha\in\Sigma^+\cap v\Sigma^+}(1 - e^{\alpha})$.

Proof. – (1) and (3) follow from the support condition.

Now we prove (2). For the negative chamber, we have

$$\begin{aligned}
A - \text{weights in } N_{v,+} &= \{e^{-v\beta} | \beta > 0, v\beta > 0\} \cup \{q^{-1}e^{v\beta} | \beta > 0, v\beta < 0\},\\
A - \text{weights in } N_{v,-} &= \{e^{-v\beta} | \beta > 0, v\beta < 0\} \cup \{q^{-1}e^{v\beta} | \beta > 0, v\beta > 0\},\\
A - \text{weights in } N_v^{\frac{1}{2}} &= \{q^{-1}e^{v\beta} | \beta > 0\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{stab}_{-,T^*G/B,\mathcal{L}}(v)|_v &= (-1)^{\mathrm{rank}\, N_{v,+}^{\frac{1}{2}}} \left(\frac{\det N_{v,-}}{\det N_v^{\frac{1}{2}}}\right)^{\frac{1}{2}} \mathcal{O}_{\mathrm{Leaf}_{\mathfrak{C}}(v)}|_v\\
&= (-1)^{\ell(v)} \left(\frac{\prod_{\beta>0, v\beta<0} e^{-v\beta} \prod_{\beta>0, v\beta>0} q^{-1}e^{v\beta}}{\prod_{\beta>0} q^{-1}e^{v\beta}}\right)^{\frac{1}{2}}\\
&\quad \times \prod_{\beta>0, v\beta<0}(1 - e^{v\beta}) \prod_{\beta>0, v\beta>0}(1 - qe^{-v\beta})\\
&\overset{\sharp_1}{=} (-1)^{\ell(v)} q_v^{\frac{1}{2}} \prod_{\beta>0, v\beta<0}(e^{-v\beta} - 1) \prod_{\beta>0, v\beta>0}(1 - qe^{-v\beta}).
\end{aligned}$$

We comment on the proof of the equality $\sharp_1$. In $\left(\frac{\prod_{\beta>0,v\beta<0} e^{-v\beta}\prod_{\beta>0,v\beta>0} q^{-1}e^{v\beta}}{\prod_{\beta>0} q^{-1}e^{v\beta}}\right)^{\frac{1}{2}}$, the factors involving powers of e can be regrouped into two copies of $\Sigma^+\cap v\Sigma^-$. Equality $\sharp_1$ then follows from the identity $e^{-\alpha}(1-e^{\alpha}) = e^{-\alpha}-1$.

(4) follows from a similar argument as that of (2). □

Lemma 3.2 implies that $\left(\mathrm{stab}_{+,T(G/B),\mathscr{L}}(v), \mathrm{stab}_{-,T^*G/B,\mathscr{L}}(v)\right) = 1$, keeping in mind that

$$\prod_{\alpha\in\Sigma^+\cap v\Sigma^-}(1-e^{\alpha})\cdot\prod_{\alpha\in\Sigma^-\cap v\Sigma^-}(1-qe^{\alpha})\prod_{\alpha\in\Sigma^-\cap v\Sigma^+}(q-e^{\alpha})\cdot\prod_{\alpha\in\Sigma^+\cap v\Sigma^+}(1-e^{\alpha}) = \bigwedge\nolimits^{\bullet} T_v(T^*G/B).$$

Choosing $\xi\in\operatorname{Lie}A$ as in Lemma 3.1, regarding the Laurent polynomials $\mathrm{stab}_{+,T(G/B),\mathscr{L}}(v)|_v$ and $\mathrm{stab}_{-,T^*G/B,\mathscr{L}}(v)|_v$, we have

$$\max_{\xi}(\mathrm{stab}_{+,T(G/B),\mathscr{L}}(v)|_v) = (\xi, \sum_{\beta>0,v\beta>0} v\beta), \tag{3}$$

$$\min_{\xi}(\mathrm{stab}_{+,T(G/B),\mathscr{L}}(v)|_v) = (\xi, \sum_{\beta>0,v\beta<0} v\beta), \tag{4}$$

$$\max_{\xi}(\mathrm{stab}_{-,T^*G/B,\mathscr{L}}(v)|_v) = (\xi, \sum_{\beta>0,v\beta<0} -v\beta), \tag{5}$$

$$\min_{\xi}(\mathrm{stab}_{-,T^*G/B,\mathscr{L}}(v)|_v) = (\xi, \sum_{\beta>0,v\beta>0} -v\beta). \tag{6}$$

Let ρ be half sum of all the positive roots. For any simple root α, we have

$$max_{\xi}(\mathrm{stab}_{+,T(G/B),\mathscr{L}}(v)|_v) + \max_{\xi}(\mathrm{stab}_{-,T^*G/B,\mathscr{L}}(v)|_v) = (\xi, 2\rho), \tag{7}$$

$$\min_{\xi}(\mathrm{stab}_{+,T(G/B),\mathscr{L}}(v)|_v) + \min_{\xi}(\mathrm{stab}_{-,T^*G/B,\mathscr{L}}(v)|_v) = -(\xi, 2\rho), \tag{8}$$

$$\max_{\xi}(\mathrm{stab}_{+,T(G/B),\mathscr{L}}(vs_\alpha)|_{vs_\alpha}) + \max_{\xi}(\mathrm{stab}_{-,T^*G/B,\mathscr{L}}(v)|_v) + (\xi, v\alpha) = (\xi, 2\rho), \tag{9}$$

$$\min_{\xi}(\mathrm{stab}_{+,T(G/B),\mathscr{L}}(vs_\alpha)|_{vs_\alpha}) + \min_{\xi}(\mathrm{stab}_{-,T^*G/B,\mathscr{L}}(v)|_v) + (\xi, v\alpha) = -(\xi, 2\rho). \tag{10}$$

4. The two Hecke actions

In this section, we compute the action of the affine Hecke algebra on stable bases.

4.1. Reminder on the Demazure-Lusztig operators

Let $Z = T^*G/B\times_{\mathscr{N}} T^*G/B$ be the Steinberg variety, where $\mathscr{N}$ is the nilpotent cone. Let $\mathbb{H}$ be the affine Hecke algebra (see Chapter 7 in [16]). There is an isomorphism

$$\mathbb{H}\simeq K_{G\times\mathbb{C}^*}(Z) \tag{11}$$

defined as follows. The diagonal G-orbits on $G/B\times G/B$ are indexed by the Weyl group. For each simple root $\alpha\in\Pi$, let Y_α° be the orbit corresponding to the simple reflection s_α, whose closure is

$$Y_\alpha := \overline{Y_\alpha^\circ} = G/B\times_{\mathscr{P}_\alpha} G/B,$$

where $\mathscr{P}_\alpha = G/P_\alpha$ and P_α is the minimal parabolic subgroup corresponding to α. Therefore, only two kinds of torus fixed points lie in Y_α: (w,w) and (w, ws_α). Let Ω_α be the sheaf of differentials along the first projection from Y_α to G/B. Let $T^*_{Y_\alpha} := T^*_{Y_\alpha}(G/B \times G/B)$ be the conormal bundle to Y_α, and consider Ω_α as a sheaf on $T^*_{Y_\alpha}(G/B \times G/B)$ via pullback. Then, $[\Omega_\alpha] = [\pi_2^* \mathcal{O}(\alpha)]$ as a sheaf on $T^*_{Y_\alpha}$, where π_1 and π_2 are the two projections from $T^*_{Y_\alpha}(G/B \times G/B)$ to T^*G/B respectively. The isomorphism (11) sends the simple generator τ_α to $-[\mathcal{O}_\Delta] - [\Omega_\alpha]$, where $\mathcal{O}_\Delta$ is the structure sheaf of the diagonal component of the Steinberg variety Z, and it sends $e^\lambda \in X^*(A)$ to $[\mathcal{O}_\Delta(\lambda)]$ (see [39, Prop. 6.1.5]). This morphism is conjugate to the one in the *loc. cit.* by the sheaf $\mathcal{O}(\rho)$, and it is related to the one used in [16] by an Iwahori-Matsumoto involution (without signs). (1)

There is a natural embedding of the convolution algebras $K_{G\times\mathbb{C}^*}(T^*G/B \times_{\mathcal{N}} T^*G/B)$ into $K_{A\times\mathbb{C}^*}(T^*G/B \times_{\mathcal{N}} T^*G/B)$, which in turn acts on $K_T(T^*G/B)$ by convolution from left and from right (see [16, §5.2.20]). The left action is given by

$$D_\alpha(\mathcal{F}) := \pi_{1*}(\pi_2^* \mathcal{F} \otimes \Omega_\alpha),$$

where $\mathcal{F} \in K_T(T^*G/B)$. The pushforward is understood as derived pushforward in equivariant K-theory. Similarly, the right action is

$$D'_\alpha(\mathcal{F}) := \pi_{2*}(\pi_1^* \mathcal{F} \otimes \Omega_\alpha).$$

For $\mathcal{F} \in K_T(T^*G/B)$, the left (resp. right) actions of $\tau_w \in \mathbb{H}$ on $\mathcal{F}$ is denoted by $T_w(\mathcal{F})$ (resp. and $T'_w(\mathcal{F})$). By definition, $D_\alpha = -T_\alpha - 1$ and $D'_\alpha = -T'_\alpha - 1$.

4.2. Hecke algebra action D_α on $\mathrm{stab}_{-,T^*G/B,\mathcal{L}}$

We will need the following lemma, which can be proved easily by calculating the weights.

LEMMA 4.1. – *For any $v \in W$ and simple root α, with $X = T^*G/B$, we have*

$$\textstyle\bigwedge^\bullet(T_{(v,v)}T^*_{Y_\alpha}) = \bigwedge^\bullet(T_vX)\frac{1-e^{v\alpha}}{1-qe^{-v\alpha}},$$
$$\textstyle\bigwedge^\bullet(T_{(v,vs_\alpha)}T^*_{Y_\alpha}) = \bigwedge^\bullet(T_{(vs_\alpha,v)}T^*_{Y_\alpha}) = \bigwedge^\bullet(T_vX)\frac{1-e^{-v\alpha}}{1-qe^{-v\alpha}}.$$

Among the alcoves for $\mathfrak{g}$, there is a fundamental one defined by

$$\nabla := \{\lambda \in (\mathrm{Lie}\,A)^*_{\mathbb{R}} \mid 0 < (\lambda, \alpha^\vee) < 1, \text{for all positive roots } \alpha\}.$$

If we pick the slope $\mathcal{L} \in \nabla$, we have the following lemma

LEMMA 4.2. – *Given $v > w \in W$ under the Bruhat order, then for any $\xi \in \mathrm{Lie}(A)^*$ in the positive chamber, $(\xi, \mathcal{L}|_v - \mathcal{L}|_w) < 0$.*

Proof. – By [4, §2], there exists a sequence of positive roots α_i, $1 \le i \le l$, such that $v > vs_{\alpha_1} > \cdots > vs_{\alpha_1}\cdots s_{\alpha_l} = w$. Therefore, $v\alpha_1 < 0$, $vs_{\alpha_1}\alpha_2 < 0, \dots, vs_{\alpha_1}\cdots s_{\alpha_{l-1}}\alpha_l < 0$. So

$$\begin{aligned}(\xi, \mathcal{L}|_v - \mathcal{L}|_w) &= \sum_i (\xi, \mathcal{L}|_{vs_{\alpha_1}\cdots s_{\alpha_{i-1}}} - \mathcal{L}|_{vs_{\alpha_1}\cdots s_{\alpha_{i-1}}s_{\alpha_i}}) \\ &= \sum_i (\mathcal{L}, \alpha_i^\vee)(\xi, vs_{\alpha_1}\cdots s_{\alpha_{i-1}}\alpha_i) < 0.\end{aligned}$$

□

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(1) We thank J. Schuermann for pointing this out to us.

In the remaining part of this paper, we fix $\mathcal{L} \in \operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ lying in the fundamental alcove, i.e., in the positive chamber and near 0. Denote

$$\text{stab}_-(w) = \text{stab}_{-,T^*G/B,\mathcal{L}}(w), \quad \text{stab}_+(w) = \text{stab}_{+,T(G/B),\mathcal{L}^{-1}}(w). \tag{12}$$

Proposition 4.3. – *With notations as above, we have*

$$D_\alpha(\text{stab}_-(w)) = \begin{cases} -q\,\text{stab}_-(w) - q^{\frac{1}{2}}\,\text{stab}_-(ws_\alpha), & \textit{if } ws_\alpha < w; \\ -\,\text{stab}_-(w) - q^{\frac{1}{2}}\,\text{stab}_-(ws_\alpha), & \textit{if } ws_\alpha > w. \end{cases}$$

Proof. – By Remark 2.3.(2),

$$D_\alpha(\text{stab}_-(w)) = \sum_v (D_\alpha(\text{stab}_-(w)), \text{stab}_+(v))\,\text{stab}_-(v).$$

Next, we show that $(D_\alpha(\text{stab}_-(w)), \text{stab}_+(v))$, a priori an element in Q, actually belongs to S, i.e., it is a Laurent polynomial. This is similar to [30, Theorem 4.6.1]. Indeed, the support condition in the definition of stable basis (Theorem 2.2(1)) yields that the K-theoretic stable basis has the same support as that of the cohomological stable basis. In the proof in [30, Theorem 4.6.1], it is shown that the intersection of the support of $\text{stab}_-(w)$, the support of $\text{stab}_+(v)$, and the component of the Steinberg variety corresponding to D_α, is a proper variety. Hence, $(D_\alpha(\text{stab}_-(w)), \text{stab}_+(v))$, which is defined as the direct image from the intersection above, belongs to $K_T(\text{pt}) \cong S$.

By the localization formula and Lemma 4.1,

$$\begin{aligned}(D_\alpha(\text{stab}_-(w)), \text{stab}_+(v)) &= \sum_u \frac{\text{stab}_+(v)|_u\,\text{stab}_-(w)|_u}{\bigwedge^\bullet(T_{(u,u)}T^*_{Y_\alpha})} e^{u\alpha} + \sum_u \frac{\text{stab}_+(v)|_{us_\alpha}\,\text{stab}_-(w)|_u}{\bigwedge^\bullet(T_{(us_\alpha,u)}T^*_{Y_\alpha})} e^{u\alpha} \\ &= \sum_{w \le u \le y} \frac{\text{stab}_+(v)|_u\,\text{stab}_-(w)|_u}{\bigwedge^\bullet(T_u X)} \frac{e^{u\alpha} - q}{1 - e^{u\alpha}} \\ &\quad + \sum_{w \le u, us_\alpha \le y} \frac{\text{stab}_+(v)|_{us_\alpha}\,\text{stab}_-(w)|_u}{\bigwedge^\bullet(T_u X)} \frac{1 - qe^{-u\alpha}}{1 - e^{-u\alpha}} e^{u\alpha}.\end{aligned}$$

We first show that if $v \notin \{w, ws_\alpha\}$, this is 0.

Denote

$$f_1 := \text{stab}_+(v)|_u\,\text{stab}_-(w)|_u, \quad f_2 := \bigwedge\nolimits^\bullet(T_u X),$$
$$f_3 := \text{stab}_+(v)|_{us_\alpha}\,\text{stab}_-(w)|_u e^{u\alpha}.$$

We can find a common ξ as in Lemma 3.1 for all w, u, v. Then, by the degree condition for stable bases, we have

$$\begin{aligned}\max_\xi f_1 &\le \max_\xi(\text{stab}_+(u)|_u) + \max_\xi(\text{stab}_-(u)|_u) + (\xi, \mathcal{L}|_u - \mathcal{L}|_w + \mathcal{L}^{-1}|_u - \mathcal{L}^{-1}|_v) \\ &= (\xi, 2\rho + \mathcal{L}|_v - \mathcal{L}|_w), \\ \max_\xi f_2 &= (\xi, 2\rho), \quad \max_\xi f_3 \le (\xi, 2\rho + \mathcal{L}|_u - \mathcal{L}|_w + \mathcal{L}|_v - \mathcal{L}|_{us_\alpha}),\end{aligned}$$

where the last inequality follows from the degree condition for stable bases and Equation (9).

By Lemma 4.2, $(\xi, \mathcal{L}|_v - \mathcal{L}|_w) < 0$, and $(\xi, \mathcal{L}|_u - \mathcal{L}|_w + \mathcal{L}|_v - \mathcal{L}|_{us_\alpha}) < 0$ because $u > w$ and $v > us_\alpha$. Therefore,

$$\lim_{t\to\infty} (D_\alpha(\mathrm{stab}_-(w)), \mathrm{stab}_+(v))\,(t\xi) = 0. \tag{13}$$

For the minimal degree, we have

$$\begin{aligned}\min_\xi f_1 &\geq \min_\xi(\mathrm{stab}_+(u)|_u) + \min_\xi(\mathrm{stab}_-(u)|_u)\\ &\quad + (\xi, \mathcal{L}|_u - \mathcal{L}|_w + \mathcal{L}^{-1}|_u - \mathcal{L}^{-1}|_v) = (\xi, -2\rho + \mathcal{L}|_v - \mathcal{L}|_w),\\ \min_\xi f_2 &= (\xi, -2\rho), \quad \min_\xi f_3 \geq (\xi, -2\rho + \mathcal{L}|_u - \mathcal{L}|_w + \mathcal{L}|_v - \mathcal{L}|_{us_\alpha}),\end{aligned}$$

where the last inequality follows from the degree condition for stable bases and Equation (10).

We can choose $\mathcal{L}$ sufficiently close to 0, such that

$$-1 < (\xi, \mathcal{L}|_v - \mathcal{L}|_w) < 0, \quad \text{and } -1 < (\xi, \mathcal{L}|_u - \mathcal{L}|_w + \mathcal{L}|_v - \mathcal{L}|_{us_\alpha}) < 0.$$

Then,

$$\lim_{t\to-\infty} \big(D_\alpha(\mathrm{stab}_(w)), \mathrm{stab}_+(v)\big)\,(t\xi) \text{ is bounded.} \tag{14}$$

Due to Lemma 3.1 and (13), (14), we get

$$(\mathrm{stab}_-(w)), \mathrm{stab}_+(v)) = 0, \text{ if } v \notin \{w, ws_\alpha\}.$$

Hence we only need to compute

$$(D_\alpha(\mathrm{stab}_-(w)), \mathrm{stab}_+(w)) \text{ and } (D_\alpha(\mathrm{stab}_-(w)), \mathrm{stab}_+(ws_\alpha))\,.$$

This is done by analyzing two cases below, depending on the order of w and ws_α.

(1). *Case* $ws_\alpha < w$. – There is only one term in the localization of $(D_\alpha(\mathrm{stab}_-(w)), \mathrm{stab}_+(ws_\alpha))$. Therefore, by Lemma 3.2 and Lemma 4.1, we get

$$(D_\alpha(\mathrm{stab}_-(w)), \mathrm{stab}_+(ws_\alpha)) = \frac{\mathrm{stab}_+(ws_\alpha)|_{ws_\alpha}\,\mathrm{stab}_-(w))|_w}{\bigwedge^\bullet(T_{(ws_\alpha,w)}T^*_{Y_\alpha})} e^{w\alpha} = -q^{\frac{1}{2}}.$$

There are two terms in the localization of $(D_\alpha(\mathrm{stab}_-(w)), \mathrm{stab}_+(w))$.

$$\begin{aligned}(D_\alpha(\mathrm{stab}_-(w)), \mathrm{stab}_+(w)) &= \frac{\mathrm{stab}_+(w)|_w\,\mathrm{stab}_-(w))|_w}{\bigwedge^\bullet(T_{(w,w)}T^*_{Y_\alpha})} e^{w\alpha} + \frac{\mathrm{stab}_+(w)|_{ws_\alpha}\,\mathrm{stab}_-(w)|_w}{\bigwedge^\bullet(T_{(ws_\alpha,w)}T^*_{Y_\alpha})} e^{w\alpha}\\ &= \frac{e^{w\alpha} - q}{1 - e^{w\alpha}} + \frac{\mathrm{stab}_+(w)|_{ws_\alpha}\,\mathrm{stab}_-(w)|_w}{\bigwedge^\bullet(T_w X)} \frac{1 - qe^{-w\alpha}}{1 - e^{-w\alpha}} e^{w\alpha}.\end{aligned}$$

As in the first part of the proof, we can find a $\xi \in \mathrm{Lie}\,A$ in the positive chamber, such that $-1 < (\xi, \mathcal{L}|_w - \mathcal{L}|_{ws_\alpha}) < 0$. Notice that $w\alpha < 0$. We have

$$\lim_{t\to\infty} (D_\alpha(\mathrm{stab}_-(w)), \mathrm{stab}_+(w))\,(t\xi) = -q,$$
$$\lim_{t\to-\infty} (D_\alpha(\mathrm{stab}_-(w)), \mathrm{stab}_+(w))\,(t\xi) \text{ is bounded.}$$

Therefore, due to Lemma 3.1, we get

$$(D_\alpha(\mathrm{stab}_-(w)), \mathrm{stab}_+(w)) = -q.$$

(2). *Case* $ws_\alpha > w$. – Although this case can be proved directly using the relation $D_\alpha^2 + (q+1)D_\alpha = 0$, we still give a localization proof for it.

$$(D_\alpha(\mathrm{stab}_-(w)), \mathrm{stab}_+(w)) = \frac{e^{w\alpha} - q}{1 - e^{w\alpha}} + \frac{\mathrm{stab}_+(w)|_w \, \mathrm{stab}_-(w)|_{ws_\alpha}}{\bigwedge^\bullet(T_w X)} \frac{1 - qe^{-w\alpha}}{1 - e^{-w\alpha}} e^{-w\alpha}.$$

As in the first case, the limit as $t \to +\infty$ is -1, while the limit as $t \to -\infty$ is bounded. Therefore,

$$(D_\alpha(\mathrm{stab}_-(w)), \mathrm{stab}_+(w)) = -1.$$

For the other one, we have

$$\begin{aligned}(D_\alpha(\mathrm{stab}_-(w)), \mathrm{stab}_+(ws_\alpha)) = {}& \frac{\mathrm{stab}_+(ws_\alpha)|_w \, \mathrm{stab}_-(w)|_w}{\bigwedge^\bullet(T_w X)} \frac{e^{w\alpha} - q}{1 - e^{w\alpha}} \\ &+ \frac{\mathrm{stab}_+(ws_\alpha)|_{ws_\alpha} \, \mathrm{stab}_-(w)|_{ws_\alpha}}{\bigwedge^\bullet(T_{ws_\alpha} X)} \frac{e^{-w\alpha} - q}{1 - e^{-w\alpha}} \\ &+ \frac{\mathrm{stab}_+(ws_\alpha)|_{ws_\alpha} \, \mathrm{stab}_-(w)|_w}{\bigwedge^\bullet(T_w X)} \frac{1 - qe^{-w\alpha}}{1 - e^{-w\alpha}} e^{w\alpha} \\ &+ \frac{\mathrm{stab}_+(ws_\alpha)|_w \, \mathrm{stab}_-(w)|_{ws_\alpha}}{\bigwedge^\bullet(T_w X)} \frac{1 - qe^{-w\alpha}}{1 - e^{-w\alpha}} e^{-w\alpha}.\end{aligned}$$

Because of Lemma 3.2, the third term is $-q^{-\frac{1}{2}} \frac{1-qe^{w\alpha}}{1-e^{w\alpha}} \frac{1-qe^{-w\alpha}}{1-e^{-w\alpha}}$. Since $ws_\alpha > w$, $w\alpha > 0$. As in the first case, pick good ξ, then the limit as $t \to +\infty$ is $-q^{\frac{1}{2}}$, while the limit as $t \to -\infty$ is bounded. Therefore,

$$(D_\alpha(\mathrm{stab}_-(w)), \mathrm{stab}_+(ws_\alpha)) = -q^{\frac{1}{2}}. \qquad \square$$

4.3. Hecke algebra action D'_α on $\mathrm{stab}_{+,T(G/B),\mathcal{L}^{-1}}$

In this section, we compute the second Hecke algebra $\mathbb{H}$ action on the stable bases for the positive chamber $+$. Although the method from 4.2 still works in this case, we use a different method for illustration purpose.

The relation between these two Hecke actions is the following adjoint property, which is a K-theoretic analogue of [3, Lemma 5.2].

LEMMA 4.4. – *For any* $\mathcal{F}$ *and* $\mathcal{G}$ *in* $K_T^*(T^*G/B)$*, we have*

$$(D_\alpha(\mathcal{F}), \mathcal{G}) = (\mathcal{F}, D'_\alpha(\mathcal{G})).$$

Therefore, for any $T_w \in \mathbb{H}$,

$$(T_w(\mathcal{F}), \mathcal{G}) = (\mathcal{F}, T'_{w^{-1}}(\mathcal{G})).$$

By definition, under this pairing, operators from the subalgebra $K_{G\times\mathbb{C}^*}(T^*G/B) \subseteq K_{G\times\mathbb{C}^*}(T^*G/B \times_{\mathcal{N}} T^*G/B)$ are self-adjoint.

Proof of Lemma 4.4. – We only need to prove the first one. And we can check on the fixed point basis. Using localization and Lemma 4.1, we get

(15)

$$D_\alpha(\iota_{v*}1) = \frac{e^{v\alpha} - q}{1 - e^{v\alpha}} \iota_{v*}1 + \frac{e^{v\alpha} - q}{1 - e^{-v\alpha}} \iota_{vs_\alpha *}1, \quad D'_\alpha(\iota_{v*}1) = \frac{e^{v\alpha} - q}{1 - e^{v\alpha}} \iota_{v*}1 + \frac{1 - qe^{-v\alpha}}{e^{v\alpha} - 1} \iota_{vs_\alpha *}1.$$

Therefore,

$$(D_\alpha(\iota_{v*}1), \iota_{w*}1) = \delta_{v,w}\frac{e^{v\alpha}-q}{1-e^{v\alpha}}\bigwedge\nolimits^{\bullet}(T_vT^*G/B) + \delta_{vs_\alpha,w}\frac{e^{v\alpha}-q}{1-e^{-v\alpha}}\bigwedge\nolimits^{\bullet}(T_{vs_\alpha}T^*G/B),$$
$$(\iota_{v*}1, D'_\alpha(\iota_{w*}1)) = \delta_{v,w}\frac{e^{v\alpha}-q}{1-e^{v\alpha}}\bigwedge\nolimits^{\bullet}(T_vT^*G/B) + \delta_{vs_\alpha,w}\frac{1-qe^{-w\alpha}}{e^{w\alpha}-1}\bigwedge\nolimits^{\bullet}(T_vT^*G/B).$$

Now it is easy to see they are equal to each other. □

THEOREM 4.5. – *With notations defined in* (12)*, the affine Hecke algebra* $\mathbb{H}$ *acts on the stable bases as follows:*

$$T_{s_\alpha}(\mathrm{stab}_-(w)) = \begin{cases} q^{\frac{1}{2}}\,\mathrm{stab}_-(ws_\alpha) + (q-1)\,\mathrm{stab}_-(w), & \textit{if } ws_\alpha < w;\\ q^{\frac{1}{2}}\,\mathrm{stab}_-(ws_\alpha), & \textit{if } ws_\alpha > w.\end{cases}$$

$$T'_{s_\alpha}(\mathrm{stab}_+(w)) = \begin{cases} q^{\frac{1}{2}}\,\mathrm{stab}_+(ws_\alpha) + (q-1)\,\mathrm{stab}_+(w), & \textit{if } ws_\alpha < w;\\ q^{\frac{1}{2}}\,\mathrm{stab}_+(ws_\alpha), & \textit{if } ws_\alpha > w.\end{cases}$$

Proof. – The formula concerning the T_{s_α} action comes from the identity $T_{s_\alpha} = -D_\alpha - 1$ and Proposition 4.3.

We look at the T'_{s_α} action. By the duality of stable bases (see Remark 2.3) and Proposition 4.3, we have

$$\begin{aligned} T'_{s_\alpha}(\mathrm{stab}_+(w)) &= \sum_y \big(T'_{s_\alpha}(\mathrm{stab}_+(w)), \mathrm{stab}_-(y)\big)\,\mathrm{stab}_+(y)\\ &= \sum_y (\mathrm{stab}_+(w), T_{s_\alpha}(\mathrm{stab}_-(y)))\,\mathrm{stab}_+(y)\\ &= (\mathrm{stab}_+(w), T_{s_\alpha}(\mathrm{stab}_-(w)))\,\mathrm{stab}_+(w)\\ &\quad + (\mathrm{stab}_+(w), T_{s_\alpha}(\mathrm{stab}_-(ws_\alpha)))\,\mathrm{stab}_+(ws_\alpha).\end{aligned}$$

The rest follows from the first part of this theorem and the duality property in Remark 2.3.(2). □

4.4. A recursive formula of the restriction coefficients

Using the Hecke actions, we give a recursive formula for the restriction coefficients of $\mathrm{stab}_-(w)$. In Theorem 7.5, we will give a closed formula of those coefficients.

PROPOSITION 4.6. – *With notations defined in* (12)*, the restriction coefficients* $\mathrm{stab}_-(w)|_v$ *are uniquely characterized by*

1. $\mathrm{stab}_-(w)|_v = 0$*, unless* $v \geq w$.
2. $\mathrm{stab}_-(w)|_w = q_w^{\frac{1}{2}} \prod_{\alpha\in\Sigma^-\cap w\Sigma^-}(1-qe^{\alpha}) \cdot \prod_{\alpha\in\Sigma^+\cap w\Sigma^-}(1-e^{\alpha})$.
3. $\mathrm{stab}_-(w)|_{vs_\alpha} = \begin{cases} \frac{(1-q)e^{v\alpha}}{1-qe^{-v\alpha}}\,\mathrm{stab}_-(w)|_v + q^{\frac{1}{2}}\frac{1-e^{v\alpha}}{1-qe^{-v\alpha}}\,\mathrm{stab}_-(ws_\alpha)|_v, & \textit{if } ws_\alpha < w;\\ \frac{1-q}{1-qe^{-v\alpha}}\,\mathrm{stab}_-(w)|_v + q^{\frac{1}{2}}\frac{1-e^{v\alpha}}{1-qe^{-v\alpha}}\,\mathrm{stab}_-(ws_\alpha)|_v, & \textit{if } ws_\alpha > w.\end{cases}$

This is an analogue of Corollary 3.3 in [48].

Proof. – The uniqueness can be easily proved by induction on $\ell(y)$. The first two equalities now follow directly from Lemma 3.2.

The last equality follows from Proposition 4.3 and (15) by applying D_α to the following identity

$$\mathrm{stab}_-(w) = \sum_v \mathrm{stab}_-(w)|_v \frac{\iota_{v*}1}{\bigwedge^\bullet(T_vT^*G/B)}. \qquad \square$$

A similar recursive formula for $\mathrm{stab}_+(w)|_y$ can also be obtained from Theorem 4.5.

5. More on the affine Hecke algebra

In this section we recall the definition of the affine Hecke algebra in terms of the twisted group algebra of Kostant and Kumar [23], while following notions from [12, 13]. The root system we consider will be the one associated to the group G.

5.1. The Demazure-Lusztig elements

In the ring $S = R[\Lambda]$, we use the following notations.

$$x_\alpha = 1 - e^{-\alpha},\ x_{-\alpha} = -e^\alpha x_\alpha,\ \tilde{x}_\alpha = q - e^\alpha,\ \hat{x}_\alpha = -e^{-\alpha}\tilde{x}_\alpha = 1 - qe^{-\alpha},\ q_w = q^{\ell(w)}.$$

$$x_{\pm w} = \prod_{\beta\in\Sigma_w} x_{\pm\beta},\ \tilde{x}_{\pm w} = \prod_{\beta\in\Sigma_w} \tilde{x}_{\pm\beta},\ \hat{x}_{\pm w} = \prod_{\beta\in\Sigma_w} \hat{x}_{\pm\beta}. \tag{16}$$

Note that $u(x_\lambda) = x_{u(\lambda)}$ for $u \in W, \lambda \in \Lambda$, but $u(x_w) \neq x_{uw}$.

Consider the twisted product $Q_W = Q \rtimes R[W]$, which has a Q-basis $\{\delta_w\}_{w\in W}$. The ring Q_W naturally acts on Q by

$$p\delta_w \cdot p' = pw(p'),\ p, p' \in Q.$$

For each root α, we define the push-pull element

$$Y_\alpha = \frac{1}{x_{-\alpha}} + \frac{1}{x_\alpha}\delta_\alpha = \frac{1}{1-e^\alpha} + \frac{1}{1-e^{-\alpha}}\delta_\alpha.$$

We also define the divided difference operator (or the Demazure operator)

$$\Delta_\alpha(p) := \frac{s_\alpha(p) - p}{1 - e^{-\alpha}} = (Y_\alpha - 1)\cdot p,\ p \in Q.$$

It restricts to an S^W-linear endomorphism on S. We define the Demazure-Lusztig elements:

$$\overset{+}{\tau}_\alpha = \tilde{x}_\alpha Y_\alpha - 1 = \frac{q-1}{x_{-\alpha}} + \frac{\tilde{x}_\alpha}{x_\alpha}\delta_\alpha = \frac{q-1}{1-e^\alpha} + \frac{q-e^\alpha}{1-e^{-\alpha}}\delta_\alpha, \tag{17}$$

$$\overset{-}{\tau}_\alpha = \hat{x}_\alpha Y_{-\alpha} - 1 = \frac{q-1}{x_{-\alpha}} + \frac{\hat{x}_\alpha}{x_{-\alpha}}\delta_\alpha = \frac{q-1}{1-e^\alpha} + \frac{1-qe^{-\alpha}}{1-e^\alpha}\delta_\alpha. \tag{18}$$

For simplicity, for simple root α_i, we will write $x_{\pm i}, \tilde{x}_{\pm i}, \hat{x}_{\pm i}, Y_{\pm i}, \Delta_{\pm i}, \overset{\pm}{\tau}_i$ for $x_{\pm\alpha_i}, \tilde{x}_{\pm\alpha_i}, \hat{x}_{\pm\alpha_i}, Y_{\pm\alpha_i}, \Delta_{\pm\alpha_i}$ and $\overset{\pm}{\tau}_{\alpha_i}$, respectively. For each reduced decomposition $w = s_{i_1}\cdots s_{i_k}$, define $\overset{\pm}{\tau}_w = \overset{\pm}{\tau}_{i_1}\cdots\overset{\pm}{\tau}_{i_k}$, and similarly define $Y_{\pm w}$ (e.g., Y_{-w} is a product of Y_{-i}). As shown in [23], they do not depend on the choice of the reduced decomposition.

By straightforward computations, we have the following properties:

LEMMA 5.1. – 1. $Y_i^2 = Y_i$, $Y_i p - s_i(p) Y_i = -\Delta_{-i}(p)$, $p \in Q$.

2. $\overset{\pm}{\tau}_i p - s_i(p) \overset{\pm}{\tau}_i = -(q-1)\Delta_{-i}(p)$, $p \in Q$.

3. $\overset{\pm}{\tau}{}_i^2 = (q-1)\overset{\pm}{\tau}_i + q$, $\overset{\pm}{\tau}{}_i^{-1} = q^{-1}(\overset{\pm}{\tau}_i + 1 - q)$.

4. $\delta_i = x_i Y_i - \frac{x_i}{x_{-i}} = \frac{x_{-i}}{\hat{x}_i}\overset{-}{\tau}_i - \frac{q-1}{\hat{x}_i} = \frac{x_i}{\tilde{x}_i}\overset{+}{\tau}_i - \frac{(q-1)x_i}{x_{-i}\tilde{x}_i}$.

LEMMA 5.2. – 1. *We have* $\overset{\pm}{\tau}_w = \sum_{v \le w} a^{\pm}_{w,v}\delta_v$ *and* $\delta_w = \sum_{v \le w} b^{\pm}_{w,v}\overset{\pm}{\tau}_v$ *such that*

$$a^{\pm}_{w,v} \in S[\frac{1}{x_{w_0}}], \quad b^{\pm}_{w,v} \in S[\frac{1}{\hat{x}_{w_0}}], \quad b^{+}_{w,w} = \frac{1}{a^{+}_{w,w}} = \frac{x_w}{\tilde{x}_w}, \quad b^{-}_{w,w} = \frac{1}{a^{-}_{w,w}} = \frac{x_{-w}}{\hat{x}_w}.$$

2. *We have* $\overset{\pm}{\tau}_w = \sum_{v \le w} d^{\pm}_{w,v} Y_{\pm v}$ *such that*

$$d^{\pm}_{w,v} \in S, \quad d^{+}_{w,w} = \tilde{x}_w, \quad d^{-}_{w,w} = \hat{x}_w, \quad d^{\pm}_{w,e} = (-1)^{\ell(w)}.$$

Proof. – Similar to [13, Lemma 3.2], these identities follow from Lemma 5.1. □

REMARK 5.3. – (1). There is an anti-involution on Q_W defined by

$$\iota : Q_W \to Q_W, \quad p\delta_v \mapsto \delta_{v^{-1}} p \frac{v(x_{-w_0}\hat{x}_{w_0})}{x_{-w_0}\hat{x}_{w_0}}, \quad p \in Q,$$

such that

$$\iota(\overset{\pm}{\tau}_\alpha) = \overset{\mp}{\tau}_\alpha.$$

(2). Recall the following operator of Lusztig in [26]

$$T^L_\alpha = \frac{q-1}{1-e^\alpha} + \frac{1-qe^\alpha}{1-e^\alpha}\delta_\alpha.$$

They satisfy the same relations as τ_α do. We have identities

$$e^{-\rho}\overset{+}{\tau}_\alpha e^{\rho} = e^{\rho}\overset{-}{\tau}_\alpha e^{-\rho} = -q \cdot (T^L_\alpha|_{q \to q^{-1}}). \tag{19}$$

5.2. The Hecke algebra

DEFINITION 5.4. – Define the affine 0-Hecke algebra $\mathbb{D}$ to be the R-subalgebra generated by S and Y_i for all i. Define the affine Hecke algebra $\mathbb{H}$ to be the R-subalgebra of Q_W generated by S and $\overset{+}{\tau}_i$ for all i. It is not difficult to see that all $\overset{-}{\tau}_i$ together with S also generate $\mathbb{H}$.

LEMMA 5.5 ([23, 26]). – *The sets* $\{Y_w\}_{w \in W}$, $\{\overset{+}{\tau}_w\}_{w \in W}$ *and* $\{\overset{-}{\tau}_w\}_{w \in W}$ *are* Q*-bases of* Q_W. *Moreover, the first set is a* S*-basis of* $\mathbb{D}$ *and the last two are* S*-bases of* $\mathbb{H}$.

Proof. – The first statement follows from Lemma 5.2, and the second one is from [12, Proposition 7.7] and [52, Corollary 3.4]. □

The following lemma is used to give an algebraic proof of duality of stable bases in Theorem 6.7.

LEMMA 5.6. – *Writing* $\overset{\pm}{\tau}_w(\overset{\pm}{\tau}_{w_0 u})^{-1} = \sum_{v \in W} c_v \overset{\pm}{\tau}_v$, *then* $c_{w_0} = q^{-1}_{w_0 u}\delta_{w,u}$.

Proof. – This is a special case of [34, Proposition 3]. More precisely, in loc.it., letting $t_1 = q^{\frac{1}{2}}, t_2 = -q^{-\frac{1}{2}}$, then one can identify h_i with $q^{-\frac{1}{2}}\overset{\pm}{\tau}_i$ and $\hat{h}_i$ with $q^{\frac{1}{2}}\overset{\pm}{\tau}_i^{-1}$, and the conclusion follows. □

6. Algebraic description of Stable bases

In this section, we first briefly recall the algebraic models of $K_T(G/B)$, $K_T(T^*G/B)$ and the morphisms between them (details can be found in [23, 13, 11]). We then obtain a formula of stable bases in this algebraic setting.

6.1. The dual of the twisted group algebra

Define

$$\mathbb{D}^* := \mathrm{Hom}_S(\mathbb{D}, S) \subset Q_W^* = \mathrm{Hom}(W, Q).$$

Let $\{f_w\}_{w\in W}$ be the standard basis of Q_W^*, that is, $f_w(\delta_v) = f_w(v) = \delta_{w,v}$. There is a commutative product with identity:

$$f_w \cdot f_v = \delta_{w,v} f_v, \quad \mathbf{1} := \sum_{w\in W} f_w \in \mathbb{D}^* \subset Q_W^*.$$

Indeed, Q_W^* is a commutative Q-algebra and $\mathbb{D}^*$ is a commutative S-algebra.

There is a canonical action of Q_W on Q_W^* defined as follows:

$$(z \bullet f)(z') = f(z'z), \quad z, z' \in Q_W, f \in Q_W^*.$$

We will frequently use the following identities, whose proof can be checked easily, or found in [13, §6].

(20)

$$p \bullet f_v = v(p) f_v, \quad \delta_w \bullet f_v = f_{vw^{-1}}, \quad p \bullet (f \cdot g) = (p \bullet f) \cdot g = f \cdot (p \bullet g), \quad p \in S,\ f, g \in Q_W^*.$$

The action indeed restricts to an action of $\mathbb{D}$ on $\mathbb{D}^*$. Moreover, it induces an action of $W \subset Q_W$ on Q_W^*. The W_J-invariant Q-submodule $(Q_W^*)^{W_J}$ has a basis $\{\sum_{v\in W_J} f_{wv}\}_{w\in W^J}$.

For each $J \subset \Pi$, define the following elements in Q_W:

$$Y_{\Pi/J} = \sum_{w\in W^J} \delta_w \frac{x_{-w_0^J}}{x_{-w_0}} = \sum_{w\in W^J} \delta_w \prod_{\alpha\in\Sigma^+\setminus\Sigma_J^+} \frac{1}{1-e^{\alpha}}, \tag{21}$$

$$Y_J = \sum_{w\in W_J} \delta_w \frac{1}{x_{-w_0^J}} = \sum_{w\in W_J} \delta_w \prod_{\alpha\in\Sigma_J^+} \frac{1}{1-e^{\alpha}}, \tag{22}$$

$$\hat{Y}_{\Pi/J} = \sum_{w\in W^J} \delta_w \frac{x_{-w_0^J}\hat{x}_{w_0^J}}{x_{-w_0}\hat{x}_{w_0}} = \sum_{w\in W^J} \delta_w \prod_{\alpha\in\Sigma^+\setminus\Sigma_J^+} \frac{1}{(1-e^{\alpha})(1-qe^{-\alpha})}, \tag{23}$$

$$\hat{Y}_J = \sum_{w\in W_J} \delta_w \frac{1}{x_{-w_0^J}\hat{x}_{w_0^J}} = \sum_{w\in W_J} \delta_w \prod_{\alpha\in\Sigma_J^+} \frac{1}{(1-e^{\alpha})(1-qe^{-\alpha})}. \tag{24}$$

In particular, Y_Π and $\hat{Y}_\Pi$ are defined when $J = \Pi$. Similar as [13, Lemmas 5.7 and 6.4], we have the composition rule and the Projection Formula:

$$Y_{\Pi/J} Y_J = Y_\Pi, \quad \hat{Y}_{\Pi/J} \hat{Y}_J = \hat{Y}_\Pi, \tag{25}$$

$$f \cdot (Y_J \bullet f') = Y_J(ff'), \; f \cdot (\hat{Y}_J \bullet f') = \hat{Y}_J \bullet (ff'), \; f' \in Q_W^*, \; f \in (Q_W^*)^{W_J}. \tag{26}$$

Via the embedding $\mathbb{H} \subset Q_W$, we can restrict the $\bullet$-action to a left action of $\mathbb{H}$ on Q_W^*. On the other hand, $\mathbb{H}$ also acts on the right on Q_W^*, where the action of $\overset{+}{\tau}_w \in \mathbb{H}$ on $f \in Q_W^*$ is given by

$$\overset{+}{\tau}_{w^{-1}} \bullet f.$$

The $\bullet$-action is a well-defined action of $\mathbb{H}$ on Q_W^*, which is linear with respect to the Q-module structure on Q_W^* coming from $\mathrm{Hom}(W, Q)$, hence so is the right action defined above. Indeed, $\overset{-}{\tau}_\alpha \bullet \_$ and $\overset{+}{\tau}_\alpha \bullet \_$ correspond to the T_α and T'_α actions in Section 4, respectively (see Lemma 6.5). The following lemma is the algebraic model of Lemma 4.4:

Lemma 6.1 (Adjointness). – *For any $\alpha_i \in J$, $f, g \in Q_W^*$, we have*

$$\hat{Y}_J \bullet ((\overset{+}{\tau}_i \bullet f) \cdot g) = \hat{Y}_J \bullet (f \cdot (\overset{-}{\tau}_i \bullet g)).$$

Proof. – Note that the $\bullet$-action is Q-linear, so it suffices to assume that $f = f_v, g = f_u$ with $u, v \in W$. The identity then follows from direct computation. □

We get an easy corollary of the coefficients appearing in Lemma 5.2.

Lemma 6.2. – *We have*

$$a^{\pm}_{w^{-1},v} v(x_{-w_0}) v(\hat{x}_{w_0}) = v(a^{\mp}_{w,v^{-1}}) x_{-w_0} \hat{x}_{w_0}.$$

Proof. – From Lemma 6.1 we know that

$$\hat{Y}_\Pi \bullet ((\overset{\pm}{\tau}_w \bullet f_e) \cdot f_v) = \hat{Y}_\Pi \bullet (f_e \cdot (\overset{\mp}{\tau}_{w^{-1}} \bullet f_v)).$$

Direct computations using Lemma 5.2 and (20) shows that the left hand side is $\frac{v(a^{\pm}_{w,v^{-1}})}{v(x_{-w_0})v(\hat{x}_{w_0})}\mathbf{1}$, and the right hand side is equal to $\frac{a^{\mp}_{w^{-1},v}}{x_{-w_0}\hat{x}_{w_0}}\mathbf{1}$. Hence, the conclusion follows. □

6.2. An algebraic model of stable bases

We define (e being the identity element of W)

$$\mathrm{pt} := \mathrm{pt}_e = x_{-w_0} f_e, \quad \mathrm{pt}_{w_0} := x_{-w_0} f_{w_0}. \tag{27}$$

Both of them belong to $\mathbb{D}^*$ ([13, Lemma 10.3]). They can be viewed (up to certain normalization) as the push-forward of the fundamental class in $K_A(G/B)$ along the A-fixed points e, $w_0 \in W$.

Definition 6.3. – Define $\mathrm{St}^+_w = \overset{+}{\tau}_{w^{-1}} \bullet \mathrm{pt}_e$ and $\mathrm{St}^-_u = (\overset{-}{\tau}_{w_0 u})^{-1} \bullet \mathrm{pt}_{w_0}$.

By definition it is easy to see that if $\ell(ws_i) \geq \ell(w)$, then,

$$\overset{\pm}{\tau}_i \bullet \mathrm{St}^{\pm}_w = \mathrm{St}^{\pm}_{ws_i} .$$

Therefore, for any $w \in W$, $\mathrm{St}^{\pm}_w = \overset{\pm}{\tau}_{w^{-1}} \bullet \mathrm{St}^{\pm}_e$. Note that $\mathrm{St}^{+}_e = \mathrm{pt}_e$, $\mathrm{St}^{-}_{w_0} = \mathrm{pt}_{w_0}$.

By the standard theory of Kostant-Kumar, $K_T(G/B) \cong \mathbb{D}^*$ and $K_T(T^*G/B) \otimes_S Q \cong \mathrm{Hom}(W, Q) = Q^*_W$ [23, Theorem (3.13)]. Let $p : T^*G/B \to G/B$ be the canonical projections. Then the isomorphism $Q^*_W \cong K_T(G/B) \otimes_S Q \underset{\sim}{\xrightarrow{p^*}} K_T(T^*G/B) \otimes_S Q$ is given by the formula $p^* = \hat{x}_{w_0} \bullet \_$; the map

$$K_T(T^*G/B) \underset{\simeq}{\xrightarrow{(p^*)^{-1}}} K_T(G/B) \longrightarrow K_T(\mathbb{C})$$

is given by the formula $\hat{Y}_\Pi \bullet \_$.

For any $\mathscr{F} \in K_T(T^*G/B) \subset Q^*_W$, we can write $\mathscr{F} = \sum_w \mathscr{F}|_w f_w \in Q^*_W$ with $\mathscr{F}|_w \in S$. For example, $\iota_{w*}1 = w(x_{-w_0}\hat{x}_{w_0}) f_w$, where $\iota_w : \mathrm{Spec}(\mathbb{C}) \to T^*G/B$ is the embedding of the T-fixed point corresponding to $w \in W$, and

$$w(x_{-w_0}\hat{x}_{w_0}) = \prod_{\alpha>0} (1 - e^{w\alpha})(1 - qe^{-w\alpha}) = \bigwedge\nolimits^{\bullet} T_w(T^*G/B).$$

THEOREM 6.4. – *Under the above identifications, for any $u \in W$ we have*

$$\mathrm{stab}_+(u) = q_u^{-\frac{1}{2}} \,\mathrm{St}^{+}_u, \quad \mathrm{stab}_-(u) = q_{w_0} q_u^{-\frac{1}{2}} \,\mathrm{St}^{-}_u .$$

Now we prove this theorem. First of all, we have the following relation between $\overset{\pm}{\tau}_\alpha$ in (17) and the above operators T_{s_α}, T'_{s_α}.

LEMMA 6.5. – *As operators on $K_T(T^*G/B)$, we have*

$$T'_{s_\alpha} = \overset{+}{\tau}_\alpha, \quad T_{s_\alpha} = \overset{-}{\tau}_\alpha .$$

Proof. – By (15), the operators $T_{s_\alpha} = -D_\alpha - 1$ and $T'_{s_\alpha} = -D'_\alpha - 1$ act on the basis $\{f_v = \frac{\iota_{v*}1}{\bigwedge^\bullet (T_v T^*G/B)} | v \in W\}$ as follows:

$$T_\alpha(f_v) = \frac{q-1}{1-e^{v\alpha}} f_v + \frac{1-qe^{v\alpha}}{1-e^{-v\alpha}} f_{vs_\alpha}, \quad T'_\alpha(f_v) = \frac{q-1}{1-e^{v\alpha}} f_v + \frac{q-e^{-v\alpha}}{1-e^{v\alpha}} f_{vs_\alpha}.$$

Comparing with the $\bullet$-action of $\overset{\pm}{\tau}_\alpha$ on $f_v \in Q^*_W$ using (20), we get the conclusion. □

Proof of Theorem 6.4. – Let us consider the first identity. By Lemma 3.2, Equation (27), and Definition 6.3, we have

$$\mathrm{stab}_-(w_0) = q_{w_0}^{\frac{1}{2}} \prod_{\beta>0} (1 - e^{\beta}) f_{w_0} = q_{w_0}^{\frac{1}{2}} \,\mathrm{pt}_{w_0} = q_{w_0}^{\frac{1}{2}} \,\mathrm{St}^{-}_{w_0},$$

and moreover, $\mathrm{St}^{-}_u = (\overset{-}{\tau}_{w_0 u})^{-1} \bullet \mathrm{pt}_{w_0}$. On the other hand, by Proposition 4.3, we get

$$T_{w_0 u}(\mathrm{stab}_-(u)) = q_{w_0 u}^{\frac{1}{2}} \,\mathrm{stab}_-(w_0).$$

Using Lemma 6.5, we get

$$\mathrm{St}^{-}_u = (\overset{-}{\tau}_{w_0 u})^{-1} \bullet \mathrm{pt}_{w_0} = (T_{w_0 u})^{-1} (q_{w_0}^{-\frac{1}{2}} \,\mathrm{stab}_-(w_0)) = q_{w_0}^{-\frac{1}{2}} q_{w_0 u}^{-\frac{1}{2}} \,\mathrm{stab}_-(u) = q_{w_0}^{-1} q_u^{\frac{1}{2}} \,\mathrm{stab}_-(u).$$

This proves the formula for $\mathrm{stab}_-(u)$.

Lemma 3.2, (27) and Definition 6.3 show that

$$\mathrm{stab}_+(e) = \prod_{\beta>0}(1-e^{\beta})f_e = \mathrm{St}_e^+ .$$

Moreover, Theorem 4.5 shows that $T'_{u^{-1}}(\mathrm{stab}_+(e)) = q_u^{\frac{1}{2}}\,\mathrm{stab}_+(u)$. Comparing with Definition 6.3 and using Lemma 6.5, we get the formula for $\mathrm{stab}_+(u)$. □

6.3. The duality

Let $\overset{\pm *}{\tau}_w$ be the bases of Q_W^* dual to $\overset{\pm}{\tau}_w$, then $\overset{\pm *}{\tau}_w = \sum_{v\geq w} b^{\pm}_{v,w} f_v$ by Lemma 5.2.

Lemma 6.6. – *The map $Q_W^* \times Q_W^* \to Q, (f,g) \mapsto \hat{Y}_\Pi \bullet (fg)$ defines a perfect pairing and the basis $\hat{x}_{w_0}\overset{\mp *}{\tau}_v$ is dual to the basis $\overset{\pm}{\tau}_{w^{-1}} \bullet \mathrm{pt}_e$. In particular, $\overset{- *}{\tau}_v$ is dual to St_w^+ .*

Proof. – By Lemma 5.2 we know $\overset{\mp *}{\tau}_v = \sum_{v'} b^{\mp}_{v',v} f_{v'}$ and $\overset{\pm}{\tau}_{w^{-1}} = \sum_u a^{\pm}_{w^{-1},u}\delta_u$, so

$$\begin{aligned}\overset{\pm}{\tau}_{w^{-1}} \bullet \mathrm{pt}_e &= (\sum_u a^{\pm}_{w^{-1},u}\delta_u) \bullet (x_{-w_0} f_e) = \sum_u x_{-w_0} u^{-1}(a^{\pm}_{w^{-1},u}) f_{u^{-1}} \\ &= \sum_u x_{-w_0} u(a^{\pm}_{w^{-1},u^{-1}}) f_u \overset{\sharp_1}{=} \sum_u \frac{a^{\mp}_{w,u} u(x_{-w_0}) u(\hat{x}_{w_0})}{\hat{x}_{-w_0}} f_u .\end{aligned}$$

Here $\sharp_1$ follows from Lemma 6.2. According to (20), we have

$$\begin{aligned}\hat{Y}_\Pi \bullet [(\hat{x}_{w_0}\overset{\mp *}{\tau}_v) \cdot (\overset{\pm}{\tau}_{w^{-1}} \bullet \mathrm{pt})] &= \hat{Y}_\Pi \bullet (\sum_{v'} \hat{x}_{w_0} b^{\mp}_{v',v} f_{v'} \cdot \sum_u \frac{a^{\mp}_{w,u} u(x_{-w_0}) u(\hat{x}_{w_0})}{\hat{x}_{w_0}} f_u) \\ &= \hat{Y}_\Pi \bullet (\sum_u b^{\mp}_{u,v} a^{\mp}_{w,u} u(x_{-w_0}) u(\hat{x}_{w_0}) f_u) \\ &= \sum_{w'\in W} \delta_{w'} \frac{1}{x_{-w_0}\hat{x}_{w_0}} \bullet (\sum_u b^{\mp}_{u,v} a^{\mp}_{w,u} u(x_{-w_0}) u(\hat{x}_{w_0}) f_u) \\ &= \sum_{w'}\sum_u a^{\mp}_{w,u} b^{\mp}_{u,v} \frac{u(x_{-w_0}) u(\hat{x}_{w_0})}{u(x_{-w_0}) u(\hat{x}_{w_0})} f_{uw'^{-1}} \\ &= \sum_{w'}(\sum_u a^{\mp}_{w,u} b^{\mp}_{u,v}) f_{u(w')^{-1}} = \delta_{w,v} \sum_{w'} f_{u(w')^{-1}} = \delta_{w,v}\mathbf{1}. \quad \square\end{aligned}$$

The following is the algebraic model of the duality between stable bases for the positive and negative chambers, see Remark 2.3.(2).

Theorem 6.7. – *1. Notations as above, we have*

$$\hat{Y}_\Pi \bullet \left(\mathrm{St}_w^+ \cdot \mathrm{St}_u^-\right) = \delta_{w,u} q_{w_0 u}^{-1}\mathbf{1}.$$

2. This duality coincides with the duality in Remark 2.3.(2).

Proof. – We have

$$\begin{aligned}\hat{Y}_\Pi \bullet (\mathrm{St}^+_w \cdot \mathrm{St}^-_u) &= \hat{Y}_\Pi \bullet \Big((\overset{+}{\tau}_{w^{-1}} \bullet \mathrm{pt}_e) \cdot [(\overline{\tau}_{w_0u})^{-1} \bullet \mathrm{pt}_{w_0}] \Big) \\ &\overset{\sharp}{=} \hat{Y}_\Pi \bullet \Big(\mathrm{pt}_e \cdot [\overline{\tau}_w \bullet (\overline{\tau}_{w_0u})^{-1} \bullet \mathrm{pt}_{w_0}] \Big),\end{aligned}$$

where ♯ follows from Lemma 6.1. Since $\mathrm{pt}_e = x_{-w_0} f_e$ and $f_u \cdot f_v = \delta_{u,v} f_u$, it suffices to look at the term involving f_e in $\overline{\tau}_w (\overline{\tau}_{w_0u})^{-1} \bullet \mathrm{pt}_{w_0}$. Furthermore, since $\delta_u \bullet f_v = f_{vu^{-1}}$ (see (20)) and $\mathrm{pt}_{w_0} = x_{-w_0} f_{w_0}$, it suffices to look at the term involving δ_{w_0} inside $\overline{\tau}_w (\overline{\tau}_{w_0u})^{-1}$. Lastly, from Lemma 5.2 we know that $\overline{\tau}_w = \sum_{v \le w} a^-_{w,v} \delta_v$, so it reduces to look at the term $\overline{\tau}_{w_0}$ inside $\overline{\tau}_w (\overline{\tau}_{w_0u})^{-1}$, which is $\delta_{w,u} q^{-1}_{w_0u} \overline{\tau}_{w_0}$ by Lemma 5.6. So we have

$$\begin{aligned}\hat{Y}_\Pi \bullet (\mathrm{St}^+_w \cdot \mathrm{St}^-_u) &= \hat{Y}_\Pi \bullet (\mathrm{pt}_e \cdot (\delta_{w,u} q^{-1}_{w_0u} \overline{\tau}_{w_0} \bullet \mathrm{pt}_{w_0})) \\ &\overset{\sharp_1}{=} \delta_{w,u} q^{-1}_{w_0u} \hat{Y}_\Pi \bullet \left([x_{-w_0} f_e] \cdot [\frac{\hat{x}_{w_0}}{x_{-w_0}} \delta_{w_0} \bullet (x_{-w_0} f_{w_0})] \right) \\ &= \delta_{w,u} q^{-1}_{w_0u} \sum_{v \in W} \delta_v \frac{1}{x_{-w_0} \hat{x}_{w_0}} \bullet \big([x_{-w_0} f_e] \cdot [\hat{x}_{w_0} f_e] \big) \\ &= \delta_{w,u} q^{-1}_{w_0u} \sum_{v \in W} \delta_v \frac{1}{x_{-w_0} \hat{x}_{w_0}} \bullet \big(x_{-w_0} \hat{x}_{w_0} f_e) \big) \\ &= \delta_{w,u} q^{-1}_{w_0u} \sum_{v \in W} f_{v^{-1}} = \delta_{w,u} q^{-1}_{w_0u} \mathbf{1}.\end{aligned}$$

Here $\sharp_1$ follows from Lemma 5.2 and the other identities follow from (20). □

Remark 6.8. – Recall from [23, 12] that there is an element $X_i = Y_i - 1$ (called the divided difference element) inside Q_W, and one can define X_w correspondingly. The method in Theorem 6.7 works also for the X_w and Y_w operators. More precisely, by using analogue of Lemma 5.6 and [13, Lemma 7.1], we can similarly show that

$$\begin{aligned}Y_\Pi \bullet \big([X_{w^{-1}} \bullet \mathrm{pt}_e] \cdot [Y_{u^{-1}w_0} \bullet (x_{w_0} f_{w_0})] \big) &= \delta_{w,u} \mathbf{1}, \\ Y_\Pi \bullet \big([Y_{w^{-1}} \bullet \mathrm{pt}_e] \cdot [X_{u^{-1}w_0} \bullet (x_{w_0} f_{w_0})] \big) &= \delta_{w,u} \mathbf{1}.\end{aligned}$$

Note that $Y_{w^{-1}} \bullet \mathrm{pt}_e$ gives the Schubert class corresponding to w. This proof is different from the one given in [25], moreover, it can be easily generalized to the connective K-theory case.

6.4. Remarks on wall-crossings

It is worth mentioning that the stable bases $\{\mathrm{stab}_{\mathfrak{C},T^{1/2},\mathcal{L}}(w) \mid w \in W\}$ of $K_T(T^*G/B)$ are well-defined over an algebraically closed field of sufficiently large positive characteristic. In that setting, based on the results of the present paper, it can be shown that each St^-_w is equal to the image of certain Verma modules over $U\mathfrak{g}$ under the localization functor of [6]. In other words, the positive characteristic analogue of the stable bases form a set of standard modules over the quantization of T^*G/B of *loc. cit.*. Notably, the combinatorics of the wall-crossings of the stable bases is controlled by the local affine braid group action considered in *loc. cit.*. As this is interesting on its own, we postpone the details to a sequel [49]. Nevertheless, for the completeness of the present paper, we briefly summarize the ingredients of wall-crossings here. In particular, in this section we give a description of the stable basis associated to an arbitrary chamber $\mathfrak{C}$ and arbitrary $\mathscr{L}$.

Recall that $\mathfrak{h}$ is the Lie algebra of the maximal torus A, and the alcoves are connected components of

$$\mathfrak{h}^*_{\mathbb{R}} \setminus \cup H_{\alpha^\vee,n}.$$

We call the walls the sets

$$H_{\alpha^\vee,n} := \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid (\lambda, \alpha^\vee) = n\}$$

for coroots $\alpha^\vee$ and $n \in \mathbb{Z}$. For any $w \in W$, the stable basis $\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathscr{L}}(w)$ is constant when $\mathscr{L}$ varies in the interior of any alcove.

Let the chamber $\mathfrak{C}$ be the positive chamber $+$, the polarization $T^{\frac{1}{2}} = T(G/B)$. When $\mathscr{L}$ lies in the interior of the anti-fundamental alcove

$$-\nabla := \{\lambda \mid -1 < (\lambda, \alpha^\vee) < 0, \textit{ for any positive coroot } \alpha^\vee\},$$

the formula for the stable bases are given explicitly in § 6.

Now let α be a root. In general, let λ_1 and λ_2 be two elements in two nearby alcoves separated by a wall $H_{\alpha^\vee,n}$, and $(\lambda_1, \alpha^\vee) < n < (\lambda_2, \alpha^\vee)$. It follows from [45, Theorem 1] that for any $y \in W$,

$$\mathrm{stab}_{+,T(G/B),\lambda_1}(y) = \begin{cases} \mathrm{stab}_{+,T(G/B),\lambda_2}(y) + f_y^{\lambda_1 \leftarrow \lambda_2} \cdot \mathrm{stab}_{+,T(G/B),\lambda_2}(ys_\alpha), & \textit{if} \quad ys_\alpha < y; \\ \mathrm{stab}_{+,T(G/B),\lambda_2}(y), & \text{if} \quad y < ys_\alpha, \end{cases}$$

where $f_y^{\lambda_1 \leftarrow \lambda_2} \in K_{A\times\mathbb{C}^*}(\mathrm{pt})$. Moreover, when $n = 0$, $f_y^{\lambda_1 \leftarrow \lambda_2} \in K_{\mathbb{C}^*}(\mathrm{pt})$, hence, does not depend on the equivariant parameters of the maximal torus A.

It can be shown that for any simple coroot $\alpha^\vee$, $m \in \mathbb{Z}$ and $y \in W$, we have

$$f_y^{\lambda_1 + m\varpi_\alpha \leftarrow \lambda_2 + m\varpi_\alpha} = e^{-my\alpha} f_y^{\lambda_1 \leftarrow \lambda_2},$$

where ϖ_α is the fundamental weight corresponding to α. Therefore, the wall crossings for the walls $H_{\alpha^\vee,n}$ are uniquely determined by the wall crossing for the single wall $H_{\alpha^\vee,0}$. Moreover, we have the following formula. If $\lambda_1 \in -\nabla$, and λ_2 lies in the alcove $s_\alpha(-\nabla)$, then

$$f_y^{\lambda_1 \leftarrow \lambda_2} = q^{\frac{1}{2}} - q^{-\frac{1}{2}}. \tag{28}$$

The proof of this formula, as well as the details of the statements above, lines better with the up-coming sequel. As this will not be used in the rest of the present paper, we postpone the details to the sequel.

As has been commented in Remark 2.3(4), for any integral weight λ and any fractional weight μ, we have

$$\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},\lambda+\mu}(y) = e^{-y\lambda} \mathscr{L}_\lambda \otimes \mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mu}(y). \tag{29}$$

Note that the extended affine Weyl group acts transitively on the set of alcoves, hence any alcove can be obtained from an integral translation of one alcove near the origin. In particular, in the sequel [49], when $\mathfrak{C}$ be the positive chamber $+$ and $T^{\frac{1}{2}} = T(G/B)$, the stable basis $\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathscr{L}}(w)$ for any $\mathscr{L}$ and any $w \in W$ is calculated using a general form of Equation (28), Theorem 6.4, and Remark 2.3(4).

Recall on $K_{A\times\mathbb{C}^*}(T^*(G/B))$, we have a Weyl group action (see § 9.4 for the detail). It can be proved that for any w and y in W, we have

$$w(\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathscr{L}}(y)) = \mathrm{stab}_{w\mathfrak{C},w(T^{\frac{1}{2}}),\mathscr{L}}(wy).$$

Notice that the $\mathscr{L}$-parameter is unchanged under this action. This gives $\mathrm{stab}_{\mathfrak{C},T^{\frac{1}{2}},\mathscr{L}}(w)$ for any $\mathfrak{C}$.

However, a systematic study of the affine braid group action induced by wall-crossings of K-theory stable basis is interesting on its own. It is related to the affine braid group action of [6], which in turn controls the monodromy of the quantum connections of T^*G/B.

7. The restriction formula

In this section we use the root polynomials to study the coefficients $b^{\pm}_{w,v}$ introduced in Lemma 5.2. Our method generalizes the formulation in [25]. In particular, this allows us to avoid the direct calculations in checking the dependence of root polynomials on reduced sequences.

7.1. The evaluation map

Throughout this section, we denote $Q^x \cong Q$ if variables of Q are denoted by $x_\lambda = 1-e^{-\lambda}$. Variables of $Q^y \cong Q$ will be denoted by y_λ. Let $\hat{Q} = Q^y \otimes_R Q^x$, and consider the ring $\hat{Q}_W := Q^y \otimes_R Q^x_W$ where elements of Q^y commute with elements of Q^x_W. The free $\hat{Q}$-module $\hat{Q}_W$ has basis $\{\delta^x_w\}_{w\in W}$. We define a ring homomorphism

$$\mathrm{ev} : \hat{Q} = Q^y \otimes_R Q^x \to Q^x, \quad y_\lambda \otimes x_\mu \mapsto x_\lambda x_\mu,$$

which induces a left $\hat{Q}$-module structure on Q^x_W. The map ev also induces a left $\hat{Q}$-module homomorphism $\mathrm{ev} : \hat{Q}_W \cong Q^y \otimes_R Q^x_W \to Q^x_W$. It is easy to check that

$$\mathrm{ev}(y\hat{z}z) = \mathrm{ev}(y)\,\mathrm{ev}(\hat{z})\,\mathrm{ev}(z), \quad y \in Q^y, \hat{z} \in \hat{Q}_W, z \in Q^x_W. \tag{30}$$

Given a set $\{a_\alpha, b_\alpha\}_{\alpha\in\Sigma} \subset Q$, denote $a_i = a_{\alpha_i}, b_i = b_{\alpha_i}$, and define a_w, b_w as the corresponding products of a_α and b_α, similar as in (16). We will use a^y_λ, b^y_λ (resp. a^x_λ, b^x_λ) when they are considered as inside Q^y (resp. Q^x).

For each simple root α_i, we consider $\sigma_i = a_i\delta_i + b_i \in Q_w$. They satisfy the braid relations, hence σ_v is well-defined for any v. When considering σ_v as an element in Q^x_W, we denote it by σ^x_v.

7.2. The root polynomials

Definition 7.1. – For any $w = s_{i_1}\cdots s_{i_l}$, denote $\beta_j = s_{i_1}\cdots s_{i_{j-1}}\alpha_j$, and define the root polynomial

$$\mathscr{R}^\sigma_w = \prod_{j=1}^{l} h^\sigma_{i_j}(\beta_j) \in \hat{Q}_W, \quad \text{where } h^\sigma_i(\beta) = \sigma^x_i - b^y_\beta \in \hat{Q}_W. \tag{31}$$

Denote

$$\mathscr{R}^\sigma_w = \sum_{v \le w} K^\sigma_{v,w} \sigma^x_v \in \hat{Q}_W, \quad K^\sigma_{v,w} \in Q^y \otimes Q^x. \tag{32}$$

Since σ^x_i satisfy the braid relations, we have $K^\sigma_{v,w} \in Q^y$. The following theorem generalizes [25, Lemma 3.3].

THEOREM 7.2. – 1. $\mathrm{ev}(\mathscr{R}^\sigma_w) = a^x_w \delta^x_w$.

2. *Writing* $\delta^x_w = \sum_v b^\sigma_{w,v} \sigma^x_v$, *then* $a^x_w b^\sigma_{w,v} = \mathrm{ev}(K^\sigma_{v,w})$. *In particular,* $K^\sigma_{v,w}$ *and hence* $\mathscr{R}^\sigma_w$ *do not depend on the choice of the reduced sequence of* w.

Proof. – (1). We use induction on $\ell(w)$. If $w = s_i$,

$$\mathrm{ev}(\mathscr{R}^\sigma_i) = \mathrm{ev}(\sigma^x_i - b^y_i) = \sigma^x_i - b^x_i = a^x_i \delta^x_i.$$

Assume the conclusion holds for all v such that $\ell(v) \le k$, i.e., $\mathrm{ev}(\mathscr{R}^\sigma_v) = a^x_v \delta^x_v$. Suppose that $w = v s_i$ with $\ell(v) = k = \ell(w) - 1$. Then $\Sigma_w = \Sigma_v \sqcup \{v(\alpha_i)\}$, and we have

$$\begin{aligned} \mathrm{ev}(\mathscr{R}^\sigma_w) &= \mathrm{ev}[\mathscr{R}^\sigma_v \cdot (\sigma^x_i - b^y_{v(\alpha_i)})] \overset{\sharp_1}{=} \mathrm{ev}[\mathscr{R}^\sigma_v \sigma^x_i - b^y_{v(\alpha_i)} \mathscr{R}^\sigma_v] \\ &\overset{\sharp_2}{=} a^x_v \delta^x_v \sigma^x_i - b^x_{v(\alpha_i)} a^x_v \delta^x_v = a_v \delta_v (\sigma^x_i - b^x_i) = a^x_v \delta^x_v a^x_i \delta^x_i = a^x_w \delta^x_w. \end{aligned}$$

Here identity $\sharp_1$ follows since $b^y_\beta \in Q^y$ commutes with elements of $\hat{Q}_W$, and $\sharp_2$ follows from (30).

(2). Applying ev on (31), we have

$$a^x_w \delta^x_w = \mathrm{ev}(\mathscr{R}^\sigma_w) = \sum_v \mathrm{ev}(K^\sigma_{v,w})\, \mathrm{ev}(\sigma^x_v) = \sum_v \mathrm{ev}(K^\sigma_{v,w}) \sigma^x_v.$$

So $\frac{1}{a^x_w} \mathrm{ev}(K^\sigma_{v,w}) = b^\sigma_{w,v}$. Since $K^\sigma_{v,w} \in Q^y$ and ev only changes the y-variables into x-variables, we see that $K^\sigma_{v,w} = \mathrm{ev}(K^\sigma_{v,w}) = a^x_w b^\sigma_{w,v}$ if we identify the x and y-variables. Therefore, $K^\sigma_{v,w}$ and hence $\mathscr{R}^\sigma_w$ do not depend on the choice of reduced decompositions. □

7.3. Root polynomials of Demazure-Lusztig elements

We now apply the root polynomial construction to the $\overset{\pm}{\tau}_i$ operators. We have

$$\mathscr{R}^{\overset{\pm}{\tau}}_w = \prod_{j=1}^{l} h^{\overset{\pm}{\tau}}_{i_j}(\beta_j), \quad \text{where } h^{\overset{\pm}{\tau}}_i(\beta) = \overset{\pm}{\tau}{}^x_i - \frac{q-1}{y_{-\beta}}. \tag{33}$$

Expanding in terms of $\overset{\pm}{\tau}{}^x_v$, we write $\mathscr{R}^{\overset{\pm}{\tau}}_w = \sum_v K^{\overset{\pm}{\tau}}_{v,w} \overset{\pm}{\tau}{}^x_v$.

By Lemma 5.1.(3.), the operators $\overset{+}{\tau}_i$ and $\overset{-}{\tau}_i$ both satisfy the quadratic relation of Hecke algebra. Note also that their corresponding root polynomials $h^{\overset{+}{\tau}}_i(\beta)$ and $h^{\overset{-}{\tau}}_i(\beta)$ both have the form in (33). Therefore, $K^{\overset{+}{\tau}}_{v,w} = K^{\overset{-}{\tau}}_{v,w} \in Q^y$, which will be denoted by $K^\tau_{v,w}$. Applying Theorem 7.2 to $\mathscr{R}^{\overset{\pm}{\tau}}_w$, we get the following.

THEOREM 7.3. – 1. $\mathrm{ev}(\mathscr{R}^{\overset{+}{\tau}}_w) = \frac{\tilde{x}_w}{x_w} \delta^x_w$, $\mathrm{ev}(\mathscr{R}^{\overset{-}{\tau}}_w) = \frac{\hat{x}_w}{x_{-w}} \delta^x_w$.

2. $\frac{\tilde{x}_w}{x_w} b^+_{w,v} = K^\tau_{v,w} = \frac{\hat{x}_w}{x_{-w}} b^-_{w,v}$.

REMARK 7.4. – The formal root polynomials can be defined similarly for the formal affine Hecke algebra [52]. They do not depend on the choice of reduced sequence for hyperbolic formal group law. Moreover, restricting to the connective K-theory, one gets a uniform treatment of the restriction formulas of K-theoretic stable bases in this paper and that of cohomological stable bases in [48].

7.4. Restriction formula via root polynomials

The following theorem gives the restriction formulas of stable bases of T^*G/B:

THEOREM 7.5. – 1. $\mathrm{St}_w^+ = \sum_{v\leq w} v(a^+_{w^{-1},v^{-1}})x_{-w_0} f_v$.

2. $q_{w_0w}\,\mathrm{St}_w^- = \hat{x}_{w_0}\overset{-*}{\tau}_w = \sum_{v\geq w} \hat{x}_{w_0} b^-_{v,w} f_v = \sum_{v\geq w} \frac{\hat{x}_{w_0}x_{-v}}{\hat{x}_v} K^\tau_{w,v} f_v$
$= \sum_{v\geq w} v(\hat{x}_{v^{-1}w_0})x_{-v} K^\tau_{w,v} f_v$.

Proof. – (1). This follows from definition and Lemma 5.2.

(2). Via the pairing defined in Lemma 6.6, $\hat{x}_{w_0}\overset{-*}{\tau}_w$ is dual to St_u^+. According to Theorem 6.7, $q_{w_0w}\,\mathrm{St}_w^-$ is also dual to St_u^+. Hence, $q_{w_0w}\,\mathrm{St}_w^- = \hat{x}_{w_0}\overset{-*}{\tau}_w$. The second equality in the theorem follows from the definition of $\overset{-*}{\tau}_u$; the third equality follows from Theorem 7.3; the last identity follows the identities

$$\begin{aligned}\Sigma^+\backslash(v\Sigma^-\cap\Sigma^+) &= \Sigma^+\backslash v\Sigma^- = v\left(v^{-1}\Sigma^+\backslash\Sigma^-\right) = v\left(v^{-1}\Sigma^+\cap\Sigma^+\right)\\ &= v\left(v^{-1}w_0\Sigma^-\cap\Sigma^+\right) = v(\Sigma_{v^{-1}w_0}).\square\end{aligned}$$

EXAMPLE 7.6. – From Theorem 7.5 and Lemma 5.2 we have

$$\begin{aligned}\mathrm{St}_w^+|w &= w(a^+_{w^{-1},w^{-1}})x_{-w_0} = (\prod_{\alpha<0,w^{-1}\alpha>0}\tilde{x}_\alpha)\cdot(\prod_{\beta>0,w^{-1}\beta>0}x_{-\beta})\\ &= [\prod_{\alpha<0,w^{-1}\alpha>0}(q-e^\alpha)]\cdot[\prod_{\beta>0,w^{-1}\beta>0}(1-e^\beta)],\\ q_{w_0w}\,\mathrm{St}_w^-|_w &= \hat{x}_{w_0}\frac{x_{-w}}{\hat{x}_w} = (\prod_{\alpha>0,w^{-1}\alpha>0}\hat{x}_\alpha)\cdot(\prod_{\beta>0,w^{-1}\beta<0}x_{-\beta})\\ &= [\prod_{\alpha>0,w^{-1}\alpha>0}(1-qe^{-\alpha})]\cdot[\prod_{\beta>0,w^{-1}\beta<0}(1-e^\beta)].\end{aligned}$$

8. Stable bases of partial flag varieties

Let J be a subset in the set of simple roots, let G/P_J be the partial flag variety corresponding to J. In this section, we consider the stable bases of $K_T(T^*G/P_J)$. The main result of this section (Theorem 8.3) says that such bases coincide with the image of the stable bases of $K_T(T^*G/B)$ via the map (36). We then give an algebraic formula for the stable bases in this case.

8.1. The definition of stable basis

The A-fixed points of T^*G/P_J under the maximal torus A are indexed by the left cosets W/W_J, or by W^J. Moreover, $v\Sigma_J^+ \subset \Sigma^+$ for $v \in W^J$. As in Section 2.1, we can define chambers, partial orders on the fixed points, leaves, slopes and polarizations in the setting of T^*G/P_J. The group $\mathrm{Pic}(T^*G/P_J)$ is isomorphic to the lattice $\{\lambda \in \Lambda | (\lambda, \alpha^\vee) = 0$ for any $\alpha \in J\}$.

We use the following theorem as the definition of stable bases of T^*G/P_J:

THEOREM 8.1. – [36, §9.1] *For any chamber $\mathfrak{C}$, any polarization $T^{\frac{1}{2}}$ of T^*G/P_J, and any rational line bundle $\mathcal{L}$, there exists a unique map of S-modules*

$$\mathrm{stab}^J_{\mathfrak{C},T^{\frac{1}{2}},\mathcal{L}} : K_T((T^*G/P_J)^A) \to K_T(T^*G/P_J),$$

such that for any $w \in W^J$, $\Gamma = \mathrm{stab}^J_{\mathfrak{C},T^{\frac{1}{2}},\mathcal{L}}(w)$ satisfies:

1. *(support)* $\operatorname{supp}\Gamma \subset \mathrm{Slope}_{\mathfrak{C}}(w)$*;*
2. *(normalization)* $\Gamma|_w = (-1)^{\operatorname{rank} N_{w,+}^{\frac{1}{2}}} \left(\dfrac{\det N_{w,-}}{\det N_w^{\frac{1}{2}}}\right)^{\frac{1}{2}} \mathcal{O}_{\mathrm{Leaf}_{\mathfrak{C}}(w)}|_w$*;*
3. *(degree)* $\deg_A\left(\Gamma|_v \otimes \mathcal{L}|_w\right) \subseteq \deg_A\left((\mathrm{stab}^J_{\mathfrak{C},T^{\frac{1}{2}},\mathcal{L}}(v) \otimes \mathcal{L})|_v\right)$*, for any $v \in W^J$ and $v \prec_{\mathfrak{C}} w$,*

where w in $\mathrm{stab}^J_{\mathfrak{C},T^{\frac{1}{2}},\mathcal{L}}(w)$ is the unit in $K_T^*(w)$.

Then the stable basis for $K_T(T^*G/P_J)_{loc}$ is $\{\mathrm{stab}^J_{\mathfrak{C},T^{\frac{1}{2}},\mathcal{L}}(w) | w \in W^J\}$. And we have the following duality property [45, Proposition 1]:

$$\left(\mathrm{stab}^J_{\mathfrak{C},T^{\frac{1}{2}},\mathcal{L}}(v),\ \mathrm{stab}^J_{-\mathfrak{C},T^{\frac{1}{2}}_{\mathrm{opp}},\mathcal{L}^{-1}}(w)\right) = \delta_{v,w}. \tag{34}$$

Let $\leq^J$ denote the Bruhat order on W^J, i.e., for any $v, w \in W^J$, $v \leq^J w$ if $BvP_J/P_J \subset \overline{BwP_J/P_J}$. Similarly to Lemma 3.2, we have

LEMMA 8.2. – *For any $v, w \in W^J$, we have*

1. $\mathrm{stab}^J_{-,T^*G/P_J,\mathcal{L}}(v)|_w = 0$*, unless $v \leq^J w$.*
2. $\mathrm{stab}^J_{-,T^*G/P_J,\mathcal{L}}(v)|_v = q_v^{\frac{1}{2}} \displaystyle\prod_{\beta\in\Sigma^+\setminus\Sigma_J^+, -v\beta\in\Sigma^+} (1 - e^{-v\beta}) \prod_{\beta\in\Sigma^+\setminus\Sigma_J^+, v\beta\in\Sigma^+} (1 - qe^{-v\beta})$.

Proof. – (1) follows from the support condition.

(2). Since we choose the negative chamber $-$, we have

$$A - \text{weights in } N_{v,+} = \{e^{-v\beta} | \beta \in \Sigma^+ \setminus \Sigma_J^+, v\beta > 0\} \cup \{q^{-1}e^{v\beta} | \beta \in \Sigma^+ \setminus \Sigma_J^+, v\beta < 0\},$$

$$A - \text{weights in } N_{v,-} = \{e^{-v\beta} | \beta \in \Sigma^+ \setminus \Sigma_J^+, v\beta < 0\} \cup \{q^{-1}e^{v\beta} | \beta \in \Sigma^+ \setminus \Sigma_J^+, v\beta > 0\},$$

$$A - \text{weights in } N_v^{\frac{1}{2}} = \{q^{-1}e^{v\beta} | \beta \in \Sigma^+ \setminus \Sigma_J^+\}.$$

And $\mathrm{Leaf}(w) = T^*_{B^- vP_J/P_J}(G/P_J)$, where P_J is the corresponding parabolic subgroup. Therefore

$$\begin{aligned}
\mathrm{stab}_-(v)|_v &= (-1)^{\operatorname{rank} N^{\frac{1}{2}}_{v,+}} \left(\frac{\det N_{v,-}}{\det N_v^{\frac{1}{2}}}\right)^{\frac{1}{2}} \mathcal{O}_{\mathrm{Leaf}_{\mathfrak{C}}(v)}|_v \\
&= (-1)^{\ell(v)} \left(\frac{\prod_{\beta\in\Sigma^+\setminus\Sigma_J^+, v\beta<0} e^{-v\beta} \prod_{\beta\in\Sigma^+\setminus\Sigma_J^+, v\beta>0} q^{-1}e^{v\beta}}{\prod_{\beta\in\Sigma^+\setminus\Sigma_J^+} q^{-1}e^{v\beta}}\right)^{\frac{1}{2}} \\
&\quad\times \prod_{\beta\in\Sigma^+\setminus\Sigma_J^+, v\beta<0} (1-e^{v\beta}) \prod_{\beta\in\Sigma^+\setminus\Sigma_J^+, v\beta>0} (1-qe^{-v\beta}) \\
&= q_v^{\frac{1}{2}} \prod_{\beta\in\Sigma^+\setminus\Sigma_J^+, v\beta<0} (1-e^{-v\beta}) \prod_{\beta\in\Sigma^+\setminus\Sigma_J^+, v\beta>0} (1-qe^{-v\beta}). \qquad \square
\end{aligned}$$

Therefore, as in Section 3, we have

$$\begin{aligned}
&\max_\xi(\mathrm{stab}^J_{-,T^*G/P_J,\mathcal{L}}(v)|_v) = (\xi, \sum_{\beta\in\Sigma^+\setminus\Sigma_J^+, v\beta<0} -v\beta), \\
&\min_\xi(\mathrm{stab}^J_{-,T^*G/P_J,\mathcal{L}}(v)|_v) = (\xi, \sum_{\beta\in\Sigma^+\setminus\Sigma_J^+, v\beta>0} -v\beta).
\end{aligned} \tag{35}$$

We have a projection $\pi : G/B \to G/P_J$ and a Lagrangian correspondence $G/B \times_{G/P_J} T^*G/P_J$ in $T^*G/B \times T^*G/P_J$:

$$\begin{tikzcd} T^*G/B & G/B \times_{G/P_J} T^*G/P_J \arrow[l, "p_1"'] \arrow[r, "p_2"] & T^*G/P_J \end{tikzcd} .$$

Therefore, we have the following map:

$$p_{2*}p_1^* : K_T(T^*G/B) \to K_T(T^*G/P_J). \tag{36}$$

Recall $\mathcal{L} = \mathcal{O}(\lambda) \in \operatorname{Pic}(T^*G/B) \otimes_{\mathbb{Z}} \mathbb{Q}$, where λ lies in the fundamental alcove and λ is sufficiently near 0. Let $\mathcal{L}_J := \mathcal{O}(\lambda - \sum_{\alpha\in J}(\lambda, \alpha^\vee)\varpi_\alpha) \in \operatorname{Pic}(T^*G/P_J) \otimes_{\mathbb{Z}} \mathbb{Q}$, where ϖ_α is the fundamental weight associated to the simple root α.

For any $v \in W^J$, denote

$$\mathrm{stab}^J_+(v) = \mathrm{stab}^J_{+,T(G/P_J),\mathcal{L}_J^{-1}}(v), \quad \mathrm{stab}^J_-(v) = \mathrm{stab}^J_{-,T^*(G/P_J),\mathcal{L}_J}(v).$$

The image of the stable bases under the map (36) is given as follows.

Theorem 8.3. – *For any $v \in W^J$, we have*

$$p_{2*}p_1^*(\mathrm{stab}_+(v)) = \mathrm{stab}^J_+(v),$$

and

$$p_{2*}p_1^*(\mathrm{stab}_-(v)) = \mathrm{stab}^J_-(v).$$

We use the rigidity technique from Section 3.

Proof. – Thanks to the duality property (34), the first identity is equivalent to

$$\left(p_{2*}p_1^*(\mathrm{stab}_+(v)), \mathrm{stab}_-^J(u)\right) = \delta_{v,u},$$

for any $u \in W^J$, which we now prove.

Similar to the argument in Proposition 4.3, the support condition of stable basis Theorem 2.2(1) and the properness of the intersection of support [30, Theorem 4.6.1] together imply that $\left(p_{2*}p_1^*(\mathrm{stab}_+(v)), \mathrm{stab}_-^J(u)\right)$ is an element in $K_T(\mathrm{pt})$, i.e., a Laurent polynomial. Localizing to the T-fixed points, we get

$$\left(p_{2*}p_1^*(\mathrm{stab}_+(v)), \mathrm{stab}_-^J(u)\right) = \left(p_1^*(\mathrm{stab}_+(v)), p_2^*(\mathrm{stab}_-^J(u))\right) \tag{37}$$

$$= \sum_{\substack{w\in W^J\\ y\in wW_J\\ y\leq v, u\leq^J w}} \frac{\mathrm{stab}_+(v)|_y\, \mathrm{stab}_-^J(u)|_w}{\bigwedge^\bullet T_{(y,w)}(G/B\times_{G/P_J} T^*G/P_J)}. \tag{38}$$

Note that

$$\bigwedge\nolimits^\bullet T_{(y,w)}(G/B\times_{G/P_J} T^*G/P_J) = \prod_{\beta>0}(1-e^{y\beta}) \prod_{\beta\in\Sigma^+\setminus\Sigma_J^+}(1-qe^{-w\beta}). \tag{39}$$

Let ξ be as in Lemma 3.1. In particular,

$$\max_\xi\left(\bigwedge\nolimits^\bullet T_{(y,w)}(G/B\times_{G/P_J} T^*G/P_J)\right) = \sum_{\beta>0, y\beta>0}(\xi, y\beta) + \sum_{\beta\in\Sigma^+\setminus\Sigma_J^+, w\beta<0}(\xi, -w\beta).$$

By the third conditions of Theorem 2.2 and Theorem 8.1, and Equations (3) and (35), we have

$$\max_\xi(\mathrm{stab}_+(v)|_y\, \mathrm{stab}_-^J(u)|_w) \leq \sum_{\substack{\beta>0\\ y\beta>0}}(\xi, y\beta) + \sum_{\substack{\beta\in\Sigma^+\setminus\Sigma_J^+\\ w\beta<0}}(\xi, -w\beta) + (\xi, \mathcal{L}|_v - \mathcal{L}|_y + \mathcal{L}_J|_w - \mathcal{L}_J|_u).$$

Since $v \geq y$ and $w \geq^J u$, Lemma 4.2 shows that

$$(\xi, \mathcal{L}|_v - \mathcal{L}|_y + \mathcal{L}_J|_w - \mathcal{L}_J|_u) \leq 0,$$

with strict inequality if $u \neq v$.

Now we analyze separately the following two cases: $u \neq v$ and $u = v$. In the case when $u \neq v$, we have

$$\lim_{t\to\infty}\left(p_{2*}p_1^*(\mathrm{stab}_+(v)), \mathrm{stab}_-^J(u)\right)(t\xi) = 0.$$

To analyze the limit as t goes to $-\infty$, we may assume λ sufficiently small so that

$$(\xi, \mathcal{L}|_v - \mathcal{L}|_y + \mathcal{L}_J|_w - \mathcal{L}_J|_u) > -1.$$

Here $\mathcal{L} = \mathcal{O}(\lambda)$. Under this condition, keeping in mind that $u \neq v$, we have

$$\lim_{t\to-\infty}\left(p_{2*}p_1^*(\mathrm{stab}_+(v)), \mathrm{stab}_-^J(u)\right)(t\xi) \text{ is bounded.}$$

Hence, by Lemma 3.1,

$$\left(p_{2*}p_1^*(\mathrm{stab}_+(v)), \mathrm{stab}_-^J(u)\right) = 0.$$

In the case when $u = v \in W^J$, we have

$$\{(y,w) \mid y \in W, w \in W^J, y \in wW_J, y \leq v, u \leq^J w\} = \{(u,v)\}.$$

Indeed, any (y,w) in the left hand side satisfies

$$\ell(y) \leq \ell(v) = \ell(u) \leq \ell(w) \leq \ell(y),$$

hence also in the right hand side. Therefore, the summation (37) has only one term. Using Lemma 3.2, Lemma 8.2, Equation (39), keeping in mind that $v\Sigma_J^+ \subset \Sigma^+$, we get

$$\left(p_{2*}p_1^*(\mathrm{stab}_+(v)), \mathrm{stab}_-^J(v)\right) = \frac{\mathrm{stab}_+(v)|_v \, \mathrm{stab}_-^J(v)|_v}{\bigwedge^\bullet T_{(v,u)}(G/B \times_{G/P_J} T^*G/P_J)} = 1.$$

This proves the identity $p_{2*}p_1^*(\mathrm{stab}_+(v)) = \mathrm{stab}_+^J(v)$. The identity $p_{2*}p_1^*(\mathrm{stab}_-(v)) = \mathrm{stab}_-^J(v)$ is proved using the same argument. ☐

8.2. More on the twisted group algebra

Let $\pi : G/B \to G/P_J$ be the canonical map. By [13, Lemma 10.12], $Y_J \in \mathbb{D}$. Indeed, $Y_{\{\alpha_i\}} = Y_i$. It follows from Kostant-Kumar (or see [11, Theorem 8.2 and Corollary 8.7] for more details) that we have commutative diagrams

$$\begin{tikzcd}
K_T(G/B) \arrow[r, "\pi^*\pi_*"] \arrow[d, "\sim"] & K_T(G/B) \arrow[d, "\sim"] \\
\mathbb{D}^* \arrow[r, "Y_J \bullet \_"] & \mathbb{D}^*
\end{tikzcd}
\qquad
\begin{tikzcd}
K_T(G/P_J) \arrow[r] \arrow[d, "\sim"] & K_T(\mathbb{C}) \arrow[d, "\sim"] \\
(\mathbb{D}^*)^{W_J} \arrow[r, "Y_{\Pi/J} \bullet \_"] & (\mathbb{D}^*)^W.
\end{tikzcd}$$

Here the top horizontal map in the second diagram is induced by the structure map $G/P_J \to \mathrm{Spec}(\mathbb{C})$.

Let $p : T^*G/B \to G/B$ and $p_J : T^*G/P_J \to G/P_J$ be the canonical projections. Then

$$Q_W^* \cong K_T(G/B) \otimes_S Q \underset{\sim}{\xrightarrow{p^*}} K_T(T^*G/B) \otimes_S Q,$$

$$(Q_W^*)^{W_J} \cong K_T(G/P_J) \otimes_S Q \underset{\sim}{\xrightarrow{p_J^*}} K_T(T^*G/P_J) \otimes_S Q.$$

Moreover, $p^* = \hat{x}_{w_0} \bullet \_$ and $p_J^* = \frac{\hat{x}_{w_0}}{\hat{x}_{w_0^J}} \bullet \_$.

Via these isomorphisms, $\hat{Y}_\Pi \bullet \_$ and $\hat{Y}_{\Pi/J} \bullet \_$ coincide with the following composites, respectively:

$$K_T(T^*G/B) \underset{\sim}{\xrightarrow{(p^*)^{-1}}} K_T(G/B) \longrightarrow K_T(\mathbb{C}), \quad K_T(T^*G/P_J) \underset{\sim}{\xrightarrow{(p_J^*)^{-1}}} K_T(G/P_J) \longrightarrow K_T(\mathbb{C}).$$

Concerning the map $p_{2*}p_1^*$ in (36), we have

LEMMA 8.4. – *1. We have a commutative diagram*

$$\begin{tikzcd}
K_T(G/B) \arrow[r, "p^*"] \arrow[d, "\pi_*"] & K_T(T^*G/B) \arrow[d, "p_{2*}p_1^*"] \\
K_T(G/P_J) \arrow[r, "p_J^*"] & K_T(T^*G/P_J).
\end{tikzcd}$$

That is, $p_J^* \circ \pi_* = (p_{2*}p_1^*) \circ p^*$.

2. *Via the* $\bullet$*-action of* Q_W *on* Q_W^**, we have* $p_{2*}p_1* = \hat{Y}_J$.

Proof. – (1). This follows from the proper base change property of K-theory.
(2). We know that $\pi_* = Y_J \bullet \_$, $p_J^* = \frac{\hat{x}_{w_0}}{\hat{x}_{w_0^J}} \bullet \_$ and $p^* = \hat{x}_{w_0} \bullet \_$. So

$$p_{2*}p_1^* = p_J^* \circ \pi_* \circ (p^*)^{-1} = \frac{\hat{x}_{w_0}}{\hat{x}_{w_0^J}} \sum_{w \in W_J} \delta_w \frac{1}{x_{-w_0^J}} \frac{1}{\hat{x}_{w_0}} \overset{\sharp}{=} \sum_{w \in W_J} \delta_w \frac{1}{x_{-w_0^J}\hat{x}_{w_0^J}} = \hat{Y}_J.$$

Here to show $\sharp$, note that in the case $w \in W_J$, we have $w(\Sigma^+ \backslash \Sigma_J^+) = \Sigma^+ \backslash \Sigma_J^+$ and consequently $w(\frac{\hat{x}_{w_0}}{\hat{x}_{w_0^J}}) = \frac{\hat{x}_{w_0}}{\hat{x}_{w_0^J}}$. ☐

8.3. The algebraic descriptions

Definition 8.5. – Let $w \in W^J$. We define elements in Q_W^* by

$$\mathrm{St}_w^{+,J} := \hat{Y}_J \bullet \mathrm{St}_w^+ = \hat{Y}_J \bullet (\overset{+}{\tau}_{w^{-1}} \bullet \mathrm{pt}_e), \quad \mathrm{St}_w^{-,J} := \hat{Y}_J \bullet \mathrm{St}_w^- = \hat{Y}_J \bullet ((\overline{\tau}_{w_0 w})^{-1} \bullet \mathrm{pt}_{w_0}).$$

Theorem 8.6. – *For any* $w \in W^J$*, denote* $g_w = \sum_{u \in W_J} f_{wu}$*, then we have*

$$\mathrm{St}_w^{+,J} = \sum_{v \le w, v \in W^J} x_{w_0} v(\frac{a^+_{w^{-1},v^{-1}}}{x_{-w_0^J}\hat{x}_{w_0^J}}) g_v,$$

$$q_{w_0 v}\,\mathrm{St}_v^{-,J} = \sum_{v \ge w, v \in W^J} x_{-v} K^\tau_{w,v} v(\frac{\hat{x}_{v^{-1}w_0}}{x_{-w_0^J}\hat{x}_{w_0^J}}) g_v = \sum_{v \ge w, v \in W^J} \frac{\hat{x}_{w_0} b^-_{v,w}}{v(x_{-w_0^J}\hat{x}_{w_0^J})} g_v.$$

Proof. – It follows from the definition of $\hat{Y}_J$ in (24), the identities in (20), and Theorem 7.5. ☐

The following give a purely algebraic description of the geometrically defined stable bases.

Corollary 8.7. – *We have*

$$\mathrm{stab}_+^J(w) = q_w^{-\frac{1}{2}}\,\mathrm{St}_w^{+,J}, \qquad \mathrm{stab}_-^J(w) = q_{w_0} q_w^{-\frac{1}{2}}\,\mathrm{St}_w^{-,J}.$$

Proof. – According to Lemma 8.4.(1), $p_{2*}p_1^* = \hat{Y}_J \bullet \_$. This corollary now follows from Corollary 6.4. ☐

Corollary 8.8. – $\hat{Y}_{\Pi/J} \bullet (\mathrm{St}_w^{+,J} \cdot \mathrm{St}_v^{-,J}) = \delta_{w,v} q_{w_0 v}^{-1} \mathbf{1}$.

Proof. – Let $\pi^J : G/P_J \to \mathbb{C}$ be the structure map. The composition

$$K_T(T^*G/P_J) \xrightarrow{(p_J^*)^{-1}} K_T(G/P_J) \overset{\pi_*^J}{\to} K_T(\mathbb{C})$$

is given by the formula $\hat{Y}_{\Pi/J} \bullet \_ : (Q_W^*)^{W_J} \to (Q_W^*)^W$. By (34) and Corollary 8.7, we have

$$\delta_{w,v}\mathbf{1} = \hat{Y}_{\Pi/J} \bullet (\mathrm{stab}_+^J(w) \cdot \mathrm{stab}_-^J(v)) = \hat{Y}_{\Pi/J} \bullet (q_w^{-\frac{1}{2}}\,\mathrm{St}_w^{+,J} \cdot q_{w_0} q_v^{-\frac{1}{2}}\,\mathrm{St}_v^{-,J}),$$

the conclusion then follows. ☐

Corollary 8.9. – *Expressing* $\overline{\tau}_w \hat{Y}_J (\overline{\tau}_{w_0 v})^{-1}$ *in terms of* $\overline{\tau}_u, u \in W$*, the coefficient in front of* $\overline{\tau}_{w_0}$ *is equal to* $\delta_{w,v} q_{w_0 v}^{-1}$.

Proof. – Denote this coefficient by c. By definition of $\hat{Y}_J$ we know that $\mathrm{St}_w^{\pm,J} \in (Q_W^*)^{W_J}$. By Corollary 8.8 we have

$$\begin{aligned} q_{w_0v}^{-1}\delta_{w,v}\mathbf{1} &\overset{\sharp_1}{=} \hat{Y}_{\Pi/J} \bullet [(\hat{Y}_J \bullet \mathrm{St}_w^+) \cdot (\hat{Y}_J \bullet \mathrm{St}_v^-)] \overset{\sharp_2}{=} \hat{Y}_{\Pi/J} \bullet [\hat{Y}_J \bullet (\mathrm{St}_w^+ \cdot (\hat{Y}_J \bullet \mathrm{St}_v^-))] \\ &\overset{\sharp_3}{=} \hat{Y}_\Pi \bullet [\mathrm{St}_w^+ \cdot (\hat{Y}_J \bullet \mathrm{St}_v^-)] = \hat{Y}_\Pi \bullet [(\overset{+}{\tau}_{w^{-1}} \bullet \mathrm{pt}_e) \cdot (\hat{Y}_J (\overline{\tau}_{w_0v})^{-1} \bullet \mathrm{pt}_{w_0})] \\ &\overset{\sharp_4}{=} \hat{Y}_\Pi \bullet [\mathrm{pt}_e \cdot (\overline{\tau}_w \hat{Y}_J (\overline{\tau}_{w_0v})^{-1} \bullet \mathrm{pt}_{w_0})] \\ &\overset{\sharp_5}{=} \hat{Y}_\Pi \bullet [(x_{-w_0} f_e) \cdot (c\overline{\tau}_{w_0} \bullet x_{-w_0} f_{w_0})] \overset{\sharp_6}{=} c\frac{x_{-w_0}\hat{x}_{-w_0}}{x_{-w_0}\hat{x}_{w_0}}\mathbf{1} = c\mathbf{1}. \end{aligned}$$

Here $\sharp_1$ follows from the definition of $\mathrm{St}_w^{\pm,J}$; $\sharp_2$ follows from the projection Formula (26); $\sharp_3$ follows from (25); $\sharp_4$ follows from Lemma 6.1; $\sharp_5$ follows from similar idea in the proof of Theorem 6.7; $\sharp_6$ follows from (20). We then have $c = q_{w_0v}^{-1}$. □

This corollary is the parabolic version of Lemma 5.6 and hence a generalization of [34, Proposition 3]. Geometrically, this corollary (resp. Lemma 5.6) reflects the fact that the stable bases of $K_T(T^*G/P_J)$ (resp. of $K_T(T^*G/B)$) corresponding to the opposite chambers are dual via the K-theory pairing.

9. Relations with p-adic unramified principal series representations

In this section we compare the K-theory stable basis and the T-fixed point basis with certain bases in unramified principal series of p-adic groups. The nature of such a connection is the local geometric Langlands duality. Hence, the stable bases of $T^*(G^\vee/(B^{\vee,-}))$ will be considered in comparison with representations of $G_{\mathbb{Q}_p}$.

For the convenience of the readers, we also give a K-theory interpretation of the intertwiners, Macdonald's formula for the spherical functions [29, 14], and the Casselman-Shalika formula for Whittaker functions [15] from p-adic representations.

9.1. Results from p-adic representations

9.1.1. *Notations.* – First, we recall some notions from p-adic representations, following [37].

Let F be a nonarchimedean local field, with ring of integers $\mathcal{O}$, a uniformizer $\varpi \in \mathcal{O}$, and residue field $\mathbb{F}_q$. Let G_F be a split reductive group over F, with maximal torus A_F and Borel subgroup $B_F = A_F N_F$. Let I be an Iwahori subgroup, i.e., the inverse image of $B(\mathbb{F}_q)$ under the evaluation map $G(\mathcal{O}) \to G(\mathbb{F}_q)$. Note that the notations here differ from [37], where in *loc. cit.*, B is used to denote the Iwahori subgroup, and P denotes the Borel subgroup. To simplify notations, we let α, β denote the coroots of G. We also have the following decomposition

$$G_F = \bigsqcup_{w\in W} B_F w I.$$

Let $\mathbb{H} = \mathbb{C}_c[I\backslash G_F/I]$ be the Iwahori Hecke algebra. It has two subalgebras, the finite Hecke algebra H_W, and a commutative subalgebra Θ which is isomorphic to the coordinate ring $\mathbb{C}[A^\vee]$ of the complex dual torus $A^\vee = \mathbb{C}^* \otimes X^*(A)$. More precisely, Θ has a $\mathbb{C}$-linear basis $\{\theta_a \mid a \in A_F/A_\mathcal{O}\}$. For any coroot α of G, let $h_\alpha : F^\times \to A_F$ be the corresponding one parameter subgroup. The isomorphism $\Theta \simeq \mathbb{C}[A^\vee]$ maps $\theta_{h_\alpha(\varpi)}$ to $e^\alpha \in X^*(A^\vee) \subset \mathbb{C}[A^\vee]$. So for any character τ of A, we have $e^\alpha(\tau) = \tau(h_\alpha(\varpi))$. We have the following pairing

$$\langle\, ,\rangle : A_F/A_{\mathcal{O}} \times A^\vee \to \mathbb{C}^*$$

given by

$$\langle a, z \otimes \lambda\rangle = z^{\mathrm{val}(\lambda(a))}.$$

This induces an isomorphism between $A_F/A_{\mathcal{O}}$ and the group $X^*(A^\vee)$ of rational characters of $A^\vee$. It also induces an identification between $A^\vee$ and unramified characters of A, i.e., characters which are trivial on $A_{\mathcal{O}}$. As a $\mathbb{C}$-vector space, we have

$$\mathbb{H} = \Theta \otimes_{\mathbb{C}} H_W.$$

Let τ be an unramified character of A avoiding all the root hyperplanes. We consider the induced representation $I(\tau) = \mathrm{Ind}_B^G(\tau)$. As a $\mathbb{C}$-vector space, $\mathrm{Ind}_B^G(\tau)$ consists of locally constant functions f on G_F such that $f(bg) = \tau(b)\delta^{\frac{1}{2}}(b)f(g)$ for any $b \in B_F$, where $\delta(b) := \prod_{\alpha>0} |\alpha^\vee(a)|_F$ is the modulus function on the Borel subgroup. The algebra $\mathbb{H}$ acts through convolution from the right on the Iwahori invariant subspace $I(\tau)^I$, so that the restriction of this action to H_W is a regular representation. This right action is denoted by $\pi : \mathbb{H} \to \mathrm{End}_{\mathbb{C}}(I(\tau)^I)$.

9.1.2. *Interwiners.* – For any character τ and $x \in W$, we can define $x\tau \in X^*(A)$ by the formula $x\tau(a) := \tau(x^{-1}ax)$ for any $a \in A$. Since we assume τ is unramified and has trivial stabilizer under the Weyl group action, the space $\mathrm{Hom}_G(I(\tau), I(x^{-1}\tau))$ is one dimensional, spanned by an operator $\mathscr{A}_x = \mathscr{A}_x^\tau$ (2) defined by

$$\mathscr{A}_x(\varphi)(g) := \int_{N_x} \varphi(\dot{x}ng)dn,$$

where $\dot{x}$ is a representative of $x \in W$, $N_x = N \cap \dot{x}^{-1}N^-\dot{x}$ with N (resp. N^-) being the unipotent radical of the (opposite) Borel subgroup B, and the measure on N_x is normalized by the condition that *vol*$(N_x \cap G(\mathcal{O})) = 1$ [37]. If $x, y \in W$ satisfy $\ell(x) + \ell(y) = \ell(xy)$, then $\mathscr{A}_y^{x^{-1}\tau}\mathscr{A}_x^\tau = \mathscr{A}_{xy}^\tau$.

For any coroot α, let

$$c_\alpha = \frac{1 - q^{-1}e^\alpha(\tau)}{1 - e^\alpha(\tau)}. \tag{40}$$

We normalize the intertwiner as in [20, Section 2.2] as follows:

$$I_w^\tau := \prod_{\alpha>0, w^{-1}\alpha<0} \frac{1}{c_\alpha}\mathscr{A}_w^\tau.$$

Then for any simple coroot α and any $y, w \in W$, we have

$$I_{s_\alpha}^{s_\alpha\tau} I_{s_\alpha}^\tau = 1, \quad \textit{and} \quad I_w^{y^{-1}\tau} I_y^\tau = I_{yw}^\tau. \tag{41}$$

---

(2) This intertwiner $\mathscr{A}_x$ is related to the one T_x in [14] by the formula $\mathscr{A}_x = T_{x^{-1}}$.

9.1.3. *Bases in Iwahori-invariants.* – There are two bases of interest in $I(\tau)^I$. One of these bases consists of normalized characteristic functions on the orbits, denoted by $\{\varphi_w^\tau \mid w \in W\}$. Here for any $w \in W$ the element φ_w^τ is characterized by the two conditions [37, pg. 319]:

1. φ_w^τ is supported on $B_F wI$;
2. $\varphi_w^\tau(bwg) = \tau(b)\delta^{\frac{1}{2}}(b)$ for any $b \in B_F$ and $g \in I$.

The action of $\mathbb{H}$ on $I(\tau)^I$ has explicit formula under this basis [37, pg. 325]. For any simple coroot α, we have

$$\pi(T_{s_\alpha})(\varphi_w^\tau) = \begin{cases} q\varphi_{ws_\alpha}^\tau + (q-1)\varphi_w^\tau, & \text{if } ws_\alpha < w; \\ \varphi_{ws_\alpha}^\tau, & \text{if } ws_\alpha > w. \end{cases} \tag{42}$$

Under the intertwiner $I_{s_\alpha}^\tau$, this basis behaves as follows [14, Theorem 3.4]

$$I_{s_\alpha}^\tau(\varphi_w^\tau) = \begin{cases} \frac{1}{qc_\alpha}\varphi_{s_\alpha w}^{s_\alpha\tau} + (1-\frac{1}{c_\alpha})\varphi_w^{s_\alpha\tau}, & \text{if } s_\alpha w > w; \\ \frac{1}{c_\alpha}\varphi_{s_\alpha w}^{s_\alpha\tau} + (1-\frac{1}{qc_\alpha})\varphi_w^{s_\alpha\tau}, & \text{if } s_\alpha w < w. \end{cases} \tag{43}$$

The second basis is called the Casselman's basis, denoted by $\{f_w^\tau \mid w \in W\}$. It consists of Θ-eigenvectors in $I(\tau)^I$, and is further characterized in terms of the intertwining operators by the following formula ⁽³⁾

$$\mathscr{A}_x^\tau(f_w^\tau)(1) = \delta_{x,w}.$$

The formula of the Hecke algebra action under this basis is also known [37, Lemma 4.1 and Proposition 4.9]. For any simple coroot α and $w \in W$, write

$$J_{\alpha,w} = \begin{cases} c_{w\alpha}c_{-w\alpha}, & \text{if } ws_\alpha > w; \\ 1, & \text{if } ws_\alpha < w. \end{cases}$$

Then, we have

$$\pi(T_{s_\alpha})(f_w^\tau) = q(1-c_{w\alpha})f_w^\tau + qJ_{\alpha,w}f_{ws_\alpha}^\tau, \tag{44}$$

$$\pi(\theta_a)f_w^\tau = \tau(waw^{-1})f_w^\tau = (w^{-1}\tau(a))f_w^\tau, \text{ for any } a \in A, \tag{45}$$

and ([37, Theorem 4.2])

$$I_{s_\alpha}^\tau(f_w^\tau) = \begin{cases} c_{-\alpha}f_{s_\alpha w}^{s_\alpha\tau}, & \text{if } s_\alpha w > w; \\ \frac{1}{c_\alpha}f_{s_\alpha w}^{s_\alpha\tau}, & \text{if } s_\alpha w < w. \end{cases} \tag{46}$$

⁽³⁾ This basis is related to the one in [14] by an inversion of the index w.

9.1.4. *Transition matrices.* – The change of bases matrix

$$f_w^\tau = \sum_{y \geq w} a_{w,y}(\tau)\varphi_y^\tau \tag{47}$$

is interesting (see, e.g., [34]). It is clear that $a_{w,w} = 1$. A formula for the generating series of general $a_{w,y}$ is given in [37, Proposition 5.2], in terms of the canonical basis of [21]. The inverse matrix $b_{w,y}(\tau)$ defined by

$$\varphi_w^\tau = \sum_{y \geq w} b_{w,y}(\tau) f_y^\tau \tag{48}$$

plays an important role in explicit computations of the Whittaker function on φ_w^τ (see, e.g., [38]).

In Corollary 9.6, we explain how Theorem 1.2 gives a closed formula for these two matrices.

9.1.5. *Macdonald's formula for spherical function.* – In this section, we review the Macdonald's formula for spherical functions.

According to the Iwasawa decomposition $G = BG(\mathcal{O})$, the vector space $I(\tau)^{G(\mathcal{O})}$ is one dimensional. Let ϕ^τ be the basis normalized by the condition that $\phi^\tau(1) = 1$. Then we have ([14])

$$\phi^\tau = \sum_w \varphi_w^\tau = \sum_w \prod_{\alpha>0, w^{-1}\alpha<0} \frac{1-q^{-1}e^\alpha(\tau)}{1-e^\alpha(\tau)} f_w^\tau.$$

It follows from either (43) or (46) that

$$I_w^\tau(\phi^\tau) = \phi^{w^{-1}\tau}. \tag{49}$$

This formula is refereed to as the Gindikin-Karpelevich formula in literature.

We consider a sesquilinear paring $\langle -, - \rangle : I(\tau^{-1}) \otimes I(\tau) \to \mathbb{C}$ [20, § 1.9]. For any $g \in G(F)$, we consider the following matrix coefficient

$$\Gamma_\tau(g) = \langle g \cdot \phi^\tau, \phi^{\tau^{-1}} \rangle. \tag{50}$$

It satisfies

$$\Gamma_\tau(1) = 1, \Gamma_\tau = \Gamma_{w\tau},$$

and

$$\Gamma_\tau(k_1 g k_2) = \Gamma_\tau(g)$$

for any $k_1, k_2 \in G(\mathcal{O})$, and $g \in G$. This gives a well-defined $\mathbb{C}$-valued function on $G(\mathcal{O})\backslash G(F)/G(\mathcal{O})$. This function Γ_τ is called the zonal spherical function corresponding to τ.

Let $X_*(A)_+$ be the dominant coweights. By the Cartan decomposition,

$$G(F) = \bigsqcup_{\mu \in X_*(A)_+} G(\mathcal{O})\varpi^\mu G(\mathcal{O}),$$

in order to know this function, it sufficed to know the value of Γ_τ at the ϖ^μ's, where $\varpi^\mu = h_\mu(\varpi)$.

For any dominant coweight μ of G, the characteristic function $1_{I\varpi^{\mu}I}$ is an element in the affine Hecke algebra $\mathbb{H} = \mathbb{C}_c[I\backslash G/I]$. Let

$$e_{I\varpi^{\mu}I} = \frac{1_{I\varpi^{\mu}I}}{\operatorname{vol}(I\varpi^{\mu}I)} = \delta_B(\varpi^{\mu})^{\frac{1}{2}}\theta_{\mu} \in \mathbb{H}. \tag{51}$$

Then by the definition of Γ_τ, we have

$$\Gamma_\tau(\varpi^{\mu}) = \langle \pi(e_{I\varpi^{\mu}I})(\phi^{\tau}), \phi^{\tau^{-1}}\rangle. \tag{52}$$

The following is the Macdonald formula.

Theorem 9.1 ([29]). – *Let Q be the volume of Bw_0B and $\mu \in X_*(A)_+$. We have*

$$\Gamma_\tau(\varpi^{\mu}) = \frac{\delta_B^{\frac{1}{2}}(\varpi^{\mu})}{Q}\sum_{w\in W} e^{w\mu}(\tau)\prod_{\beta>0}\frac{1-q^{-1}e^{-w\beta}(\tau)}{1-e^{-w\beta}(\tau)},$$

where δ_B is the modulus function on the Borel subgroup B.

The proof by Casselman [14] uses the W-invariance of the function and the eigenbasis f_w^{τ}. This formula gives the Satake transform for the spherical Hecke algebra [20, Theorem 5.6.1].

In Theorem 9.9, we give an equivariant K-theoretic interpretation of this formula.

9.1.6. *Casselman-Shalika formula.* – In this section, we review the Casselman-Shalika formula for the Whittaker functions, see [15].

Recall N is the unipotent radical of the Borel subgroup B, and $\prod_{\alpha\in\Pi} N_\alpha$ is a quotient of N, where the product runs over all simple roots, and N_α is the corresponding root subgroup, all of which are isomorphic to the additive group. Given characters σ_α of N_α, the product $\sigma := \prod \sigma_\alpha$ is a character of N. We say σ is principle if all the σ_α are non-trivial. We say σ is unramified if all the characters σ_α are trivial on $\mathcal{O}$, but nontrivial on $\varpi^{-1}\mathcal{O}$. From now on, we assume σ is principal and unramified. Let $\operatorname{Ind}_N^G \sigma$ be the induced representation.

For every unramified character τ as before, a *Whittaker functional* on $I(\tau)$ is a $\mathbb{C}$-module map

$$L : I(\tau) \to \mathbb{C},$$

such that $L(n\phi) = \sigma(n)L(\phi)$ for any $n \in N$ and $\phi \in I(\tau)$. It is proved in [43] (see also [15]) that the space of Whittaker functionals is one-dimensional. For any $f \in I(\tau)$, define $\mathcal{W}_\tau(f) : G \to \mathbb{C}$ by

$$\mathcal{W}_\tau(f)(g) := L(gf).$$

Then $\mathcal{W}_\tau(f)$ is a function on G satisfying

$$\mathcal{W}_\tau(f)(ng) = \sigma(n)\mathcal{W}_\tau(f)(g), \quad \textit{if} \quad n \in N.$$

And $f \mapsto \mathcal{W}_\tau(f)$ is a G-map from $I(\tau)$ to $\operatorname{Ind}_N^G \sigma$, denoted by $\mathcal{W}_\tau$.(4) It follows from [15, Proposition 2.1] that for fixed $g \in G$ and $f \in I(\tau)$, the function $\tau \to \mathcal{W}_\tau(f)(g)$ is a polynomial function on the dual torus $A^\vee$.

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(4) This is the notation used in [38]. We normalize our L such that our $\mathcal{W}_\tau(f)(g)$ coincides with the one in *loc. cit.*

The Whittaker functional $\mathcal{W}_\tau$ enjoys the following properties (see [38, Equation (1.3), Proposition 3.1]) :

$$\mathcal{W}_{w^{-1}\tau} I_w^\tau = \prod_{\beta>0, w^{-1}\beta<0} \frac{1-q^{-1}e^{-\beta}(\tau)}{1-q^{-1}e^{\beta}(\tau)} \mathcal{W}_\tau, \tag{53}$$

and for every dominant coweight μ,

$$\mathcal{W}_\tau(f_w^\tau)(\varpi^\mu) = \delta_B^{\frac{1}{2}}(\varpi^\mu) e^{w\mu}(\tau) \prod_{\beta>0, w^{-1}\beta>0} \frac{1-q^{-1}e^{\beta}(\tau)}{1-e^{-\beta}(\tau)}. \tag{54}$$

Recall we have the spherical function $\phi^\tau \in I(\tau)^{G(\mathcal{O})}$. We define the Whittaker function

$$W_\tau(g) := \mathcal{W}_\tau(\phi^\tau)(g) = L(g\phi^\tau).$$

The Casselman-Shalika formula is an explicit formula for W_τ. Since W_τ is right $G(\mathcal{O})$-invariant, and for any $n \in N$,

$$W_\tau(ng) = \sigma(n) W_\tau(g),$$

we only need to determine the value of W_τ at the elements ϖ^μ for any coweight $\mu \in X_*(A)$. Moreover, $W_\tau(\varpi^\mu) = 0$, unless μ is dominant, cause if not, there exists some $x \in N_\alpha \cap G(\mathcal{O})$, such that $\sigma_\alpha(\varpi^\mu x \varpi^{-\mu})$ is nontrivial. However,

$$W_\tau(\varpi^\mu) = W_\tau(\varpi^\mu x) = \sigma_\alpha(\varpi^\mu x \varpi^{-\mu}) W_\tau(\varpi^\mu),$$

forcing $W_\tau(\varpi^\mu) = 0$.

Assume μ is dominant. By the Iwahori factorization $I = (I \cap \bar{B})(I \cap N)$, we have $I\varpi^{-\mu}I = I\varpi^{-\mu}(I \cap N)$. Since σ is trivial on $I \cap N$ and ϕ^τ is invariant under I, we have [20, Theorem 6.5.1]

$$W_\tau(\varpi^\mu) = L(\pi(e_{I\varpi^{-\mu}I})\phi^\tau). \tag{55}$$

The Casselman-Shalika formula is given by the following theorem.

THEOREM 9.2. – [15, Theorem 5.4] *Let μ be a dominant coweight of G, then*

$$\begin{aligned} W_\tau(\varpi^\mu) &= \delta_B^{\frac{1}{2}}(\varpi^\mu) \prod_{\beta>0} (1-q^{-1}e^{\beta}(\tau)) \sum_w \frac{e^{w\mu}(\tau)}{\prod_{\beta>0}(1-e^{-w\beta}(\tau))} \\ &= \delta_B^{\frac{1}{2}}(\varpi^\mu) \prod_{\beta>0} (1-q^{-1}e^{\beta}(\tau)) E_\mu(\tau), \end{aligned}$$

where E_μ is the character of the representation of the Langlands dual group $G^\vee$ having highest weight μ.

9.2. Bases in equivariant K-theory for the complex dual group $G^\vee$

From now on we only consider K-theory with $\mathbb{C}$ coefficients. The Iwahori-Hecke algebra $\mathbb{H}$ (with $\mathbb{C}$ coefficients) of G_F can be expressed in terms of the complex reductive group $G^\vee$, whose root datum is Langlands dual to that of G_F.

More precisely, let $G^\vee$ be the complex reductive group , whose root datum is Langlands dual to that of G_F. We fix the set of positive simple roots of $G^\vee$ determined by B_F. In particular, the maximal torus of $G^\vee$ is naturally isomorphic to $A^\vee$, the complex dual of $A_F \subset G_F$. Let $B^\vee = B^{\vee,-}$ be the Borel subgroup of $G^\vee$ associated to the negative roots.

Remark 9.3. – We honor the conventions adapted in the literature of local geometric Langlands duality (see, e.g., [39, 5]), and consider complex-valued functions on G_F and coherent sheaves on $T^*G^\vee/(B^{\vee,-})$. In particular, the isomorphism $\mathbb{H} \simeq K_{G^\vee\times\mathbb{C}^*}(Z)$ used in this section is the same as [39, Prop. 6.1.5]. Note that this is different from § 5, where the stable bases considered were associated to T^*G/B where G is complex reductive, and B is the Borel subgroup associated to the positive roots. Therefore, we describe the modified isomorphism below, and hence modify the formulas from § 4 correspondingly.

Under the modified isomorphism [39, Prop. 6.1.5], $e^\lambda \in X^*(A^\vee)$ is mapped to $\pi_\Delta^*(\mathcal{O}(\lambda))$ with $\mathcal{O}(\lambda)$ being the line bundle on $T^*(G^\vee/B^\vee)$; $\pi_\Delta : Z_\Delta = \Delta(T^*(G^\vee/B^\vee)) \to T^*(G^\vee/B^\vee)$; the operator $T_\alpha \in \mathbb{H}$ for simple root α is mapped to

$$-[\mathcal{O}_\Delta] - [\mathcal{O}_{T^*_{Y_\alpha}}(-\rho, \rho-\alpha)].$$

The above isomorphism has symmetry$[\mathcal{O}_{T^*_{Y_\alpha}}(-\rho, \rho-\alpha)] \simeq [\mathcal{O}_{T^*_{Y_\alpha}}(\rho-\alpha, -\rho)]$ [39, Lemma 1.5.1]. Hence, the right convolution and the left convolution by T_α will give the same operator on $K_{A^\vee\times\mathbb{C}^*}(T^*(G^\vee/B^\vee))$. In what follows, the right convolution action of $\mathbb{H}$ on $K_{A^\vee\times\mathbb{C}^*}(T^*(G^\vee/B^\vee))$ will be denoted by π. As in § 4, we use T_α (resp. T'_α) to denote the left (resp. right) convolution by the simple generator of $\mathbb{H}$. The relations between these convolution operators are

$$\pi(T_\alpha) = \mathcal{O}(-\rho)T_\alpha\mathcal{O}(\rho) = \mathcal{O}(\rho)T'_\alpha\mathcal{O}(-\rho), \tag{56}$$

where $\mathcal{O}(\pm\rho)$ is the operator of multiplication by the line bundle. Note that the second equality also gives a geometric proof of the first equality in (19).

For the bases of $K_{G^\vee\times\mathbb{C}^*}(Z)$, we will then consider the following instead:

$$(\iota_{w*}1)_{-\rho} := \iota_{w*}1 \otimes \mathcal{O}(-\rho), \quad (\mathrm{stab}_-(w))_{-\rho} := \mathrm{stab}_-(w) \otimes \mathcal{O}(-\rho).$$

The fixed point basis is an eigenbasis for the action of the lattice part Θ of $\mathbb{H}$. Therefore, for any $e^\lambda \in X^*(A^\vee)$,

$$\pi(e^\lambda)(\iota_{w*}1 \otimes \mathcal{O}(-\rho)) = e^{w\lambda}\iota_{w*}1 \otimes \mathcal{O}(-\rho). \tag{57}$$

From the proof of Lemma 4.4 and Equation (56) (we switch the positive and negative roots), we have

$$\pi(T_{s_\alpha})(\iota_{w*}1)_{-\rho} = T_{s_\alpha}(\iota_{w*}1) \otimes \mathcal{O}(-\rho) = \frac{q-1}{1-e^{-w\alpha}}(\iota_{w*}1)_{-\rho} + \frac{q-e^{-w\alpha}}{1-e^{w\alpha}}(\iota_{ws*}1)_{-\rho}. \tag{58}$$

As for the second basis, we get from Theorem 4.5 the following

$$\pi(T_{s_\alpha})(\mathrm{stab}_-(w)_{-\rho}) = \begin{cases} q^{\frac{1}{2}}(\mathrm{stab}_-(ws_\alpha))_{-\rho} + (q-1)(\mathrm{stab}_-(w)_{-\rho}, & \text{if } ws_\alpha < w; \\ q^{\frac{1}{2}}(\mathrm{stab}_-(ws_\alpha))_{-\rho}, & \text{if } ws_\alpha > w. \end{cases} \tag{59}$$

By definition, the fixed point basis $\iota_{w*}1$ is supported at w with restriction

$$\iota_{w*}1|_w = \prod_{\beta>0}(1-e^{-w\beta})(1-qe^{w\beta}).$$

Hence, by the definition of $\mathrm{stab}_-(w)$ (Theorem 2.2), the second part of Remark 2.3 and Lemma 3.2, we can write

$$\iota_{w*}1 = q^{-\frac{\ell(w)}{2}} \prod_{\beta>0, w\beta>0}(1-e^{-w\beta}) \prod_{\beta>0, w\beta<0}(q-e^{-w\beta})\,\mathrm{stab}_-(w) + \sum_{y>w} \mathrm{stab}_+(y)|_w\,\mathrm{stab}_-(y), \tag{60}$$

where $\mathrm{stab}_+(y)|_w$ is given by Theorem 1.2.

Since an unramified character τ of A_F corresponds to a maximal ideal in $K_{A^\vee}(pt)$, we have the evaluation map $K_{A^\vee}(pt) \to \mathbb{C}_\tau$. Consequently, we have the tensor product

$$K_{A^\vee\times\mathbb{C}^*}(T^*(G^\vee/B^\vee)) \otimes_{K_{A^\vee}(pt)} \mathbb{C}_\tau$$

which without raising any confusion will also be denoted by

$$K_\tau := K_{A^\vee\times\mathbb{C}^*}(T^*(G^\vee/B^\vee)) \otimes_{K_{A^\vee\times\mathbb{C}^*}(pt)} \mathbb{C}_\tau.$$

For any $f \in K_{A^\vee\times\mathbb{C}^*}(T^*(G^\vee/B^\vee))$, the corresponding class $f \otimes 1 \in K_\tau$ will also be denoted by f for simplicity. We further assume that the values of the roots of $G^\vee$ at τ do not equal $q^{\pm 1}$, so that the above tensor product has the following two bases

$$\{(\iota_{w*}1)_{-\rho} \mid w \in W\} \quad \text{and} \quad \{(\mathrm{stab}_-(w))_{-\rho} \mid w \in W\}.$$

9.3. The comparison

The main result of this section is the following.

Theorem 9.4. – *Fix an unramified character τ of A. There is a unique isomorphism between the following two right $\mathbb{H}$-modules*

$$\Psi : K_{A^\vee\times\mathbb{C}^*}(T^*(G^\vee/B^\vee)) \otimes_{K_{A^\vee\times\mathbb{C}^*}(pt)} \mathbb{C}_\tau \to I(\tau)^I,$$

with the equivariant parameter q for $\mathbb{C}^*$ evaluated to the cardinality of the residue field of $\mathcal{O}_F$, satisfying the following properties:

1. *for any $w \in W$,*

$$g_w := \frac{q^{\ell(w)}}{\prod_{\beta>0,w\beta>0}(1-e^{-w\beta})\prod_{\beta>0,w\beta<0}(q-e^{-w\beta})}(\iota_{w*}1)_{-\rho} \quad \mapsto \quad f_w^\tau,$$

2. $(\mathrm{stab}_-(w))_{-\rho} \mapsto q^{-\frac{\ell(w)}{2}}\varphi_w^\tau$.

Remark 9.5. – 1. Such an isomorphism has been studied by Lusztig [28] and Braverman-Kazhdan [9] from different points of view. However, the present paper explicitly identifies different bases from K-theory and from p-adic representation theory, which had been previously unknown.

2. Under this isomorphism, the spherical function ϕ^τ corresponds to the following element on the K-theory side:

$$\tilde{\phi}^\tau := \sum_w q^{\frac{\ell(w)}{2}} (\mathrm{stab}_-(w))_{-\rho} = \sum_w \frac{(\iota_{w*}1)_{-\rho}}{\prod_{\beta>0}(1-e^{-w\beta})} \tag{61}$$

$$= [\mathcal{O}_{G^\vee/B^\vee} \otimes \mathcal{O}(-\rho)] \in K_{A^\vee\times\mathbb{C}^*}(T^*(G^\vee/B^\vee)) \otimes_{K_{A^\vee\times\mathbb{C}^*}(pt)} \mathbb{C}_\tau, \tag{62}$$

where the last equality follows from localization. Hence, in what follow we refer to $\tilde{\phi}^\tau$ as the K-theoretic spherical class.

Proof. – Condition (1) uniquely defines Ψ as a map of $\mathbb{C}$-vector spaces. We need to check that Ψ is a map of $\mathbb{H}$-modules, and that it satisfies Condition (2).

First we verify that Ψ is a map of $\mathbb{H}$-modules. For any simple root α, by (58), we have

$$\begin{aligned}
\pi(T_{s_\alpha})(g_w) &= \frac{q^{\ell(w)}}{\prod_{\beta>0,w\beta>0}(1-e^{-w\beta})\prod_{\beta>0,w\beta<0}(q-e^{-w\beta})} \\
&\quad\times\left(\frac{q-1}{1-e^{-w\alpha}}\iota_{w*}1\otimes\mathcal{O}(-\rho)+\frac{q-e^{-w\alpha}}{1-e^{w\alpha}}\iota_{ws*}1\otimes\mathcal{O}(-\rho)\right) \\
&= \frac{q-1}{1-e^{-w\alpha}}g_w + \begin{cases} q g_{ws_\alpha}, & \text{if } ws_\alpha < w; \\ \frac{q-e^{-w\alpha}}{1-e^{w\alpha}}\frac{q-e^{w\alpha}}{1-e^{-w\alpha}}q^{-1}g_{ws_\alpha}, & \text{if } ws_\alpha > w. \end{cases}
\end{aligned}$$

Applying the map Ψ, we get

$$\begin{aligned}
\Psi(\pi(T_{s_\alpha})(g_w)) &= \frac{q-1}{1-e^{-w\alpha}(\tau)}f_w^\tau + \begin{cases} q f_{ws}^\tau, & \text{if } ws_\alpha < w, \\ \frac{(1-q^{-1}e^{-w\alpha}(\tau))}{1-e^{w\alpha}(\tau)}\frac{1-q^{-1}e^{w\alpha}(\tau)}{1-e^{-w\alpha}(\tau)}q f_{ws}^\tau, & \text{if } ws_\alpha > w, \end{cases} \\
&= q(1-c_{w\alpha})f_w^\tau + qJ_{\alpha,w}f_{ws}^\tau \\
&= \pi(T_{s_\alpha})(f_w^\tau) \\
&= \pi(T_{s_\alpha})(\Psi(g_w)).
\end{aligned}$$

Therefore, Ψ commutes with the H_W-actions. Next we consider the action of Θ. For any $e^\lambda \in X^*(A^\vee)$ We have

$$\Psi((\pi(e^\lambda)g_w) = \Psi(e^{w\lambda}g_w) = e^{w\lambda}(\tau)f_w^\tau = \pi(\theta_\lambda)f_w^\tau = \pi(\theta_\lambda)\Psi(g_w).$$

This proves that Ψ is a map of $\mathbb{H}$-modules.

We now prove Condition (2) by descending induction on $\ell(w)$. For the longest element $w = w_0 \in W$, we have $f_{w_0}^\tau = \varphi_{w_0}^\tau$, and

$$\iota_{w_0*}1 = q^{-\frac{\ell(w_0)}{2}} \prod_{\beta>0,w_0\beta>0}(1-e^{-w_0\beta}) \prod_{\beta>0,w_0\beta<0}(q-e^{-w_0\beta})\,\mathrm{stab}_-(w_0).$$

Therefore,

$$(\mathrm{stab}_-(w_0))_{-\rho} = q^{-\frac{\ell(w_0)}{2}} g_{w_0}.$$

This proves Condition (2) for $w = w_0$. The inductive step follows directly from (42) and (59). □

One immediate corollary is the following relation between the restriction formulas in Theorem 1.1 and the transition matrix in Equations (47) and (48).

Corollary 9.6. – *For any $y, w \in W$ with $w \leq y$, we have*

$$\operatorname{stab}_+(y)|_w = q^{\frac{\ell(y)}{2}-\ell(w)} a_{w,y} \prod_{\beta>0, w\beta>0} (1-e^{-w\beta}) \prod_{\beta>0, w\beta<0} (q-e^{-w\beta}), \tag{63}$$

and

$$\operatorname{stab}_-(w)|_y = q^{\ell(y)-\frac{\ell(w)}{2}} b_{w,y} \prod_{\beta>0, y\beta>0} (1-qe^{y\beta}) \prod_{\beta>0, y\beta<0} (1-e^{y\beta}). \tag{64}$$

Proof. – By the definition of $a_{w,y}$ in Equation (47) and the above theorem, we have

$$g_w = \sum_z a_{w,z} q^{\frac{\ell(z)}{2}} (\operatorname{stab}_-(z))_{-\rho}.$$

Pairing with $\operatorname{stab}_+(y) \otimes \mathcal{O}(\rho)$ on both sides and using the duality between the opposite stable bases (see Remark 2.3), we get the first equation. The other equation follows immediately from the theorem and the localization formula. □

9.4. Weyl group action and intertwiners

In this section, we compare, under Theorem 9.4, the Weyl group action on the equivariant K-theory side, and the intertwiner action I_x^τ on the p-adic side. Recall for any $G^\vee \times \mathbb{C}^*$-variety Y, we have a Weyl group action on $K_{A^\vee \times \mathbb{C}^*}(Y)$ defined as follows. For any $w \in W$, pick a representative $\dot{w} \in N_{G^\vee}(A^\vee)$, the normalizers of $A^\vee$ in $G^\vee$. Then left multiplication by $\dot{w}^{-1}$ defines a morphism from Y to itself, which is not $A^\vee \times \mathbb{C}^*$-equivariant. For any $\mathcal{F} \in K_{A^\vee \times \mathbb{C}^*}(Y)$, the pullback sheaf $(\dot{w}^{-1})^* \mathcal{F}$ has a natural $A \times \mathbb{C}^*$-equivariant coherent sheaf structure. Hence $(\dot{w}^{-1})^* \mathcal{F} \in K_{A^\vee \times \mathbb{C}^*}(Y)$, and this construction does not depend on the choice of the representative. So we get a Weyl group action on $K_{A^\vee \times \mathbb{C}^*}(Y)$. Note that W acts on the base ring $K_{A^\vee \otimes \mathbb{C}^*}(pt)$ by $w(e^\lambda) = e^{w\lambda}$ for any $e^\lambda \in K_{A^\vee \otimes \mathbb{C}^*}(pt)$. The action on $K_{A^\vee \times \mathbb{C}^*}(Y)$ makes it a W-equivariant $K_{A^\vee \otimes \mathbb{C}^*}(pt)$-module.

More explicitly, for $Y = T^*(G^\vee/B^\vee)$, the action can be written under localization as follows. For any $\mathcal{F} \in K_{A^\vee \times \mathbb{C}^*}(T^*(G^\vee/B^\vee))$, we have

$$w(\mathcal{F})|_y = w(\mathcal{F}|_{w^{-1}y}).$$

In particular, we have

$$w(\iota_{y*}1) = \iota_{wy*}1.$$

Let us use Ψ_τ to denote the isomorphism Ψ in Theorem 9.4. Then we have the following compatibility result, which is also studied by Braverman-Kazhdan from a different point of view [9, Corollary 5.7].

Corollary 9.7. – *For any $w \in W$, the following diagram is commutative*

$$\begin{tikzcd} K_\tau \arrow[r, "\Psi_\tau"] \arrow[d, "w\otimes 1"] & I(\tau)^I \arrow[d, "I^\tau_{w^{-1}}"] \\ K_{w\tau} \arrow[r, "\Psi_{w\tau}"] & I(w\tau)^I. \end{tikzcd}$$

Note that the map $w \otimes 1$ is well-defined. For any $\mathscr{F} \in K_{A^\vee \times \mathbb{C}^*}(T^*(G^\vee/B^\vee))$, $e^\lambda \in K_{A^\vee \times \mathbb{C}^*}(pt)$ and $z \in \mathbb{C}_\tau$, $e^\lambda \mathscr{F} \otimes z = \mathscr{F} \otimes ze^\lambda(\tau)$. And $(w \otimes 1)(e^\lambda \mathscr{F} \otimes z) = w(e^\lambda \mathscr{F}) \otimes z = e^{w\lambda} w(\mathscr{F}) \otimes z = w(\mathscr{F}) \otimes ze^{w\lambda}(\tau) = (w \otimes 1)(\mathscr{F} \otimes ze^\lambda(\tau))$.

Proof. – By the properties of the intertwiners I_w^τ (41), we only need to prove the corollary in the case when w is a simple reflection s_α. Using the notations in Theorem 9.4, we check the commutativity using the basis g_w.

First of all, we have the following easy identities

$$\begin{aligned}
&\prod_{\beta>0, s_\alpha w\beta>0} (1-e^{-s_\alpha w\beta}(s_\alpha\tau)) \prod_{\beta>0, s_\alpha w\beta<0} (q-e^{-s_\alpha w\beta}(s_\alpha\tau)) \\
&\quad= \prod_{\beta>0, s_\alpha w\beta>0} (1-e^{-w\beta}(\tau)) \prod_{\beta>0, s_\alpha w\beta<0} (q-e^{-w\beta}(\tau)) \\
&\quad= \begin{cases} qc_{-\alpha} \prod_{\beta>0,w\beta>0}(1-e^{-w\beta}(\tau)) \prod_{\beta>0,w\beta<0}(q-e^{-w\beta}(\tau)), & \text{if } w^{-1}\alpha>0; \\ \frac{1}{qc_\alpha} \prod_{\beta>0,w\beta>0}(1-e^{-w\beta}(\tau)) \prod_{\beta>0,w\beta<0}(q-e^{-w\beta}(\tau)), & \text{if } w^{-1}\alpha<0, \end{cases}
\end{aligned}$$

where c_α is defined in (40). For example, if $w^{-1}\alpha > 0$, then

$$\begin{aligned}
\prod_{\beta>0, s_\alpha w\beta>0} (1-e^{-w\beta}) &= \prod_{\beta\in R^+\setminus\{w^{-1}\alpha\}, s_\alpha w\beta>0} (1-e^{-w\beta}) \\
&= \prod_{\beta\in R^+\setminus\{w^{-1}\alpha\}, w\beta>0} (1-e^{-w\beta}) = \frac{1}{1-e^{-\alpha}} \prod_{\beta>0, w\beta>0} (1-e^{-w\beta}).
\end{aligned}$$

Notice that $w^{-1}\alpha > 0$ iff $s_\alpha w > w$, and $w^{-1}\alpha < 0$ iff $s_\alpha w < w$.

Using these, we have

$$\begin{aligned}
(s_\alpha \otimes 1)(g_w) &= (s_\alpha \otimes 1)\left(q^{\ell(w)} \iota_{w*}1 \otimes \frac{e^{-w\rho}(\tau)}{\prod_{\beta>0,w\beta>0}(1-e^{-w\beta}(\tau)) \prod_{\beta>0,w\beta<0}(q-e^{-w\beta}(\tau))} \right) \\
&= q^{\ell(w)} \iota_{s_\alpha w*}1 \otimes \frac{e^{-s_\alpha w\rho}(s_\alpha\tau)}{\prod_{\beta>0,w\beta>0}(1-e^{-w\beta}(\tau)) \prod_{\beta>0,w\beta<0}(q-e^{-w\beta}(\tau))} \\
&= \begin{cases} g_{s_\alpha w} \otimes c_{-\alpha}, & \text{if } w^{-1}\alpha>0; \\ g_{s_\alpha w} \otimes \frac{1}{c_\alpha}, & \text{if } w^{-1}\alpha<0. \end{cases} \\
&\in K_{s_\alpha\tau}.
\end{aligned}$$

Comparing with (46), we get

$$\Psi_{s_\alpha\tau}(s_\alpha \otimes 1)(g_w) = I_{s_\alpha}^\tau \left(\Psi_\tau(g_w)\right),$$

which finishes the proof. □

Remark 9.8. – As an application of this corollary, we reprove the Gindikin-Karpelevich Formula (49). According to Remark 9.5, the two sides of the Gindikin-Karpelevich formula become $w(\tilde{\phi}^\tau)$ and $\tilde{\phi}^{w\tau}$, where $\tilde{\phi}^\tau$ is the K-theoretic spherical vector defined in (61). The equality between these two K-theory classes follows directly from the definition of $w \otimes 1$.

9.5. Macdonald's formula in equivariant K-theory

In this section, we give a K-theory interpretation of the Macdonald's formula from Theorem 9.1.

We define the K-theory analogue of the pairing $\langle -,-\rangle : I(\tau)\otimes I(\tau^{-1}) \to \mathbb{C}$ [20, § 1.9] to be the following. Let $\iota : A^\vee \to A^\vee$ be the endomorphism of abelian groups sending an element to its inverse. It induced a map $\iota : K_{A^\vee}(Y) \to K_{A^\vee}(Y)$ for any $A^\vee$-variety Y. Explicitly, on $K_{A^\vee\times\mathbb{C}^*}(T^*(G^\vee/B^\vee))$, using localization we have $\iota(\mathcal{F})|_w(\tau) = \mathcal{F}|_w(\tau^{-1})$, for any point $\tau \in A^\vee$. We consider the pairing

$$\langle -,-\rangle_\tau : K_\tau \times K_{\tau^{-1}} \to \mathbb{C}$$

defined as

$$(\mathcal{F},\mathcal{G}) \mapsto (p_*\iota(\mathcal{H}om(\mathcal{F},\mathcal{G})))(\tau),$$

with $p : T^*(G^\vee/B^\vee) \to \mathfrak{g}^\vee$ being the Springer map. Here $(-)(\tau)$ means evaluating the K-theory classes using the map $K_{A^\vee\times\mathbb{C}^*}(\mathrm{pt}) \to \mathbb{C}_\tau$ induced by the character τ and the $\mathbb{C}^*$-equivariant parameter q is evaluated to be the cardinality of the residue field of $\mathcal{O}_F$. It is easy to see this pairing is well defined. Using localization, the above pairing can be written as

$$(65) \qquad (\mathcal{F},\mathcal{G}) \mapsto \sum_{w\in W} \frac{\mathcal{F}|_w(\tau)\mathcal{G}|_w(\tau^{-1})}{\bigwedge^\bullet T_w(T^*(G^\vee/B^\vee))(\tau^{-1})}.$$

We will show in Remark 9.10 that under the isomorphism in Theorem 9.4, this pairing differs from $\langle -,-\rangle : I(\tau)\otimes I(\tau^{-1}) \to \mathbb{C}$ [20, § 1.9] by a scalar.

For any μ in the dominant coweights $X_*(A)_+$ of G, we have the element

$$e_{I\varpi^\mu I} = \delta_B(\varpi^\mu)^{\frac{1}{2}}\theta_\mu \in \mathbb{H}.$$

We define the K-theoretic analogue of the spherical function (52) as follows

$$\tilde{\Gamma}_\tau : X_*(A)_+ \to \mathbb{C}, \quad \mu \to \langle \pi(e_{I\varpi^\mu I})\tilde{\phi}^\tau, \tilde{\phi}^{\tau^{-1}}\rangle_\tau,$$

where recall that $\pi(e_{I\varpi^\mu I})$ is the action of Θ on the equivariant K-theory.

Then our K-theoretic interpretation of Macdonald's formula is the following

Theorem 9.9. – *For any dominant coweight μ of G, we have*

$$\tilde{\Gamma}_\tau(\mu) = q^{\dim G/B} Q \cdot \Gamma_\tau(\varpi^\mu),$$

where Q is defined as the volume of Bw_0B.

Remark 9.10. – By the linear independence of the coweights μ, it follows from the theorem that if we normalize the pairing in (65) by multiplying by $q^{\dim G/B}Q$, then the isomorphism Ψ_τ in Theorem 9.4 respects this pairing and the pairing between the contragredient modules $I(\tau)$ and $I(\tau^{-1})$.

Proof. – We use localization formula. First of all, we have

$$(66) \qquad \pi(e_{I\varpi^\mu I})\tilde{\phi}^\tau = \delta_B(\varpi^\mu)^{\frac{1}{2}} \sum_w e^{w\mu - w\rho}(\tau) \frac{(\iota_{w*}1)_{-\rho}\otimes 1}{\prod_{\beta>0}(1-e^{-w\beta}(\tau))}.$$

Then

$$\begin{aligned}\tilde{\Gamma}_\tau(\mu) &= \langle \pi(e_{I\varpi^\mu I})\tilde{\phi}^\tau, \tilde{\phi}^{\tau^{-1}}\rangle_\tau \\ &= \delta_B(\varpi^\mu)^{\frac{1}{2}}\sum_w e^{w\mu}(\tau)\frac{\bigwedge^\bullet T_w(T^*(G^\vee/B^\vee))(\tau)}{\prod_{\beta>0}(1-e^{-w\beta}(\tau))(1-e^{w\beta}(\tau))} \\ &= \delta_B(\varpi^\mu)^{\frac{1}{2}}\sum_w e^{w\mu}(\tau)\prod_{\beta>0}\frac{1-qe^{w\beta}(\tau)}{1-e^{w\beta}(\tau)} \\ &= q^{\dim G/B}\delta_B(\varpi^\mu)^{\frac{1}{2}}\sum_w e^{w\mu}(\tau)\prod_{\beta>0}\frac{1-q^{-1}e^{-w\beta}(\tau)}{1-e^{-w\beta}(\tau)} \\ &= q^{\dim G/B}Q\cdot\Gamma_\tau(\varpi^\mu). \qquad\square\end{aligned}$$

9.6. Casselman-Shalika formula in equivariant K-theory

In this section, we investigate the K-theoretic meaning of the Casselman-Shalika formula, see Theorem 9.2.

First of all, we define the K-theoretic analogue of the Whittaker functional. On the space K_τ, we have the equivariant Euler characteristic map

$$p_* : K_\tau \to \mathbb{C}$$

induced by $p : T^*(G^\vee/B^\vee) \to \mathfrak{g}^\vee$. Via localization, for any $\mathscr{F} \in K_\tau$, we have

$$p_*(\mathscr{F}) := \sum_i (-1)^i H^i(T^*(G^\vee/B^\vee), \mathscr{F})(\tau) = \sum_{w\in W}\frac{\mathscr{F}|_w(\tau)}{\bigwedge^\bullet(T_w(T^*G/B))(\tau)} \in \mathbb{C}.$$

In particular, $p_*(\iota_{w*}1) = 1$ for any $w \in W$.

The K-theoretic analogue of the Whittaker functional is defined to be a modification of the above

$$\tilde{L}_\tau(-) := \prod_{\beta>0}(1-q^{-1}e^\beta(\tau))\cdot p_*(-\otimes \mathcal{O}(\rho)) : K_\tau \to \mathbb{C}.$$

For any $\mathscr{F} \in K_\tau$ and any dominant coweight μ of G, we consider

$$\tilde{\mathscr{W}}_\tau(\mathscr{F})(\varpi^\mu) := \tilde{L}_\tau\left(\pi(e_{I\varpi^\mu I})\mathscr{F}\right),$$

where $e_{I\varpi^\mu I}$ is defined in (51). Here recall that $e_{I\varpi^\mu I} = \delta_B(\varpi^\mu)^{\frac{1}{2}}\theta_\mu \in \mathbb{H}$, and $\pi(e_{I\varpi^\mu I})$ is the action of Θ on the equivariant K-theory. The K-theory analogue of the Whittaker function is defined to be

$$\tilde{W}_\tau : X_*(A)_+ \to \mathbb{C}, \quad \tilde{W}_\tau(\mu) := \tilde{L}_\tau\left(\pi(e_{I\varpi^\mu I})\tilde{\phi}^\tau\right),$$

where $\tilde{\phi}^\tau$ is our K-theoretic spherical class defined in (61).

Under the isomorphism in Theorem 9.4, properties of the Whittaker functions in (53) and (54) correspond to the following lemma, according to Corollary 9.7.

LEMMA 9.11. – *For any $w \in W$, we have*

1. *The following diagram is commutative*

$$\begin{tikzcd} K_\tau \arrow[r, "\tilde{L}_\tau"] \arrow[d, "w^{-1}\otimes 1"] & \mathbb{C} \arrow[d, "\prod_{\beta>0, w^{-1}\beta<0} \frac{1-q^{-1}e^{-\beta}(\tau)}{1-q^{-1}e^{\beta}(\tau)}"] \\ K_{w^{-1}\tau} \arrow[r, "\tilde{L}_{w^{-1}\tau}"] & \mathbb{C}. \end{tikzcd}$$

2. *For the g_w defined in Theorem 9.4, we have*

$$\tilde{\mathcal{W}}_\tau(g_w)(\varpi^\mu) = \mathcal{W}_\tau(f_w^\tau)(\varpi^\mu).$$

Proof. – 1. Since any $\mathcal{J} \in K_\tau$ can be written as $\mathbb{C}$-linear combination of the fixed point basis $\iota_{y*}1 \otimes 1 \in K_\tau$, we only need to check for these basis elements. Then it follows from the following easy identity

$$\prod_{\beta>0}(1-q^{-1}e^{\beta}) \prod_{\beta>0, w^{-1}\beta<0} \frac{1-q^{-1}e^{-\beta}}{1-q^{-1}e^{\beta}} = \prod_{\beta>0}(1-q^{-1}e^{w\beta}).$$

2. The second one is verified immediately once we know $\pi(e_{I\varpi^\mu I})(g_w) = \delta_B^{\frac{1}{2}}(\varpi^\mu)e^{w\mu}g_w$. □

Then the following is the K-theoretic interpretation of the Casselman-Shalika formula.

Theorem 9.12. – *For any dominant coweight μ of G, we have*

$$\tilde{W}_\tau(\mu) = W_\tau(\varpi^\mu).$$

Proof. – From (66) and the fact $p_*(\iota_{w*}1) = 1$, we get

$$\begin{aligned} \tilde{W}_\tau(\mu) &= \tilde{L}_\tau\left(\pi(e_{I\varpi^\mu I})\tilde{\phi}^\tau\right) \\ &= \delta_B^{\frac{1}{2}}(\varpi^\mu)\prod_{\beta>0}(1-q^{-1}e^{\beta}(\tau))\sum_{w\in W} e^{w\mu}(\tau)\frac{1}{\prod_{\beta>0}(1-e^{-w\beta}(\tau))} \\ &= W_\tau(\varpi^\mu). \end{aligned}$$

□

BIBLIOGRAPHY

[1] M. Aganagic, A. Okounkov, Elliptic stable envelope, preprint arXiv:1604.00423.

[2] P. Aluffi, L. C. Mihalcea, Chern-Schwartz-MacPherson classes for Schubert cells in flag manifolds, *Compos. Math.* **152** (2016), 2603–2625.

[3] P. Aluffi, L. C. Mihalcea, J. Schuermann, C. Su, Shadows of characteristic cycles, Verma modules, and positivity of Chern-Schwartz-MacPherson classes of Schubert cells, preprint arXiv:1709:08697.

[4] I. N. Bernšteĭn, I. M. Gel′fand, S. I. Gel′fand, Schubert cells, and the cohomology of the spaces G/P, *Uspehi Mat. Nauk* **28** (1973), 3–26.

[5] R. Bezrukavnikov, On two geometric realizations of an affine Hecke algebra, *Publ. Math. Inst. Hautes Études Sci.* **123** (2016), 1–67.

[6] R. Bezrukavnikov, I. Mirković, Representations of semisimple Lie algebras in prime characteristic and the noncommutative Springer resolution, *Ann. of Math.* **178** (2013), 835–919.

[7] S. C. Billey, Kostant polynomials and the cohomology ring for G/B, *Duke Math. J.* **96** (1999), 205–224.

[8] A. Braverman, M. Finkelberg, H. Nakajima, *Instanton moduli spaces and W-algebras*, Astérisque **385**, 2016.

[9] A. Braverman, D. Kazhdan, On the Schwartz space of the basic affine space, *Selecta Math. (N.S.)* **5** (1999), 1–28.

[10] A. Braverman, D. Maulik, A. Okounkov, Quantum cohomology of the Springer resolution, *Adv. Math.* **227** (2011), 421–458.

[11] B. Calmès, K. Zainoulline, C. Zhong, Equivariant oriented cohomology of flag varieties, *Doc. Math.* Extra vol.: Alexander S. Merkurjev's sixtieth birthday (2015), 113–144.

[12] B. Calmès, K. Zainoulline, C. Zhong, A coproduct structure on the formal affine Demazure algebra, *Math. Z.* **282** (2016), 1191–1218.

[13] B. Calmès, K. Zainoulline, C. Zhong, Push-pull operators on the formal affine Demazure algebra and its dual, *Manuscripta Math.* **160** (2019), 9–50.

[14] W. Casselman, The unramified principal series of $\mathfrak{p}$-adic groups. I. The spherical function, *Compos. math.* **40** (1980), 387–406.

[15] W. Casselman, J. Shalika, The unramified principal series of p-adic groups. II. The Whittaker function, *Compos. math.* **41** (1980), 207–231.

[16] N. Chriss, V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser, 1997.

[17] I. B. Frenkel, N. Y. Reshetikhin, Quantum affine algebras and holonomic difference equations, *Comm. Math. Phys.* **146** (1992), 1–60.

[18] E. Gorsky, A. Neguţ, Infinitesimal change of stable basis, *Selecta Math. (N.S.)* **23** (2017), 1909–1930.

[19] W. Graham, Equivariant K-theory and Schubert varieties, preprint arXiv:2002.02076.

[20] T. J. Haines, R. E. Kottwitz, A. Prasad, Iwahori-Hecke algebras, *J. Ramanujan Math. Soc.* **25** (2010), 113–145.

[21] D. Kazhdan, G. Lusztig, Representations of Coxeter groups and Hecke algebras, *Invent. math.* **53** (1979), 165–184.

[22] D. Kazhdan, G. Lusztig, Proof of the Deligne-Langlands conjecture for Hecke algebras, *Invent. math.* **87** (1987), 153–215.

[23] B. Kostant, S. Kumar, T-equivariant K-theory of generalized flag varieties, *J. Differential Geom.* **32** (1990), 549–603.

[24] K.-H. Lee, C. Lenart, D. Liu, Whittaker functions and Demazure characters, *J. Inst. Math. Jussieu* **18** (2019), 759–781.

[25] C. Lenart, K. Zainoulline, Towards generalized cohmology Schubert calculus via formal root polynomials, *Math. Res. Lett.* **24** (2017), 839–877.

[26] G. Lusztig, Equivariant K-theory and representations of Hecke algebras, *Proc. Amer. Math. Soc.* **94** (1985), 337–342.

[27] G. Lusztig, Periodic W-graphs, *Represent. Theory* **1** (1997), 207–279.

[28] G. Lusztig, Bases in equivariant K-theory, *Represent. Theory* **2** (1998), 298–369.

[29] I. G. Macdonald, Spherical functions on a $\mathfrak{p}$-adic Chevalley group, *Bull. Amer. Math. Soc.* **74** (1968), 520–525.

[30] D. Maulik, A. Okounkov, *Quantum groups and quantum cohomology*, Astérisque **408**, 2019.

[31] D. Maulik, A. Okounkov, in preparation.

[32] M. B. McBreen, D. K. Shenfeld, Quantum cohomology of hypertoric varieties, *Lett. Math. Phys.* **103** (2013), 1273–1291.

[33] H. Nakajima, Quiver varieties and tensor products, II, in *Symmetries, integrable systems and representations*, Springer Proc. Math. Stat. **40**, Springer, 2013, 403–428.

[34] M. Nakasuji, H. Naruse, Yang-Baxter basis of Hecke algebra and Casselman's problem, in *FPSAC 2016 Vancouver, Canada DMTCS proc. BC*, 2016 (extended abstract), 935–946.

[35] A. Neguţ, The $\frac{m}{n}$ Pieri rule, *Int. Math. Res. Not.* **2016** (2016), 219–257.

[36] A. Okounkov, Lectures on K-theoretic computations in enumerative geometry, in *Geometry of moduli spaces and representation theory*, IAS/Park City Math. Ser. **24**, Amer. Math. Soc., 2017, 251–380.

[37] M. Reeder, On certain Iwahori invariants in the unramified principal series, *Pacific J. Math.* **153** (1992), 313–342.

[38] M. Reeder, p-adic Whittaker functions and vector bundles on flag manifolds, *Compos. math.* **85** (1993), 9–36.

[39] S. Riche, Geometric braid group action on derived categories of coherent sheaves, *Represent. Theory* **12** (2008), 131–169.

[40] R. Rimányi, V. Tarasov, A. Varchenko, Trigonometric weight functions as K-theoretic stable envelope maps for the cotangent bundle of a flag variety, *J. Geom. Phys.* **94** (2015), 81–119.

[41] R. Rimányi, V. Tarasov, A. Varchenko, Elliptic and K-theoretic stable envelopes and Newton polytopes, preprint arXiv:1705.09344.

[42] R. Rimányi, A. Varchenko, Equivariant Chern-Schwartz-MacPherson classes in partial flag varieties: interpolation and formulae, in *Schubert Varieties, Equivariant Cohomology and Characteristic Classes*, EMS Series of Congress Reports **15**, EMS Publishing House, 2018, 225–235.

[43] F. Rodier, Whittaker models for admissible representations of reductive p-adic split groups, in *Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972)*, 1973, 425–430.

[44] A. Smirnov, On the instanton R-matrix, *Comm. Math. Phys.* **345** (2016), 703–740.

[45] A. Smirnov, Quantum difference equations for quiver varieties, Ph.D. Thesis, Columbia University, 2016.

[46] A. Smirnov, Polynomials associated with fixed points on the instanton moduli space, preprint arXiv:1404.5304.

[47] C. Su, Equivariant quantum cohomology of cotangent bundle of G/P, *Adv. Math.* **289** (2016), 362–383.

[48] C. Su, Restriction formula for stable basis of the Springer resolution, *Selecta Math. (N.S.)* **23** (2017), 497–518.

[49] C. Su, G. Zhao, Z. Zhong, Wall-crossings and a categorification of K-theory stable bases of the Springer resolution, preprint arXiv:1904.03769.

[50] R. W. Thomason, Les K-groupes d'un schéma éclaté et une formule d'intersection excédentaire, *Invent. math.* **112** (1993), 195–215.

[51] M. Willems, Cohomologie et K-théorie équivariantes des variétés de Bott-Samelson et des variétés de drapeaux, *Bull. Soc. Math. France* **132** (2004), 569–589.

[52] G. Zhao, C. Zhong, Geometric representations of the formal affine Hecke algebra, *Adv. Math.* **317** (2017), 50–90.

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Changjian Su
IHÉS
35 Route de Chartres
91440 Bures-sur-Yvette, France
E-mail: `changjiansu@gmail.com`

Gufang Zhao
University of Massachusetts Amherst
Department of Mathematics and Statistics
Amherst MA 01003
Institute of Science and Technology Austria
Am Campus, 1
Klosterneuburg 3400, Austria
E-mail: `gufang.zhao@ist.ac.at`

Changlong Zhong
State University of New York at Albany
1400 Washington Ave
ES 110, Albany, NY, 12222
E-mail: `czhong@albany.edu`