Mathematische Annalen (2019) 374:1845–1882
https://doi.org/10.1007/s00208-019-01845-0

Mathematische Annalen

Uniqueness of stable capillary hypersurfaces in a ball

Guofang Wang[1] · Chao Xia[2]

Received: 20 February 2018 / Revised: 5 December 2018 / Published online: 20 May 2019

Abstract

In this paper we prove that any immersed stable capillary hypersurfaces in a ball in space forms are totally umbilical. Our result also provides a proof of a conjecture proposed by Sternberg and Zumbrun (J Reine Angew Math 503:63–85, 1998). We also prove a Heintze–Karcher–Ros type inequality for hypersurfaces with free boundary in a ball, which, together with the new Minkowski formula, yields a new proof of Alexandrov's Theorem for embedded CMC hypersurfaces in a ball with free boundary.

Contents

Part of this work was done while CX was visiting the mathematical institute of Albert-Ludwigs-Universität Freiburg. He would like to thank the institute for its hospitality. Research of CX is supported in part by NSFC (Grant no. 11501480) and the Natural Science Foundation of Fujian Province of China (Grant no. 2017J06003), and the Fundamental Research Funds for the Central Universities (Grant no. 20720180009).

✉ Chao Xia
chaoxia@xmu.edu.cn

Guofang Wang
guofang.wang@math.uni-freiburg.de

1 Mathematisches Institut, Universität Freiburg, Ernst-Zermelo-Str. 1, 79104 Freiburg, Germany

2 School of Mathematical Sciences, Xiamen University, Xiamen 361005, People's Republic of China

1 Introduction

Let $(\bar{M}^{n+1}, \bar{g})$ be an oriented $(n+1)$-dimensional Riemannian manifold and B be a smooth compact domain in $\bar{M}$ with non-empty boundary ∂B. We are interested in capillary hypersurfaces, namely minimal or constant mean curvature (CMC) hypersurfaces in B with boundary on ∂B and intersecting ∂B at a constant angle $\theta \in (0, \pi)$. Minimal or CMC hypersurfaces with free boundary, namely, intersecting ∂B orthogonally, are special and important examples of capillary hypersufaces. Capillary hypersurfaces are critical points of some geometric variational functional under certain volume constraint. It has a very long history. It was Young who first considered capillary surfaces mathematically in 1805 and introduced the mathematical concept of mean curvature of a surface [66]. His work was followed by Laplace and later by Gauss. For the reader who are interested in the history of capillary surfaces, we refer to an article of Finn et al. [19]. See also Finn's book [18] for a survey about the mathematical theory of capillary surfaces.

The stability of minimal or CMC hypersurfaces plays an important role in differential geometry. For closed hypersurfaces (i.e. compact without boundary), there is a classical uniqueness result proved by Barbosa and do Carmo [5] and Barbosa et al. [6]: *any stable closed CMC hypersurfaces in space forms are geodesic spheres.* In this paper we are concerned with stable capillary hypersurfaces in a ball in space forms. It is known that totally geodesic balls and spherical caps are stable and even area-minimizing. In fact, these are only isoperimetric hypersurfaces in a ball which was first proved by Burago and Maz'ya[1] [13] and later also by Bokowsky and Sperner [7] and Almgren [2]. Ros and Souam [53] showed that totally geodesic balls and spherical caps are capillary stable. Conversely, the uniqueness problem was first studied by Ros and Vergasta [54] for minimal or CMC hypersurfaces in free boundary case, i.e., $\theta = \frac{\pi}{2}$ and later Ros and Souam [53] for general capillary ones. Their works have been followed by many mathematicians. Comparing to the uniqueness result for stable closed hypersurfaces [5,6], there is a natural and long standing open problem on the uniqueness of stable capillary hypersurfaces since the work of Ros–Vergasta and Ros–Souam:

Are any immersed stable capillary hypersurfaces in a ball in space forms totally umbilical?

The main objective of this paper is to give a complete answer to this open problem. For convenience, we discuss in the introduction mainly on the case of hypersurfaces in *a Euclidean ball with free boundary* and give a brief discussion about general capillary hypersurfaces in a ball in any space forms later. It is surprising that this problem leaves quite open except in the following special cases.

(1) When $n \geq 2$, $H = 0$ and $\theta = \frac{\pi}{2}$, i.e., in the case of minimal hypersurfaces with free boundary, Ros and Vergasta gave an affirmative answer in [54] (1995).
(2) When $n = 2$, $H = const.$ and $\theta = \frac{\pi}{2}$, i.e., in the case of 2-dimensional CMC surfaces with free boundary, Ros and Vergata [54] and Nunes [45] (2017) gave an affirmative answer. See also the work of Barbosa [4].

[1] The authors would like to thank Professor Frank Morgan for this information.

The stability for CMC hypersurfaces is defined by using variations with a volume constraint. For minimal ones we also use this stability, which is also called the weak stability. A general way to utilize the stability condition is to find admissible test functions. For a volume constraint problem, such an admissible function φ should satisfy $\int_M \varphi dA = 0$, i.e., its average is zero. In the work of Barbosa and do Carmo [5] for closed hypersurfaces, the test function is defined by using the classical Minkowski formula, that is, for a closed immersion $x : M \to \mathbb{R}^{n+1}$,

$$\int_M H\langle x, \nu\rangle dA = n\int_M dA, \tag{1.1}$$

where H is the mean curvature and ν is the outward unit normal of M. In fact, in this case the test function is $\psi = n - H\langle x, \nu\rangle$. The Minkowski formula (1.1) implies that this is an admissible function. For a hypersurface M in a ball with free boundary, Ros and Vergasta [54] obtained the following Minkowski formula

$$|\partial M| = n|M| - \int_M H\langle x, \nu\rangle dA. \tag{1.2}$$

Unlike (1.1), the Minkowski formula (1.2) provides a relationship among three geometric quantities, the area of the boundary ∂M, the area of M and an integral involving the mean curvature. It is this complication that makes free boundary problems more difficult than problems for closed hypersurfaces. In the minimal case, the Minkowski formula (1.2) relates only two geometric quantities, since the term involving the mean curvature vanishes. The proof of Result (1) relies on this fact.

There is another way to find admissible test functions, which is called a Hersch type balancing argument. This argument is extremely useful, especially in two-dimensional problems, see for example the work of Li and Yau [36] and Montiel and Ros [43]. Using such an argument, together with the Minkowski formula (1.2), Ros and Vergasta proved in [54] the following partial result.

If $M \subset \bar{\mathbb{B}}^3$ is an immersed compact stable CMC surface with free boundary, then ∂M is embedded and the only possibilities are

(i) *M is a totally geodesic disk;*
(ii) *M is a spherical cap;*
(iii) *M has genus 1 with at most two boundary components.*

Case (iii) was excluded very recently by Nunes [45] by using a new stability criterion and a modified Hersch type balancing argument. See also the work of Barbosa [4] without using the modified Hersch type balancing argument. Therefore, when $n = 2$ this open problem was solved. This is result (2).

There are several partial results on the uniqueness of stable CMC hypersurfaces in a Euclidean ball with free boundary, see e.g., [4,31,40,54].

We remark that there are many embedded or non-embedded non-spherical examples. In fact, for any constant $H > 0$ there is a piece of an unduloid of mean curvature H in the Euclidean unit ball $\mathbb{B}^{n+1}$ with free boundary, which is however unstable. In fact, Ros [52] proved that neither catenoid nor unduloid pieces, which intersect $\partial\mathbb{B}^{n+1}$ orthogonally, are stable. The following uniqueness result classifies all stable immersed CMC hypersurfaces with free boundary in a Euclidean ball.

Theorem 1.1 *Any stable immersed CMC hypersurface with free boundary in a Euclidean ball is either a totally geodesic ball or a spherical cap.*

One of crucial ingredients to prove this result is a new Minkowski type formula. For an immersion $x : M \to \bar{\mathbb{B}}^{n+1}$ with free boundary, we establish a weighted Minkowski formula

$$n\int_M V_a dA = \int_M H\langle X_a, \nu\rangle dA, \tag{1.3}$$

which is one of a family of Minkowski's formulae proved in Sect. 3. Here $a \in \mathbb{R}^{n+1}$ is any constant vector field, V_a and X_a are defined by

$$V_a := \langle x, a\rangle, \qquad X_a := \langle x, a\rangle x - \frac{1}{2}(1 + |x|^2)a.$$

The key feature of X_a is its conformal Killing property. For the details about V_a and X_a see Sect. 3 below.

Different to (1.2), this new Minkowski formula (1.3) gives a relation between two (weighted) geometric quantities. More important is that there is no boundary integral in this new Minkowski formula. It is clear to see from (1.3) that $nV_a - H\langle X_a, \nu\rangle$ is an admissible test function for the stability for any $a \in \mathbb{R}^{n+1}$. These admissible functions play an essential role in the proof of Theorem 1.1.

It is interesting that our proof works for stable CMC hypersurfaces with free boundary in $\bar{\mathbb{B}}^{n+1}$ with a singular set of sufficiently low Hausdorff dimension and therefore gives a proof of a conjecture proposed by Sternberg and Zumbrun ([58] p. 77). As an application of their stability formula (Theorem 2.2 in [58]), which they called Poincaré inequality for stable hypersurfaces with a singular set with Hausdorff measure $\mathcal{H}^{n-2} = 0$, they proved in [58] (Theorem 3.5) that any local minimizer of perimeter under the volume constraint in $\bar{\mathbb{B}}^{n+1}$ is either a totally geodesic ball or a regular graph over $\partial\mathbb{B}^{n+1}$, provided that $H = 0$ or

$$\int_M \langle x, \nu\rangle d\mathcal{H}^n < 0. \tag{1.4}$$

Condition (1.4) is equivalent to $n\mathcal{H}^n(M) < \mathcal{H}^{n-1}(M \cap \partial\mathbb{B}^{n+1})$. This condition is almost the same as that in one of results of Ros and Vergasta (Theorem 8 in [54]). They conjectured that (1.4) holds always for stable hypersurfaces with boundary and all local minimizers in $\bar{\mathbb{B}}^{n+1}$ are regular. Here we prove more, namely all local minimizers in $\bar{\mathbb{B}}^{n+1}$ are totally geodesic balls or spherical caps.

Theorem 1.2 [2] *Let Ω be a local minimizer of perimeter with respect to fixed volume in $\bar{\mathbb{B}}^{n+1}$. Then $M = \overline{\partial\Omega \cap \mathbb{B}^{n+1}}$ is the intersection of $\bar{\mathbb{B}}^{n+1}$ with either a plane through the origin or a sphere.*

We remark that, Barbosa [4] also gave a proof of the conjecture proposed by Sternberg and Zumbrun [58] about the regularity of the local minimizers in $\bar{\mathbb{B}}^{n+1}$.

The minimal or CMC hypersurfaces with free boundary attract much attention of many mathematicians. In 1980s there are many existence results obtained from geometric variational methods, see for example, [14,26,29,61,64]. The corresponding regularity problem has been studied by Grüter and Jost [27]. Recently one of inspiring work is a series of papers of Fraser and Schoen [22–24] about minimal hypersurfaces with free boundary in a ball and the first Steklov eigenvalue. See also [3,11,15,20,25, 35,63]. Our research on the stability on CMC hypersurfaces are motivated by these results.

There are many interesting properties of closed surfaces in a space form that are valid also for surfaces with free boundary. However, in many cases the proof for the case of surfaces with free boundary is quite different and becomes more difficult, while in other cases the counterpart for surfaces with free boundary is still open. It means that the free boundary problems for surfaces are in general more difficult. Here we just mention several good examples. Comparing to the result of Montiel and Ros [43]: *Any minimal torus immersed in $\mathbb{S}^3$ by the first eigenfunctions is the Clifford torus*, Fraser and Schoen [23] took much more effort to obtain: *any minimal annulus with free boundary, which is immersed by the first Steklov eigenvalue, is the critical catenoid.* While the Lawson conjecture about uniqueness of embedded torus in $\mathbb{S}^3$ was solved recently by Brendle [9] with a clever use of the maximum principle on a two-point function, the free boundary version of the Lawson conjecture is still open. See [21] and also [46], where it was claimed without providing a proof. Even if any minimal surfaces with free boundary with index 4 is the critical catenoid is also open.

Let us turn to the general case, the capillary hypersurfaces in a ball (in space forms). There are only partial results. See for example the work of Ros and Souam [53] mentioned already above, and also [31,40,57]. Our approach to prove Theorem 1.1 is powerful enough to work for immersed capillary hypersurfaces in a ball in any space forms after establishing appropriate weighted Minkowski formulae, see Propositions 3.2 and 4.4. In other words, we can give a complete affirmative answer to the open problem mentioned above.

Theorem 1.3 *Any stable immersed capillary hypersurface in a ball in space forms is totally umbilical.*

For this theorem, though the ideas of proof are essentially the same as the one for Theorem 1.1, the proof becomes more involving.

By going through the proof, we see that our approach also works for closed hypersurfaces. Namely, we provide a new proof of the uniqueness results of Barbosa and do

[2] In view of this result Sternberg and Zumbrun asked in their new paper [60], whether volume-constrained local minimizers in a convex domain remain regular in arbitrary dimension n and not just for $n \leq 7$. For a further discussion, see [60].

Carmo and Barbosa et al. mentioned above, see Remark 3.2 and Remark 3.3 below. Furthermore, our approach works for the corresponding exterior problem. To be precise, we are able to prove the following

Theorem 1.4 *Any compact stable immersed capillary hypersurface outside a ball in space forms is totally umbilical.*

Remark 1.1 From the proof we can easily see that we *do not* need the immersed hypersurface is contained *in* or *outside* a ball, but only need the assumption $x(\partial M) \subset \partial B$.

There are many interesting uniqueness results on stable capillary hypersurfaces within other types of domains, e.g.,

a wedge, a slab, a cone, a cylinder or a half space, see e.g. [1,16,32,38,39,41,44,47,50,55,62].

There are other important uniqueness results concerning capillary hypersurfaces. One is Hopf type theorem which says any CMC 2-*sphere* in $\mathbb{R}^3$ is a round sphere. Nitsche [46] proved that any disk type capillary surface in $\mathbb{B}^3$ is either a totally geodesic disk or a spherical cap by using Hopf type argument, see also Fraser and Schoen [24] for recent development. Another is Alexandrov type theorem which says that any *embedded* CMC closed hypersurface is a round sphere. For capillary hypersurfaces, if it is embedded with its boundary ∂M lying in a half sphere, then Ros and Souam [53] (Proposition 1.2) showed that it is either a totally geodesic ball or a spherical cap by Alexandrov's reflection method.

In the last section we will give a new proof of the Alexandrov type theorem [53] for CMC hypersurface with free boundary by using integral method in the spirit as Reilly [49] and Ros [51]. The key ingredients are the new Minkowski formula as well as a Heintze–Karcher–Ros type inequality we will establish. This is another objective of this paper. For such an inequality we also use the weight function V_a.

The Heintze–Karcher–Ros inequality for an embedded closed hypersurface Σ of positive mean curvature in $\mathbb{R}^{n+1}$ is

$$\int_\Sigma \frac{1}{H} dA \geq \frac{n+1}{n} \int_\Omega d\Omega, \tag{1.5}$$

where Ω is the enclosed body by Σ. Equality in (1.5) holds if and only if Σ is a round sphere. (1.5) is a sharp inequality for hypersurfaces of positive mean curvature inspired by a classical inequality of Heintze and Karcher [28]. In 1987, Ros [51] provided a proof of the above inequality by using a remarkable Reilly formula (see [49]), and applied it to show Alexandrov's rigidity theorem for high order mean curvatures. Recently, Brendle [12] established such an inequality in a large class of warped product spaces, including the space forms and the (Anti-de Sitter-)Schwarzschild manifold. A geometric flow method, which is quite different from Ros' proof, was used by Brendle. Motivated by Brendle's work, new Reilly type formulae have been established by the second named author and his collaborators in [33,34,48]. These formulae will be used to establish the following Heintze–Karcher–Ros inequality:

for an embedded hypersurface Σ lying in a half ball B_+ in any space forms with its boundary $\partial\Sigma \subset \partial B_+$, there holds

$$\int_\Sigma \frac{V_a}{H} dA \geq \frac{n+1}{n} \int_\Omega V_a d\Omega, \tag{1.6}$$

where Ω is the enclosed body by Σ and ∂B_+. Equality in (1.6) holds if and only if Σ is totally umbilical and intersects ∂B_+ orthogonally.

See Theorem 5.2 below. The Alexandrov rigidity theorem for embedded CMC hypersurfaces with free boundary follows from this inequality and the Minkowski formula (1.3). We believe that there is a sharp version of Heintze–Karcher–Ros type inequality for hypersurfaces in a ball, whose equality case is achieved by capillary hypersurfaces with a fixed contact angle $\theta \in (0, \pi)$.

The remaining part of this paper is organized as follows. In Sect. 2 we review the definition and basic properties of capillary hypersurfaces. Since we are concerned with the immersions, a suitable notion of volume and the so-called wetting area functional is needed to study capillary hypersurfaces. In Sect. 3 we give a proof of Theorem 1.3 for capillary hypersurfaces in a ball in $\mathbb{R}^{n+1}$ after establishing the Minkowski formula (3.4). Theorem 1.1 is a special case of Theorem 1.3. The same proof works for singular hypersurfaces, hence we have Theorem 1.2. In Sect. 4 we provide a detailed proof of Theorem 1.3 for capillary hypersurfaces in a ball in $\mathbb{H}^{n+1}$ and sketch a proof for capillary hypersurfaces in a ball in $\mathbb{S}^{n+1}$. For the corresponding exterior problem, we sketch its proof at the end of Sect. 4. In Sect. 5, we prove the Heintze–Karcher–Ros type inequality and the Alexandrov theorem for hypersurfaces in a ball with free boundary.

2 Preliminaries on capillary hypersurfaces

Let $(\bar{M}^{n+1}, \bar{g})$ be an oriented $(n+1)$-dimensional Riemannian manifold and B be a smooth compact domain in $\bar{M}$ that is diffeomorphic to a Euclidean ball. Let $x : (M^n, g) \to B$ be an isometric immersion of an orientable n-dimensional compact manifold M with boundary ∂M into B that maps $\text{int}M$ into $\text{int}B$ and ∂M into ∂B.

We denote by $\bar{\nabla}$, $\bar{\Delta}$ and $\bar{\nabla}^2$ the gradient, the Laplacian and the Hessian on $\bar{M}$ respectively, while by ∇, Δ and ∇^2 the gradient, the Laplacian and the Hessian on M respectively. We will use the following terminology for four normal vector fields. We choose one of the unit normal vector field along x and denote it by ν. We denote by $\bar{N}$ the unit outward normal to ∂B in B and μ be the unit outward normal to ∂M in M. Let $\bar{\nu}$ be the unit normal to ∂M in ∂B such that the bases $\{\nu, \mu\}$ and $\{\bar{\nu}, \bar{N}\}$ have the same orientation in the normal bundle of $\partial M \subset \bar{M}$. See Fig. 1. Denote by h and H the second fundamental form and the mean curvature of the immersion x respectively. Precisely, $h(X, Y) = \bar{g}(\bar{\nabla}_X \nu, Y)$ and $H = \text{tr}_g(h)$. For constant mean curvature hypersurfaces which are our main concern, we always choose ν to be one of the unit normal vector fields so that $H \geq 0$.

Since in this paper we consider immersions, we need to introduce generalized definitions of area, volume and a wetting area for an isometric immersion. For embedded

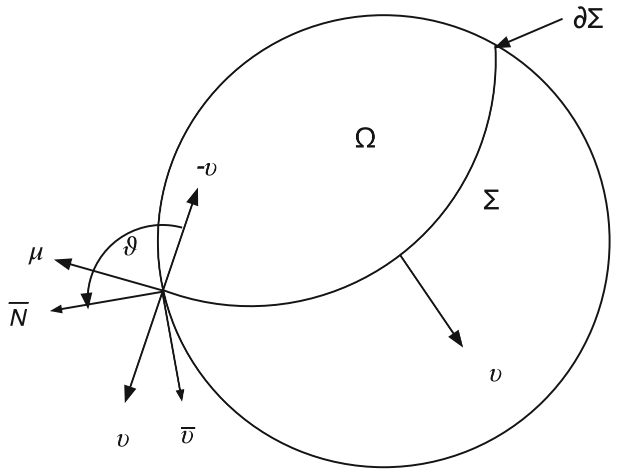

Fig. 1 $\Sigma = x(M)$ and $\partial\Sigma = x(\partial M)$

hypersurfaces, these generalized definitions are certainly equivalent to the usual definitions (see [53,54]).

By an admissible variation of x we mean a differentiable map $x : (-\epsilon, \epsilon) \times M \to B \subset \bar{M}$ such that $x(t, \cdot) : M \to B$ is an immersion satisfying $x(t, \text{int}M) \subset \text{int}B$ and $x(t, \partial M) \subset \partial B$ for every $t \in (-\epsilon, \epsilon)$ and $x(0, \cdot) = x$. For this variation, the area functional $A : (-\epsilon, \epsilon) \to \mathbb{R}$ and the volume functional $V : (-\epsilon, \epsilon) \to \mathbb{R}$ are defined by

$$A(t) = \int_M dA_t,$$
$$V(t) = \int_{[0,t]\times M} x^* dV_{\bar{M}},$$

where dA_t is the area element of M with respect to the metric induced by $x(t, \cdot)$ and $dV_{\bar{M}}$ is the volume element of $\bar{M}$. A variation is said to be volume-preserving if $V(t) = V(0) = 0$ for each $t \in (-\epsilon, \epsilon)$. Another area functional, which is called wetting area functional, $W(t) : (-\epsilon, \epsilon) \to \mathbb{R}$ is defined by

$$W(t) = \int_{[0,t]\times\partial M} x^* dA_{\partial B},$$

where $dA_{\partial B}$ is the area element of ∂B.

Fix a real number $\theta \in (0, \pi)$. The energy functional $E(t) : (-\epsilon, \epsilon) \to \mathbb{R}$ is defined by

$$E(t) = A(t) - \cos\theta\, W(t).$$

The first variation formulae of $V(t)$ and $E(t)$ for an admissible variation with a variation vector field $Y = \frac{\partial}{\partial t}x(t,\cdot)|_{t=0}$ are given by

$$V'(0) = \int_M \bar{g}(Y,\nu)dA,$$
$$E'(0) = \int_M H\bar{g}(Y,\nu)dA + \int_{\partial M} \bar{g}(Y,\mu - \cos\theta\,\bar{\nu})ds,$$

where dA and ds are the area element of M and ∂M respectively, see e.g. [53].

An immersion $x : M \to B$ is said to be *capillary* if it is a critical point of the energy function E for any volume-preserving variation of x. It follows from the above first variation formulae that x is capillary if and only if x has constant mean curvature and ∂M intersects ∂B at the constant angle θ. We make a convention on the choice of ν to be the opposite direction of mean curvature vector so that the mean curvature of a spherical cap is positive. Under this convention, along ∂M, the angle between $-\nu$ and $\bar{N}$ or equivalently between μ and $\bar{\nu}$ is everywhere equal to θ (see Figure 1). To be more precise, in the normal bundle of ∂M, we have the following relations:

$$\mu = \sin\theta\,\bar{N} + \cos\theta\,\bar{\nu}, \tag{2.1}$$
$$\nu = -\cos\theta\,\bar{N} + \sin\theta\,\bar{\nu}. \tag{2.2}$$

For each smooth function φ on M with $\int_M \varphi dA_M = 0$, there exists an admissible volume-preserving variation of x with the variation vector field having $\varphi\nu$ as normal part (see [53], page 348). When x is a capillary hypersurface, for an admissible volume-preserving variation with respect to φ, the second variational formula of E is given by

$$E''(0) = \int_M -\varphi(\Delta\varphi + (|h|^2 + \overline{\mathrm{Ric}}(\nu,\nu))\varphi)dA + \int_{\partial M} \varphi(\nabla_\mu\varphi - q\varphi)ds. \tag{2.3}$$

Here

$$q = \frac{1}{\sin\theta}h^{\partial B}(\bar{\nu},\bar{\nu}) + \cot\theta\,h(\mu,\mu),$$

$\overline{\mathrm{Ric}}$ is the Ricci curvature tensor of $\bar{M}$, and $h^{\partial B}$ is the second fundamental form of ∂B in $\bar{M}$ given by $h^{\partial B}(X,Y) = \bar{g}(\bar{\nabla}_X\bar{N}, Y)$, see e.g. [53].

A capillary hypersurface is called stable if $E''(0) \geq 0$ for all volume-preserving variations, that is,

$$E''(0) \geq 0, \qquad \forall\varphi \in \mathcal{F} := \left\{\varphi \in C^\infty(M) | \int_M \varphi dA = 0\right\}.$$

The following proposition is a well-known and fundamental fact for capillary hypersurfaces when ∂B is umbilical in $\bar{M}$.

Proposition 2.1 *Assume ∂B is umbilical in $\bar{M}$. Let $x : M \to B$ be an immersion whose boundary ∂M intersects ∂B at a constant angle θ. Then μ is a principal direction of ∂M in M. Namely, $h(e, \mu) = 0$ for any $e \in T(\partial M)$. In turn,*

$$\bar{\nabla}_\mu \nu = h(\mu, \mu)\mu.$$

Proof For $e \in T(\partial M)$, by using (2.1) and (2.2), we have

$$\begin{aligned} h(e, \mu) &= \bar{g}(\bar{\nabla}_e \nu, \mu) = \bar{g}(\bar{\nabla}_e(-\cos\theta\, \bar{N} + \sin\theta\, \bar{\nu}), \sin\theta\, \bar{N} + \cos\theta\, \bar{\nu}) \\ &= -\bar{g}(\bar{\nabla}_e \bar{N}, \bar{\nu}) = -h^{\partial B}(e, \bar{\nu}) = 0. \end{aligned}$$

□

3 Capillary hypersurfaces in a euclidean ball

In this section, we consider the case $(\bar{M}, \bar{g}) = (\mathbb{R}^{n+1}, \delta)$ and $B = \bar{\mathbb{B}}^{n+1}$ is the Euclidean unit ball (in our notation, $\mathbb{B}^{n+1}$ is the Euclidean unit open ball). In this case, $\overline{\mathrm{Ric}} \equiv 0$, $h^{\partial\mathbb{B}} = g^{\partial\mathbb{B}}$ and $\bar{N}(x) = x$. Abuse of notation, we use x to denote the position vector in $\mathbb{R}^{n+1}$. We use $\langle\cdot,\cdot\rangle$ to denote the Euclidean inner product.

3.1 A new Minkowski type formula in $\mathbb{R}^{n+1}$

In this subsection we establish a new Minkowski type formula, which is very powerful for hypersurfaces in $\mathbb{B}^{n+1}$ with free boundary or intersecting $\partial\mathbb{B}^{n+1}$ with a constant angle.

We first consider a conformal Killing vector field. For each constant vector field $a \in \mathbb{R}^{n+1}$, define a corresponding smooth vector field X_a in $\mathbb{R}^{n+1}$ by

$$X_a = \langle x, a\rangle x - \frac{1}{2}(|x|^2 + 1)a. \tag{3.1}$$

Define $f : \mathbb{R}^{n+1}\backslash(0, -1) \to \mathbb{R}^{n+1}$ by

$$f(u, v) = \frac{2(u, 0) + (|u|^2 + v^2 - 1)e_{n+1}}{|u|^2 + (1 + v)^2},$$

where $(u, v) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ and $e_{n+1} = (0, 1)$. One can check that f maps $\mathbb{R}^{n+1}_+ \to \mathbb{B}^{n+1}$ and $\partial\mathbb{R}^{n+1}_+ \to \mathbb{S}^n$. Moreover f is conformal. In fact

$$f^*(\delta_{\mathbb{B}^{n+1}}) = \frac{4}{(|u|^2 + (1 + v)^2)^2}\delta_{\mathbb{R}^{n+1}_+}.$$

If one transfers the free boundary problem in $\mathbb{B}^{n+1}$ to the free boundary problem in $\mathbb{R}^{n+1}_+$ with the pull back metric $f^*(\delta_{\mathbb{B}^{n+1}})$, one obtains an equivalent problem.

The vector field X_a with $a = e_{n+1}$ is the push-forward of the radial vector field (or the position vector field) (u, v) with respect to the origin in $\mathbb{R}^{n+1}_+$, which is usually important in such problems. This is the way we found that this vector field should be useful in the capillary problems. From this observation, it is clear that X_a is conformal Killing and tangential to $\partial\mathbb{B}^{n+1}$. Namely, we have the following two simple but crucial properties of X_a.

Proposition 3.1 *X_a is a conformal Killing vector field and its restriction on $\partial\mathbb{B}^{n+1}$ is a tangential vector field on $\partial\mathbb{B}^{n+1}$, i.e.,*

(i) *X_a is a conformal Killing vector field in $\mathbb{R}^{n+1}$ with $\mathcal{L}_{X_a}\bar{g} = \langle x, a\rangle \bar{g}$, namely,*

$$\frac{1}{2}\left[\bar{\nabla}_i (X_a)_j + \bar{\nabla}_j (X_a)_i\right] = \langle x, a\rangle \delta_{ij}. \tag{3.2}$$

(ii) *$X_a|_{\partial\mathbb{B}}$ is a tangential vector field on $\partial\mathbb{B}$. I.e.,*

$$\langle X_a, x\rangle|_{\partial\mathbb{B}} = 0. \tag{3.3}$$

Proof It is a well-known fact and one can check by a direct computation. □

Remark 3.1 The conformal Killing property of X_a is well-known in conformal geometry. For each $a \in \mathbb{R}^{n+1}$, X_a generates a 1-parameter family of conformal automorphism of $\mathbb{B}^{n+1}$ onto itself, see [36], page 274. The restriction of X_a to $\mathbb{S}^n$ gives a conformal Killing vector field on $\mathbb{S}^n$ generating an associated 1-parameter family of conformal automorphism of $\mathbb{S}^n$, which has been widely used in differential geometry and conformal geometry, see e.g. [10,17,42,43]. This vector field was used by Fraser and Schoen in their study of free boundary to show the result mentioned in the Introduction about the first Steklov eigenvalue [56]. We also realized that this vector field has already been used in the capillary problems implicitly by Ros and Vergasta [53] and explicitly by Marinov [40] in 2-dimension and Li and Xiong [31] in any dimensions.

Utilizing the conformal Killing vector field X_a, we show the following Minkowski type formula.

Proposition 3.2 *Let $x : M \to \bar{\mathbb{B}}^{n+1}$ be an isometric immersion into the Euclidean unit ball, whose boundary ∂M intersects $\partial\mathbb{B}^{n+1}$ at a constant angle $\theta \in (0, \pi)$. Let $a \in \mathbb{R}^{n+1}$ be a constant vector field and X_a be defined by (3.1). Then*

$$\int_M n\langle x + \cos\theta\, \nu, a\rangle dA = \int_M H\langle X_a, \nu\rangle dA. \tag{3.4}$$

Proof Denote by X_a^T the tangential projection of X_a on M. Let $\{e_\alpha\}_{\alpha=1}^n$ be an orthonormal frame on M. We claim that

$$\frac{1}{2}\left[\nabla_\alpha (X_a^T)_\beta + \nabla_\beta (X_a^T)_\alpha\right] = \langle x, a\rangle g_{\alpha\beta} - h_{\alpha\beta}\langle X_a, \nu\rangle. \tag{3.5}$$

Here $\nabla_\alpha (X_a^T)_\beta := \langle \nabla_{e_\alpha} X_a^T, e_\beta \rangle$. In fact,

$$\begin{aligned}\nabla_\alpha (X_a^T)_\beta &= \langle \bar\nabla_{e_\alpha} X_a^T, e_\beta \rangle \\ &= \langle \bar\nabla_{e_\alpha} X_a, e_\beta \rangle - \langle \bar\nabla_{e_\alpha} (\langle X_a, \nu \rangle \nu), e_\beta \rangle \\ &= \bar\nabla_\alpha (X_a)_\beta - \langle X_a, \nu \rangle \langle \bar\nabla_{e_\alpha} \nu, e_\beta \rangle \\ &= \bar\nabla_\alpha (X_a)_\beta - h_{\alpha\beta} \langle X_a, \nu \rangle.\end{aligned}$$

By using (3.2), we get the claim.

Taking trace of (3.5) with respect to the induced metric g and integrating over M, we have

$$\int_M n\langle x, a\rangle - H\langle X_a, \nu\rangle dA = \int_M \mathrm{div}_M (X_a^T) dA = \int_{\partial M} \langle X_a^T, \mu \rangle ds. \tag{3.6}$$

Note that on ∂M, $\bar N = x$ and $X_a = \langle x, a\rangle x - a$. By using (2.1), (2.2) and (3.3), we deduce

$$\begin{aligned}\langle X_a^T, \mu\rangle = \langle X_a, \mu \rangle &= \langle X_a, \sin\theta\, \bar N + \cos\theta\, \bar\nu \rangle = \cos\theta\, \langle X_a, \bar\nu \rangle \\ &= \cos\theta\, (\langle x, a\rangle \langle x, \bar\nu\rangle - \langle a, \bar\nu\rangle) = -\cos\theta\, \langle a, \bar\nu \rangle.\end{aligned}$$

It follows from (3.6)

$$\int_M n\langle x, a\rangle - H\langle X_a, \nu\rangle dA = -\cos\theta \int_{\partial M} \langle \bar\nu, a\rangle ds. \tag{3.7}$$

When $\theta = \frac{\pi}{2}$, i.e., if we are in the free boundary case, the Minkowski formula (3.4) follows already from (3.7). For the general case, we claim

$$n \int_M \langle \nu, a \rangle dA = \int_{\partial M} \langle \bar\nu, a \rangle ds. \tag{3.8}$$

It is easy to see that the Minikowski formula (3.4) follows from the claim and (3.7).

It remains to show this claim. It has been shown in [1] that

$$n \int_M \langle \nu, a\rangle dA = \int_{\partial M} \langle x, \mu\rangle \langle \nu, a\rangle - \langle x, \nu \rangle \langle \mu, a \rangle ds. \tag{3.9}$$

For the convenience of reader, we give a proof of (3.9). Set $Z_a = \langle \nu, a\rangle x - \langle x, \nu\rangle a$. Then

$$\begin{aligned}\mathrm{div}_M[(Z_a)^T] &= [h(a^T, x^T) + \langle \nu, a\rangle (n - \langle x, \nu\rangle H)] - [h(x^T, a^T) - \langle x, \nu\rangle \langle \nu, a\rangle H] \\ &= n\langle \nu, a\rangle.\end{aligned}$$

Then (3.9) follows by integration by parts. From (2.1) and (2.2), we deduce

$$\begin{aligned}&\langle x,\mu\rangle\langle\nu,a\rangle-\langle x,\nu\rangle\langle\mu,a\rangle\\&\quad=\sin\theta\langle-\cos\theta\,\bar N+\sin\theta\,\bar\nu,a\rangle+\cos\theta\langle\sin\theta\,\bar N+\cos\theta\,\bar\nu,a\rangle\\&\quad=\langle\bar\nu,a\rangle.\end{aligned}$$

Therefore, we get the claim (3.8) and the proof is completed. □

Remark 3.2 For the free boundary problem, i.e., $\theta=\pi/2$, we obtain the Minkowski formula discussed in the Introduction:

$$n\int_M\langle x,a\rangle dA=\int_M H\langle X_a,\nu\rangle dA. \tag{3.10}$$

We remark that (3.10) holds also for any compact hypersurfaces without boundary in $\mathbb{R}^{n+1}$ with the same proof, just ignoring the boundary integral. To our best knowledge it is also new for any compact hypersurfaces without boundary and we believe that it has its own interest.

Minkowski formula (3.4) plays a crucial role in the proof of uniqueness of stable capillary hypersurfaces in a Euclidean ball in the next subsection. Its further interesting applications will be presented in Sect. 5.

3.2 Uniqueness of stable capillary hypersurfaces in a Euclidean ball

Proposition 3.3 *Let $x: M\to\bar{\mathbb{B}}^{n+1}$ be an isometric immersion into the Euclidean unit ball, whose boundary ∂M intersects $\partial\mathbb{B}^{n+1}$ at a constant angle $\theta\in(0,\pi)$. Let $a\in\mathbb{R}^{n+1}$ be a constant vector field. Then along ∂M,*

$$\bar\nabla_\mu\langle x+\cos\theta\,\nu,a\rangle=q\langle x+\cos\theta\,\nu,a\rangle, \tag{3.11}$$

$$\bar\nabla_\mu\langle X_a,\nu\rangle=q\langle X_a,\nu\rangle, \tag{3.12}$$

where

$$q=\frac{1}{\sin\theta}+\cot\theta\,h(\mu,\mu). \tag{3.13}$$

Proof Using Proposition 2.1,

$$\bar\nabla_\mu\langle x+\cos\theta\,\nu,a\rangle=\langle\mu+\cos\theta\,h(\mu,\mu)\mu,a\rangle=q\sin\theta\langle\mu,a\rangle.$$

On the other hand, using (2.1) and (2.2),

$$\begin{aligned}\langle x+\cos\theta\,\nu,a\rangle&=\langle x+\cos\theta(-\cos\theta\bar N+\sin\theta\bar\nu),a\rangle\\&=\sin\theta\langle\sin\theta\bar N+\cos\theta\bar\nu,a\rangle=\sin\theta\langle\mu,a\rangle.\end{aligned}$$

Thus we get (3.11).

Using the definition (3.1) of X_a and again Proposition 2.1,

$$\begin{aligned}\bar{\nabla}_\mu\langle X_a,\nu\rangle &= \langle\bar{\nabla}_\mu X_a,\nu\rangle+\langle X_a,\bar{\nabla}_\mu\nu\rangle\\ &=\left\langle\bar{\nabla}_\mu\left(\langle x,a\rangle x-\frac{1}{2}(|x|^2+1)a\right),\nu\right\rangle+h(\mu,\mu)\langle X_a,\mu\rangle\\ &=\langle\langle\mu,a\rangle x+\langle x,a\rangle\mu-\langle x,\mu\rangle a,\nu\rangle+h(\mu,\mu)\langle\langle x,a\rangle x-a,\mu\rangle\\ &=-\cos\theta\langle\mu,a\rangle-\sin\theta\langle\nu,a\rangle+h(\mu,\mu)(\sin\theta\langle x,a\rangle-\langle\mu,a\rangle).\end{aligned}$$

Note that $x=\bar{N}=\sin\theta\,\mu-\cos\theta\,\nu$ and in turn $\mu=\frac{1}{\sin\theta}x+\cot\theta\,\nu$, we deduce further

$$\begin{aligned}\bar{\nabla}_\mu\langle X_a,\nu\rangle &=-\cot\theta(1+h(\mu,\mu)\cos\theta)\langle x,a\rangle-\frac{1}{\sin\theta}(1+h(\mu,\mu)\cos\theta)\langle\nu,a\rangle\\ &=-q(\cos\theta\langle x,a\rangle+\langle\nu,a\rangle).\end{aligned}$$

On the other hand,

$$\langle X_a,\nu\rangle|_{\partial M}=\langle x,a\rangle\langle x,\nu\rangle-\langle a,\nu\rangle=-(\cos\theta\langle x,a\rangle+\langle\nu,a\rangle).$$

(3.12) follows. □

Proposition 3.4 *Let $x: M\to\mathbb{R}^{n+1}$ be an isometric immersion into the Euclidean space. Let $a\in\mathbb{R}^{n+1}$ be a constant vector field. The following identities hold along M:*

$$\Delta x=-H\nu, \tag{3.14}$$
$$\Delta\frac{1}{2}|x|^2=n-H\langle x,\nu\rangle, \tag{3.15}$$
$$\Delta\nu=\nabla H-|h|^2\nu, \tag{3.16}$$
$$\Delta\langle x,\nu\rangle=\langle x,\nabla H\rangle+H-|h|^2\langle x,\nu\rangle, \tag{3.17}$$
$$\Delta\langle X_a,\nu\rangle=\langle X_a,\nabla H\rangle+\langle x,a\rangle H-|h|^2\langle X_a,\nu\rangle-n\langle\nu,a\rangle. \tag{3.18}$$

Proof Equations (3.14)–(3.17) are well-known. We now prove (3.18). First,

$$\Delta\langle X_a,\nu\rangle=\langle\Delta X_a,\nu\rangle+2\langle\nabla X_a,\nabla\nu\rangle+\langle X_a,\Delta\nu\rangle.$$

Using the definition (3.1) of X_a, (3.14) and (3.15), we see

$$\begin{aligned}\langle\Delta X_a,\nu\rangle &=\left\langle\Delta\left(\langle x,a\rangle x-\frac{1}{2}(|x|^2+1)a\right),\nu\right\rangle\\ &=\big\langle-H\langle\nu,a\rangle x-\langle x,a\rangle H\nu-(n-H\langle x,\nu\rangle)a,\nu\big\rangle\\ &=-H\langle x,a\rangle-n\langle\nu,a\rangle.\end{aligned}$$

Also,

$$\begin{aligned}\langle \nabla X_a, \nabla \nu \rangle &= \left\langle \nabla \left(\langle x, a \rangle x - \frac{1}{2}(|x|^2 + 1)a \right), \nabla \nu \right\rangle \\ &= \langle e_\alpha, a \rangle h(e_\alpha, x^T) + \langle x, a \rangle H - \langle x, e_\alpha \rangle h(e_\alpha, a^T) \\ &= H \langle x, a \rangle.\end{aligned}$$

Using (3.16),

$$\langle X_a, \Delta \nu \rangle = \langle X_a, \nabla H \rangle - |h|^2 \langle X_a, \nu \rangle.$$

Combining above, we get (3.18). □

Proposition 3.5 *Let $x : M \to \bar{\mathbb{B}}^{n+1}$ be an isometric immersion into the Euclidean unit ball, whose boundary ∂M intersects $\partial \mathbb{B}^{n+1}$ at a constant angle $\theta \in (0, \pi)$. For each constant vector field $a \in \mathbb{R}^{n+1}$ define*

$$\varphi_a = n \langle x + \cos \theta \, \nu, a \rangle - H \langle X_a, \nu \rangle$$

along M. Then φ_a satisfies

$$\int_M \varphi_a dA = 0, \tag{3.19}$$

$$\nabla_\mu \varphi_a - q \varphi_a = 0. \tag{3.20}$$

If, in addition, that M has constant mean curvature, then φ_a satisfies also

$$\Delta \varphi_a + |h|^2 \varphi_a = (n|h|^2 - H^2) \langle x, a \rangle. \tag{3.21}$$

Proof (3.19) and (3.20) follow from Propositions 3.2 and 3.3 respectively. If H is constant, Proposition 3.4 implies

$$\begin{aligned}(\Delta + |h|^2) \langle x, a \rangle &= |h|^2 \langle x, a \rangle - H \langle \nu, a \rangle, \\ (\Delta + |h|^2) \langle X_a, \nu \rangle &= H \langle x, a \rangle - n \langle \nu, a \rangle, \\ (\Delta + |h|^2) \langle \nu, a \rangle &= 0.\end{aligned}$$

Then (3.21) follows. □

Now we prove the uniqueness for stable capillary hypersurfaces in a Euclidean ball.

Theorem 3.1 *Assume $x : M \to \bar{\mathbb{B}}^{n+1}$ is an immersed stable capillary hypersurface in the Euclidean unit ball $\mathbb{B}^{n+1}$ with constant mean curvature $H \geq 0$ and constant contact angle $\theta \in (0, \pi)$. Then x is either a totally geodesic ball or a spherical cap.*

Proof The stability condition states as

$$-\int_M \varphi(\Delta\varphi + |h|^2\varphi)dA + \int_{\partial M} \varphi(\nabla_\mu\varphi - q\varphi)ds \geq 0 \tag{3.22}$$

for all function $\varphi \in \mathcal{F}$, where q is given by (3.13).

For each constant vector field $a \in \mathbb{R}^{n+1}$, we consider φ_a, which is defined in Proposition 3.5. Proposition 3.5 implies that $\varphi_a \in \mathcal{F}$ and is an admissible function for testing stability. Inserting (3.19) and (3.21) into the stability condition (3.22), we get

$$\int_M (n\langle x + \cos\theta\, \nu, a\rangle - H\langle X_a, \nu\rangle)\, \langle x, a\rangle(n|h|^2 - H^2)\, dA \leq 0 \text{ for any } a \in \mathbb{R}^{n+1}. \tag{3.23}$$

We take a to be the $n+1$ coordinate vectors $\{E_i\}_{i=1}^{n+1}$ in $\mathbb{R}^{n+1}$, and add (3.23) for all a to get

$$\int_M \left(n|x|^2 + n\cos\theta\langle x, \nu\rangle - \frac{1}{2}(|x|^2 - 1)H\langle x, \nu\rangle\right)(n|h|^2 - H^2)\, dA \leq 0. \tag{3.24}$$

Here we have used

$$\sum_{i=1}^{n+1}\langle x, E_i\rangle X_{E_i} = \frac{1}{2}(|x|^2 - 1)x.$$

Now, if $H = 0$ and $\theta = \frac{\pi}{2}$, (3.24) gives $\int_M |x|^2|h|^2 dA \leq 0$, which implies that $h \equiv 0$, i.e., $x : M \to \bar{\mathbb{B}}^{n+1}$ is totally geodesic. This gives a new proof of a result of Ros and Vergasta [54].

When $H \neq 0$ or $\theta \neq \frac{\pi}{2}$, the proof does not follow from (3.24) directly. In fact the term

$$\gamma := n|x|^2 + n\cos\theta\langle x, \nu\rangle - \frac{1}{2}(|x|^2 - 1)H\langle x, \nu\rangle \tag{3.25}$$

may have no definite sign. In order to handle this problem, we introduce the following function

$$\Phi = \frac{1}{2}(|x|^2 - 1)H - n(\langle x, \nu\rangle + \cos\theta).$$

Using (3.15) and (3.17), one can check that Φ satisfies

$$\Delta\Phi = (n|h|^2 - H^2)\langle x, \nu\rangle. \tag{3.26}$$

Since $|x|^2 = 1$ and $\langle x, \nu\rangle = -\cos\theta$ on ∂M, we have $\Phi = 0$ on ∂M. Consequently,

$$\int_M \Delta\frac{1}{2}\Phi^2 = \int_{\partial M} \Phi\nabla_\mu\Phi\, dA = 0. \tag{3.27}$$

Adding (3.27) to (3.24) and using (3.26), we obtain

$$\begin{aligned}
0 &\geq \int_M \left(n(|x|^2 + \cos\theta \langle x, \nu\rangle) - \frac{1}{2}(|x|^2 - 1) H \langle x, \nu\rangle \right) (n|h|^2 - H^2) + \Delta \frac{1}{2} \Phi^2 \, dA \\
&= \int_M \left(n(|x|^2 + \cos\theta \langle x, \nu\rangle) - \frac{1}{2}(|x|^2 - 1) H \langle x, \nu\rangle \right) (n|h|^2 - H^2) \\
&\quad + \Phi \Delta \Phi + |\nabla \Phi|^2 \, dA \\
&= \int_M n|x^T|^2 (n|h|^2 - H^2) + |\nabla \Phi|^2 \, dA \\
&\geq 0,
\end{aligned}$$

where x^T is the tangential part of x. The last inequality holds since $n|h|^2 \geq H^2$ which follows from Cauchy's inequality. It follows that $|x^T|^2(n|h|^2 - H^2) = 0$ on M and $\nabla\Phi = 0$. The latter implies that Φ is a constant. This fact, together with (3.26), implies that $\langle x, \nu\rangle(n|h|^2 - H^2) = 0$ on M. Together with $|x^T|^2(n|h|^2 - H^2) = 0$, it implies that $|x|^2(n|h|^2 - H^2) = 0$ on M. Hence we have $n|h|^2 - H^2 = 0$ on M, which means that M is umbilical and is a spherical cap. The proof is completed. □

Remark 3.3 In the case of free boundary, i.e., $\cos\theta = 0$, Barbosa [4] proved that $\langle x, \nu\rangle$ has a fixed sign, namely, $\langle x, \nu\rangle \leq 0$ in M in our notation. By our convention of the choice of ν, we have $H > 0$. It is not clear whether γ has a sign from these information. However, one can show the non-negativity of γ with the help of Φ used in the proof as follows. In this case, by (3.26) we have $\Delta\Phi \leq 0$ in M and $\Phi = 0$ on ∂M, which implies that $\Phi \geq 0$ by the maximum principle, and hence $-\langle x, \nu\rangle\Phi \geq 0$ in M. It follows that

$$\gamma \geq n\langle x, \nu\rangle^2 - \frac{1}{2}(|x|^2 - 1) H \langle x, \nu\rangle = -\langle x, \nu\rangle \Phi \geq 0.$$

Therefore, in the case of free boundary, with the help of the non-negativity of γ, we can get $n|h|^2 - H^2 = 0$ from (3.24).

Remark 3.4 Since the new Minkowski formula holds also for closed hypersurfaces in $\mathbb{R}^{n+1}$, (Remark 3.2), the above proof for the stability of capillary surfaces works without any changes for closed hypersurfaces. This means that we give a new proof of the result of Barbosa and do Carmo [5] mentioned above. This works also for closed hypersurfaces in space forms. See the next section.

4 Capillary hypersurfaces in a ball in space forms

In this section we handle the case when $\bar{M}$ is a space form $\mathbb{H}^{n+1}$ or $\mathbb{S}^{n+1}$ and B is a ball in $\bar{M}$. Since these two cases are quite similar, we will prove the hyperbolic case and indicate the minor modifications for the spherical case in Sect. 4.3 below.

4.1 A new Minkowski type formula in $\mathbb{H}^{n+1}$

Let $\mathbb{H}^{n+1}$ be the simply connected hyperbolic space with curvature -1. We use here the Poincaré ball model, which is given by

$$\mathbb{H}^{n+1} = \left(\mathbb{B}^{n+1}, \bar{g} = e^{2u}\delta\right), \quad e^{2u} = \frac{4}{(1-|x|^2)^2}. \tag{4.1}$$

One can also use other models. The advantage to use the Poincaré ball model for us is that for this model it is relatively easy to find the corresponding conformal Killing vector field X_a.

In this section we use δ or $\langle\cdot,\cdot\rangle$ to denote the Euclidean metric and the Cartesian coordinate in $\mathbb{B}^{n+1} \subset \mathbb{R}^{n+1}$. Sometimes we also represent the hyperbolic metric, in terms of the polar coordinate with respect to the origin, as

$$\bar{g} = dr^2 + \sinh^2 r g_{\mathbb{S}^n}.$$

We use $r = r(x)$ to denote the hyperbolic distance from the origin and denote $V_0 = \cosh r$. It is easy to verify that

$$V_0 = \cosh r = \frac{1+|x|^2}{1-|x|^2}, \quad \sinh r = \frac{2|x|}{1-|x|^2}. \tag{4.2}$$

The position function x, in terms of polar coordinate, can be represented by

$$x = \sinh r \partial_r. \tag{4.3}$$

It is well-known that x is a conformal Killing vector field with

$$\bar{\nabla} x = V_0 \bar{g}. \tag{4.4}$$

Let $B_R^{\mathbb{H}}$ be a ball in $\mathbb{H}^{n+1}$ with hyperbolic radius $R \in (0,\infty)$. By an isometry of $\mathbb{H}^{n+1}$, we may assume $B_R^{\mathbb{H}}$ is centered at the origin. $B_R^{\mathbb{H}}$, when viewed as a set in $\mathbb{B}^{n+1} \subset \mathbb{R}^{n+1}$, is the Euclidean ball with radius $R_{\mathbb{R}} := \sqrt{\frac{1-\operatorname{arccosh} R}{1+\operatorname{arccosh} R}} \in (0,1)$. The principal curvatures of $\partial B_R^{\mathbb{H}}$ are $\coth R$. The unit normal $\bar{N}$ to $\partial B_R^{\mathbb{H}}$ with respect to $\bar{g}$ is given by

$$\bar{N} = \frac{1}{\sinh R} x.$$

As in the Euclidean case, for each constant vector field $a \in \mathbb{R}^{n+1}$, define a corresponding smooth vector field X_a in $\mathbb{H}^{n+1}$ by

$$X_a = \frac{2}{1-R_{\mathbb{R}}^2}\left[\langle x, a\rangle x - \frac{1}{2}(|x|^2 + R_{\mathbb{R}}^2)a\right]. \tag{4.5}$$

Moreover, we define another smooth vector field Y_a in $\mathbb{H}^{n+1}$ by

$$Y_a = \frac{1}{2}(|x|^2 + 1)a - \langle x, a\rangle x. \tag{4.6}$$

Proposition 4.1 (i) *X_a is a conformal Killing vector field in $\mathbb{H}^{n+1}$ with*

$$\frac{1}{2}(\bar{\nabla}_i (X_a)_j + \bar{\nabla}_j (X_a)_i) = V_a \bar{g}_{ij}, \ \textit{where } V_a = \frac{2\langle x, a\rangle}{1 - |x|^2}. \tag{4.7}$$

(ii) *$X_a|_{\partial B_R^{\mathbb{H}}}$ is a tangential vector field on $\partial B_R^{\mathbb{H}}$. In particular,*

$$\bar{g}(X_a, \bar{N}) = 0.$$

(iii) *Y_a is a Killing vector field in $\mathbb{H}^{n+1}$, i.e.,*

$$\frac{1}{2}(\bar{\nabla}_i (Y_a)_j + \bar{\nabla}_j (Y_a)_i) = 0. \tag{4.8}$$

Remark 4.1 $Y_a = \lim_{R_{\mathbb{R}} \to 1}(R_{\mathbb{R}} - 1)X_a$. Though X_a and Y_a look very similar, they are quite different. Y_a is the Killing vector field induced by the isometry of "translation" in $\mathbb{H}^{n+1}$, while X_a is a special conformal vector field added by a translation as in the Euclidean case. For our purpose, Y_a in $\mathbb{H}^{n+1}$ plays a similar role as a constant vector field a in $\mathbb{R}^{n+1}$.

Proof These are known facts. For the convenience of reader we give a proof.

(i) Recall that X_a is a conformal Killing vector field in the Euclidean unit ball $\mathbb{B}^{n+1}$ with respect to the Euclidean metric (Proposition 3.1). A well known fact is that a conformal Killing vector field is still a conformal one with respect to a conformal metric, see e.g. [8]. To be precise,

$$\frac{1}{2}(\bar{\nabla}_i (X_a)_j + \bar{\nabla}_j (X_a)_i) = \frac{1}{n+1}\mathrm{div}_{\bar{g}}(X_a)\bar{g}_{ij},$$

where

$$\begin{aligned}
\mathrm{div}_{\bar{g}}(X_a) &= \mathrm{div}_\delta(X_a) + (n+1)du(X_a)\\
&= \frac{2}{1 - R_{\mathbb{R}}^2}(n+1)\langle x, a\rangle\\
&\quad +(n+1)\left\langle \frac{2x}{1 - |x|^2}, \frac{2}{1 - R_{\mathbb{R}}^2}\left[\langle x, a\rangle x - \frac{1}{2}(|x|^2 + R_{\mathbb{R}}^2)a\right]\right\rangle\\
&= (n+1)\frac{2\langle x, a\rangle}{1 - |x|^2}.
\end{aligned}$$

(ii) This is because $\langle X_a, x\rangle|_{\partial B^{n+1}_{R_\mathbb{R}}} = 0$ in the Euclidean metric and the fact that a conformal transformation preserves the angle.

(iii) As in (i), we know that Y_a is a conformal Killing vector field in $\mathbb{B}^{n+1}$ with respect to the Euclidean metric. Thus Y_a is again a conformal Killing one with respect to the conformal metric $\bar{g}$ with

$$\frac{1}{2}(\bar{\nabla}_i (Y_a)_j + \bar{\nabla}_j (Y_a)_i) = \frac{1}{n+1}\mathrm{div}_{\bar{g}}(Y_a)\bar{g}_{ij},$$

where

$$\begin{aligned}\mathrm{div}_{\bar{g}}(Y_a) &= \mathrm{div}_\delta(Y_a) + (n+1)du(Y_a)\\ &= -(n+1)\langle x, a\rangle - (n+1)\left\langle \frac{2x}{1-|x|^2}, \langle x, a\rangle x - \frac{1}{2}(|x|^2+1)a\right\rangle\\ &= 0.\end{aligned}$$

□

Proposition 4.2 *The functions V_0 and V_a satisfy*

$$\bar{\nabla}^2 V_0 = V_0\bar{g}, \tag{4.9}$$
$$\bar{\nabla}^2 V_a = V_a\bar{g}. \tag{4.10}$$

Proof Identity (4.9) is clear because $V_0 = \cosh r$. We verify next (4.10). Using the conformal transformation law of the Laplacian, one can compute directly that

$$\bar{\Delta} V_a = e^{-2u}(\Delta_\delta V_a + (n-1)du(V_a)) = (n+1)V_a. \tag{4.11}$$

Using (4.7) and the commutation formula

$$\bar{R}_{ijkl} = \bar{g}(\bar{R}(\partial_i, \partial_j)\partial_k, \partial_l) = \bar{g}(\bar{\nabla}_i\bar{\nabla}_j\partial_k - \bar{\nabla}_j\bar{\nabla}_i\partial_k, \partial_l)$$

and

$$\bar{R}_{ijkl} = -(g_{il}g_{jk} - g_{ik}g_{jl}),$$

we compute

$$\begin{aligned}(n+1)\bar{\nabla}_i\bar{\nabla}_j V_a &= \bar{\nabla}_i\bar{\nabla}_j\bar{\nabla}_k(X_a)^k\\ &= \bar{\nabla}_i(\bar{\nabla}_k\bar{\nabla}_j(X_a)^k + (X_a)^l\bar{R}_{jkl}{}^k)\\ &= \bar{\nabla}_i\bar{\nabla}_k(2V_a\delta_j^k - \bar{\nabla}^k(X_a)_j) + n\bar{\nabla}_i(X_a)_j\\ &= 2\bar{\nabla}_i\bar{\nabla}_j V_a - \bar{\nabla}_i\bar{\nabla}_k\bar{\nabla}^k(X_a)_j + n\bar{\nabla}_i(X_a)_j.\end{aligned}$$

Further,

$$\begin{aligned}\bar\nabla_i\bar\nabla_k\bar\nabla^k(X_a)_j &= \bar\nabla_k\bar\nabla_i\bar\nabla^k(X_a)_j - \bar\nabla^k(X_a)_l\bar R_{ik\ j}^{\ \ l} + \bar\nabla^l(X_a)_j\bar R_{ikl}^{\ \ \ k}\\ &= \bar\nabla_k(\bar\nabla^k\bar\nabla_i(X_a)_j + (X_a)_l\bar R_{i\ \ j}^{\ kl}) + \bar\nabla^k(X_a)_l\bar R_{ik\ j}^{\ \ l} + \bar\nabla^l(X_a)_j\bar R_{ikl}^{\ \ \ k}\\ &= \bar\Delta\bar\nabla_i(X_a)_j - 2\bar\nabla^k(X_a)_l(\bar g_{ij}\delta_k^l - \delta_i^l\bar g_{kj}) + \bar\nabla^l(X_a)_j n\bar g_{li}\\ &= \bar\Delta\bar\nabla_i(X_a)_j - 2\bar{\mathrm{div}}(X_a)\bar g_{ij} + 2\bar\nabla_j(X_a)_i + n\bar\nabla_i(X_a)_j\end{aligned}$$

and

$$(n-1)\bar\nabla_i\bar\nabla_j V_a = -\bar\Delta\bar\nabla_i(X_a)_j + 2\bar{\mathrm{div}}(X_a)\bar g_{ij} - 2\bar\nabla_j(X_a)_i. \tag{4.12}$$

Commutating the indices i and j in (4.12), summing up, and using (4.11) we obtain

$$\begin{aligned}2(n-1)\bar\nabla_i\bar\nabla_j V_a &= -\bar\Delta(\bar\nabla_i(X_a)_j + \bar\nabla_j(X_a)_i) + 4\bar{\mathrm{div}}(X_a)\bar g_{ij} - 2(\bar\nabla_i(X_a)_j + \bar\nabla_j(X_a)_i)\\ &= -2\bar\Delta V_a\bar g_{ij} + 4(n+1)V_a\bar g_{ij} - 4V_a\bar g_{ij}\\ &= 2(n-1)V_a\bar g_{ij}.\end{aligned}$$

Identity (4.10) follows. □

Remark 4.2 We remark that in $\mathbb{H}^{n+1}$, the vector space $\{V \in C^2(\mathbb{H}^{n+1}) : \bar\nabla^2 V = V\bar g\}$ is spanned by V_0 and V_a, $a \in \mathbb{R}^{n+1}$. Thus it has dimension $n+2$.

Note that the vector field a is not a constant (or parallel) with respect to the hyperbolic metric. In the following we derive formulae of covariant derivatives of several functions and vector fields associated with a. We will frequently use (4.1) and (4.2).

Proposition 4.3 *For any tangential vector field Z on $\mathbb{H}^{n+1}$,*

$$\bar\nabla_Z a = e^{-u}\left[\bar g(x,Z)a + \bar g(x,a)Z - \bar g(Z,a)x\right], \tag{4.13}$$
$$\bar\nabla_Z(e^{-u}a) = e^{-u}\left[\bar g(x,e^{-u}a)Z - \bar g(Z,e^{-u}a)x\right]. \tag{4.14}$$
$$\bar\nabla_Z V_0 = \bar g(x,Z), \tag{4.15}$$
$$\bar\nabla_Z V_a = \bar g(Z,e^{-u}a) + e^{-u}\bar g(x,e^{-u}a)\bar g(Z,x), \tag{4.16}$$
$$\bar\nabla_Z Y_a = e^{-u}\bar g(x,Z)a - e^{-u}\bar g(Z,a)x. \tag{4.17}$$
$$\bar\nabla_Z X_a = -\cosh R[e^{-u}\bar g(x,Z)a - e^{-u}\bar g(Z,a)x] + e^{-u}\bar g(x,a)Z. \tag{4.18}$$

Proof Let $\{E_i\}_{i=1}^{n+1}$ be the coordinate unit vector in $\mathbb{R}^{n+1}$. Let $Z = Z^iE_i$ and $a = a^iE_i$. Under the conformal transformation,

$$\begin{aligned}\bar\nabla_{E_i}E_j &= E_i(u)E_j + E_j(u)E_i - \langle E_k(u), E_k\rangle\delta_{ij}\\ &= \frac{2}{1-|x|^2}(x_iE_j + x_jE_i - x\delta_{ij}).\end{aligned}$$

It follows that

$$\begin{aligned}\bar\nabla_Z a &= Z^i a^j \bar\nabla_{E_i} E_j\\ &= Z^i a^j \frac{2}{1-|x|^2}(x_i E_j + x_j E_i - x\delta_{ij})\\ &= e^{-u}\left[\bar g(x,Z)a + \bar g(x,a)Z - \bar g(Z,a)x\right],\end{aligned}$$

where we have used $e^{-u} = \frac{1-|x|^2}{2}$ and $\bar g = e^{2u}\langle\cdot,\cdot\rangle$. It is easy to check

$$\bar\nabla_Z(e^{-u}) = -e^{-u}Z(u) = -e^{-2u}\bar g(x,Z). \tag{4.19}$$

Equation (4.14) follows then from (4.13) and (4.19). Equation (4.15) follow easily from $V_0 = \cosh r$ and $x = \sinh r\partial_r$.

We rewrite V_a as

$$V_a = \frac{2\langle x,a\rangle}{1-|x|^2} = \bar g(x, e^{-u}a). \tag{4.20}$$

We compute V_a using (4.20). Using (4.4) and (4.14), we get

$$\begin{aligned}\bar\nabla_Z V_a &= \bar g(\bar\nabla_Z x, e^{-u}a) + \bar g(x, \bar\nabla_Z(e^{-u}a))\\ &= V_0 e^{-u}\bar g(Z,a) + e^{-2u}\left[\bar g(x,a)\bar g(x,Z) - \bar g(Z,a)\bar g(x,x)\right]\\ &= e^{-u}\bar g(Z,a) + e^{-2u}\bar g(x,a)\bar g(Z,x).\end{aligned}$$

This is (4.16). In the last equality, we have used $V_0 - e^{-u}\bar g(x,x) = \cosh r - \frac{1}{1+\cosh r}\sinh^2 r = 1$.

Recall $Y_a = \frac12(|x|^2+1)a - \langle x,a\rangle x$. Using (4.13) and (4.4), we have

$$\begin{aligned}\bar\nabla_Z Y_a &= \langle x,Z\rangle a + \frac12(|x|^2+1)\bar\nabla_Z a - \langle Z,a\rangle x - \langle x,a\rangle\bar\nabla_Z x\\ &= e^{-2u}\bar g(x,Z)a + \frac12(|x|^2+1)e^{-u}\left[\bar g(x,Z)a + \bar g(x,a)Z - \bar g(Z,a)x\right]\\ &\quad -e^{-2u}\bar g(Z,a)x - e^{-2u}\bar g(x,a)V_0 Z\\ &= e^{-u}\bar g(x,Z)a - e^{-u}\bar g(Z,a)x.\end{aligned}$$

The proof of equation (4.18) is similar to that of (4.17). □

Let $x : M \to B_R^{\mathbb{H}}$ be an isometrically immersed hypersurface which intersects $\partial B_R^{\mathbb{H}}$ at a constant angle θ. As in the Euclidean space, by using properties of X_a and Y_a in Proposition 4.1 and the fact that $\partial B_R^{\mathbb{H}}$ is umbilical in $\mathbb{H}^{n+1}$, we have the following Minkowski type formula.

Proposition 4.4 (Minkowski formula) *Let $x : M \to B_R^{\mathbb{H}}$ be an isometric immersion into the hyperbolic ball $B_R^{\mathbb{H}}$, whose boundary ∂M intersects $\partial B_R^{\mathbb{H}}$ at a constant angle $\theta \in (0,\pi)$. Let $a \in \mathbb{R}^{n+1}$ be a constant vector field and X_a, Y_a are defined by (4.5) and (4.6). Then*

$$\int_M n(V_a + \sinh R \cos\theta\, \bar{g}(Y_a, \nu))dA = \int_M H\bar{g}(X_a, \nu)dA. \tag{4.21}$$

Proof Similarly as in the proof of Proposition 3.2, by using the two properties of X_a in Proposition 4.1, we get

$$\begin{aligned}\int_M nV_a - H\bar{g}(X_a, \nu)dA &= \int_M \mathrm{div}_M(X_a^T)dA \\ &= \int_{\partial M} \bar{g}(X_a^T, \mu)ds = -\frac{2R_{\mathbb{R}}^2}{1-R_{\mathbb{R}}^2}\cos\theta \int_{\partial M} \bar{g}(a, \bar{\nu})ds.\end{aligned} \tag{4.22}$$

Set

$$Z_a = \bar{g}(\nu, e^{-u}a)x - \bar{g}(x, \nu)(e^{-u}a).$$

We claim that

$$\mathrm{div}_M Z_a = n\bar{g}(Y_a, \nu). \tag{4.23}$$

Indeed, by a direct computation we have

$$\begin{aligned}\mathrm{div}_M[\bar{g}(\nu, e^{-u}a)x] = h(a^T, x^T) - e^{-u}\bar{g}(x^T, e^{-u}a)\bar{g}(x, \nu) \\ +\bar{g}(\nu, e^{-u}a)(nV_0 - H\bar{g}(x, \nu)),\end{aligned}$$

and

$$\begin{aligned}\mathrm{div}_M[\bar{g}(x, \nu)(e^{-u}a)] = h(x^T, a^T) + \bar{g}(x, \nu)[e^{-u}(n\bar{g}(x, e^{-u}a) \\ -\bar{g}(x^T, e^{-u}a)) - H\bar{g}(\nu, e^{-u}a)].\end{aligned}$$

It follows that

$$\begin{aligned}\mathrm{div}_M Z_a &= nV_0\bar{g}(\nu, e^{-u}a) - ne^{-u}\bar{g}(x, e^{-u}a)\bar{g}(x, \nu) \\ &= n\bar{g}(\nu, \frac{1}{2}(|x|^2+1)a - \langle x, a\rangle x) \\ &= n\bar{g}(Y_a, \nu),\end{aligned}$$

where we have used $V_0 = \frac{1+|x|^2}{1-|x|^2}$, $e^{-u} = \frac{1-|x|^2}{2}$ and $\bar{g} = e^{2u}\langle\cdot,\cdot\rangle$. Thus we proved the claim.

Integrating (4.23) over M and using integration by parts, we have

$$\int_M n\bar{g}(Y_a, \nu)dA = \int_{\partial M} \bar{g}(Z_a, \mu)ds. \tag{4.24}$$

Using (2.1) and (2.2), It is easy to check that

$$\bar{g}(Z_a, \mu)|_{\partial M} = R_{\mathbb{R}}\bar{g}(\bar{\nu}, a). \tag{4.25}$$

The Minwokski formual (4.21) follows from (4.22), (4.24) and (4.25). □

Proposition 4.5 *Along ∂M, we have*

$$\bar\nabla_\mu (V_a + \sinh R \cos\theta\, \bar g(Y_a,\nu)) = q(V_a + \sinh R\cos\theta\, \bar g(Y_a,\nu)), \tag{4.26}$$

$$\bar\nabla_\mu \bar g(X_a,\nu) = q\, \bar g(X_a,\nu), \tag{4.27}$$

where

$$q = \frac{1}{\sin\theta}\coth R + \cot\theta\, h(\mu,\mu). \tag{4.28}$$

Proof In this proof we always take value along ∂M and use (2.1) and (2.2). First, note that

$$\bar g(Y_a, x) = e^{2u}\langle Y_a, x\rangle = e^{2u}\frac{1}{2}(|x|^2-1)\langle x, a\rangle = e^{-u}\bar g(x,a) = V_a.$$

Thus we have

$$\begin{aligned} V_a + \sinh R\cos\theta\, \bar g(Y_a,\nu) &= \bar g(Y_a, x + \sinh R\cos\theta\,\nu)\\ &= \bar g(Y_a, \sinh R\,\bar N + \sinh R\cos\theta\,(-\cos\theta \bar N + \sin\theta\bar\nu))\\ &= \sinh R\sin\theta \bar g(Y_a,\mu). \end{aligned} \tag{4.29}$$

By (4.16) and (4.17), we compute

$$\begin{aligned} &\bar\nabla_\mu (V_a + \sinh R\cos\theta\, \bar g(Y_a,\nu))\\ &\quad = e^{-u}\bar g(\mu,a) + e^{-2u}\bar g(x,a)\bar g(\mu,x)\\ &\qquad + \sinh R\cos\theta\, \bar g(Y_a, h(\mu,\mu)\mu) + \sinh R\cos\theta\, e^{-u}[\bar g(x,\mu)\bar g(\nu,a)\\ &\qquad - \bar g(\mu,a)\bar g(x,\nu)]. \end{aligned}$$

Using $\nu = -\frac{1}{\cos\theta}\bar N + \tan\theta\,\mu$ and $x = \sinh R\,\bar N$, we obtain

$$\begin{aligned} &\sinh R\cos\theta\, e^{-u}[\bar g(x,\mu)\bar g(\nu,a) - \bar g(\mu,a)\bar g(x,\nu)]\\ &\quad = \sinh^2 R\, e^{-u}\bar g(\mu,a) - e^{-u}\bar g(x,\mu)\,\bar g(x,a). \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\bar\nabla_\mu (V_a + \sinh R\cos\theta\, \bar g(Y_a,\nu))\\ &\quad = e^{-u}\cosh^2 R\,\bar g(\mu,a) + (e^{-2u} - e^{-u})\bar g(x,a)\bar g(x,\mu)\\ &\qquad + \sinh R\cos\theta\, \bar g(Y_a, h(\mu,\mu)\mu)\\ &\quad = \cosh R\,\bar g\left(\frac{1}{2}(|x|^2+1)a - \langle x,a\rangle x, \mu\right) + \sinh R\cos\theta\,\bar g(Y_a, h(\mu,\mu)\mu)\\ &\quad = (\cosh R + \sinh R\cos\theta\, h(\mu,\mu))\bar g(Y_a,\mu). \end{aligned} \tag{4.30}$$

The first formula (4.26) follows from (4.29) and (4.30).

Next, using $\bar N = \sin\theta\,\mu - \cos\theta\,\nu$, we get

$$\begin{aligned}\bar g(X_a,\nu) &= \frac{2}{1-R_{\mathbb R}^2}\left[e^{-2u}\bar g(x,a)\bar g(x,\nu) - \frac12(|x|^2+R_{\mathbb R}^2)\bar g(a,\nu)\right]\\ &= \frac{2}{1-R_{\mathbb R}^2}\left[-\cos\theta\,R_{\mathbb R}^2\bar g(\sin\theta\,\mu-\cos\theta\,\nu,a)+R_{\mathbb R}^2\bar g(a,\nu)\right]\\ &= -\frac{2R_{\mathbb R}^2}{1-R_{\mathbb R}^2}\sin\theta\,\bar g(\cos\theta\,\mu+\sin\theta\,\nu,a). \qquad (4.31)\end{aligned}$$

Since $\nu = -\frac{1}{\cos\theta}\bar N + \tan\theta\,\mu$ and $X_a \perp \bar N$, we have

$$\bar g(X_a,\nu) = \tan\theta\,\bar g(X_a,\mu). \qquad (4.32)$$

In view of (4.18) and Proposition 2.1, we have

$$\begin{aligned}\bar\nabla_\mu \bar g(X_a,\nu) &= -\cosh R[e^{-u}\bar g(x,\mu)\bar g(a,\nu) - e^{-u}\bar g(\mu,a)\bar g(x,\nu)]\\ &\quad + e^{-u}\bar g(x,a)\bar g(\mu,\nu) + \bar g(X_a, h(\mu,\mu)\mu)\\ &= -\cosh R\,R_{\mathbb R}[\sin\theta\,\bar g(a,\nu)+\cos\theta\,\bar g(\mu,a)] + h(\mu,\mu)\bar g(X_a,\mu)\\ &= -\coth R\,\frac{2R_{\mathbb R}^2}{1-R_{\mathbb R}^2}\bar g(\cos\theta\,\mu+\sin\theta\,\nu,a) + h(\mu,\mu)\bar g(X_a,\mu)\\ &= \frac{1}{\sin\theta}\coth R\,\bar g(X_a,\nu) + h(\mu,\mu)\bar g(X_a,\mu), \qquad (4.33)\end{aligned}$$

where in the last equality, we have used (4.31). The second assertion (4.27) follows. The proof is completed. □

Proposition 4.6 *Let $x : M \to (\mathbb B^{n+1}, \bar g)$ be an isometric immersion into the hyperbolic Poincaré ball. Let a be a constant vector field in $\mathbb R^{n+1}$. The following identities hold along M:*

$$\begin{aligned}\Delta V_0 &= nV_0 - H\bar g(x,\nu), \qquad (4.34)\\ \Delta V_a &= nV_a - H\bar\nabla_\nu V_a, \qquad (4.35)\\ \Delta \bar g(x,\nu) &= HV_0 + \bar g(x,\nabla H) - |h|^2\bar g(x,\nu), \qquad (4.36)\\ \Delta \bar g(X_a,\nu) &= HV_a + \bar g(X_a,\nabla H) - |h|^2\bar g(X_a,\nu) - n\bar\nabla_\nu V_a + n\bar g(\nu,X_a), \qquad (4.37)\\ \Delta \bar g(Y_a,\nu) &= -|h|^2\bar g(Y_a,\nu) + \bar g(Y_a,\nabla H) + n\bar g(Y_a,\nu). \qquad (4.38)\end{aligned}$$

Proof (4.34) and (4.35) follow from (4.9) and (4.10) respectively and the Weingarten formula.

We prove next (4.37). Choose an local normal frame $\{e_\alpha\}_{\alpha=1}^n$ at a given point p, i.e., $\nabla_{e_\alpha} e_\beta|_p = 0$. Denote by $\mathcal W : TM \to TM$ the Weingarten map. We will frequently use the conformal property (4.7) of X_a. We compute at p,

$$
\begin{aligned}
e_\alpha \bar g(X_a,\nu) &= \bar g(X_a,\mathcal{W}(e_\alpha)) + \bar g(\bar\nabla_{e_\alpha}X_a,\nu)\\
&= \bar g(X_a,\mathcal{W}(e_\alpha)) - \bar g(\bar\nabla_\nu X_a, e_\alpha),
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta\bar g(X_a,\nu) &= e_\alpha e_\alpha \bar g(X_a,\nu)\\
&= \bar g(\bar\nabla_{e_\alpha}X_a,\mathcal{W}(e_\alpha)) + \bar g(X_a,\bar\nabla_{e_\alpha}(\mathcal{W}(e_\alpha)))\\
&\quad -\bar g(\bar\nabla_{e_\alpha}(\bar\nabla_\nu X_a),e_\alpha) - \bar g(\bar\nabla_\nu X_a,-H\nu)\\
&= h_{\alpha\beta}\bar g(\bar\nabla_{e_\alpha}X_a,e_\beta) + \bar g(X_a,(\nabla_{e_\alpha}\mathcal{W})(e_\alpha) - h(e_\alpha,\mathcal{W}(e_\alpha))\nu)\\
&\quad -\bar g(\bar\nabla_{e_\alpha}\bar\nabla_\nu X_a,e_\alpha) + HV_a\\
&= HV_a + \bar g(X_a,\nabla H) - |h|^2\bar g(X_a,\nu) - \bar g(\bar\nabla_{e_\alpha}\bar\nabla_\nu X_a,e_\alpha).
\end{aligned}
$$

Using the definition of Riemannian curvature tensor and the fact that the ambient space has curvature -1, we get

$$
\begin{aligned}
&-\bar g(\bar\nabla_{e_\alpha}\bar\nabla_\nu X_a,e_\alpha)\\
&\quad = -\bar g(\bar\nabla_\nu\bar\nabla_{e_\alpha}X_a,e_\alpha) + \bar g(\bar\nabla_{[\nu,e_\alpha]}X_a,e_\alpha) + \bar g(\bar R(\nu,e_\alpha)X_a,e_\alpha)\\
&\quad = -\bar\nabla_\nu\bar g(\bar\nabla_{e_\alpha}X_a,e_\alpha) + \bar g(\bar\nabla_{e_\alpha}X_a,\bar\nabla_\nu e_\alpha) + \bar g(\bar\nabla_{[\nu,e_\alpha]}X_a,e_\alpha) + n\bar g(\nu,X_a)\\
&\quad = -n\bar\nabla_\nu V_a + \bar g(\bar\nabla_{e_\alpha}X_a,\bar\nabla_{e_\alpha}\nu + [\nu,e_\alpha]) + \bar g(\bar\nabla_{[\nu,e_\alpha]}X_a,e_\alpha) + n\bar g(\nu,X_a)\\
&\quad = -n\bar\nabla_\nu V_a + HV_a + \bar g([\nu,e_\alpha],e_\alpha)V_a + n\bar g(\nu,X_a).
\end{aligned}
$$

Furthermore the Koszul formula gives

$$
2\bar g(\bar\nabla_{e_\alpha}\nu,e_\alpha) = -\bar g([\nu,e_\alpha],e_\alpha) - \bar g([e_\alpha,e_\alpha],\nu) + \bar g([e_\alpha,\nu],e_\alpha),
$$

which implies

$$
\bar g([\nu,e_\alpha],e_\alpha) = -H.
$$

Combining the above, we get (4.37).

By taking account of the fact that x has the conformal Killing property (4.4) and Y_a has the Killing property (4.8), (4.36) and (4.38) follow similarly as (4.37). □

4.2 Uniqueness of stable capillary hypersurfaces in a hyperbolic ball

Theorem 4.1 *Assume $x : M \to B_R^{\mathbb{H}} \subset (\mathbb{B}^{n+1},\bar g)$ is an immersed stable capillary in the ball $B_R^{\mathbb{H}}$ with constant mean curvature $H \geq 0$ and constant contact angle $\theta \in (0,\pi)$. Then x is totally umbilical.*

Proof The stability inequality (2.3) reduces to

$$
-\int_M \varphi(\Delta\varphi + |h|^2\varphi - n\varphi) - \int_{\partial M}(\nabla_\mu\varphi - q\,\varphi)\varphi \geq 0 \tag{4.39}
$$

for all function $\varphi \in \mathcal{F}$, where q is given by (4.28) since $\partial B_R^{\mathbb{H}}$ has constant principal curvature $\coth R$.

For each constant vector field $a \in \mathbb{R}^{n+1}$, we consider a test function

$$\varphi_a = n(V_a + \sinh R \cos\theta \bar{g}(Y_a, \nu)) - H\bar{g}(X_a, \nu)$$

along M. The Minkowski type formula (4.21) tells us that $\int_M \varphi_a dA = 0$. Therefore, $\varphi_a \in \mathcal{F}$ and is an admissible function for testing stability. Using (4.35), (4.37) and (4.38), noting that H is a constant, we easily see that

$$\Delta\varphi_a + |h|^2\varphi_a - n\varphi_a = (n|h|^2 - H^2)V_a. \tag{4.40}$$

From (4.26) and (4.27), we know

$$\nabla_\mu\varphi_a - q\varphi_a = 0. \tag{4.41}$$

Inserting (4.40) and (4.41) into the stability condition (4.39), we get for any $a \in \mathbb{R}^{n+1}$,

$$\int_M [n(V_a + \sinh R \cos\theta \bar{g}(Y_a, \nu)) - H\bar{g}(X_a, \nu)]\, V_a(n|h|^2 - H^2)\, dA \le 0. \tag{4.42}$$

We take a to be the $n+1$ coordinate vectors $\{E_i\}_{i=1}^{n+1}$ in $\mathbb{R}^{n+1}$. Noticing that $V_a = \frac{2\langle x,a\rangle}{1-|x|^2}$, $X_a = \frac{2}{1-R_{\mathbb{R}}^2}\left(\langle x,a\rangle x - \frac{1}{2}(|x|^2 + R_{\mathbb{R}}^2)a\right)$ and $Y_a = \frac{1}{2}(|x|^2+1)a - \langle x,a\rangle x$, we have

$$\sum_{a=1}^{n+1} V_a^2 = \frac{4|x|^2}{(1-|x|^2)^2} = \bar{g}(x,x),$$
$$\sum_{a=1}^{n+1} V_a X_a = \frac{2}{1-R_{\mathbb{R}}^2}\frac{|x|^2 - R_{\mathbb{R}}^2}{1-|x|^2}x = (V_0 - \cosh R)x,$$
$$\sum_{a=1}^{n+1} V_a Y_a = x.$$

Therefore, by summing (4.42) for all a, we get

$$\int_\Sigma [n(\bar{g}(x,x) + \sinh R \cos\theta \bar{g}(x,\nu)) - (V_0 - \cosh R)H\bar{g}(x,\nu)]\,(n|h|^2 - H^2) \le 0. \tag{4.43}$$

As in the Euclidean case, we introduce an auxiliary function

$$\Phi = (V_0 - \cosh R)\, H - n(\bar{g}(x,\nu) + \cos\theta \sinh R).$$

From (4.34) and (4.36), we get

$$\Delta\Phi = (n|h|^2 - H^2)\bar{g}(x, \nu). \tag{4.44}$$

Note that $\Phi|_{\partial M} = 0$. Thus we have

$$\int_M \Delta\frac{1}{2}\Phi^2 dA = \int_{\partial M} \Phi\nabla_\mu\Phi ds = 0.$$

Adding this to (4.43), using (4.44), we have

$$\begin{aligned}
0 &\geq \int_M (n\bar{g}(x, x) - (\cosh r - \cosh R)\, H\bar{g}(x, \nu))\, (n|h|^2 - H^2) + \Delta\frac{1}{2}\Phi^2 \\
&= \int_M n\bar{g}(x^T, x^T)(n|h|^2 - H^2) + |\nabla\Phi|^2 \\
&\geq 0.
\end{aligned}$$

The same argument as before yields the umbilicy of the immersion x. This implies $x : M \to B_R^{\mathbb{H}}$ is either part of a totally geodesic hypersurface or part of a geodesic ball. The proof is completed. □

4.3 The case $\mathbb{S}^{n+1}$

In this subsection, we sketch the necessary modifications in the case that the ambient space is the spherical space form $\mathbb{S}^{n+1}$. We use the model

$$(\mathbb{R}^{n+1}, \bar{g}_{\mathbb{S}} = e^{2u}\delta) \quad \text{with } u(x) = \frac{4}{(1 + |x|^2)^2},$$

to represent $\mathbb{S}^{n+1}\backslash\{\mathcal{S}\}$, the unit sphere without the south pole. Let $B_R^{\mathbb{S}}$ be a ball in $\mathbb{S}^{n+1}$ with radius $R \in (0, \pi)$ centered at the north pole. The corresponding $R_{\mathbb{R}} = \sqrt{\frac{1-\cos R}{1+\cos R}} \in (0, \infty)$. The crucial conformal Killing vector field X_a and the Killing vector field Y_a in this case are

$$X_a = \frac{2}{1 + R_{\mathbb{R}}^2}\left[\langle x, a\rangle x - \frac{1}{2}(|x|^2 + R_{\mathbb{R}}^2)a\right], \tag{4.45}$$

$$Y_a = \frac{1}{2}(1 - |x|^2)a + \langle x, a\rangle x. \tag{4.46}$$

The crucial functions V_0 and V_a in this case are

$$V_0 = \cos r = \frac{1 - |x|^2}{1 + |x|^2}, \quad V_a = \frac{2\langle x, a\rangle}{1 + |x|^2}.$$

Similarly as the hyperbolic case, these $(n+2)$ functions span the vector space

$$\{V \in C^2(\mathbb{S}^{n+1} \backslash \{\mathcal{S}\}) : \bar{\nabla}^2 V = -V\bar{g}\}.$$

Using X_a, Y_a, V_0 and V_a, the proof goes through parallel to the hyperbolic case. The method works for balls with any radius $R \in (0, \pi)$. Compare to the hyperbolic case, in this case $V_0 = \cos r$ can be negative when $R \in (\frac{\pi}{2}, \pi)$. Nevertheless, by going through the proof, we see this does not affect the issue on stability. We leave the details to the interested reader. □

4.4 Exterior problem

To end this section, we give a sketch of proof for the exterior problem, Theorem 1.4. We take the hyperbolic case as an example.

Theorem 4.2 *Assume $x : M \to \mathbb{H}^{n+1} \backslash B_R^{\mathbb{H}}$ is a compact immersed stable capillary hypersurface outside the hyperbolic ball $B_R^{\mathbb{H}}$ with constant mean curvature $H \geq 0$ and constant contact angle $\theta \in (0, \pi)$. Then x is totally umbilical.*

Proof In this case, the differences occur that $x = -\sinh R \bar{N}$ and the term q in the stability inequality (2.3) is given by

$$q = -\frac{1}{\sin\theta} \coth R + \cot\theta\, h(\mu, \mu).$$

By checking the proof of Proposition 4.4, we see the Minkowski formula is

$$\int_M n(V_a - \sinh R \cos\theta\, \bar{g}(Y_a, \nu))dA = \int_M H\bar{g}(X_a, \nu)dA.$$

We take the test function to be

$$\varphi_a = n(V_a - \sinh R \cos\theta\, \bar{g}(Y_a, \nu)) - H\bar{g}(X_a, \nu). \tag{4.47}$$

Then $\int_M \varphi_a dA = 0$. Also, by checking the proof of Proposition 4.5, we see that $\bar{\nabla}_\mu \varphi_a = q\varphi_a$ along ∂M. From Proposition 4.6, φ_a in (4.47) still satisfies (4.40). Then the proof is exactly the same as the interior problem, Theorem 4.1. □

5 Heintze–Karcher–Ros type inequality and Alexandrov theorem

Let $K = 0$ or ± 1. Denote by $\bar{\mathbb{M}}^{n+1}(K)$ the space form with sectional curvature K. As in previous section, we use the Poincaré ball model $(\mathbb{B}^{n+1}, \bar{g}_{\mathbb{H}})$ for $\bar{\mathbb{M}}^{n+1}(-1)$ and the model $(\mathbb{R}^{n+1}, \bar{g}_{\mathbb{S}})$ for $\bar{\mathbb{M}}^{n+1}(1)$.

In this section we consider an *isometric embedding* $x : M \to \bar{\mathbb{M}}^{n+1}(K)$ into a ball B in a space form with free boundary, i.e., $\theta = \pi/2$. To unify the notation, we use B to mean the unit ball $\mathbb{B}^{n+1}$ in the Euclidean case, the ball $B_R^{\mathbb{H}}$ with radius R

($R \in (0, \infty)$) in the hyperbolic case and the ball $B_R^{\mathbb{S}}$ with radius R ($R \in (0, \pi)$) in the spherical case. We denote $\Sigma = x(M)$. Let B be decomposed by Σ into two connected components. We choose one and denote it by Ω. Denote by T the part of $\partial\Omega$ lying on ∂B. Thus, $\partial\Omega = \Sigma \cup T$.

We also unify the following notations:

$$V_0 = \begin{cases} 1, & K = 0, \\ \cosh r, & K = -1, \\ \cos r, & K = 1, \end{cases}$$

and

$$V_a = \begin{cases} \langle x, a\rangle, & K = 0, \\ \frac{2\langle x,a\rangle}{1-|x|^2}, & K = -1, \\ \frac{2\langle x,a\rangle}{1+|x|^2}, & K = 1. \end{cases}$$

and X_a is conformal vector field in (3.1), (4.5) and (4.45) in each case respectively.

We first prove other Minkowski type formulae.

Proposition 5.1 *Let $x : M \to \bar{\mathbb{M}}^{n+1}(K)$ be an embedded smooth hypersurface into B which meets B orthogonally. Let σ_k, $k = 1, \ldots, n$ be the k-th mean curvatures, i.e., the elementary symmetric functions acting on the principal curvatures. Then*

$$\int_\Omega V_a d\Omega = \frac{1}{n+1}\int_\Sigma \bar{g}(X_a, \nu)dA. \tag{5.1}$$

$$\int_\Sigma V_a \sigma_{k-1} dA = \frac{k}{n+1-k}\int_\Sigma \sigma_k \bar{g}(X_a, \nu)dA, \quad \forall k = 1, \ldots, n. \tag{5.2}$$

Remark 5.1 Formula (5.2) is still true if x is only an immersion.

Proof Due to the perpendicularity condition, $\mu = \bar{N}$. Since $X_a \perp \bar{N}$ along ∂B, we see $X_a \perp \mu$ along $\partial\Sigma$. From the conformal property, we have

$$\mathrm{div}_{\bar{g}} X_a = (n+1)V_a.$$

Integrating it over Ω and using Stokes' theorem, we have

$$\begin{aligned}(n+1)\int_\Omega V_a d\Omega = \int_\Omega \mathrm{div}_{\bar{g}} X_a d\Omega &= \int_\Sigma \bar{g}(X_a, \nu)dA + \int_T \bar{g}(X_a, \bar{N})dA \\ &= \int_\Sigma \bar{g}(X_a, \nu)dA.\end{aligned}$$

This is (5.1). Denote by X_a^T the tangential projection of X_a on Σ. From above we know that $X_a^T \perp \mu$ along $\partial\Sigma$. Let $\{e_\alpha\}_{\alpha=1}^n$ be an orthonormal frame on Σ. From the conformal property, we have that

$$\frac{1}{2}\left[\nabla_\alpha (X_a^T)_\beta + \nabla_\beta (X_a^T)_\alpha\right] = V_a g_{\alpha\beta} - h_{\alpha\beta}\bar{g}(X_a, \nu). \tag{5.3}$$

(cf. (3.5)). Denote $T_{k-1}(h) = \frac{\partial \sigma_k}{\partial h}$ the Newton transformation. Multiplying $T^{\alpha\beta}_{k-1}(h)$ to (5.3) and integrating by parts on Σ, we get

$$\begin{aligned}
&\int_\Sigma (n+1-k)V_a\sigma_{k-1}(h) - k\sigma_k(h)\bar{g}(X_a, \nu)dA \\
&= \int_\Sigma T^{\alpha\beta}_{k-1}(h)\nabla_\alpha(X_a^T)_\beta dA = \int_{\partial\Sigma} T^{\alpha\beta}_{k-1}(h)\bar{g}(X_a^T, e_\beta)\bar{g}(\mu, e_\alpha)ds \\
&= \int_{\partial\Sigma} T_{k-1}(X_a^T, \mu)ds = 0.
\end{aligned}$$

In the last equality, we have used Proposition 2.1 and the fact that $X_a^T \perp \mu$ along $\partial\Sigma$. In fact, since μ is a principal direction of h, it is also a principal direction of the Newton tensor T_{k-1} of h, which implies that $T_{k-1}(X_a^T, \mu) = 0$. The proof is completed. □

Next we prove a Heintze–Karcher–Ros type inequality. In order to prove the Heintze–Karcher–Ros type inequality, we need a generalized Reilly formula, which has been proved by Qiu and Xia [48], Li and Xia [33,34].

Theorem 5.1 [33,48] *Let Ω be a bounded domain in a Riemannian manifold $(\bar{M}, \bar{g})$ with piecewise smooth boundary $\partial\Omega$. Assume that $\partial\Omega$ is decomposed into two smooth pieces $\partial_1\Omega$ and $\partial_2\Omega$ with a common boundary Γ. Let V be a non-negative smooth function on $\bar{\Omega}$ such that $\frac{\bar{\nabla}^2 V}{V}$ is continuous up to $\partial\Omega$. Then for any function $f \in C^\infty(\bar{\Omega}\backslash\Gamma)$, we have*

$$\begin{aligned}
&\int_\Omega V\left(\left(\bar{\Delta} f - \frac{\bar{\Delta} V}{V} f\right)^2 - \left|\bar{\nabla}^2 f - \frac{\bar{\nabla}^2 V}{V} f\right|^2\right) d\Omega \\
&= \int_\Omega \left(\bar{\Delta} V \bar{g} - \bar{\nabla}^2 V + V\mathrm{Ric}\right)\left(\bar{\nabla} f - \frac{\bar{\nabla} V}{V} f, \bar{\nabla} f - \frac{\bar{\nabla} V}{V} f\right) d\Omega \\
&\quad + \int_{\partial\Omega} V\left(f_\nu - \frac{V_\nu}{V} f\right)\left(\Delta f - \frac{\Delta V}{V} f\right) dA \\
&\quad - \int_{\partial\Omega} Vg\left(\nabla\left(f_\nu - \frac{V_\nu}{V} f\right), \nabla f - \frac{\nabla V}{V} f\right) dA \\
&\quad + \int_{\partial\Omega} VH\left(f_\nu - \frac{V_\nu}{V} f\right)^2 + \left(h - \frac{V_\nu}{V} g\right)\left(\nabla f - \frac{\nabla V}{V} f, \nabla f - \frac{\nabla V}{V} f\right) dA.
\end{aligned} \tag{5.4}$$

Remark 5.2 The formula (5.4) here is a bit different with that in [33]. We do not do integration by parts on $\partial\Omega$ in the last step of the proof as [33,48].

Theorem 5.2 *Let $x : M \to \bar{\mathbb{M}}^{n+1}(K)$ be an embedded smooth hypersurface into B with $\partial\Sigma \subset \partial B$. Assume Σ lies in a half ball*

$$B_{a+} = \{V_a \geq 0\} = \{\langle x, a\rangle \geq 0\}.$$

If Σ has positive mean curvature, then

$$\int_\Sigma \frac{V_a}{H} dA \geq \frac{n+1}{n} \int_\Omega V_a d\Omega. \tag{5.5}$$

Moreover, equality in (5.5) holds if and only if Σ is a spherical cap which meets ∂B orthogonally.

Proof Recall $\partial\Omega = \Sigma \cup T$, where T is the boundary part lying in ∂B. See Figure 1. Let f be a solution of the mixed boundary value problem

$$\begin{cases} \bar{\Delta} f + K(n+1) f = 1 & \text{in } \Omega, \\ f = 0 & \text{on } \Sigma, \\ V_a f_{\bar{N}} - f(V_a)_{\bar{N}} = 0 & \text{on } T. \end{cases} \tag{5.6}$$

Since the existence of (5.6) has its own interest, we give a proof in Appendix A. From the Appendix we have $f \in W^{1,2}(\Omega)$ satisfying (5.6) in the weak sense, i.e., $f = 0$ on Σ and

$$\int_\Omega [\bar{g}(\bar{\nabla} f, \bar{\nabla}\varphi) - K(n+1) f\varphi + \varphi] d\Omega = \int_T f\varphi dA, \tag{5.7}$$

for all $\varphi \in W^{1,2}(\Omega)$ with $\varphi = 0$ on Σ. Moreover the regularity of f, $f \in C^\infty(\bar{\Omega}\backslash\Gamma)$ follows from standard linear elliptic PDE theory.

From the fact $\bar{\nabla}^2 V_a = -K V_a \bar{g}$, we see

$$\bar{\Delta} V_a + K(n+1) V_a = 0, \quad \bar{\Delta} V_a \bar{g} - \bar{\nabla}^2 V_a + V_a \overline{\mathrm{Ric}} = 0. \tag{5.8}$$

By using Green's formula, (5.6) and (5.8), we have

$$\begin{aligned} \int_\Omega V_a d\Omega &= \int_\Omega V_a(\bar{\Delta} f + K(n+1) f) - (\bar{\Delta} V_a + K(n+1) V_a) f d\Omega \\ &= \int_{\partial\Omega} V_a f_\nu - (V_a)_\nu f dA \\ &= \int_\Sigma V_a f_\nu dA + \int_T V_a f_{\bar{N}} - f(V_a)_{\bar{N}} dA \\ &= \int_\Sigma V_a f_\nu dA. \end{aligned} \tag{5.9}$$

Using Hölder's inequality for the RHS of (5.9), we have

$$\left(\int_\Omega V_a d\Omega\right)^2 \leq \int_\Sigma V_a H f_\nu^2 dA \int_\Sigma \frac{V_a}{H} dA. \tag{5.10}$$

Next, we use formula (5.4) in our situation with $V = V_a$. Because of (5.8) and (5.6), formula (5.4) gives

$$
\begin{aligned}
\frac{n}{n+1}\int_\Omega V_a d\Omega &= \int_\Omega V_a(\bar\Delta f + K(n+1)f)^2 d\Omega \\
&\quad -\frac{1}{n+1}\int_\Omega V_a(\bar\Delta f + K(n+1)f)^2 d\Omega \\
&\geq \int_\Omega V_a\left((\bar\Delta f + K(n+1)f)^2 - |\bar\nabla^2 f + K(n+1)f\bar g|^2\right)d\Omega \\
&= \int_\Sigma V_a H f_\nu^2 dA \\
&\quad + \int_T \left(h^{\partial B} - \frac{(V_a)_{\bar N}}{V_a} g^{\partial B}\right)\left(\nabla f - \frac{\nabla V_a}{V_a} f, \nabla f - \frac{\nabla V_a}{V_a} f\right) dA \\
&= \mathrm{I} + \mathrm{II}.
\end{aligned}
\tag{5.11}
$$

We claim that

$$
h^{\partial B} - \frac{(V_a)_{\bar N}}{V_a} g^{\partial B} = 0, \ \text{ on } T \subset \partial B, \tag{5.12}
$$

which implies that term II vanishes. We take the hyperbolic case for instance. First, ∂B is umbilical in $\mathbb{H}^{n+1}$ with all principal curvatures $\coth R = \frac{1+R_{\mathbb{R}}^2}{2R_{\mathbb{R}}}$. Second, since $\bar N = \frac{1-R_{\mathbb{R}}^2}{2}\frac{x}{R_{\mathbb{R}}}$ and $V_a = \frac{2\langle x,a\rangle}{1-|x|^2}$, a direct computation gives

$$
\frac{(V_a)_{\bar N}}{V_a}\bigg|_{|x|=R_{\mathbb{R}}} = \left\langle \bar\nabla^{\mathbb{R}} \log\left(\frac{2\langle x,a\rangle}{1-|x|^2}\right), \frac{1-R_{\mathbb{R}}^2}{2}\frac{x}{R_{\mathbb{R}}}\right\rangle\bigg|_{|x|=R_{\mathbb{R}}} = \frac{1+R_{\mathbb{R}}^2}{2R_{\mathbb{R}}}.
$$

Thus (5.12) follows for the hyperbolic case. For other two cases (5.12) follows similarly.

Taking account of the above information in (5.11), we obtain

$$
\frac{n}{n+1}\int_\Omega V_a d\Omega \geq \int_\Sigma V_a H f_\nu^2 dA. \tag{5.13}
$$

Combining (5.10) and (5.13), we conclude (5.5).

We are remained to consider the equality case. If Σ a spherical cap which meets ∂B orthogonally, the Minkowski formula (5.1) implies that equality in (5.5) holds, for Σ has constant mean curvature. Conversely, if equality in (5.5) holds, then equality in (5.11) holds, which implies that $\bar\nabla^2 f + Kf\bar g = 0$ holds in Ω. Restricting this equation on Σ, in view of $f = 0$ on Σ we know that Σ must be umbilical. Thus it is a spherical cap and Ω is the intersection of two geodesic balls. It is easy to show the contact angle must be $\frac{\pi}{2}$. Indeed, we have an explicit form for f:

$$f(x) = \begin{cases} \frac{1}{2(n+1)} d_p(x)^2 + A, & K = 0, \\ A \cosh d_p(x) - \frac{1}{n+1}, & K = -1, \\ A \cos d_p(x) + \frac{1}{n+1}, & K = 1, \end{cases}$$

where $p \in \bar{M}^{n+1}(K)$, $A \in \mathbb{R}$ and d_p is the distance function from p. From the boundary condition, we see $\bar{g}(\bar{\nabla} f, \bar{N}) = 0$ on $\Gamma = \Sigma \cap T$, and then $\bar{g}(\bar{\nabla} d_p, \bar{N}) = 0$. This implies that these two intersecting geodesic balls are perpendicular. The proof is completed. □

As an application we give an integral geometric proof of the Alexandrov Theorem, which was obtained by Ros and Souam in [53] by using the method of moving plane. Our proof has the same flavor of Reilly [49] and Ros [51], see also [33,48].

Theorem 5.3 *Let $x : M \to \bar{\mathbb{M}}^{n+1}(K)$ be an embedded smooth CMC hypersurface into B which meets B orthogonally. Assume Σ lies in a half ball. Then Σ is either a spherical cap or part of totally geodesic hyperplane.*

Remark 5.3 The condition that Σ lies in a half ball cannot be removed because there are other embedded CMC hypersurfaces, like the Denaulay hypersurfaces in a ball which meets ∂B orthogonally which does not lie in a half ball.

Proof We take the hyperbolic case for instance.

We claim first that the constant mean curvature H is non-negative. To prove this claim, let the totally geodesic hyperplane $\{\langle x, a\rangle = 0\}$ move upward along a direction along the totally geodesic foliation of $\mathbb{H}^{n+1}$, until it touches Σ at some point p at a first time. It is clear that $H = H(p) \geq 0$. If $H = 0$, then the boundary point lemma or the interior maximum principle implies that Σ must be some totally geodesic hyperplane.

Next we assume $H > 0$. In this case the two Minkowski formulae (5.1) and (5.2) yield

$$\begin{aligned} (n+1)\int_\Omega V_a dA &= \int_\Sigma \bar{g}(X_a, \nu) dA = \frac{1}{H}\int_\Sigma H \bar{g}(X_a, \nu) dA \\ &= \frac{n}{H}\int_\Sigma V_a dA = n\int_\Sigma \frac{V_a}{H} dA. \end{aligned}$$

The above equation means, for the constant mean curvature hypersurface Σ, the Heintze–Karcher–Ros inequality is indeed an equality. By the classification of equality case in (5.5), we conclude that Σ must be a spherical cap. The proof is completed. □

Using the higher order Minkowski formulae (5.2) and the Heintze–Karcher–Ros inequality (5.5), we can also prove the rigidity when Σ has constant higher order mean curvatures or mean curvature quotients as Ros [51] and Koh and Lee [30].

Theorem 5.4 *Let $x : M \to \bar{\mathbb{M}}^{n+1}(K)$ be an isometric immersion into a ball with free boundary. Assume that Σ lies in a half ball.*

(i) *Assume x is an embedding and has nonzero constant higher order mean curvatures σ_k, $1 \le k \le n$. Then Σ is a spherical cap.*
(ii) *Assume x has nonzero constant curvature quotient, i.e.,*

$$\frac{\sigma_k}{\sigma_l} = const., \quad \sigma_l > 0, \quad 1 \le l < k \le n.$$

Then Σ is a spherical cap.

Note that in Theorem 5.4(ii), we do not need assume the embeddedness of x, since in the proof we need only use the higher order Minkowski formulae (5.2) (without use of the Heintze–Karcher–Ros inequality), which is true for immersions, see Remark 5.1. On the other hand, the condition of embeddedness may not be removed in Theorem 5.4(i) in view of Wente's counterexample. The proof is similar, we leave it to the interested reader.

Appendix A. Existence of weak solution of (5.6)

In this Appendix we discuss the existence of weak solution of (5.6), namely

$$\begin{cases} \bar{\Delta} f + K(n+1) f = 1 & \text{in } \Omega, \\ f = 0 & \text{on } \Sigma, \\ V_a f_{\bar{N}} - f (V_a)_{\bar{N}} = 0 & \text{on } T \end{cases} \tag{A.1}$$

in the weak sense (5.7). Since our Robin boundary condition has a different sign, i.e. $\frac{(V_a)_{\bar{N}}}{V_a} < 0$, we can not apply known results about the existence for mixed boundary problem, for example, [37]. Here we use the Fredholm alternative theorem. In order to use it, we have to show that

$$\begin{cases} \bar{\Delta} \phi + K(n+1) \phi = 0 & \text{in } \Omega, \\ \phi = 0 & \text{on } \Sigma, \\ V_a \phi_{\bar{N}} - \phi (V_a)_{\bar{N}} = 0 & \text{on } T \end{cases} \tag{A.2}$$

has only the trivial solution $\phi = 0$ in $W^{1,2}(\Omega)$. For a general domain Ω we do not know how to prove it. Nevertheless, we can prove it for domains under the conditions given in Theorem 5.2.

Proposition A.1 *Let $x : M \to \bar{\mathbb{M}}^{n+1}(K)$ be an embedded smooth hypersurface into B with $\partial\Sigma \subset \partial B$. Assume Σ lies in a half ball $B_{a+} = \{V_a \ge 0\} = \{\langle x, a\rangle \ge 0\}$. If Σ has positive mean curvature, then* (A.2) *has only the trivial solution $\phi = 0$.*

Proof Let $\phi \in W^{1,2}(\Omega)$ be a weak solution of (A.2), i.e., $\phi = 0$ on Σ and

$$\int_\Omega [\bar{g}(\bar{\nabla}\phi, \bar{\nabla}\varphi) - K(n+1)\phi\varphi]\, d\Omega = \int_T \phi\varphi dA,$$

for all $\varphi \in W^{1,2}(\Omega)$ with $\varphi = 0$ on Σ. The classical elliptic PDE theory gives the regularity $\phi \in C^\infty(\bar{\Omega}\backslash\Gamma)$. Now we can use the Reilly type formula, (5.4), as in the proof of Theorem 5.2. Replacing f by ϕ in (5.4), using (5.12), we get

$$-\int_\Omega V_a|\bar{\nabla}^2\phi + K\phi\bar{g}|^2 d\Omega = \int_\Sigma V_a H(\phi_\nu)^2 dA.$$

Since V_a and H is positive, it follows that

$$\bar{\nabla}^2\phi + K\phi\bar{g} = 0 \text{ in } \Omega \tag{A.3}$$

and $\phi_\nu = 0$ on Σ. From (A.3) and Remark 4.2, we see ϕ must be of form

$$\phi = \sum_{i=0}^{n+1} b_i V_i,$$

for $b_i \in \mathbb{R}, i = 0, 1, \ldots n+1$. By checking the boundary condition $V_a\phi_{\bar{N}} - \phi(V_a)_{\bar{N}} = 0$ on T, we see $b_0 = 0$. Moreover, since $\phi = 0$ on Σ, Σ must be the totally geodesic hyperplane through the origin if one of $b_i \neq 0$, which is a contraction to Σ having positive mean curvature. We get the assertion. □

With this Proposition one can use the Fredholm alternative to get a unique weak solution of (A.1).

References

1. Ainouz, A., Souam, R.: Stable capillary hypersurfaces in a half-space or a slab. Indiana Univ. Math. J. **65**(3), 813–831 (2016)
2. Almgren, F.J.: Spherical symmetrization. In: Proceedings of International Workshop on Integral Functions in the Calculus of Variations, Trieste, 1985, Red. Circ. Mat. Palermo 2 Supple. pp. 11–25 (1987)
3. Ambrozio, L.: Rigidity of area-minimizing free boundary surfaces in mean convex three-manifolds. J. Geom. Anal. **25**(2), 1001–1017 (2015)
4. Barbosa, E.: On stable CMC hypersurfaces with free-boundary in a Euclidean ball. Math. Ann. **372**(1–2), 17–187 (2018)
5. Barbosa, J.L., do Carmo, M.: Stability of hypersurfaces with constant mean curvature. Math. Z. **185**, 339–353 (1984)
6. Barbosa, J.L., do Carmo, M., Eschenburg, J.: Stability of hypersurfaces of constant mean curvature in Riemannian manifolds. Math. Z. **197**, 123–138 (1988)
7. Bokowsky, J., Sperner, E.: Zerlegung konvexer Körper durch minimale Trennflächen. J. Reine Angew. Math. **311**(312), 80–100 (1979)
8. Bourguignon, J.P., Ezin, J.P.: Scalar curvature functions in a conformal class of metrics and conformal transformations. Trans. Am. Math. Soc. **301**, 723–736 (1987)
9. Brendle, S.: Embedded minimal tori in $\mathbb{S}^3$ and the Lawson conjecture. Acta Math. **211**(2), 177–190 (2013)
10. Brendle, S.: Convergence of the Q-curvature flow on $\mathbb{S}^4$. Adv. Math. **205**(1), 1–32 (2006)
11. Brendle, S.: A sharp bound for the area of minimal surfaces in the unit ball. Geom. Funct. Anal. **22**(3), 621–626 (2012)
12. Brendle, S.: Constant mean curvature surfaces in warped product manifolds. Publ. Math. Inst. Hautes Études Sci. **117**, 247–269 (2013)

13. Burago, Yu.D., Maz'ya, V.G.: Certain questions of potential theory and function theory for regions with irregular boundaries (Russian) Zap. Naucn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 3 1967 152 ; English translation: Potential theory and function theory for irregular regions. Translated from Russian. Seminars in mathematics, V. A. Steklov Mathematical Institute, Leningrad, vol. 3. Consultants Bureau, New York (1969)
14. Bürger, W., Kuwert, E.: Area-minimizing disks with free boundary and prescribed enclosed volume. J. Reine Angew. Math. **621**, 1–27 (2008)
15. Chen, J., Fraser, A., Pang, C.: Minimal immersions of compact bordered Riemann surfaces with free boundary. Trans. Am. Math. Soc. **367**(4), 2487–2507 (2015)
16. Choe, J., Koiso, M.: Stable capillary hypersurfaces in a wedge. Pac. J. Math. **280**(1), 1–15 (2016)
17. Chang, S.-Y.A., Yang, P.C.: Prescribing Gaussian curvature on S2. Acta Math. **159**(3–4), 215–259 (1987)
18. Finn, R.: Equilibrium Capillary Surfaces. Springer, New York (1986)
19. Finn, R., McCuan, J., Wente, H.: Thomas Young's surface tension diagram: its history, legacy, and irreconcilabilities. J. Math. Fluid Mech. **14**(3), 445–453 (2012)
20. Folha, A., Pacard, F., Zolotareva, T.: Free boundary minimal surfaces in the unit 3-ball. Manuscr. Math. **154**(3–4), 359–409 (2017)
21. Fraser, A., Li, M.: Compactness of the space of embedded minimal surfaces with free boundary in three-manifolds with nonnegative Ricci curvature and convex boundary. J. Differ. Geom. **96**(2), 183–200 (2014)
22. Fraser, A., Schoen, R.: The first Steklov eigenvalue, conformal geometry, and minimal surfaces. Adv. Math. **226**(5), 4011–4030 (2011)
23. Fraser, A., Schoen, R.: Sharp eigenvalue bounds and minimal surfaces in the ball. Invent. Math. **203**(3), 823–890 (2016)
24. Fraser, A., Schoen, R.: Uniqueness theorems for free boundary minimal disks in space forms. Int. Math. Res. Not. **17**, 8268–8274 (2015)
25. Freidin, B., Gulian, M., McGrath, P.: Free boundary minimal surfaces in the unit ball with low cohomogeneity. Proc. Am. Math. Soc. **145**(4), 1671–1683 (2017)
26. Grüter, M., Jost, J.: On embedded minimal disks in convex bodies. Ann. Inst. H. Poincaré Anal. Non Linéaire **3**(5), 345–390 (1986)
27. Grüter, M., Jost, J.: Allard type regularity results for varifolds with free boundaries. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **13**(1), 129–169 (1986)
28. Heintze, E., Karcher, H.: A general comparison theorem with applications to volume estimates for submanifolds. Ann. Sci. École Norm. Sup. (4) **11**(4), 451–470 (1978)
29. Jost, J.: Embedded minimal surfaces in manifolds diffeomorphic to the three-dimensional ball or sphere. J. Differ. Geom. **30**(2), 555–577 (1989)
30. Koh, S.-E., Lee, S.-W.: Addendum to the paper: "Sphere theorem by means of the ratio of mean curvature functions" by Koh. Glasg. Math. J. **43**(2), 275–276 (2001)
31. Li, H., Xiong, C.: Stability of capillary hypersurfaces in a Euclidean ball. Pac. J. Math. **297**(1), 131–146 (2018)
32. Li, H., Xiong, C.: Stability of capillary hypersurfaces with planar boundaries. J. Geom. Anal. **27**(1), 79–94 (2017)
33. Li, J., Xia, C.: An integral formula and its applications on sub-static manifolds. J. Differ. Geom. arXiv:1603.02201v1 **(to appear)**
34. Li, J., Xia, C.: An integral formula for affine connections. J. Geom. Anal. **27**(3), 2539–2556 (2017)
35. Li, M.: A general existence theorem for embedded minimal surfaces with free boundary. Commun. Pure Appl. Math. **68**(2), 286–331 (2015)
36. Li, P., Yau, S.T.: A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces. Invent. Math. **69**(2), 269–291 (1982)
37. Lieberman, G.M.: Mixed boundary value problems for elliptic and parabolic differential equations of second order. J. Math. Anal. Appl. **113**(2), 422–440 (1986)
38. López, R.: Stability and bifurcation of a capillary surface on a cylinder. SIAM J. Appl. Math. **77**(1), 108–127 (2017)
39. López, R.: Capillary surfaces modeling liquid drops on wetting phenomena. In: The Role and Importance of Mathematics in Innovation. Springer, Berlin (2017)
40. Marinov, P.: Stability analysis of capillary surfaces with planar or spherical boundary in the absence of gravity. PhD thesis, University of Toledo

41. Marinov, P.: Stability of capillary surfaces with planar boundary in the absence of gravity. Pac. J. Math. **255**(1), 177–190 (2012)
42. Marques, F., Neves, A.: Min–max theory and the Willmore conjecture. Ann. Math. (2) **179**(2), 683–782 (2014)
43. Montiel, S., Ros, A.: Minimal immersions of surfaces by the first eigenfunctions and conformal area. Invent. Math. **83**(1), 153–166 (1986)
44. Morabito, F.: Higher genus capillary surfaces in the unit ball of $\mathbb{R}^3$. Bound. Value Probl. **2014**, 130 (2014)
45. Nunes, I.: On stable constant mean curvature surfaces with free boundary. Math. Z. **287**(1–2), 473–479 (2017)
46. Nitsche, J.: Stationary partitioning of convex bodies. Arch. Rat. Mech. Anal. **89**, 1–19 (1985)
47. Park, S.H.: Every ring type spanner in a wedge is spherical. Math. Ann. **332**(3), 475–482 (2005)
48. Qiu, G., Xia, C.: A generalization of Reilly's formula and its applications to a new Heintze–Karcher type inequality. Int. Math. Res. Not. IMRN **17**, 7608–7619 (2015)
49. Reilly, R.C.: Applications of the Hessian operator in a Riemannian manifold. Indiana Univ. Math. J. **26**(3), 459–472 (1977)
50. Ritoré, M., Rosales, C.: Existence and characterization of regions minimizing perimeter under a volume constraint inside Euclidean cones. Trans. Am. Math. Soc. **356**(11), 4601–4622 (2004)
51. Ros, A.: Compact hypersurfaces with constant higher order mean curvatures. Revista Matemática Iberoamericana **3**, 447–453 (1987)
52. Ros, A.: Stability of minimal and constant mean curvature surfaces with free boundary. Mat. Contemp. **35**, 221–240 (2008)
53. Ros, A., Souam, R.: On stability of capillary surfaces in a ball. Pac. J. Math. **178**(2), 345–361 (1997)
54. Ros, A., Vergasta, E.: Stability for hypersurfaces of constant mean curvature with free boundary. Geometriae Dedicata **56**, 19–33 (1995)
55. Rosales, C.: Stable Constant Mean Curvature Hypersurfaces Inside Convex Domains. Differential Geometry and Its Applications, pp. 165–177. MatfyzPress, Prague (2005)
56. Schoen, R.: An extremal eigenvalue problem for surfaces with boundary. https://www.math.uci.edu/~scgas/scgas-2012/Talks/Schoen.pdf. Accessed Jan 2012
57. Souam, R.: On stability of stationary hypersurfaces for the partitioning problem for balls in space forms. Math. Z. **224**(2), 195–208 (1997)
58. Sternberg, P., Zumbrun, K.: A Poincaré inequality with applications to volume-constrained area-minimizing surfaces. J. Reine Angew. Math. **503**, 63–85 (1998)
59. Sternberg, P., Zumbrun, K.: On the connectivity of boundaries of sets minimizing perimeter subject to a volume constraint. Commun. Anal. Geom. **7**(1), 199–220 (1999)
60. Sternberg, P., Zumbrun, K.: A singular local minimizer for the volume constrained minimal surface problem in a nonconvex domain. Proc. Am. Math. Soc. **146**(12), 5141–5146 (2018)
61. Struwe, M.: The existence of surfaces of constant mean curvature with free boundaries. Acta Math. **160**(1–2), 19–64 (1988)
62. Vogel, T.I.: Stability of a liquid drop trapped between two parallel planes. II. General contact angles. SIAM J. Appl. Math. **49**(4), 1009–1028 (1989)
63. Volkmann, A.: A monotonicity formula for free boundary surfaces with respect to the unit ball. Commun. Anal. Geom. **24**(1), 195–221 (2016)
64. Wang, G.: Birkhoff minimax principle for minimal surfaces with a free boundary. Math. Ann. **314**(1), 89–107 (1999)
65. Wente, H.C.: The symmetry of sessile and pendent drops. Pac. J. Math. **88**(2), 387–397 (1980)
66. Young, T.: An essay on the cohesion of fluids. Philos. Trans. R. Soc. Lond. **95**, 65–87 (1805)