WEYL-TYPE ASYMPTOTIC FORMULA FOR SCHRÖDINGER OPERATORS ON UNBOUNDED FRACTAL SPACES

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Abstract. We study an asymptotic estimate on the number of negative eigenvalues of the Schrödinger operators on unbounded fractal spaces which admit a cellular decomposition. We first give some sufficient conditions for Weyl-type asymptotic formula to hold. Second, we verify these conditions for the infinite blowup of Sierpiński gasket and unbounded generalized Sierpiński carpets. Final, we demonstrate how the result can be applied to the infinite blowup of certain fractals associated with iterated function systems with overlaps, including those defining the classical infinite Bernoulli convolution with golden ratio.

Contents

1. Introduction

Non-relativistic quantum mechanics may be viewed as the study of the Schrödinger operator $-\Delta + V$ on Euclidean space $\mathbb{R}^d$, where Δ is the Laplacian on $\mathbb{R}^d$ and V is a potential. Negative eigenvalues of the Schrödinger operator are referred to by physicists as *bound state energies*. Let $N^-(V)$ be the number of negative eigenvalues, counted with multiplicities, of the Schrödinger operator $-\Delta + V$. As a rule, one considers the quantity $N^-(V)$ not for an "individual" potential V, but rather for the family βV, where $\beta > 0$ is a large parameter (coupling constant). Here one is interested in the behavior of the function $N^-(\beta V)$ as $\beta \to \infty$. It is well-known (see, e.g., [36])

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that the *Weyl-type (or semi-classical) asymptotic formula*

$$N^-(\beta V) = \frac{\omega_d}{(2\pi)^d}\Big(\int_{D_d^-(V)}\big(-V(x)\big)^{d/2}\,dx\Big)\beta^{d/2}(1+o(1)), \qquad \text{as } \beta\to\infty, \tag{1.1}$$

is satisfied, under some appropriate assumptions on the potential V, where $D_d^-(V) := \{x\in\mathbb{R}^d : V(x)\le 0\}$ and ω_d is the volume of the unit ball in $\mathbb{R}^d$. The conditions on the potential V, guaranteeing the validity of (1.1), depend on the dimension d. For $d\ge 3$, Rozenblum [38], Lieb [30] and Cwikel [11] proved the following *Cwikel-Lieb-Rozenblum (CLR)* inequality independently:

$$N^-(V)\le C(d)\int_{D_d^-(V)}(-V)^{d/2}dx, \qquad \text{for all } V\in L^{d/2}(\mathbb{R}^d), \tag{1.2}$$

where $C(d)$ is a positive constant. Afterwards, other proof of (1.2) were given by Li and Yau [29], and by Conlon [10]. The proof of Li and Yau is the most remarkable, because it relies only upon a few basic facts, such as the "gobal Sobolev inequality" for functions $u\in C_0^\infty(\mathbb{R}^d)$ and the positivity of the heat kernel. Using (1.2), Rozenblum [38] showed that the Weyl-type asymptotic formula (1.1) holds for all potentials $V\in L^{d/2}(\mathbb{R}^d)$. For $d=1,2$, the situation is different. In particular, for $d=1$ an estimate similar to (1.2) is impossible, since $V\in L^1_{loc}(\mathbb{R})$ necessary for the Schrödinger operator $-\Delta+V$ to be well-defined. The necessary and sufficient conditions for the validity of (1.2) for $d=1$ can be easily derived from [31, Theorem D], and they are much stronger than $V\in L^{1/2}(\mathbb{R})$. The CLR inequality for $d=2$ fails. Instead, the opposite inequality

$$N^-(V)\ge c\int_{\mathbb{R}^2} V\,dx$$

is established by Grigor'yan, Netrusov and Yau in [15], where $V\ge 0$. On the other hand, for $d=2$ the recent results by Grigor'yan and Nadirashvili [14] and Shargorodsky [39] give the desirable (though not simple) estimate.

On any fractal space where a Laplacian may be defined, it makes sense mathematically to consider the analog of Schrödinger operator. Strichartz [42] studied the counting function for the negative eigenvalues of the Schrödinger operator $-\Delta+V$ on the product of two copies of an infinite blowup of the Sierpiński gasket, where Δ is the Laplacian on the product and V is a Coulomb potential. He showed that the number of eigenvalues that are less than $-\epsilon$ is of the order $\epsilon^{-\delta}$ as $\epsilon\to 0^+$, where $\delta = (\ln(25/9)\ln 9)/(\ln(9/5)\ln 5)$. Under suitable conditions, Chen *et al.* [9] proved the Bohr's formula for eigenvalue counting function of Schrödinger operators with some unbounded potentials on several types of unbounded fractal spaces supporting a measure and having a well-defined Laplacian. Moreover, these conditions are verified for fractafolds and fractal fields based on nested fractals. Recently, Ngai and first author [33] obtained an analog of (1.1) for Schrödinger operators defined on domains by a class of self-similar measures with overlaps, and proved the Bohr's formula for Schrödinger operators on blowups of fractals associated with iterated function systems with overlaps. A main goal of this paper is to obtain a crude analogue of Weyl-type asymptotic formula (1.1) for Schrödinger operators on unbounded fractal spaces (see Theorem 1.3), which admit a cellular decomposition.

Let K be a compact set in $\mathbb{R}^d$ with a positive finite Borel measure μ and a "well-defined boundary" ∂K which has μ-measure zero. Note that ∂K might not coincide with the boundary of K in the topological sense. We now consider an unbounded set K_∞, which admits a cellular decomposition into copies of K. Formally, let $K_\infty := \bigcup_{i\in I} K_i$, where

(C1) I is a countably infinite index set containing 0;

(C2) for each $i \in I$, there corresponds a similitude $\tau_i : K \to K_i$ of the form $\tau_i(x) = x + b_i$, with $b_i \in \mathbb{R}^d$ such that τ_0 is the identity map on $\mathbb{R}^d$ and $\tau_i(K) = K_i$;

(C3) for any distinct $i, j \in I$, $K_i \cap K_j = \partial K_i \cap \partial K_j$.

Condition (C3) implies that for any distinct $i, j \in I$, the interiors of K_i and K_j are disjoint. For each $i \in I$, let $\mu_i := \mu \circ \tau_i^{-1}$ be the push forward measure of μ on K_i. We remark that $\mu_0 = \mu$, and each μ_i and μ have the same measure structure. In a natural way, we can define a glued measure μ_∞ on K_∞ by

$$\mu_\infty(E) := \sum_{i\in I} \mu_i(E \cap K_i) \qquad \text{for all Borel subsets } E \subseteq K_\infty. \tag{1.3}$$

In this case, we call μ_∞ the *natural extension measure* on K_∞ of μ. Note that $\mu_\infty(K_i \cap K_j) = 0$ for any distinct $i, j \in I$.

We shall assume that there exists a well-defined non-negative self-adjoint Laplacian $-\Delta_{K_\infty}$ in $L^2(K_\infty, \mu_\infty)$. In this paper, we present an asymptotic behavior for the number of negative eigenvalues of $-\Delta_{K_\infty} + \beta V$ as $\beta \to \infty$. The main technique we used is the *Dirichlet-Neumann bracketing technique* [18, 22, 36], which is a basic and useful technique for deriving various asymptotic formulas of Schrödinger operators. Using the idea of Dirichlet-Neumann bracketing, one can bound the Laplacian by the Dirichlet and Neumann Laplacians with conditions on the gluing boundary $\bigcup_{i\in I} \partial K_i$ (see, e.g., [36, Section XIII.15]). Condition (C3) above allows us to decouple the Dirichlet (or Neumann) Laplacian in $L^2(K_\infty, \mu_\infty)$ into the direct sum of the Dirichlet (or Neumann) Laplacians on the individual components. In order to state the precise results, we will impose some mild conditions on the measure space (K, μ).

We first introduce some definitions, notation and assumptions that will be used. We call a μ-measurable compact subset B of K a *cell (in K)* if $\mu(B) > 0$ and B has a well-defined boundary ∂B with $\mu(\partial B) = 0$. Clearly, K itself is a cell. We call a finite family $\mathbf{P}$ of interior disjoint cells a *partition* of K if $K = \bigcup_{B\in\mathbf{P}} B$. In particular, $\{K\}$ is a partition of K.

Definition 1.1. *We say a partition $\mathbf{P}$ of K satisfies* condition (DN) *if for each $B \in \mathbf{P}$,*

1. *there exists a well-defined non-negative self-adjoint Laplacian operator $-\Delta_B^D$ (resp. $-\Delta_B^N$) in $L^2(B, \mu|_B)$ satisfying the Dirichlet (resp. Neumann) condition on ∂B;*
2. *for any $B \in \mathbf{P}$, $-\Delta_B^D$ and $-\Delta_B^N$ have compact resolvent.*

Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$ that is semi-bounded below. If A has compact resolvent, then the number of negative eigenvalues, counting multiplicity, is finite; moreover, each eigenspace is finite dimensional. We define the *eigenvalue counting function* as

$$N(\lambda, A) := \#\{n : \lambda_n(A) < \lambda\},$$

where $\lambda_n(A)$ is the n-th eigenvalue of A counted according to their multiplicities, and $\#F$ denotes the cardinality of a finite set F. Condition (2) implies that $-\Delta_B^D$ and $-\Delta_B^N$ have pure point spectrum, and then $N(\lambda, -\Delta_B^b)$ is well-defined for all $\lambda > 0$ and $b \in \{D, N\}$.

Let $(\mathbf{P}_k)_{k\geq 0}$ be a sequence of partitions of K, and let ν be a positive finite Borel measure on K. We say that $(\mathbf{P}_k)_{k\geq 0}$ is *refining* with respect to ν if (1) $\mathbf{P}_0 = \{K\}$; (2) for any $B \in \mathbf{P}_k$ and any $B' \in \mathbf{P}_{k+1}$, either $B' \subseteq B$ or $B' \cap B = \partial B \cap \partial B'$; (3) $\lim_{k\to\infty} \max\{\nu(B) : B \in \mathbf{P}_k\} = 0$. Condition (2) means that each member of $\mathbf{P}_{k+1}$ is a subset of some member of $\mathbf{P}_k$.

Assumption 1.2. *There exist a sequence of partitions $(\mathbf{P}_k)_{k\geq 0}$ of K and a positive finite Borel measure ν on K satisfying the following conditions:*

(A1) $(\mathbf{P}_k)_{k\geq 0}$ *is refining with respect to ν and each $\mathbf{P}_k$ satisfies condition (DN).*

(A2) *there exist positive constants α, c_1, and c_2 such that*

$$c_1\nu(B)\lambda^{\alpha/2}\big(1 + o(1)\big) \leq N(\lambda, -\Delta_B^b) \leq c_2\nu(B)\lambda^{\alpha/2}\big(1 + o(1)\big) \qquad \text{as } \lambda \to \infty, \tag{1.4}$$

for all $b \in \{D, N\}$, $k \geq 0$ and $B \in \mathbf{P}_k$.

We refer the reader to Section 2 for the definition of the notation $\bigoplus$. Condition (A1) implies that for each $k \geq 0$, the Laplacians $-\bigoplus_{B\in\mathbf{P}_k} \Delta_B^D$ and $-\bigoplus_{B\in\mathbf{P}_k} \Delta_B^N$ in $L^2(K, \mu)$ with the Dirichlet and Neumann conditions on $\bigcup_{B\in\mathbf{P}_k} \partial B$, respectively, are well-defined. In particular, the Dirichlet (resp. Neumann) Laplacian $-\Delta_K^D$ (resp. $-\Delta_K^N$) in $L^2(K, \mu)$ with boundary condition on ∂K is well-defined. Thus $-\Delta_{K_i}^D$ and $-\Delta_{K_i}^N$ also are well-defined for all $i \in I$, and $-\Delta_{K_\infty}^D := -\bigoplus_{i\in I} \Delta_{K_i}^D$ and $-\Delta_{K_\infty}^N := -\bigoplus_{i\in I} \Delta_{K_i}^N$ are the Dirichlet and Neumann Laplacians in $L^2(K_\infty, \mu_\infty)$ with the boundary $\bigcup_{i\in I} \partial K_i$. Consequently, condition (A1) gives the structure of Dirichlet-Neumann bracketing technique. One uses the monotonicity under addition of Dirichlet or Neumann boundaries and their decoupling properties to reduce a problem to one about cells which is then solved by condition (A2). We remark that condition (A2) (or (1.4)) is a *generalized Weyl law*, and the parameter α is often called the *spectral dimension* of the Laplacians (or K). We remark that, in general, $\alpha \leq d$. If each $B \in \bigcup_{k\geq 0} \mathbf{P}_k$ is a domain in $\mathbb{R}^d$ with a nice boundary, and μ is the Lebesgue measure, then condition (A2) holds with $\alpha = d$, $\nu = \mu$, and $c_1 = c_2 = \boldsymbol{\omega}_d\mu(B)/(2\pi)^d$. In this case, condition (A2) is the *Weyl law* [45], which gives a relationship between the analytic and geometric properties of a domain, which has popular applications for deriving various asymptotic formulas of Schrödinger operators (see, e.g., [36]).

In order to state the precise results, we introduce the following associated *Weyl's asymptotic function*: for any $X \subseteq \mathbb{R}^d$, positive Borel measure ν on X, $\alpha > 0$, and $f \in L^{\alpha/2}(X, \nu)$, define

$$W(X, \nu, f, \alpha) := \int_{D^-(f)} (-f)^{\alpha/2} \, d\nu, \tag{1.5}$$

where $D^-(f) := \{x \in X : f(x) \leq 0\}$.

Theorem 1.3. *Let K, μ, K_∞, μ_∞ be given as above. Assume that there exists a non-negative self-adjoint Laplacian $-\Delta_{K_\infty}$ in $L^2(K_\infty, \mu_\infty)$, and Assumption 1.2 holds with a sequence of partitions $(\mathbf{P}_k)_{k\geq 0}$ of K and a positive finite Borel measure ν on K.*

(a) *Let V be a continuous function on K_∞ that has compact support. Then as $\beta \to \infty$,*

$$c_1 W(K_\infty, \nu_\infty, V, \alpha)\beta^{\alpha/2}(1+o(1)) \leq N(0, -\Delta_{K_\infty} + \beta V) \leq c_2 W(K_\infty, \nu_\infty, V, \alpha)\beta^{\alpha/2}(1+o(1)), \tag{1.6}$$

where $W(\cdot,\cdot,\cdot,\cdot)$ is defined as in (1.5), constants α, c_1 and c_2 come from (1.4), and ν_∞ is the natural extension measure on K_∞ of ν.

(b) *Let $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ be the closed quadratic form in $L^2(K_\infty, \mu_\infty)$ associated with $-\Delta_{K_\infty}$. Assume that $\alpha > 2$ and there exists constant $C > 0$ such that*

$$\|u\|_{L^p(K_\infty, \mu_\infty)} \leq C\mathcal{E}^{1/2}(u, u) \qquad \text{for all } u \in \operatorname{dom}\mathcal{E}, \tag{1.7}$$

where $p := 2\alpha/(\alpha - 2)$. If $V \in L^{\alpha/2}(K_\infty, \mu_\infty) \cap L^{\alpha/2}(K_\infty, \nu_\infty)$, then the Weyl-type asymptotic formula (1.6) also holds.

The main ingredients in the proof of Theorem 1.3(a) are the Dirichlet-Neumann bracketing and the generalized Weyl law (1.4). We remark that the method used to prove Theorem 1.3(a) does not extend to V's that are not at least locally bounded. However, Theorem 1.3(b) tells us that one can extend (1.6) by approximating non-smooth V's by V's in $C_0^\infty(K_\infty)$ provided by the *Sobolev inequality* (1.7). In fact, (1.7) implies the existence of an upper bound for $N(0, -\Delta_{K_\infty} + V)$ that have the right coupling constant behavior, which is the critical element in this approximation argument. The assumption $V \in L^{\alpha/2}(K_\infty, \nu_\infty)$ guarantees that $W(K_\infty, \nu_\infty, V, \alpha) < \infty$.

We are interested in unbounded fractal spaces. Let $\{S_i\}_{i=0}^m$, $m \geq 1$, be an iterated function system (IFS) on $\mathbb{R}^d$. For $k \geq 0$ and $\boldsymbol{\omega} = w_1 \dots w_k \in \{0, \dots, m\}^k$, where $\{0, \dots, m\}^0 := \{\emptyset\}$, we use the standard notation

$$S_{\boldsymbol{\omega}} := S_{w_1} \circ \cdots \circ S_{w_k}$$

with $S_\emptyset := id$, where id is the identity map on $\mathbb{R}^d$. We say IFS $\{S_i\}_{i=0}^m$ satisfy the *open set condition (OSC)* if there exists a nonempty bounded open set $U \subset \mathbb{R}^d$ such that $\bigcup_{i=0}^m S_i(U) \subseteq U$ and $S_i(U) \cap S_j(U) = \emptyset$ if $i \neq j$. OSC is an separation condition. An IFS that does not satisfy OSC is said to have *overlaps*; in this case, we also say that an associated self-similar measure has overlaps. Another important separation condition is post-critically finite (PCF) condition, which is introduced first in [21]. The PCF condtion and the OSC are key conditions in studying the analysis and geometry of fractals. Since 1980s, the Laplacian operators and energy on fractals defined by IFS's satisfy these conditions, such as Sierpiński gasket and Sierpiński carpet, have been constructed and studied extensively (see, e.g., [1, 3, 7, 17, 20–22, 41] and the references therein). In particular, the Weyl law for the Laplacians on Sierpiński gasket and Sierpiński carpet have been obtained (see e.g., [17, 20–22]). Based on these results, we can apply Theorem 1.3 to the infinite blowup of Sierpiński gasket and unbounded generalized Sierpiński carpet (see Corollaries 1.4 and 1.5 below).

Let K denote the Sierpiński gasket, the unique nonempty compact set in the plane satisfying $K = \bigcup_{i=0}^{2} S_i(K)$ where $S_i(x) = (x + q_i)/2$ and (q_0, q_1, q_2) are the vertices of an equilateral triangle. For each infinite word $\boldsymbol{\omega} = w_1 w_2 \cdots$ with $w_i \in \{0, 1, 2\}$, define

$$K_\infty^{\boldsymbol{\omega}} := \bigcup_{m=1}^{\infty} \left(S_{w_1}^{-1} \circ S_{w_2}^{-1} \circ \cdots S_{w_m}^{-1}\right)(K) \tag{1.8}$$

to be the *infinite blowup* of K associated with the word $\boldsymbol{\omega}$. We will assume that the blowup word $\boldsymbol{\omega}$ is not eventually constant. This means $K_\infty^{\boldsymbol{\omega}}$ is non-compact and has no boundary. Here we point out that by construction, $K_\infty^{\boldsymbol{\omega}}$ admits a cellular decomposition into copies of K which intersect on the boundary only. Let μ be the self-similar measure defined by the IFS $\{S_i\}_{i=0}^{2}$ and probability weights $\{1/3, 1/3, 1/3\}$. Thus we can extend μ to an infinite measure $\mu_\infty^{\boldsymbol{\omega}}$ on $K_\infty^{\boldsymbol{\omega}}$ as in (1.3). The theory of Kigami [21] allows us to define the standard Laplacian $-\Delta_K$ in $L^2(K, \mu)$ with either Dirichlet or Neumann condition on the boundary $\partial K = \{q_0, q_1, q_2\}$. Moreover, the definition of the Laplacian can be transferred from K to $K_\infty^{\boldsymbol{\omega}}$, which we denote by $-\Delta_{K_\infty^{\boldsymbol{\omega}}}$ (see, e.g., [40, 42]).

Corollary 1.4. *Let K, μ, $\boldsymbol{\omega}$, $K_\infty^{\boldsymbol{\omega}}$, $\mu_\infty^{\boldsymbol{\omega}}$ and $-\Delta_{K_\infty}^{\boldsymbol{\omega}}$ be given as above, where word $\boldsymbol{\omega}$ is not eventually constant. Assume that V is a continuous function on $K_\infty^{\boldsymbol{\omega}}$ that has compact support. Then there exist positive constants c_1, c_2 such that as $\beta \to \infty$*

$$c_1 W(K_\infty^{\boldsymbol{\omega}}, \mu_\infty^{\boldsymbol{\omega}}, V, d_s)\beta^{d_s/2}(1 + o(1)) \le N(0, -\Delta_{K_\infty^{\boldsymbol{\omega}}} + \beta V) \le c_2 W(K_\infty^{\boldsymbol{\omega}}, \mu_\infty^{\boldsymbol{\omega}}, V, d_s)\beta^{d_s/2}(1 + o(1)),$$

where $W(\cdot, \cdot, \cdot, \cdot)$ is defined as in (1.5) *and $d_s = \log 9/\log 5$.*

The two main ingredients used in the proof of Corollary 1.4 is the self-similarity property and Weyl law of the Laplacian on Sierpiński gasket, which also hold for the Laplacians on PCF fractals with regular harmonic structure (see e.g., [21, 22]). Thus Theorem 1.3(a) also can be applied to the infinite blowup of PCF fractals with regular harmonic structure.

Let $K \subset \mathbb{R}^d$ $(d \ge 2)$ be a generalized Sierpiński carpet (GSC) in the sense of [6, 7]. The definitions of the associated *unbounded generalized Sierpiński carpets* K_∞, as well as the corresponding Laplacians $-\Delta_{K_\infty}$, are given in Section 3.3. It is known that the spectral dimension d_s and the Hausdorff dimension d_f of K satisfy $d_s \le d_f < d$ (see, e.g., [6]). Generalized Sierpiński carpets with $d_s > 2$ can be found in [6, 23].

Corollary 1.5. *Let K be a GSC, K_∞ and μ be the associated unbounded GSC and self-similar measure with equal probability weights. Let $-\Delta_{K_\infty}$ be the Laplacian in $L^2(K_\infty, \mu_\infty)$ as given in [4,6], where μ_∞ is the natural extension measure on K_∞ of μ. Then the following two results hold.*

(a) *If V is a continuous function on K_∞ that has compact support, then the Weyl-type asymptotic formula* (1.6) *holds with $\alpha = d_s$ and $\nu_\infty = \mu_\infty$, where d_s is the spectral dimension of K.*
(b) *If $d_s > 2$ and $V \in L^{d_s/2}(K_\infty, \mu_\infty)$, then the conclusion of part (a) also holds.*

A main motivation of this work is to present an asymptotic behavior for Schrödinger operators on the infinite blowup of certain fractals defined by IFSs with overlaps together with the associated self-similar measures. The structure of the IFSs with overlaps and the associated self-similar measures are in general complicated and intractable. Despite difficulties due to overlaps, many interesting IFSs with overlaps have been studied for a long time. Next, we illustrate Theorem 1.3 by the infinite blowup of fractals defined by two classes of IFSs with overlaps, including those defining the classical infinite Bernoulli convolutions with golden ratio. These IFSs and the associated Laplacians have been studied very extensively (see, e.g., [12,16,25–27,32,33,35,44] and the references therein). They define Laplacians that exhibit many behaviors analogous to Laplacians on PCF fractals, such as non-integer spectral dimension [32], sub-Gaussian heat kernel estimates [16] and infinite wave propagation speed [35].

Let μ be a continuous, positive, finite Borel measure on $\mathbb{R}$ with $\operatorname{supp}(\mu) \subseteq [0,a] =: K$. Define the standard bilinear form in $L^2((0,a),\mu)$ by

$$\mathcal{E}(u,v) := \int_0^a u'(x)v'(x)\,dx$$

with domain $\operatorname{dom}\mathcal{E}$ equal to the Sobolev space $H_0^1(0,a)$ (Dirichlet boundary condition) or $H^1(0,a)$ (Neumann boundary condition). It is well-known that one can define a Dirichlet (resp. Neumann) Laplacian $-\Delta^D_{\mu|_K}$ (resp. $-\Delta^N_{\mu|_K}$) in $L^2(K,\mu)$ with boundary $\partial K = \{0,a\}$. Throughout this paper, we call $-\Delta^D_{\mu|_K}$ (resp. $-\Delta^N_{\mu|_K}$) the *Dirichlet* (resp. *Neumann*) *Laplacian with respect to μ*. We remark that $-\Delta^D_{\mu|_K}$ and $-\Delta^N_{\mu|_K}$ have compact resolvents (see, e.g., [8,19]). For each $i \in \mathbb{N}$, define $\tau_i(x) := x + ia$, and then let

$$K_\infty := \bigcup_{i=0}^\infty \tau_i(K).$$

It is easy to check that conditions (C1)–(C3) above hold. Note that $K_\infty = [0,+\infty) =: \mathbb{R}_+$. Let μ_∞ be the natural extension measure on $\mathbb{R}_+$ of μ. Consider the non-negative bilinear form $\widetilde{\mathcal{E}}(\cdot,\cdot)$ in $L^2(\mathbb{R}_+,\mu_\infty)$ given by

$$\widetilde{\mathcal{E}}(u,v) := \int_0^\infty u'v'\,dx \tag{1.9}$$

with domain $\operatorname{dom}\widetilde{\mathcal{E}} = \{u \in H^1_{loc}(\mathbb{R}_+) : \int_0^\infty |u'|^2 dx < \infty, u(0) = 0\}$. It is well-known that $(\widetilde{\mathcal{E}}, \operatorname{dom}\widetilde{\mathcal{E}})$ is a closed quadratic form in $L^2(\mathbb{R}_+, \mu_\infty)$, since $\mu_\infty(F) < \infty$ for any compact $F \subseteq \mathbb{R}_+$. Hence, there exists a unique non-negative self-adjoint operator $-\Delta_{\mu_\infty}$ in $L^2(\mathbb{R}_+, \mu_\infty)$ associated with $(\widetilde{\mathcal{E}}, \operatorname{dom}\widetilde{\mathcal{E}})$.

The first IFS with overlaps we study is defined by

$$S_1(x) = \rho x, \quad S_2(x) = \rho x + (1-\rho), \quad \rho = (\sqrt{5}-1)/2. \tag{1.10}$$

This IFS defines the infinite Bernoulli convolution associated with the golden ratio. The corresponding self-similar identity is

$$\mu = \frac{1}{2}\mu \circ S_1^{-1} + \frac{1}{2}\mu \circ S_2^{-1}, \tag{1.11}$$

with $\operatorname{supp}(\mu) = [0,1]$. Strichartz *et al.* [43] showed that μ satisfies a family of second-order identities with respect to the following auxiliary IFS:

$$T_0(x) := \rho^2 x, \qquad T_1(x) := \rho^3 x + \rho^2, \qquad T_2(x) := \rho^2 x + \rho. \tag{1.12}$$

We remark that $\{T_i\}_{i=0}^2$ satisfies the OSC. Let $\mathbf{P}_k := \{T_{\boldsymbol{\omega}}[0,1] : \boldsymbol{\omega} \in \{0,1,2\}^k\}$ for all $k \geq 0$. Then $(\mathbf{P}_k)_{k\geq 0}$ is a sequence of partitions of $[0,1]$. We will introduce a measure ν on $[0,1]$ in Section 4 such that Assumption 1.2 holds with $(\mathbf{P}_k)_{k\geq 0}$ and ν.

Theorem 1.6. *Let μ be defined as in* (1.11) *and let $-\Delta^D_{\mu|_B}$ denote the Dirichlet Laplacian with respect to $\mu|_B$ for any cell B in $[0,1]$. Let ν be defined by* (4.7)*.*

(a) *Then for all $k \geq 0$ and $\boldsymbol{\omega} = w_1 \cdots w_k \in \{0,1,2\}^k$, we have*

$$N(\lambda, -\Delta^D_{\mu|_{T_{\boldsymbol{\omega}}[0,1]}}) = \nu(T_{\boldsymbol{\omega}}[0,1])\lambda^{d_s/2}(1+o(1)), \qquad \text{as } \lambda \to \infty,$$

where $\{T_i\}_{i=0}^2$ and d_s are defined in (1.12) *and* (4.2) *respectively.*

(b) *Let μ_∞ and ν_∞ be the natural extension measures on $\mathbb{R}_+$ of μ and ν, respectively. Let $-\Delta_{\mu_\infty}$ be the non-negative self-adjoint operator in $L^2(\mathbb{R}_+, \mu_\infty)$ associated with $(\widetilde{\mathcal{E}}, \operatorname{dom}\widetilde{\mathcal{E}})$ given in* (1.9)*. If V is a continuous function on $\mathbb{R}_+$ that has compact support, then*

$$N(0, -\Delta_{\mu_\infty} + \beta V) = W(\mathbb{R}_+, \nu_\infty, V, d_s)\beta^{d_s/2}(1+o(1)) \qquad \text{as } \beta \to \infty,$$

where $W(\cdot,\cdot,\cdot,\cdot)$ is defined as in (1.5)*.*

The second classes of IFSs with overlaps we study are defined by

$$S_i(x) = \frac{1}{m}x + \frac{m-1}{m}i, \qquad i = 0, 1, \ldots, m. \tag{1.13}$$

Let μ_m be the self-similar measure defined by the IFS $\{S_i\}_{i=0}^m$ together with probability weights $p_i := \binom{m}{i}/2^m$, $i = 0, 1, \ldots, m$, that is,

$$\mu_m = \sum_{i=0}^m p_i \mu_m \circ S_i^{-1}. \tag{1.14}$$

Then μ_m is the m-fold convolution of the Cantor-type measure (see [26, 32]). We will assume that m is an odd integer and $m \geq 3$. Note that $\operatorname{supp}(\mu_m) = [0, m]$. It is shown in [26] that μ_m satisfies a family of second-order identities with respect to the auxiliary IFS:

$$T_i(x) = \frac{1}{m}x + i, \qquad i = 0, 1, \ldots, m-1. \tag{1.15}$$

We remark that $\{T_i\}_{i=0}^{m-1}$ also satisfies the OSC. Similarly, we will introduce a measure ν on $[0, m]$ in Section 5 such that Assumption 1.2 holds with $(\mathbf{P}_k)_{k\geq 0}$ and ν, where $\mathbf{P}_k := \{T_{\boldsymbol{\omega}}[0,1] : \boldsymbol{\omega} \in \{0, 1, \cdots, m-1\}^k\}$ for all $k \geq 0$.

Theorem 1.7. *For any odd integer $m \geq 3$, let $\mu := \mu_m$ be defined by (1.14), and let $-\Delta^D_{\mu|_B}$ denote the Dirichlet Laplacian with respect to $\mu|_B$ for any cell B in $[0, m]$. Let ν be defined by (5.10).*

(a) *Then for all $k \geq 0$ and $\boldsymbol{\omega} = w_1 \cdots w_k \in \{0, 1, \cdots, m-1\}^k$, we have*

$$N(\lambda, -\Delta^D_{\mu|_{T_{\boldsymbol{\omega}}[0,m]}}) = \nu(T_{\boldsymbol{\omega}}[0, m])\lambda^{d_s/2}(1 + o(1)) \qquad \text{as } \lambda \to \infty,$$

where $\{T_i\}_{i=0}^{m-1}$ and d_s are defined by (1.15) and (5.5), respectively.

(b) *Let μ_∞ and ν_∞ be the natural extension measures on $\mathbb{R}_+$ of μ and ν, respectively. Let $-\Delta_{\mu_\infty}$ be the non-negative self-adjoint operator in $L^2(\mathbb{R}_+, \mu_\infty)$ associated with $(\widetilde{\mathcal{E}}, \operatorname{dom} \widetilde{\mathcal{E}})$ defined in (1.9). Assume that V is continuous on $\mathbb{R}_+$ with compact support. Then*

$$N(0, -\Delta_{\mu_\infty} + \beta V) = W(\mathbb{R}_+, \nu_\infty, V, d_s)\beta^{d_s/2}(1 + o(1)) \qquad \text{as } \beta \to \infty,$$

where $W(\cdot, \cdot, \cdot, \cdot)$ is defined as in (1.5).

For the measures μ in Theorems 1.6 and 1.7, the spectral dimension of $-\Delta^D_{\mu|_K}$ are computed in [32], where $K := \operatorname{supp}(\mu)$. One of our main efforts is in constructing the measures ν on K for the two classes of IFSs with overlaps above and proving the generalized Weyl law (1.4), which is crucial in obtaining the Weyl-type asymptotic formula for Schrodinger operators $-\Delta_{\mu_\infty} + \beta V$.

We outline the proof of Theorems 1.6 and 1.7 here. First, we use second-order identities and spectral dimension formulas to construct the above-mentioned measure ν on K (see (4.7) and (5.10)). Second, using the definition of ν and the known Weyl asymptotic of the Laplacians on some cells in $\mathbf{P}_1$ (see (4.3) and (5.6)), we obtain a generalized Weyl law on these cells (see Lemmas 4.2 and 5.3). Third, applying the variational formula to show the non-arithmetic (or non-lattice) case holds on other cells in $\mathbf{P}_1$ (see Lemmas 4.3 and 5.5), and then prove a generalized Weyl law on these cells (see Lemmas 4.4 and 5.7). Final, we prove Theorems 1.6 and 1.7 by combining obtained generalized Weyl law, Remark 2.4, and Theorem 1.3(a).

The rest of this paper is organized as follows. Section 2 summarizes some of the definitions, notation and results that will be needed throughout the paper. In Section 3, we prove Theorem 1.3, and apply it to the infinite blowup of Sierpiński gasket and unbounded generalized Sierpiński carpets. Section 4 is devoted to the proof of Theorem 1.6. Finally, we prove Theorem 1.7 in Section 5.

2. Preliminaries

In this section, we summarize some notation, definitions and preliminary results that will be used throughout the rest of this paper. Let $\mathcal{H}$ be a (real or complex) Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. A self-adjoint operator $(A, \operatorname{dom} A)$ in $\mathcal{H}$ is said to be *semi-bounded below* if there exists some constant $C \geq 0$ such that $(Au,u) \geq -C\|u\|^2$ for all $u \in \operatorname{dom} A$. It is well known that for each semi-bounded below self-adjoint operator A, there exists a unique closed quadratic from $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ such that $\operatorname{dom} A \subseteq \operatorname{dom}\mathcal{E}$, and

$$\mathcal{E}(u,v) = (Au,v) \qquad \text{for all } u \in \operatorname{dom} A \text{ and } v \in \operatorname{dom}\mathcal{E}$$

(see e.g., [13, Section 1.3]).

Definition 2.1. *For $i = 1,2$, let A_i be a self-adjoint operator in a Hilbert space $\mathcal{H}_i$ that is semi-bounded below, and $(\mathcal{E}_i, \operatorname{dom}\mathcal{E}_i)$ be the associated closed quadratic form. We say $A_1 \preccurlyeq A_2$* (in the sense of quadratic forms) *if $\mathcal{H}_2 \subseteq \mathcal{H}_1$, $\operatorname{dom}\mathcal{E}_2 \subseteq \operatorname{dom}\mathcal{E}_1$, and $\mathcal{E}_1(u,u) \leq \mathcal{E}_2(u,u)$ for all $u \in \operatorname{dom}\mathcal{E}_2$.*

We state a simple proposition, which will be used repeatedly. A proof can be found in [36, Section XIII].

Proposition 2.2. (a) *For $i = 1,2$, let A_i be a self-adjoint operator in a Hilbert space $\mathcal{H}_i$ that is semi-bounded below. Assume $A_1 \preccurlyeq A_2$. If A_1 has compact resolvent, then so does A_2; moreover, $N(\lambda, A_1) \geq N(\lambda, A_2)$ for all $\lambda \in \mathbb{R}$.*
(b) *For $i = 1,2$, let A_i be a self-adjoint operator in a Hilbert space $\mathcal{H}$ that is semi-bounded below, and $(\mathcal{E}_i, \operatorname{dom}\mathcal{E}_i)$ be the associated closed quadratic form. If A_1 and A_2 have compact resolvent, and $\operatorname{dom}\mathcal{E}_1 \cap \operatorname{dom}\mathcal{E}_2$ is dense in $\mathcal{H}$, then $N(0, A_1 + A_2) \leq N(0, A_1) + N(0, A_2)$, where $A_1 + A_2$ is defined as a sum of quadratic forms.*

Let $(\mathcal{H}_i)_{i\in I}$ be a finite or countably infinite family of Hilbert spaces. Define a Hilbert space

$$\mathcal{H} = \bigoplus_{i\in I} \mathcal{H}_i := \Big\{ u = (u_i)_{i\in I} : u_i \in \mathcal{H}_i \text{ for all } i \in I \text{ and } \|u\|_{\mathcal{H}}^2 := \sum_{i\in I} \|u_i\|_{\mathcal{H}_i}^2 < \infty \Big\}.$$

Assume that each A_i is a self-adjoint operator in $\mathcal{H}_i$. We write $A := \bigoplus_{i\in I} A_i$ if $Au := (A_i u_i)_{i\in I}$ with domain $\operatorname{dom} A := \{u = (u_i)_{i\in I} \in \mathcal{H} : u_i \in \operatorname{dom} A_i \text{ for all } i \in I \text{ and } Au \in \mathcal{H}\}$ (see [37]). We remark that $(A, \operatorname{dom} A)$ is a self-adjoint operator in $\mathcal{H}$.

Let $(\mathcal{H}_1, \|\cdot\|_1)$ and $(\mathcal{H}_2, \|\cdot\|_2)$ be Hilbert spaces. Let A_1, A_2 be linear operators in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. A_1 and A_2 are said to be *unitarily equivalent*, denoted $A_1 \approx A_2$, if there exists a unitary operator $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\varphi(\operatorname{dom} A_1) = \operatorname{dom} A_2 \quad \text{and} \quad \varphi(A_1(u)) = A_2(\varphi(u)) \quad \text{for all } u \in \operatorname{dom} A_1.$$

Note that u is a λ-eigenvector of A_1 if and only if $\varphi(u)$ is a λ-eigenvector of A_2. In particular, unitarily equivalent operators have the same set of eigenvalues.

For the convenience of the reader, we state a slightly modified version of [32, Proposition 2.2(b)] below, which will be used later in this paper.

Proposition 2.3. ([32, Proposition 2.2]) *Let $S:\mathbb{R}\to\mathbb{R}$ be a similitude with Lipschitz constant r and ν be a continuous positive finite Borel measure on $\mathbb{R}$ with $\operatorname{supp}(\nu)\subseteq[0,a]$. Let $B\subseteq[0,a]$ be a closed interval. Assume that $B':=S(B)\subseteq[0,a]$ and $\nu|_{B'}=w\nu\circ S^{-1}$ on B' for some $w>0$. Then $-\Delta^D_{\nu|_{B'}}\approx(rw)^{-1}\cdot\big(-\Delta^D_{\nu|_B}\big)$, where $-\Delta^D_{\nu|_B}$ and $-\Delta^D_{\nu|_{B'}}$ be the Dirichlet Laplacians with respect to $\nu|_B$ and $\nu|_{B'}$ respectively. Moreover, $N(\lambda,-\Delta^D_{\nu|_{B'}})=N\big(rw\lambda,-\Delta^D_{\nu|_B}\big)$ for all $\lambda>0$.*

Let μ be a continuous positive finite Borel measure on $\mathbb{R}$ with $\operatorname{supp}(\mu)\subseteq[0,a]=:K$. For each cell B in K, let $-\Delta^D_{\mu|_B}$ and $-\Delta^N_{\mu|_B}$ be the Dirichlet and Neumann Laplacians with respect to $\mu|_B$ respectively. We remark that if a cell $B\subseteq[0,a]$ is a closed interval, then $N(\lambda,-\Delta^D_{\mu|_B})\le N(\lambda,-\Delta^N_{\mu|_B})\le N(\lambda,-\Delta^D_{\mu|_B})+2$ for all $\lambda\ge0$ (see, e.g., [32]). Thus $N(\lambda,-\Delta^D_{\mu|_B})$ and $N(\lambda,-\Delta^N_{\mu|_B})$ have the same asymptotic behavior as $\lambda\to\infty$. Consequently, we have the following result holds.

Remark 2.4. *Let μ, K, and $-\Delta^D_{\mu|_B}$ be given as above, where B is a cell in K. Assume that there exist a sequence of partitions $(\mathbf{P}_k)_{k\ge0}$ of K and a positive finite Borel measure ν on K satisfying the following conditions:*

1. *$(\mathbf{P}_k)_{k\ge0}$ is refining with respect to ν and each $\mathbf{P}_k$ consists of closed intervals.*
2. *there exist positive constants α, c_1, and c_2 such that*
$$c_1\lambda^{\alpha/2}\nu(B)\big(1+o(1)\big)\le N(\lambda,-\Delta^D_{\mu|_B})\le c_2\lambda^{\alpha/2}\nu(B)\big(1+o(1)\big)\qquad \text{as } \lambda\to\infty,$$
for all $B\in\mathbf{P}_k$ and $k\ge0$.

Then Assumption 1.2 holds with $(\mathbf{P}_k)_{k\ge0}$ and ν.

Proof. Since each $\mathbf{P}_k$ consists of closed intervals, $\mathbf{P}_k$ satisfies condition (DN). Thus condition (A1) in Assumption 1.2 holds. Condition (A2) in Assumption 1.2 follows by combining the assumption (2) and the fact $N(\lambda,-\Delta^D_{\mu|_B})$ and $N(\lambda,-\Delta^N_{\mu|_B})$ have the same asymptotic behavior as $\lambda\to\infty$. Hence, Assumption 1.2 holds. □

The following proposition follows from [34, Proposition 4.1] and the variational formula (see, e.g., [21, Theorem 4.1.7 or Corollary 4.1.18]), which will be used repeatedly. We omit the proof.

Proposition 2.5. *Let μ, K, and $-\Delta^D_{\mu|_B}$ be given as above, where B is a cell in K. If $\mathbf{P}=\{B_i\}_{i=1}^n$ is a partition of K and each B_i is a closed interval, then for each $\lambda\ge0$, there exists some positive constant $\epsilon(\mathbf{P},\lambda)\le2(n-1)$ such that*
$$N(\lambda,-\Delta^D_{\mu|_K})=\sum_{i=1}^n N(\lambda,-\Delta^D_{\mu|_{B_i}})+\epsilon(\mathbf{P},\lambda).$$

3. Fractal analog of Weyl-type asymptotic formula for the number of negative eigenvalues

In this section, we prove Theorem 1.3, and apply it to the infinite blowup of Sierpiński gasket and unbounded generalized Sierpiński carpets.

3.1. **Proof of Theorem 1.3.** Let K be a compact set in $\mathbb{R}^d$ with a positive finite Borel measure μ and a "well-defined boundary" ∂K which has μ-measure zero. Let $K_\infty := \bigcup_{i\in I} K_i$ satisfy conditions (C1)–(C3) in Section 1. Then for each $i \in I$, there exists a similitude $\tau_i : K \to K_i$ of the form $\tau_i(x) = x + b_i$, with $b_i \in \mathbb{R}^d$ such that τ_0 is the identity map on $\mathbb{R}^d$ and $\tau_i(K) = K_i$. Let μ_∞ be the natural extension measure on K_∞ of μ.

In the rest of this subsection, we assume that there exists a non-negative self-adjoint Laplacian $-\Delta_{K_\infty}$ in $L^2(K_\infty, \mu_\infty)$, and Assumption 1.2 holds with a sequence of partitions $(\mathbf{P}_k)_{k\geq 0}$ of K and a positive finite Borel measure ν on K. Then $\mathbf{P}_0 = \{K\}$ satisfies condition (DN). Thus we can denote $-\Delta^D_{K_i}$ (resp. $-\Delta^N_{K_i}$) by the Laplacian in $L^2(K_i, \mu_i)$ with Dirichlet (resp. Neumann) boundary on ∂K_i for all $i \in I$, where $\mu_i := \mu \circ \tau_i^{-1}$ on K_i. We remark that $-\Delta^D_{K_\infty} := -\bigoplus_{i\in I} \Delta^D_{K_i}$ and $-\Delta^N_{K_\infty} := -\bigoplus_{i\in I} \Delta^N_{K_i}$ are the Dirichlet and Neumann Laplacians in $L^2(K_\infty, \mu_\infty)$ with boundary condition on $\bigcup_{i\in I} \partial K_i$ respectively. Let V be a continuous function on K_∞ with compact support. Since $-\Delta^N_{K_\infty} \preccurlyeq -\Delta_{K_\infty} \preccurlyeq -\Delta^D_{K_\infty}$, we have $-\Delta^N_{K_\infty} + \beta V \preccurlyeq -\Delta_{K_\infty} + \beta V \preccurlyeq -\Delta^D_{K_\infty} + \beta V$ for all $\beta \geq 0$. It follows from Proposition 2.2 that

$$\begin{aligned}\sum_{i\in I} N(0, -\Delta^D_{K_i} + \beta V_i) = N(0, -\Delta^D_{K_\infty} + \beta V) &\leq N(0, -\Delta_{K_\infty} + \beta V) \\ &\leq N(0, -\Delta^N_{K_\infty} + \beta V) = \sum_{i\in I} N(0, -\Delta^N_{K_i} + \beta V_i),\end{aligned} \tag{3.1}$$

where $V_i := V|_{K_i}$ for all $i \in I$.

We now prove Theorem 1.3 (b) by modifying a method in [36, Theorem XIII 80].

Proof of Theorem 1.3. (a) Use the notation above. We first claim that as $\beta \to \infty$,

$$N(0, -\Delta^D_K + \beta V_0) \geq c_1 \beta^{\alpha/2} W(K, \nu, V_0, \alpha)\big(1 + o(1)\big). \tag{3.2}$$

Fix any $\ell \geq 0$. We denote $V^\wedge_{0,\ell}$ (resp. $V^\vee_{0,\ell}$) by the piecewise constant function over each cell $B \in \mathbf{P}_\ell$ with the value $\max\{V(x) : x \in B\}$ (resp. $\min\{V(x) : x \in B\}$). Since Assumption 1.2 holds, $\mathbf{P}_\ell$ satisfies condition (DN). Let $-\Delta^D_B$ and $-\Delta^N_B$ be the Dirichlet and Neumann Lapalcians in $L^2(B, \mu|_B)$ for all $B \in \mathbf{P}_\ell$, respectively. Define $\Delta^{\ell,b}_K := \bigoplus_{B\in\mathbf{P}_\ell} \Delta^b_B$ and $b \in \{D, N\}$. Then $-\Delta^{\ell,D}_K$ (resp. $-\Delta^{\ell,N}_K$) is the Laplacian in $L^2(K, \mu)$ with Dirichlet (resp. Neumann) boundary on $\bigcup_{B\in\mathbf{P}_\ell} \partial B$. Since $-\Delta^{\ell,N}_K \preccurlyeq -\Delta^D_K \preccurlyeq -\Delta^{\ell,D}_K$ and $V^\vee_{0,\ell} \leq V \leq V^\wedge_{0,\ell}$, we get $-\Delta^{\ell,N}_K + \beta V^\vee_{0,\ell} \preccurlyeq$

$-\Delta_K^D + \beta V_0 \preccurlyeq -\Delta_K^{\ell,D} + \beta V_{0,\ell}^{\wedge}$ for all $\beta \geq 0$. It follows from Proposition 2.2 that for all $\beta > 0$,

$$\begin{aligned} N(0, -\Delta_K^D + \beta V_0) &\geq N(0, -\Delta_K^{\ell,D} + \beta V_{0,\ell}^{\wedge}) = \sum_{B\in\mathbf{P}_\ell} N(0, -\Delta_B^D + \beta V_{0,\ell}^{\wedge}|_B) \\ &= \sum_{B\in\mathbf{P}_\ell} N(-\beta V_{0,\ell}^{\wedge}|_B, -\Delta_B^D) = \sum_{\{B\in\mathbf{P}_\ell : V_{0,\ell}^{\wedge}|_B \leq 0\}} N(-\beta V_{0,\ell}^{\wedge}|_B, -\Delta_B^D). \end{aligned} \tag{3.3}$$

Substituting (1.4) into (3.3), we deduce as $\beta \to \infty$,

$$N(0, -\Delta_K^D + \beta V_0) \geq c_1 \beta^{\alpha/2} \left(\sum_{\{B\in\mathbf{P}_\ell : V_{0,\ell}^{\wedge}|_B \leq 0\}} \big(- V_{0,\ell}^{\wedge}|_B\big)^{\alpha/2} \nu(B) \right) (1 + o(1)). \tag{3.4}$$

The definition of refining implies that $\lim_{k\to\infty} \max\{\nu(B) : B \in \mathbf{P}_k\} = 0$. Moreover, it follows from the continuity of V that for $b \in \{\vee, \wedge\}$,

$$\lim_{\ell\to\infty} \sum_{\{B\in\mathbf{P}_\ell : V_{0,\ell}^{b}|_B \leq 0\}} \big(- V_{0,\ell}^{b}|_B\big)^{\alpha/2} \nu(B) = W(K, \nu, V_0, \alpha),$$

which, together with (3.4), yields the claim holds. Since each measure space (K_i, μ_i) and (K, μ) have same structure, we can deduce from (3.2) that

$$N(0, -\Delta_{K_i}^D + \beta V_i) \geq c_1 \beta^{\alpha/2} W(K_i, \nu_i, V_i, \alpha)\big(1 + o(1)\big), \tag{3.5}$$

for all $i \in I$, where $\nu_i := \nu \circ \tau_i^{-1}$ on K_i.

Similarly, we can check that as $\beta \to \infty$,

$$\begin{aligned} N(0, -\Delta_K^N + \beta V_0) &\leq \sum_{\{B\in\mathbf{P}_\ell : V_{0,\ell}^{\vee}|_B \leq 0\}} N(-\beta V_{0,\ell}^{\vee}|_B, -\Delta_B^N) \\ &\leq c_2 \beta^{\alpha/2} \left(\sum_{\{B\in\mathbf{P}_\ell : V_{0,\ell}^{\vee}|_B \leq 0\}} \big(- V_{0,\ell}^{\vee}|_B\big)^{\alpha/2} \nu(B) \right) (1 + o(1)), \end{aligned}$$

and then obtain as $\beta \to \infty$,

$$N(0, -\Delta_{K_i}^N + \beta V_i) \leq c_2 \beta^{\alpha/2} W(K_i, \nu_i, V_i, \alpha)\big(1 + o(1)\big) \quad \text{for all } i \in I. \tag{3.6}$$

Hence, the desired inequality (1.6) follows by combining (3.1), (3.5) and (3.6), which completes the proof.

(b) Fix any $\varepsilon > 0$, small. Choose $V_k \in C_0^\infty(K_\infty)$ so that $V_k \to V$ in $L^{\alpha/2}(K_\infty, \mu_\infty)$ and $L^{\alpha/2}(K_\infty, \nu_\infty)$. Combining the Sobolev inequality (1.7) and [28, Theorem 1.2], we have the following general CLR inequality holds:

$$N\big(0, -\Delta_{K_\infty} + \sigma(V - V_k)\big) \leq C\sigma^{\alpha/2} W(K_\infty, \mu_\infty, V - V_k, \alpha) \qquad \text{for all } \sigma > 0, \tag{3.7}$$

where C is a positive constant. We remark that

$$-\Delta_{K_\infty} + \beta V = \big(- (1-\varepsilon)\Delta_{K_\infty} + \beta V_k\big) + \big(\varepsilon(-\Delta_{K_\infty}) + \beta(V - V_k)\big).$$

It follows from Proposition 2.3(b), part (a) and (3.7) that

$$\begin{aligned}N(0,-\Delta_{K_\infty}+\beta V)&\le N\big(0,-(1-\varepsilon)\Delta_{K_\infty}+\beta V_k\big)+N\big(0,\varepsilon(-\Delta_{K_\infty})+\beta(V-V_k)\big)\\&=N\big(0,-\Delta_{K_\infty}+(1-\varepsilon)^{-1}\beta V_k\big)+N\big(0,-\Delta_{K_\infty}+\varepsilon^{-1}\beta(V-V_k)\big)\\&\le c_2(1-\varepsilon)^{-\alpha/2}\beta^{\alpha/2}W(K_\infty,\nu_\infty,V_k,\alpha)(1+o(1))\\&\quad+C\varepsilon^{-\alpha/2}\beta^{\alpha/2}W(K_\infty,\mu_\infty,V-V_k,\alpha),\qquad\text{as }\beta\to\infty,\end{aligned}$$

where the fact $N(0,-c\Delta_{K_\infty}+V)=N(0,-\Delta_{K_\infty}+c^{-1}V)$ is used in the first equality. Taking $k\to\infty$ and then $\varepsilon\to 0$, we find that

$$N(0,-\Delta_{K_\infty}+\beta V)\le c_2W(K_\infty,\nu_\infty,V,\alpha)\beta^{\alpha/2}(1+o(1))\qquad\text{as }\beta\to\infty. \tag{3.8}$$

On the other hand, we have $-\Delta+\beta V_k=\big(-(1-\varepsilon)\Delta+\beta V\big)+\big(-\varepsilon\Delta+\beta(V_k-V)\big)$. So using Proposition 2.3(b), part(a) and (3.7) as above, we get as $\beta\to\infty$

$$\begin{aligned}c_1\beta^{\alpha/2}W(K_\infty,\nu_\infty,V_k,\alpha)(1+o(1))&\le N(0,-\Delta_{K_\infty}+\beta V_k)\\&\le N(0,-(1-\varepsilon)\Delta_{K_\infty}+\beta V)+N(0,-\varepsilon\Delta_{K_\infty}+\beta(V_k-V))\\&=N\big(0,-\Delta_{K_\infty}+(1-\varepsilon)^{-1}\beta V\big)+N\big(0,-\Delta_{K_\infty}+\varepsilon^{-1}\beta(V_k-V)\big)\\&\le N\big(0,-\Delta_{K_\infty}+(1-\varepsilon)^{-1}\beta V\big)+C\varepsilon^{-\alpha/2}\beta^{\alpha/2}W(K_\infty,\mu_\infty,V-V_k,\alpha).\end{aligned}$$

Again taking $k\to\infty$ and then $\varepsilon\to 0$, we obtain

$$N(0,-\Delta_{K_\infty}+\beta V)\ge c_1W(K_\infty,\nu_\infty,V,\alpha)\beta^{\alpha/2}(1+o(1)). \tag{3.9}$$

Hence, (3.8) and (3.9) imply the desired result. □

3.2. **Infinite blowup of Sierpiński gasket.** In this subsection, we illustrate Theorem 1.3 by the infinite blowup of Sierpiński gasket.

Proof of Corollary 1.4. For $k\ge 0$, let $\mathcal{M}^k:=\{0,1,2\}^k$ and $\mathbf{P}_k:=\{K_{\boldsymbol\omega}:=S_{\boldsymbol\omega}(K):\boldsymbol\omega\in\mathcal{M}^k\}$. It is easy to check that $(\mathbf{P}_k)_{k\ge0}$ is a sequence partitions of K that is refining with respect to μ. Let $-\Delta_K$ be the standard Laplacian constructed by Kigami [21] in $L^2(K,\mu)$ and $(\mathcal{E},\mathcal{F})$ be the associated Dirichlet form in $L^2(K,\mu)$. It is known that $(\mathcal{E},\mathcal{F})$ has the self-similarity property in the following sense: for any $u\in\mathcal{F}$, $u\circ S_i\in\mathcal{F}$ for all $i\in\{0,1,2\}$ and

$$\mathcal{E}(u,v)=\frac{5}{3}\sum_{i=0}^{2}\mathcal{E}(u\circ S_i,v\circ S_i)\qquad\text{fro all }v\in\mathcal{F}. \tag{3.10}$$

Fix any $\boldsymbol i\in\mathcal{M}^k$, where $k\ge 0$. Define $\mathcal{D}_{\boldsymbol i}:=\{u\in\mathcal{F}:u|_{K\setminus K_{\boldsymbol i}}\equiv 0\}$. Let $\mathcal{F}_{\boldsymbol i}$ be the closure of $\mathcal{D}_{\boldsymbol i}$ with respect to the inner product $\mathcal{E}_*(u,v)=\mathcal{E}(u,v)+(u,v)_{L^2(K,\mu)}$, and let $\mathcal{E}_{\boldsymbol i}=\mathcal{E}|_{\mathcal{F}_{\boldsymbol i}\times\mathcal{F}_{\boldsymbol i}}$. It is easy to check that $(\mathcal{E}_{\boldsymbol i},\mathcal{F}_{\boldsymbol i})$ is a Dirichlet form in $L^2(K_{\boldsymbol i},\mu|_{K_{\boldsymbol i}})$. Using the self-similarity property of $(\mathcal{E},\mathcal{F})$ and iteration, we can deduce that if $u\in\operatorname{dom}(-\Delta_{K_{\boldsymbol i}})$, then $u\circ S_{\boldsymbol i}\in\operatorname{dom}(-\Delta_K)$, and

$$-\Delta_K(u\circ S_{\boldsymbol i})=5^{-k}\big(-\Delta_{K_{\boldsymbol i}}u\big)\circ S_{\boldsymbol i}, \tag{3.11}$$

where $-\Delta_{K_{\boldsymbol{i}}}$ is the Laplacian in $L^2(K_{\boldsymbol{i}}, \mu|_{K_{\boldsymbol{i}}})$ associated with $(\mathcal{E}_{\boldsymbol{i}}, \mathcal{F}_{\boldsymbol{i}})$. By the theory of Kigami in [21], we have $\mathbf{P}_0 = \{K\}$ satisfies condition (DN). Combining (3.11) and the method in [21], we can check that $\mathbf{P}_k$ satisfies condition (DN) for all $k \geq 0$. Hence, condition (A1) in Assumption 1.2 holds.

Let $-\Delta_K^D$ and $-\Delta_K^N$ be the Dirichlet and Neumann Laplacians in $L^2(K, \mu)$ associated with $(\mathcal{E}, \mathcal{F})$, respectively. Kigami and Lapidus [22] proved that the eigenvalue counting function $N(\lambda, -\Delta_K^b)$ satisfies the following asymptotic formula:

$$C_1 \lambda^{d_s/2}(1 + o(1)) \leq N(\lambda, -\Delta_K^b) \leq C_2 \lambda^{d_s/2}(1 + o(1)) \quad \text{as } \lambda \to \infty, \tag{3.12}$$

where $b \in \{D, N\}$, $d_s = \log 9/\log 5$, C_1 and C_2 are positive constants. Let $-\Delta_{K_{\boldsymbol{i}}}^D$ and $-\Delta_{K_{\boldsymbol{i}}}^N$ be the Dirichlet and Neumann Laplacians in $L^2(K_{\boldsymbol{i}}, \mu|_{K_{\boldsymbol{i}}})$ associated with $(\mathcal{E}_{\boldsymbol{i}}, \mathcal{F}_{\boldsymbol{i}})$, respectively. W can verify that (3.11) also holds replacing $-\Delta_{K_{\boldsymbol{i}}}$ and $-\Delta_K$ by $-\Delta_{K_{\boldsymbol{i}}}^b$ and $-\Delta_K^b$, respectively, for all $b \in \{D, N\}$. It follows that

$$N(\lambda, -\Delta_{K_{\boldsymbol{i}}}^b) = N(5^{-k}\lambda, -\Delta_K^b) \qquad \text{for } \lambda > 0, \tag{3.13}$$

where $b \in \{D, N\}$. We remark that $\mu(K_{\boldsymbol{i}}) = 3^{-k}$. Combining it with (3.12) and (3.13), we have

$$C_1 \mu(K_{\boldsymbol{i}}) \lambda^{d_s/2}(1 + o(1)) \leq N(\lambda, -\Delta_{K_{\boldsymbol{i}}}^b) \leq C_2 \mu(K_{\boldsymbol{i}}) \lambda^{d_s/2}(1 + o(1)) \qquad \text{as } \lambda \to \infty, \tag{3.14}$$

where $b \in \{D, N\}$. Since $\boldsymbol{i}$ is arbitrary, (3.14) implies condition (A2) in Assumption 1.2 holds. Consequently, Assumption 1.2 holds with $(\mathbf{P}_k)_{k \geq 0}$ and μ.

By the definition of $K_\infty^{\boldsymbol{\omega}}$ (see (1.8)), there exists a sequence of similitudes $(\tau_i)_{i \geq 0}$ such that $K_\infty^{\boldsymbol{\omega}} = \bigcup_{i \geq 0} \tau_i(K)$ satisfies conditions (C1)–(C3) in Section 1. The existence of non-negative self-adjoint Laplacian $-\Delta_{K_\infty^{\boldsymbol{\omega}}}$ in $L^2(K_\infty^{\boldsymbol{\omega}}, \mu_\infty^{\boldsymbol{\omega}})$ have been obtained in [40, 42]. Final, the desired result follows from Theorem 1.3 (a). □

3.3. **Unbounded generalized Sierpiński carpets.** In this subsection, we illustrate Theorem 1.3 by using a class of unbounded generalized Sierpiński carpets. The following definition is given in [6, 7].

Let $d \geq 2$, $K_0 = [0, 1]^d$, and let $\ell_K \in \mathbb{N}$ with $\ell_K \geq 3$ being fixed. For $n \in \mathbb{Z}$, let $\mathcal{Q}_n$ be the collection of closed cubes with side length ℓ_K^{-n} and with vertices at $\ell_F^{-n}\mathbb{Z}^d$. For $E \subseteq \mathbb{R}^d$, let

$$\mathcal{Q}_n(E) := \{Q \in \mathcal{Q}_n : \mathrm{int}(Q) \cap E \neq \emptyset\}. \tag{3.15}$$

For $Q \in \mathcal{Q}_n$, let Ψ_Q be the orientation preserving affine map (i.e., similitude with no rotation part) which maps K_0 onto Q. Define a decreasing sequence $\{K_n\}$ of closed subsets of K_0. Let m_K be an integer satisfying $1 \leq m_K \leq \ell_K^d$, and K_1 be the union of m_K distinct elements of $\mathcal{Q}_1(K_0)$. We impose the following conditions on K_1.

(H1) (*Symmetry*) K_1 is preserved by all the isometries of the unit cube K_0.
(H2) (*Connectedness*) $\mathrm{int}(K_1)$ is connected.

(H3) (*Non-diagonality*) Let $m \geq 1$ and $B \subseteq K_0$ be a cube of side length $2\ell_K^{-m}$, which is the union of 2^d distinct elements of $\mathcal{Q}_m$. Then if $\mathrm{int}(K_1 \cap B)$ is non-empty, it is connected.

(H4) (*Border included*) K_1 contains the line segment $\{x : 0 \leq x_1 \leq 1, x_2 = \cdots = x_n = 0\}$.

One may think of K_1 as being derived from K_0 by removing the interiors of $\ell_K^d - m_K$ cubes in $\mathcal{Q}_1(K_0)$. Iterating this, we obtain a sequence $\{K_n\}$, where K_n is the union of m_K^n cubes in $\mathcal{Q}_n(K_0)$. Formally, we define

$$K_{n+1} = \bigcup_{Q \in \mathcal{Q}_n(K_n)} \Psi_Q(K_1) = \bigcup_{Q \in \mathcal{Q}_1(K_1)} \Psi_Q(K_n), \quad n \geq 1. \tag{3.16}$$

We call $K := \bigcap_{n=0}^{\infty} K_n$ a *generalized Sierpiński carpet (GSC)* and $K_\infty := \bigcup_{n=0}^{\infty} \ell_K^n K$ an *unbounded generalized Sierpiński carpet*, where $rA := \{rx : x \in A\}$ for all $A \subseteq \mathbb{R}^d$. Let $\partial K := [0,1]^d \setminus (0,1)^d$, which should be regarded as the boundary of K. It is easy to check that for each GSC K, there exists an IFS $\{F_i\}_{i=1}^{m_K}$ such that $K = \bigcup_{i=1}^{m_K} F_i(K)$, and we can rewrite the unbounded generalized Sierpiński carpet $K_\infty = \bigcup_{i=0}^{\infty} \tau_i(K)$ satisfying conditions (C1)–(C3) in Section 1.

Example 3.1. (Standard Sierpiński carpet) *Let* $q_1 = 0$, $q_2 = 1/2$, $q_3 = 1$, $q_4 = 1 + \sqrt{-1}/2$, $q_5 = 1 + \sqrt{-1}$, $q_6 = 1/2 + \sqrt{-1}$, $q_7 = \sqrt{-1}$ *and* $q_8 = \sqrt{-1}/2$. *Define* $S_i : \mathbb{C} \to \mathbb{C}$ *as* $F_i(z) = (z - q_i)/3 + q_i$ *for* $i \in \{1, \ldots, 8\}$. *Then there exists a unique nonempty compact subset* K, *which satisfies* $K = \bigcup_{i=1}^{8} F_i(K)$. K *is called the standard Sierpiński carpet.*

The standard Sierpiński carpet in the above Example is a GSC with $n = 2$, $l_F = 3$, $m_F = 8$ and with F_1 being obtained from F_0 by removing the middle cube.

In the rest of this subsection, we fix a GSC K associated an IFS $\{F_i\}_{i=1}^{m_K}$. Let μ be the self-similar measure defined by the IFS $\{F_i\}_{i=1}^{m_K}$ together with probability weights $(1/m_K, \cdots, 1/m_K)$. Barlow and Bass [1–6] have constructed a diffusion process on K and studied it extensively. For example, they extended the diffusion process on K to K_∞ and proved Sobolev inequality. Unfortunately, the Dirichlet form on $L^2(K,\mu)$ associated with their diffusion process is not necessarily self-similar. On the other hand, Kusuoka and Zhou [23] have given a different construction of a self-similar Dirichlet form on a GSC. Recently, Barlow *et al.* [7] showed that, up to scalar multiples of the time parameter, there exists only one such Dirichlet form on a GSC. Consequently, there exists a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ in $L^2(K,\mu)$, which has the self-similarity in the following sense: for any $u \in \mathcal{F}$, $u \circ F_i \in \mathcal{F}$ for all $i \in \{1, \ldots, m_K\}$,and

$$\mathcal{E}(u,v) = \rho_K \sum_{i=1}^{m_K} \mathcal{E}(u \circ F_i, v \circ F_i) \tag{3.17}$$

for all $u \in \mathcal{F}$, where ρ_K is a constant that is determined by the scaling in the resistance of the Sierpiński carpet. We remark that $t_K := \rho_K m_K \geq \ell_K^2 > 1$. Barlow *et al.* [7] showed that, up to scalar multiples of the time parameter, there exists only one such Brownian motion on a GSC. Hambly [17] and Kajino [20] showed the Dirichlet Laplacian $-\Delta_K^D$ and Neumann Laplacian $-\Delta_K^N$ in $L^2(K,\mu)$ with the boundary condition on ∂K are well-defined and have compact resolvents. In other words, $\{K\}$ satisfies condition (DN). Moreover, their results imply that the eigenvalue counting function satisfies the asymptotics

$$c_1\lambda^{d_s/2}(1+o(1)) \le N(\lambda, -\Delta_K^b) \le c_2\lambda^{d_s/2}(1+o(1)) \qquad \text{as } \lambda\to\infty, \tag{3.18}$$

where $b\in\{D,N\}$, $d_s = 2\log m_K/\log t_K$ is called the spectral dimension of K, c_1 and c_2 are positive constants.

Proof of Corollary 1.5. (a) The proof of part (a) is similar to that of Corollary 1.4; we use (3.17) and (3.18) instead of (3.10) and (3.12), respectively.

(b) follows by combining Theorem 1.3(b) and Sobolev inequality [6, Theorem 7.2]. □

4. Infinite Bernoulli convolution associated with the golden ratio

Let $K := [0,1]$ and μ be given by (1.10) and (1.11). Let $\{T_i\}_{i=0}^2$ be the auxiliary IFS defined as in (1.12) (see Figure 1). In this section, we introduce a new measure ν on K, and prove Theorem 1.6. We remark that the contraction ratios of T_0, T_1, T_2 are ρ^2, ρ^3, ρ^2, respectively.

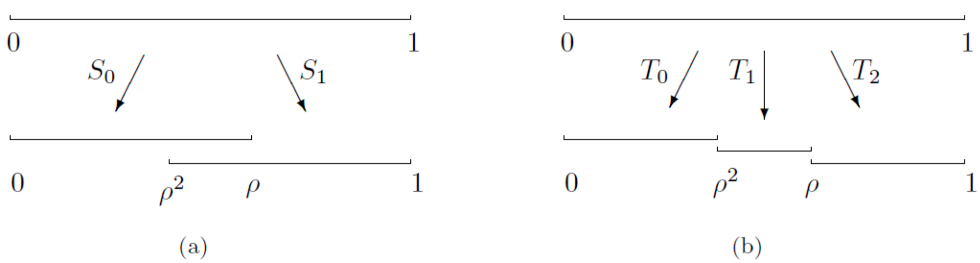

Figure 1. (a) The IFS $\{S_0, S_1\}$ in (1.10) has overlaps. (b) The auxiliary IFS $\{T_0, T_1, T_2\}$ does not have overlaps.

For convenience, we introduce some notation. Let $\mathscr{T} := \{0,1,2\}$ and $\mathscr{T}_0 := \{0,2\}$. For each $n\in\mathbb{N}$, define

$$\mathscr{T}^n := \{0,1,2\}^n, \qquad \mathscr{T}_0^n := \{0,2\}^n, \qquad \mathscr{T}^* := \bigcup_{n=0}^\infty \mathscr{T}^n, \qquad \mathscr{T}_0^* := \bigcup_{n=0}^\infty \mathscr{T}_0^n,$$

where $\mathscr{T}^0$ and $\mathscr{T}_0^0$ are defined to be the singleton $\{\emptyset\}$ of the empty word $\emptyset$. For any $\boldsymbol{\omega} = w_1w_2\cdots w_n \in \mathscr{T}^n$, let $K_{\boldsymbol{\omega}} := T_{\boldsymbol{\omega}}(K) = T_{w_1}\cdots T_{w_n}(K)$ and $|\boldsymbol{\omega}| := n$ be the length of $\boldsymbol{\omega}$. We use the convention that

$$\boldsymbol{\omega}\emptyset = \emptyset\boldsymbol{\omega} = \boldsymbol{\omega} \qquad \text{for any word } \boldsymbol{\omega}\in\mathscr{T}^*.$$

If $\boldsymbol{\omega} = w\cdots w\in\mathscr{T}^n$, then we denote $\boldsymbol{\omega} = w^n$. In [26], they showed that μ satisfies the following second-order identities: for each Borel subset $A\subseteq K$ and $j\in\mathscr{T}$,

$$\begin{bmatrix}\mu(T_0T_jA)\\ \mu(T_1T_jA)\\ \mu(T_2T_jA)\end{bmatrix} = M_j\begin{bmatrix}\mu(T_0A)\\ \mu(T_1A)\\ \mu(T_2A)\end{bmatrix}, \tag{4.1}$$

where

$$M_0 = \frac{1}{8}\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 4 & 0 \end{bmatrix}, \quad M_1 = \frac{1}{4}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_2 = \frac{1}{8}\begin{bmatrix} 0 & 4 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

This can be used to compute the measure of $K_{\boldsymbol{\omega}}$ for all finite words $\boldsymbol{\omega} \in \mathscr{T}^*$. For any $k \geq 0$ and finite word $J = j_1 j_2 \ldots j_k \in \mathscr{T}_0^k$, let

$$c_J := \frac{1}{4}\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} M_J \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

where $M_J := M_{j_1} M_{j_2} \cdots M_{j_k}$. In the rest of this section, we denote

$$\rho_* := \rho/2.$$

Let $-\Delta^D_{\mu|_{K_{\boldsymbol{\omega}}}}$ be the Dirichlet Laplacian with respect to $\mu|_{K_{\boldsymbol{\omega}}}$ for all finite words $\boldsymbol{\omega} \in \mathscr{T}^*$. Combining Proposition 2.3 and second-order identities (4.1), we can obtain the following proposition, which will be used repeatedly.

Proposition 4.1. *Use the notation above. Let $\boldsymbol{\omega} \in \mathscr{T}^*$ be a finite word (possibly the empty word) and $\lambda > 0$. Then*

(a) *for any finite word $J \in \mathscr{T}_0^*$, we have*

$$N\big(\lambda, -\Delta^D_{\mu|_{K_{1J1\boldsymbol{\omega}}}}\big) = N\big(\rho^{2|J|+3} c_J \lambda, -\Delta^D_{\mu|_{K_{1\boldsymbol{\omega}}}}\big).$$

(b) *for any $n \geq 1$, we have*

$$N\big(\lambda, -\Delta^D_{\mu|_{K_{0^n 1\boldsymbol{\omega}}}}\big) = N\big(\rho_*^{2n} \lambda, -\Delta^D_{\mu|_{K_{1\boldsymbol{\omega}}}}\big) \quad \text{and} \quad N\big(\lambda, -\Delta^D_{\mu|_{K_{0^n 2\boldsymbol{\omega}}}}\big) = N\big(\rho_*^{2n-1} \lambda, -\Delta^D_{\mu|_{K_{1\boldsymbol{\omega}}}}\big).$$

Proof. (a) Let $J \in \mathscr{T}_0^*$ be a finite word. It is shown in [25, Proposition 2.1(i)] that for any Borel subset $A \subseteq K$ and finite word $J \in \mathscr{T}_0^*$,

$$\mu(T_{1J1}A) = c_J \mu(T_1 A).$$

It follows that $\mu|_{K_{1J1\boldsymbol{\omega}}} = c_J \mu \circ T_{1J}^{-1}$ on $K_{1J1\boldsymbol{\omega}}$. Noting that the contraction ratio of T_{1J} equals $\rho^{2|J|+3}$. Thus the desired result follows from Proposition 2.3.

(b) Fix any $n \geq 1$. Use the second-order identities (4.1), we can check that

$$\mu(T_{0^n 1}A) = 2^{-2n} \mu(T_1 A) \quad \text{and} \quad \mu(T_{0^n 2}A) = 2^{-2n+1} \mu(T_1 A).$$

for all Borel subsets $A \subseteq K$. Then $\mu|_{K_{0^n 1\boldsymbol{\omega}}} = 2^{-2n} \mu \circ T_{0^n}^{-1}$ on $K_{0^n 1\boldsymbol{\omega}}$ and $\mu|_{K_{0^n 2\boldsymbol{\omega}}} = 2^{-2n+1} \mu \circ \big(T_{0^{n-1}} S_1\big)^{-1}$ on $K_{0^n 2\boldsymbol{\omega}}$, where the fact $T_{0^n 2} = T_{0^{n-1}} S_1 T_1$ is used in second equality. It follows from Proposition 2.3 that the assertions holds. $\square$

The spectral dimension of $-\Delta^D_{\mu|_K}$ is computed in [32]. Their approach is to use the second-order identities (4.1) to derive a scalar-valued renewal equation for the eigenvalue counting function on

K_1. And then applying the vector-valued renewal theorem proved by Lau *et al.* [24], they showed the spectral dimension d_s of $-\Delta^D_{\mu|_K}$ is the unique positive solution of

$$\sum_{k=0}^{\infty} \sum_{J\in\mathscr{T}_0^k} (\rho^{2|J|+3}c_J)^{d_s/2} = 1, \tag{4.2}$$

and there exist positive constants C_1, C_2 such that $C_1\lambda^{d_s/2} \le N(\lambda, -\Delta^D_{\mu|_K}) \le C_2\lambda^{d_s/2}$ for sufficiently large λ. Ngai and first author [33, Proposition 5.2] recently proved that the non-arithmetic (or non-lattice) case holds for $-\Delta^D_{\mu|_{K_1}}$: there exists constant $C > 0$ such that

$$N(\lambda, -\Delta^D_{\mu|_{K_1}}) = C\lambda^{d_s/2}\big(1 + o(1)\big) \qquad \text{as } \lambda \to \infty. \tag{4.3}$$

Together with the scalar-valued renewal equation and the error estimates obtained in [32, section 5], it yields

$$N(\lambda, -\Delta^D_{\mu|_{K_{1J}}}) = C\Big(\sum_{n=0}^{\infty}\sum_{I\in\mathscr{T}_0^n} (\rho^{2|JI|+3}c_{JI})^{d_s/2}\Big)\lambda^{d_s/2}(1 + o(1)) \qquad \text{as } \lambda \to \infty \tag{4.4}$$

for all finite words $J \in \mathscr{T}_0^*$ (possibly the empty word).

We now introduce a measure ν on K by repeated subdivision as follows. First, define

$$\nu(K_1) = C, \quad \nu(K_{1J1\boldsymbol{\omega}}) = (\rho^{2|J|+3}c_J)^{d_s/2}\nu(K_{1\boldsymbol{\omega}}), \text{ and } \quad \nu(K_{1J}) = \sum_{n=0}^{\infty}\sum_{I\in\mathscr{T}_0^n} \nu(K_{1JI1}) \tag{4.5}$$

for all finite words $J \in \mathscr{T}_0^*$ and $\boldsymbol{\omega} \in \mathscr{T}^*$, where C comes from (4.3). We remark that the value of $\nu(K_{1\boldsymbol{\omega}})$ is well-defined for all finite words $\boldsymbol{\omega} \in \mathscr{T}^*$, since

$$\mathscr{T}^n = \mathscr{T}_0^n \bigcup \big\{J1\boldsymbol{\omega} : J \in \mathscr{T}_0^i, \boldsymbol{\omega} \in \mathscr{T}^{n-1-i}, 0 \le i \le n-1\big\} \quad \text{ for all } n \ge 1.$$

Let $\varphi(x) = -x + 1$. For any finite word $\boldsymbol{\omega} \in \mathscr{T}^*$, there exists a unique $\boldsymbol{\omega}_* \in \mathscr{T}^{|\boldsymbol{\omega}|}$ such that

$$\varphi(K_{2\boldsymbol{\omega}}) = K_{0\boldsymbol{\omega}_*}.$$

By symmetric of μ, we have $\mu|_{K_{2\boldsymbol{\omega}}} = \mu \circ \varphi^{-1}$ on $K_{2\boldsymbol{\omega}}$, which, by combining Proposition 2.3, implies

$$N(\lambda, -\Delta^D_{\mu|_{K_{2\boldsymbol{\omega}}}}) = N(\lambda, -\Delta^D_{\mu|_{K_{0\boldsymbol{\omega}_*}}}) \qquad \text{for all } \lambda > 0.$$

Second, define

$$\begin{aligned}
&\nu(K_{0^n1\boldsymbol{\omega}}) = \rho_*^{nd_s}\nu(K_{1\boldsymbol{\omega}}), \qquad \nu(K_{0^n2\boldsymbol{\omega}}) = \nu(K_{2^n0\boldsymbol{\omega}}) = \rho_*^{(n-1/2)d_s}\nu(K_{1\boldsymbol{\omega}}),\\
&\nu(K_{0^n}) := \sum_{k=0}^{\infty}\Big(\nu(K_{0^{n+k}1}) + \nu(K_{0^{n+k}2})\Big), \text{ and}\\
&\nu(K_{2\boldsymbol{\omega}}) = \nu(K_{0\boldsymbol{\omega}_*})
\end{aligned} \tag{4.6}$$

for all $n \ge 1$ and finite words $\boldsymbol{\omega} \in \mathscr{T}^*$. Note that the value of $\nu(K_{i\boldsymbol{\omega}})$ is well-defined for all finite words $\boldsymbol{\omega} \in \mathscr{T}^*$ and $i \in \mathscr{T}_0$, and

$$\nu(K_{0^n})=\nu(K_{2^n})=\sum_{k=0}^{\infty}\Big(\rho_*^{(n+k)d_s}+\rho_*^{(n+k-1/2)d_s}\Big)\nu(K_1)=C\rho_*^{(n-1/2)d_s}(1-\rho_*^{d_s/2})^{-1}.$$

Final, define

$$\nu(K)=\nu(K_0)+\nu(K_1)+\nu(K_2).$$

Consequently, we can define

$$\nu(A)=\inf\Big\{\sum_{\boldsymbol{\omega}\in\Lambda}\nu(K_{\boldsymbol{\omega}}):A\subseteq\bigcup_{\boldsymbol{\omega}\in\Lambda}K_{\boldsymbol{\omega}},\Lambda\subseteq\mathscr{T}^*\Big\}\quad\text{for all } A\subseteq K. \tag{4.7}$$

We remark that ν is a well-defined measure on K (see Proposition 4.6 below for details). Moreover, μ and ν have the same symmetric.

In order to prove Theorem 1.6(a), we divide $\mathscr{T}^*$ into two parts, namely,

$$\big\{1\boldsymbol{\omega}:\boldsymbol{\omega}\in\mathscr{T}^n,n\geq 0\big\}\text{ and }\bigcup_{i\in\mathscr{T}_0}\big\{i\boldsymbol{\omega}:\boldsymbol{\omega}\in\mathscr{T}^n,n\geq 0\big\}.$$

We begin with the first case.

Lemma 4.2. *Use the notation above. Let ν be given as (4.7). Then*

$$N(\lambda,-\Delta^D_{\mu|_{K_{1\boldsymbol{\omega}}}})=\nu\big(K_{1\boldsymbol{\omega}}\big)\lambda^{d_s/2}(1+o(1))\qquad as\ \lambda\to\infty,$$

for all finite words $\boldsymbol{\omega}\in\mathscr{T}^*$ (possibly the empty word).

Proof. We use induction. In view of (4.3) and (4.5), we observe that

$$N(\lambda,-\Delta^D_{\mu|_{K_1}})=\nu(K_1)\lambda^{d_s/2}\big(1+o(1)\big)\qquad\text{as }\lambda\to\infty.$$

Assume that

$$N(\lambda,-\Delta^D_{\mu|_{K_{1\boldsymbol{\omega}}}})=\nu\big(K_{1\boldsymbol{\omega}}\big)\lambda^{d_s/2}(1+o(1))\qquad\text{as }\lambda\to\infty, \tag{4.8}$$

for all $\boldsymbol{\omega}\in\bigcup_{i=0}^n\mathscr{T}^i$, where $n\geq 0$. Let $\boldsymbol{\omega}\in\mathscr{T}^{n+1}$. If $\boldsymbol{\omega}\in\mathscr{T}_0^{n+1}$, then (4.4) and (4.5) imply

$$N(\lambda,-\Delta^D_{\mu|_{K_{1\boldsymbol{\omega}}}})=\nu\big(K_{1\boldsymbol{\omega}}\big)\lambda^{d_s/2}(1+o(1))\qquad\text{as }\lambda\to\infty. \tag{4.9}$$

On the other hand, if $\boldsymbol{\omega}\in\mathscr{T}^{n+1}\setminus\mathscr{T}_0^{n+1}$, then there exist $J\in\mathscr{T}_0^*$ and $\boldsymbol{\tau}\in\mathscr{T}^*$ such that $\boldsymbol{\omega}=J1\boldsymbol{\tau}$. It follows from Proposition 4.1 and the assumption (4.8) that as $\lambda\to\infty$,

$$\begin{aligned}N(\lambda,-\Delta^D_{\mu|_{K_{1\boldsymbol{\omega}}}})&=N(\lambda,-\Delta^D_{\mu|_{K_{1J1\boldsymbol{\tau}}}})=N(\rho^{2|J|+3}c_J\lambda,-\Delta^D_{\mu|_{K_{1\boldsymbol{\tau}}}})\\&=\big(\rho^{2|J|+3}c_J\big)^{d_s/2}\nu\big(K_{1\boldsymbol{\tau}}\big)\lambda^{d_s/2}(1+o(1))=\nu\big(K_{1J1\boldsymbol{\tau}}\big)\lambda^{d_s/2}(1+o(1))\\&=\nu\big(K_{1\boldsymbol{\omega}}\big)\lambda^{d_s/2}(1+o(1)),\end{aligned} \tag{4.10}$$

where fourth equality follows from (4.5). Combining (4.9) and (4.10), we have (4.8) also holds for all $\boldsymbol{\omega}\in\mathscr{T}^{n+1}$. By induction, the desired result follows. $\square$

We now turn to consider the case: $\bigcup_{i\in\mathscr{T}_0}\{i\boldsymbol{\omega}:\boldsymbol{\omega}\in\mathscr{T}^n, n\geq 0\}$. To end this, we first develop a lemma, which shows an asymptotic behavior of eigenvalue counting function on K_0 and K_2.

Lemma 4.3. *Use the notation above. Let ν be defined as in (4.7). Then*

$$N(\lambda,-\Delta^D_{\mu|_{K_i}})=\nu(K_i)\lambda^{d_s/2}\big(1+o(1)\big)\qquad \text{as } \lambda\to\infty,$$

for all $i\in\mathscr{T}_0$.

Proof. By the symmetry of measures μ, ν, and $\{T_i\}_{i=0}^2$, it suffices to show that

$$N(\lambda,-\Delta^D_{\mu|_{K_0}})=\nu(K_0)\lambda^{d_s/2}\big(1+o(1)\big)\qquad \text{as } \lambda\to\infty.$$

Using second-order identities (4.1) again, we can deduce that $\mu(T_{0^{n+1}}A)=2^{-2n}\mu(T_0A)$ for all $A\subseteq K$ and $n\geq 1$. It follows from Proposition 2.3 that

$$N(\lambda,-\Delta^D_{\mu|_{K_{0^{n+1}\boldsymbol{\omega}}}})=N(\rho_*^{2n}\lambda,-\Delta^D_{\mu|_{K_{0\boldsymbol{\omega}}}})\qquad \text{for all } \lambda>0,\ \boldsymbol{\omega}\in\mathscr{T}^* \text{ and } n\geq 1. \tag{4.11}$$

Let $\mathbf{P}_n:=\big\{K_{0^i\ell}:\ell\in\{1,2\},1\leq i\leq n\big\}\cup\{K_{0^{n+1}}\}$ for all $n\geq 1$. We remark that $(\mathbf{P}_n)_{n\geq 1}$ is a sequence of partitions of K_0. Propositions 2.5, 4.1(b) and (4.11) imply

$$\begin{aligned}
N(\lambda,-\Delta^D_{\mu|_{K_0}})&=\sum_{i=1}^{n}\Big(N(\lambda,-\Delta^D_{\mu|_{K_{0^i1}}})+N(\lambda,-\Delta^D_{\mu|_{K_{0^i2}}})\Big)+N(\lambda,-\Delta^D_{\mu|_{K_{0^{n+1}}}})+\epsilon(\mathbf{P}_n,\lambda)\\
&=\sum_{i=1}^{\infty}\Big(N(\rho_*^{2i}\lambda,-\Delta^D_{\mu|_{K_1}})+N(\rho_*^{2i-1}\lambda,-\Delta^D_{\mu|_{K_1}})\Big)+N(\rho_*^{2n}\lambda,-\Delta^D_{\mu|_{K_0}})\\
&\quad-\sum_{i=n+1}^{\infty}\Big(N(\rho_*^{2i}\lambda,-\Delta^D_{\mu|_{K_1}})+N(\rho_*^{2i-1}\lambda,-\Delta^D_{\mu|_{K_1}})\Big)+\epsilon(\mathbf{P}_n,\lambda)
\end{aligned}\tag{4.12}$$

for all $\lambda>0$, where $0\leq\epsilon(\mathbf{P}_n,\lambda)\leq 4n$. Since the first eigenvalue of $-\Delta^D_{\mu|_{K_\ell}}$ is positive for all $\ell\in\{0,1\}$, there exists $\lambda_0>0$ such that $N(\lambda,-\Delta^D_{\mu|_{K_\ell}})=0$ for all $\lambda<\lambda_0$ and all $\ell\in\{0,1\}$. For $\lambda>0$, n_λ is the smallest integer such that

$$\rho_*^{2n_\lambda-1}\lambda<\lambda_0.$$

Letting $n=n_\lambda$ in (4.12). Then the second term and third summation in (4.12) vanish and thus we get

$$N(\lambda,-\Delta^D_{\mu|_{K_0}})=\sum_{i=1}^{\infty}\Big(N(\rho_*^{2i}\lambda,-\Delta^D_{\mu|_{K_1}})+N(\rho_*^{2i-1}\lambda,-\Delta^D_{\mu|_{K_1}})\Big)+\epsilon(\mathbf{P}_{n_\lambda},\lambda). \tag{4.13}$$

It is easy to check that $\epsilon(\mathbf{P}_{n_\lambda},\lambda)=o(\lambda^{d_s/2})$ as $\lambda\to\infty$. Combining it with Lemma 4.2, (4.13) and (4.6), we have as $\lambda\to\infty$,

$$\begin{aligned}
N(\lambda,-\Delta^D_{\mu|_{K_0}})&=\Big(\sum_{i=1}^{\infty}\big(\rho_*^{id_s}\nu(K_1)+\rho_*^{(i-1/2)d_s}\nu(K_1)\big)\Big)\lambda^{d_s/2}(1+o(1))\\
&=\Big(\sum_{i=1}^{\infty}\big(\nu(K_{0^i1})+\nu(K_{0^i2})\big)\Big)\lambda^{d_s/2}(1+o(1))=\nu(K_0)\lambda^{d_s/2}(1+o(1)),
\end{aligned}$$

which completes the proof. □

Lemma 4.4. *Use the notation above. Let ν be defined as in* (4.7)*. Then as $\lambda \to \infty$,*

$$N(\lambda, -\Delta^D_{\mu|_{K_{i\boldsymbol{\omega}}}}) = \nu\big(K_{i\boldsymbol{\omega}}\big)\lambda^{d_s/2}(1+o(1))$$

for all $i \in \mathscr{T}_0$ and finite words (possibly empty word) $\boldsymbol{\omega} \in \mathscr{T}^*$.

Proof. Similarly, by symmetry, it suffices to show that as $\lambda \to \infty$,

$$N(\lambda, -\Delta^D_{\mu|_{K_{0\boldsymbol{\omega}}}}) = \nu\big(K_{0\boldsymbol{\omega}}\big)\lambda^{d_s/2}(1+o(1)) \qquad \text{for all finite words } \boldsymbol{\omega} \in \mathscr{T}^*. \tag{4.14}$$

Lemma 4.3 tells us that (4.14) holds for all $\boldsymbol{\omega} \in \mathscr{T}^0$. Assume that

$$N(\lambda, -\Delta^D_{\mu|_{K_{0\boldsymbol{\omega}}}}) = \nu\big(K_{0\boldsymbol{\omega}}\big)\lambda^{d_s/2}(1+o(1)) \qquad \text{as } \lambda \to \infty, \tag{4.15}$$

for all $\boldsymbol{\omega} \in \bigcup_{i=0}^n \mathscr{T}^i$, where $n \geq 0$. Let $\boldsymbol{\omega} \in \mathscr{T}^n$. It follows from Proposition 4.1 and Lemma 4.2 that

$$\begin{aligned} N(\lambda, -\Delta^D_{\mu|_{K_{01\boldsymbol{\omega}}}}) &= N(\rho_*^2\lambda, -\Delta^D_{\mu|_{K_{1\boldsymbol{\omega}}}}) = \rho_*^{d_s}\nu\big(K_{1\boldsymbol{\omega}}\big)\lambda^{d_s/2}(1+o(1)) \\ &= \nu\big(K_{01\boldsymbol{\omega}}\big)\lambda^{d_s/2}(1+o(1)) \qquad \text{as } \lambda \to \infty, \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} N(\lambda, -\Delta^D_{\mu|_{K_{02\boldsymbol{\omega}}}}) &= N(\rho_*\lambda, -\Delta^D_{\mu|_{K_{1\boldsymbol{\omega}}}}) = \rho_*^{d_s/2}\nu\big(K_{1\boldsymbol{\omega}}\big)\lambda^{d_s/2}(1+o(1)) \\ &= \nu\big(K_{02\boldsymbol{\omega}}\big)\lambda^{d_s/2}(1+o(1)) \qquad \text{as } \lambda \to \infty. \end{aligned} \tag{4.17}$$

where the facts $\nu\big(K_{01\boldsymbol{\omega}}\big) = \rho_*^{d_s}\nu(K_{1\boldsymbol{\omega}})$ and $\nu\big(K_{02\boldsymbol{\omega}}\big) = \rho_*^{d_s/2}\nu(K_{1\boldsymbol{\omega}})$ are used in the last equality of (4.16) and (4.17), respectively. On the other hand, using (4.11), (4.15) and the fact $\nu(K_{00\boldsymbol{\omega}}) = \rho_*^{d_s}\nu(K_{0\boldsymbol{\omega}})$, we can obtain

$$\begin{aligned} N(\lambda, -\Delta^D_{\mu|_{K_{00\boldsymbol{\omega}}}}) &= N(\rho_*^2\lambda, -\Delta^D_{\mu|_{K_{0\boldsymbol{\omega}}}}) = \rho_*^{d_s}\nu\big(K_{0\boldsymbol{\omega}}\big)\lambda^{d_s/2}(1+o(1)) \\ &= \nu\big(K_{00\boldsymbol{\omega}}\big)\lambda^{d_s/2}(1+o(1)) \qquad \text{as } \lambda \to \infty. \end{aligned} \tag{4.18}$$

Hence, by combining (4.16), (4.17) and (4.18), we have (4.15) also holds for all $\boldsymbol{\omega} \in \mathscr{T}^{n+1}$. Consequently, the assertion holds by reduction. □

The following lemma is a directly result from Proposition 2.5, Lemmas 4.2 and 4.4.

Lemma 4.5. *Use the notation above. Let ν be defined as in* (4.7)*. Then $N(\lambda, -\Delta^D_{\mu|_K}) = \nu(K)\lambda^{d_s/2}\big(1+o(1)\big)$ as $\lambda \to \infty$.*

Proof. We first remark that $\mathbf{P} := \{K_\ell : \ell \in \mathscr{T}\}$ is a partition of K. Then Proposition 2.5, Lemmas 4.2 and 4.4 imply that

$$\begin{aligned} N(\lambda, -\Delta^D_{\mu|_K}) &= \sum_{\ell\in\mathscr{T}} N(\lambda, -\Delta^D_{\mu|_{K_\ell}}) + \epsilon(\mathbf{P}, \lambda) = \Big(\sum_{\ell\in\mathscr{T}} \nu(K_\ell)\Big)\lambda^{d_s/2}(1+o(1)) \\ &= \nu(K)\lambda^{d_s/2}(1+o(1)) \qquad \text{as } \lambda \to \infty, \end{aligned}$$

where $0 \le \epsilon(\mathbf{P}, \lambda) \le 4$. □

We now show that ν given in (4.7) is a well-defined measure on K.

Proposition 4.6. *Use the notation above. Let ν be given by (4.7). Then*

(a) $\nu(K_{\boldsymbol{\omega}}) = \sum_{\ell\in\mathscr{T}} \nu(K_{\boldsymbol{\omega}\ell})$ *for all finite word* $\boldsymbol{\omega} \in \mathscr{T}^*$.
(b) $\max\big\{\nu(K_{\boldsymbol{\omega}}) : \boldsymbol{\omega} \in \mathscr{T}^n\big\} \to 0$ *as* $n \to \infty$.

Consequently, ν is a well-defined measure on K.

Proof. (a) Let $\boldsymbol{\omega} \in \mathscr{T}^*$ be a finite word. Since $\mathbf{P} := \{K_{\boldsymbol{\omega}\ell} : \ell \in \mathscr{T}\}$ is a partition of $K_{\boldsymbol{\omega}}$, Proposition 2.5 implies that

$$N(\lambda, -\Delta^D_{\mu|_{K_{\boldsymbol{\omega}}}}) = \sum_{\ell\in\mathscr{T}} N(\lambda, -\Delta^D_{\mu|_{K_{\boldsymbol{\omega}\ell}}}) + \epsilon(\mathbf{P}, \lambda)$$

for all $\lambda > 0$, where $0 \le \epsilon(\mathbf{P}, \lambda) \le 4$. Letting $\lambda \to \infty$. It follows from Lemmas 4.2 and 4.4 that $\nu(K_{\boldsymbol{\omega}}) = \sum_{\ell\in\mathscr{T}} \nu(K_{\boldsymbol{\omega}\ell})$, which completes the proof.

(b) Combining (4.5) and (4.6), it suffices to show that

$$\max\big\{\nu(K_{1J}) : J \in \mathscr{T}_0^n\big\} \to 0 \qquad \text{as } n \to \infty. \tag{4.19}$$

Using (4.5) again, we can deduce

$$\sum_{J\in\mathscr{T}_0^n} \nu(K_{1J}) = C \sum_{J\in\mathscr{T}_0^n} \sum_{k=0}^{\infty} \sum_{I\in\mathscr{T}_0^k} (\rho^{2|JI|+3} c_{JI})^{d_s/2} = C \sum_{k=n}^{\infty} \sum_{I\in\mathscr{T}_0^k} (\rho^{2|I|+3} c_I)^{d_s/2}$$

for all $n \ge 0$ and $J \in \mathscr{T}_0^n$. It follows from (4.2) that $\sum_{J\in\mathscr{T}_0^n} \nu(K_{1J}) \to 0$ as $n \to \infty$. Thus (4.19) holds, which completes the proof. □

Now we prove Theorem 1.6.

Proof of Theorem 1.6. (a) Combine Lemmas 4.2, 4.4 and 4.5, and Proposition 4.6.

(b) Let $\mathbf{P}_k = \{K_{\boldsymbol{\omega}} : \boldsymbol{\omega} \in \mathscr{T}^k\}$ for $k \ge 0$ and ν be defined as in (4.7). Using Proposition 4.6(b) and definition of ν, we deduce that a sequence of partitions $(\mathbf{P}_k)_{k\ge0}$ of K is refining with respect to ν. Combining it with part (a) and Remark 2.4, we have Assumption 1.2 holds. Thus the desired result follows from Theorem 1.3(a). □

5. M-fold convolution of Cantor-type measures

Let $\{S_i\}_{i=0}^m$ and $\mu := \mu_m$ be defined as in (1.13) and (1.14) respectively, with $m \ge 3$ being an odd integer, and let $K := [0, m]$. Let $\{T_i\}_{i=0}^{m-1}$ be the auxiliary IFS defined as in (1.15) (see Figure 2 for the case $m = 3$). In this section, we introduce a new measure ν on K, and then prove Theorem 1.7.

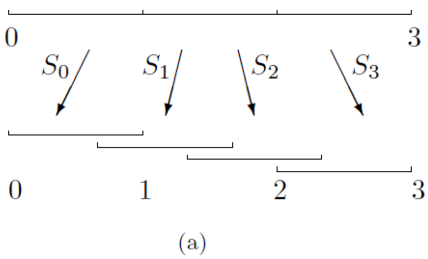

FIGURE 2. (a) The IFS $\{S_i\}_{i=0}^3$ in (1.13) with $m = 3$ has overlaps. (b) The auxiliary IFS $\{T_i\}_{i=0}^2$ does not have overlaps.

For convenience, we introduce some notation as before. Let

$$\mathscr{T} := \{0, 1, \dots, m-1\}, \qquad \mathscr{T}_0 := \{0, m-1\}, \qquad \mathscr{T}_1 := \{1, \dots, m-2\},$$

and for each $n \in \mathbb{N}$, let

$$\mathscr{T}^n := \{0, 1, \dots, m-1\}^n, \qquad \mathscr{T}_0^n = \{0, m-1\}^n, \qquad \mathscr{T}^* := \bigcup_{n=0}^{\infty} \mathscr{T}^n, \qquad \mathscr{T}_0^* = \bigcup_{n=0}^{\infty} \mathscr{T}_0^n,$$

where $\mathscr{T}^0$ and $\mathscr{T}_0^0$ are defined to be the singleton $\{\emptyset\}$ of the empty word. For $\boldsymbol{\omega} = \omega_1 \cdots \omega_n \in \mathscr{T}^n$, we use the notation $K_{\boldsymbol{\omega}} := T_{\omega_1} \circ \cdots \circ T_{\omega_n}(K)$ and $|\boldsymbol{w}| := n$ be the length of $\boldsymbol{w}$. We use the convention that

$$\boldsymbol{\omega}\emptyset = \emptyset\boldsymbol{\omega} = \boldsymbol{\omega} \qquad \text{for all finite words } \boldsymbol{\omega} \in \mathscr{T}^*.$$

If $\boldsymbol{\omega} = \omega \cdots \omega \in \mathscr{T}^n$, then we denote $\boldsymbol{\omega} = \omega^n$. For $i, j, k \in \mathscr{T}$, we define

$$a_{j,k}^{(i)} = \begin{cases} p_\ell & \text{if } \exists\ \ell \in [0, m] \text{ such that } i + mj - (m-1)\ell = k, \\ 0 & \text{otherwise,} \end{cases}$$

where $\{p_\ell\}_{\ell=0}^m$ is given as in (1.14). For $0 \le i \le m-1$, let M_i be the matrix

$$M_i := \left[a_{k-1,\ell-1}^{(i)} \right]_{k,\ell=1}^m .$$

In particular, if $m = 3$, then

$$M_0 = \begin{bmatrix} p_0 & 0 & 0 \\ 0 & p_1 & 0 \\ p_3 & 0 & p_2 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0 & p_0 & 0 \\ p_2 & 0 & p_1 \\ 0 & p_3 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} p_1 & 0 & p_0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}.$$

It is shown in [26] that μ satisfies the following second-order identities with respect to the IFS $\{T_i\}_{i=0}^{m-1}$: for all Borel subsets $A \subseteq K$,

$$\begin{bmatrix} \mu(T_0 T_i A) \\ \vdots \\ \mu(T_{m-1} T_i A) \end{bmatrix} = M_i \begin{bmatrix} \mu(T_0 A) \\ \vdots \\ \mu(T_{m-1} A) \end{bmatrix}. \tag{5.1}$$

For simplicity, we shall use the notation

$$i' := m - i - 1 \qquad \text{for all } i \in \mathscr{T}_1,$$

throughout the rest of this section. We remark that $i \in \mathscr{T}_1$ if and only if $i' \in \mathscr{T}_1$. For any integer $i \in \mathscr{T}$ and word $J = j_1 \cdots j_k \in \mathscr{T}_0^k$, where $k \geq 0$, define

$$c_{i,J} := \begin{bmatrix} p_{i+1} & p_i \end{bmatrix} P_J \begin{bmatrix} p_0 \\ p_m \end{bmatrix},$$

where $P_J := P_{j_1} \cdots P_{j_k}$,

$$P_0 := \begin{bmatrix} p_0 & 0 \\ p_m & p_{m-1} \end{bmatrix}, \quad P_{m-1} := \begin{bmatrix} p_1 & p_0 \\ 0 & p_m \end{bmatrix}.$$

For any finite word $\boldsymbol{\omega} \in \mathscr{T}^*$, let $-\Delta^D_{\mu|_{K_{\boldsymbol{\omega}}}}$ be the Dirichlet Laplacian with respect to $\mu|_{K_{\boldsymbol{\omega}}}$.

We first state two propositions, which will be used.

Proposition 5.1. *Use the notation above. Let $i \in \mathscr{T}_1$, $\boldsymbol{\omega} \in \mathscr{T}^*$ be a finite word and $\lambda > 0$. Then*

(a) *for any $j \in \mathscr{T} \setminus \{i'\}$, there exist some $p(i,j) \in \{p_i, p_{i+1}\}$ and $h(i,j) \in \mathscr{T}_1$ such that*

$$N\big(\lambda, -\Delta^D_{\mu|_{K_{ij\boldsymbol{\omega}}}}\big) = N\Big(\frac{p(i,j)}{m}\lambda, -\Delta^D_{\mu|_{K_{h(i,j)\boldsymbol{\omega}}}}\Big).$$

(b) *for any $\ell \in \mathscr{T}_1$ and finite word $J \in \mathscr{T}_0^*$, we have*

$$N\big(\lambda, -\Delta^D_{\mu|_{K_{ii'J\ell\boldsymbol{\omega}}}}\big) = N\Big(\frac{c_{i,J}}{m^{|J|+2}}\lambda, -\Delta^D_{\mu|_{K_{\ell\boldsymbol{\omega}}}}\Big).$$

Proof. (a) [26, Proposition 4.3]) tells us that for any Borel subset $A \subseteq K$, we have

$$\mu(T_{ij}A) = \begin{cases} p_i \mu(T_{j+i}A) & \text{if } 0 \leq j < i', \\ p_{i+1}\mu(T_{j-i'}A) & \text{if } i' < j \leq m-1. \end{cases} \tag{5.2}$$

We remark that $j + i \in \mathscr{T}_1$ if $0 \leq j < i'$; while $j - i' \in \mathscr{T}_1$ if $i' < j \leq m-1$. Combining it with Proposition 2.3 and (5.2), the desired result holds with

$$p(i,j) = \begin{cases} p_i & \text{if } 0 \leq j < i', \\ p_{i+1} & \text{if } i' < j \leq m-1, \end{cases} \quad \text{and} \quad h(i,j) = \begin{cases} j+i & \text{if } 0 \leq j < i', \\ j - i' & \text{if } i' < j \leq m-1. \end{cases}$$

(b) Let $\ell \in \mathscr{T}_1$ and $J \in \mathscr{T}_0^*$ be a finite word. Lau and Ngai [26, Proposition 4.4] proved that

$$\mu(T_{ii'J\ell}A) = c_{i,J}\mu(T_\ell A) \tag{5.3}$$

for all Borel subsets $A \subseteq K$. Thus the assertion follows from Proposition 2.3 and (5.3). □

Proposition 5.2. *Use the notation above. Let $\boldsymbol{\omega} \in \mathscr{T}^*$ be a finite word, $n \geq 1$ and $\lambda > 0$. Then*

$$N\big(\lambda, -\Delta^D_{\mu|_{K_{0^n\ell\boldsymbol{\omega}}}}\big) = N\Big(\Big(\frac{p_0}{m}\Big)^n\lambda, -\Delta^D_{\mu|_{K_{\ell\boldsymbol{\omega}}}}\Big) \quad \textit{for all } 0 \leq \ell \leq m-1, \textit{ and}$$

$$N\big(\lambda,-\Delta^D_{\mu|_{K_{0^n(m-1)J\ell\boldsymbol{\omega}}}}\big)=N\Big(\frac{p_0^{n-1}c_{0,J}}{m^{|J|+n+1}}\lambda,-\Delta^D_{\mu|_{K_{\ell\boldsymbol{\omega}}}}\Big)\quad \textit{for all } J\in\mathscr{T}_0^* \textit{ and } \ell\in\mathscr{T}_1.$$

Proof. Using [26, Proposition 4.2], we can deduce that

$$\mu(T_{0^n\ell}A)=p_0^n\mu(T_\ell A) \tag{5.4}$$

for all Borel subsets $A\subseteq K$, $\ell\in\mathscr{T}_1$ and $J\in\mathscr{T}_0^*$. Thus the desired results hold by combining Proposition 2.3 and (5.4). Similarly, we get

$$\mu(T_{0^n(m-1)J\ell}A)=p_0^{n-1}c_{0,J}\mu(T_\ell A)$$

for all Borel subsets $A\subseteq K$, $J\in\mathscr{T}_0^*$, and $\ell\in\mathscr{T}_1$, and then the assertions hold. □

The spectral dimension of $-\Delta^D_{\mu|_K}$ also is computed in [32]. They proved that the spectral dimension d_s of $-\Delta^D_{\mu|_K}$ is the unique solution of

$$\sum_{i\in\mathscr{T}_1}\Big(\frac{p_i}{m}\Big)^{d_s/2}+\sum_{i\in\mathscr{T}_1}\sum_{k=0}^{\infty}\sum_{J\in\mathscr{T}_0^k}\Big(\frac{c_{i,J}}{m^{k+2}}\Big)^{d_s/2}=1, \tag{5.5}$$

and for all $i\in\mathscr{T}_1$, there exist positive constants c_1,c_2 such that $c_1\lambda^{d_s/2}\le N(\lambda,-\Delta^D_{\mu|_{K_i}})\le c_2\lambda^{d_s/2}$ for sufficiently large λ (see [32, section 6] for details). Furthermore, Ngai and first author in [33, Proposition 5.4] proved that the non-arithmetic case holds for $-\Delta^D_{\mu|_{K_i}}$, $i\in\mathscr{T}_1$: there exist positive constants $(C_i)_{i\in\mathscr{T}_1}$ such that

$$N(\lambda,-\Delta^D_{\mu|_{K_i}})=C_i\lambda^{d_s/2}\big(1+o(1)\big)\qquad\text{as }\lambda\to\infty. \tag{5.6}$$

Together with the vector-valued renewal equations and the error estimates obtained in [32, Section 6], it yields

$$N(\lambda,-\Delta^D_{\mu|_{K_{ii'J}}})=\Big(\sum_{\ell\in\mathscr{T}_1}\sum_{k=0}^{\infty}\sum_{I\in\mathscr{T}_0^k}C_\ell\Big(\frac{c_{i,JI}}{m^{|JI|+2}}\Big)^{d_s/2}\Big)\lambda^{d_s/2}(1+o(1))\qquad\text{as }\lambda\to\infty \tag{5.7}$$

for all $i\in\mathscr{T}_1$ and finite words $J\in\mathscr{T}_0^*$, where, and throughout this section, $(C_\ell)_{\ell\in\mathscr{T}_1}$ come from (5.6).

Based on the results above, we now define a measure ν on K by repeated subdivision as follows. First, for any $i\in\mathscr{T}_1$ and finite word (possibly the empty word) $\boldsymbol{\omega}\in\mathscr{T}^*$, define

$$\begin{aligned}
\nu(K_i)&:=C_i;\\
\nu(K_{ij\boldsymbol{\omega}})&:=\Big(\frac{p(i,j)}{m}\Big)^{d_s/2}\nu(K_{h(i,j)\boldsymbol{\omega}})&&\text{for all } j\in\mathscr{T}\setminus\{i'\};\\
\nu(K_{ii'J\ell\boldsymbol{\omega}})&:=\Big(\frac{c_{i,J}}{m^{|J|+2}}\Big)^{d_s/2}\nu(K_{\ell\boldsymbol{\omega}})&&\text{for all finite words } J\in\mathscr{T}_0^* \text{ and } \ell\in\mathscr{T}_1;\\
\nu(K_{ii'J})&:=\sum_{\ell\in\mathscr{T}_1}\sum_{k=0}^{\infty}\sum_{I\in\mathscr{T}_0^k}\nu(K_{ii'JI\ell})&&\text{for all finite words } J\in\mathscr{T}_0^*,
\end{aligned} \tag{5.8}$$

where $p(i,j)$ and $h(i,j)$ are given as in Proposition 5.1, and $(C_i)_{i\in\mathscr{T}_1}$ come from (5.6). It is easy to check that $\nu(K_{i\boldsymbol{\omega}})$ is well-defined for all $i\in\mathscr{T}_1$ and finite words $\boldsymbol{\omega}\in\mathscr{T}^*$, since

$$\mathscr{T}^n=\mathscr{T}_0^n\bigcup\Big\{J\ell\boldsymbol{\omega}:J\in\mathscr{T}_0^k,\boldsymbol{\omega}\in\mathscr{T}^{n-1-k},0\le k\le n-1\Big\}\qquad\text{for } n\ge 1.$$

Second, for any $n\ge 1$, finite words $\boldsymbol{\omega}\in\mathscr{T}^*$ and $J\in\mathscr{T}_0^*$, we can define

$$\begin{aligned}\nu(K_{0^n\ell\boldsymbol{\omega}})&=\Big(\frac{p_0}{m}\Big)^{nd_s/2}\nu(K_{\ell\boldsymbol{\omega}})\qquad\text{for all }\ell\in\mathscr{T}_1;\\ \nu(K_{0^n(m-1)J\ell\boldsymbol{\omega}})&=\Big(\frac{p_0^{n-1}c_{0,J}}{m^{|J|+n+1}}\Big)^{d_s/2}\nu(K_{\ell\boldsymbol{\omega}})\qquad\text{for all }\ell\in\mathscr{T}_1;\\ \nu(K_{0^n(m-1)J})&=\sum_{\ell\in\mathscr{T}_1}\sum_{k=0}^{\infty}\sum_{I\in\mathscr{T}_0^k}\nu(K_{0^n(m-1)JI\ell});\\ \nu(K_{0^n})&=\sum_{\ell\in\mathscr{T}_1}\sum_{k=n}^{\infty}\nu(K_{0^k\ell})+\sum_{\ell\in\mathscr{T}_1}\sum_{k=n}^{\infty}\sum_{i=0}^{k-1}\sum_{J\in\mathscr{T}_0^{k-i-1}}\nu(K_{0^{i+1}(m-1)J\ell});\end{aligned}\tag{5.9}$$

Note that the value of $\nu(K_{0\boldsymbol{\omega}})$ is well-defined for all finite words $\boldsymbol{\omega}\in\mathscr{T}^*$. By symmetric of μ, for any finite word $\boldsymbol{\omega}\in\mathscr{T}^n$, where $n\ge 0$, there exists a unique $\boldsymbol{\omega}_*\in\mathscr{T}^n$ such that $\varphi(K_{(m-1)\boldsymbol{\omega}})=K_{0\boldsymbol{\omega}_*}$, where $\varphi(x)=-x+m$. Since $\mu|_{K_{(m-1)\boldsymbol{\omega}}}=\mu\circ\varphi^{-1}$ on $K_{(m-1)\boldsymbol{\omega}}$ It follows from Proposition 2.3 that

$$N(\lambda,-\Delta^D_{\mu|_{K_{(m-1)\boldsymbol{\omega}}}})=N(\lambda,-\Delta^D_{\mu|_{K_{0\boldsymbol{\omega}_*}}})\qquad\text{for all }\lambda>0\text{ and finite words }\boldsymbol{\omega}\in\mathscr{T}^*.$$

Third, we define

$$\nu(K_{(m-1)\boldsymbol{\omega}})=\nu(K_{1\boldsymbol{\omega}_*})\qquad\text{for all finite words }\boldsymbol{\omega}\in\mathscr{T}^*.$$

Final, define

$$\nu(K)=\sum_{i\in\mathscr{T}}\nu(K_i).$$

Consequently, we can define

$$\nu(A):=\inf\Big\{\sum_{\boldsymbol{\omega}\in\Lambda}\nu(K_{\boldsymbol{\omega}}):A\subseteq\bigcup_{\boldsymbol{\omega}\in\Lambda}K_{\boldsymbol{\omega}},\Lambda\subseteq\mathscr{T}^*\Big\}\tag{5.10}$$

for all Borel subsets $A\subseteq K$. We remark that ν is a well-defined measure on K (see Proposition 5.8 below for details). Moreover, μ and ν have the same symmetric.

Similarly, in order to prove Theorem 1.7(a), we divide $\mathscr{T}^*$ into two parts, namely,

$$\bigcup_{i\in\mathscr{T}_1}\big\{i\boldsymbol{\omega}:\boldsymbol{\omega}\in\mathscr{T}^n,n\ge 0\big\}\quad\text{ and }\quad\bigcup_{i\in\mathscr{T}_0}\big\{i\boldsymbol{\omega}:0\boldsymbol{\omega}\in\mathscr{T}^n,n\ge 0\big\}.$$

We begin with the first case.

Lemma 5.3. *Use the notation as above. Let ν be defined as in (5.10). Then*

$$N(\lambda,-\Delta^D_{\mu|_{K_{i\boldsymbol{\omega}}}})=\nu(K_{i\boldsymbol{\omega}})\lambda^{d_s/2}(1+o(1)),\qquad\text{as }\lambda\to\infty,\tag{5.11}$$

for all $i\in\mathscr{T}_1$ and finite words $\boldsymbol{\omega}\in\mathscr{T}^*$.

Proof. We use induction. Comparing (5.6) and (5.8), we can see that (5.11) holds for all $i \in \mathscr{T}_1$ and $\boldsymbol{\omega} \in \mathscr{T}^0$. We assume that as $\lambda \to \infty$

$$N(\lambda, -\Delta^D_{\mu|_{K_{i\boldsymbol{\omega}}}}) = \nu(K_{i\boldsymbol{\omega}})\lambda^{d_s/2}(1+o(1)) \qquad \text{for all } i \in \mathscr{T}_1 \text{ and } \boldsymbol{\omega} \in \bigcup_{k=0}^{n} \mathscr{T}^k, \tag{5.12}$$

where $n \geq 0$. Let $i \in \mathscr{T}_1$ and $\boldsymbol{\omega} \in \mathscr{T}^n$. Using Proposition 5.1(a), (5.8) and (5.12), we have for all $j \in \mathscr{T} \setminus \{i'\}$,

$$\begin{aligned} N(\lambda, -\Delta^D_{\mu|_{K_{ij\boldsymbol{\omega}}}}) &= N\Big(\frac{p(i,j)}{m}\lambda, -\Delta^D_{\mu|_{K_{h(i,j)\boldsymbol{\omega}}}}\Big) && \big(\text{by Proposition 5.1(a)}\big) \\ &= \Big(\frac{p(i,j)}{m}\Big)^{d_s/2}\nu(K_{h(i,j)\boldsymbol{\omega}})\lambda^{d_s/2}(1+o(1)) && \big(\text{by the fact } h(i,j) \in \mathscr{T}_1 \text{ and } (5.12)\big) \\ &= \nu(K_{ij\boldsymbol{\omega}})\lambda^{d_s/2}(1+o(1)) \quad \text{as } \lambda \to \infty, && \big(\text{by } (5.8)\big) \end{aligned} \tag{5.13}$$

where $p(i,j)$ and $h(i,j)$ are given as in Proposition 5.1. If $\boldsymbol{\omega} \in \mathscr{T}_0^n$, then (5.7) and (5.8) imply

$$N(\lambda, -\Delta^D_{\mu|_{K_{ii'\boldsymbol{\omega}}}}) = \nu(K_{ii'\boldsymbol{\omega}})\lambda^{d_s/2}(1+o(1)) \qquad \text{as } \lambda \to \infty. \tag{5.14}$$

On the other hand, if $\boldsymbol{\omega} \in \mathscr{T}^n \setminus \mathscr{T}_0^n$, then there exist finite words $J \in \mathscr{T}_0^*$, $\boldsymbol{\tau} \in \mathscr{T}^*$ and $j \in \mathscr{T}_1$ such that $\boldsymbol{\omega} = Jj\boldsymbol{\tau}$, and thus it follows from Proposition 5.1(b), (5.12) and (5.8) imply, as $\lambda \to \infty$

$$\begin{aligned} N(\lambda, -\Delta^D_{\mu|_{K_{ii'\boldsymbol{\omega}}}}) = N(\lambda, -\Delta^D_{\mu|_{K_{ii'J\ell\boldsymbol{\tau}}}}) &= N\Big(\frac{c_{i,J}}{m^{|J|+2}}\lambda, -\Delta^D_{\mu|_{K_{\ell\boldsymbol{\tau}}}}\Big) \\ &= \Big(\frac{c_{i,J}}{m^{|J|+2}}\Big)^{d_s/2}\nu(K_{\ell\boldsymbol{\tau}})\lambda^{d_s/2}(1+o(1)) \\ &= \nu(K_{ii'J\ell\boldsymbol{\tau}})\lambda^{d_s/2}(1+o(1)) = \nu(K_{ii'\boldsymbol{\omega}})\lambda^{d_s/2}(1+o(1)), \end{aligned}$$

which, together with (5.13) and (5.14), yields $N(\lambda, -\Delta^D_{\mu|_{K_{i\boldsymbol{\omega}}}}) = \nu(K_{i\boldsymbol{\omega}})\lambda^{d_s/2}(1+o(1))$ as $\lambda \to \infty$ for all words $\boldsymbol{\omega} \in \mathscr{T}^{n+1}$. This proves the Lemma by induction. □

We now turn to consider the case: $\bigcup_{i\in\mathscr{T}_0}\{i\boldsymbol{\omega} : \boldsymbol{\omega} \in \mathscr{T}^n, n \geq 0\}$. To end this, we first develop a lemma, which shows an asymptotic behavior of eigenvalue counting function on K_0 and K_{m-1}.

Proposition 5.4. *Use the notation as above. Let $\ell \in \mathscr{T}_1$. Then for any $n \geq 1$ and $\lambda > 0$,*

$$\begin{aligned} \sum_{k=0}^{n-1}\sum_{J\in\mathscr{T}_0^k} N(\lambda, -\Delta^D_{\mu|_{K_{0J\ell}}}) &= \sum_{k=0}^{n-1} N\Big(\big(\frac{p_0}{m}\big)^{k+1}\lambda, -\Delta^D_{\mu|_{K_\ell}}\Big) \\ &\quad + \sum_{k=1}^{n-1}\sum_{i=0}^{k-1}\sum_{J\in\mathscr{T}_0^{k-i-1}} N\Big(\frac{p_0^i c_{0,J}}{m^{k+1}}\lambda, -\Delta^D_{\mu|_{K_\ell}}\Big). \end{aligned}$$

Proof. Combine Proposition 5.2 and the fact $\mathscr{T}_0^k = \{0^i(m-1)J : 0 \leq i \leq k-1, J \in \mathscr{T}_0^{k-i-1}\} \bigcup \{0^k\}$ for all $k \geq 1$. □

Lemma 5.5. *Use the notation as above. Let ν be given by* (5.10)*. Then as $\lambda \to \infty$,*

$$N(\lambda, -\Delta^D_{\mu|_{K_i}}) = \nu(K_i)\lambda^{d_s/2}(1+o(1)) \qquad \textit{for all } i \in \mathscr{T}_0.$$

Proof. By symmetry of the measures μ, ν and $\{T_i\}_{i=0}^{m-1}$, it suffices to show that

$$N(\lambda, -\Delta^D_{\mu|_{K_0}}) = \nu(K_0)\lambda^{d_s/2}(1+o(1)) \qquad \text{as } \lambda \to \infty.$$

For any $n \geq 1$, we have

$$\mathbf{P}_n := \{K_{0J\ell} : J \in \mathscr{T}_0^k, 0 \leq k \leq n-1, \ell \in \mathscr{T}_1\} \bigcup \{K_{0J} : J \in \mathscr{T}_0^n\}$$

is a partition of K_0. Thus Propositions 2.5 and 5.4 imply

$$\begin{aligned}
&N(\lambda, -\Delta^D_{\mu|_{K_0}}) \\
&= \sum_{\ell \in \mathscr{T}_1} \sum_{k=0}^{n-1} \sum_{J \in \mathscr{T}_0^k} N(\lambda, -\Delta^D_{\mu|_{K_{0J\ell}}}) + \sum_{J \in \mathscr{T}_0^n} N(\lambda, -\Delta^D_{\mu|_{K_{0J}}}) + \epsilon(\mathbf{P}_n, \lambda) \\
&= \sum_{\ell \in \mathscr{T}_1} \sum_{k=0}^{n-1} N\Big(\big(\frac{p_0}{m}\big)^{k+1}\lambda, -\Delta^D_{\mu|_{K_\ell}}\Big) + \sum_{\ell \in \mathscr{T}_1} \sum_{k=1}^{n-1} \sum_{i=0}^{k-1} \sum_{J \in \mathscr{T}_0^{k-i-1}} N\Big(\frac{p_0^i c_{0,J}}{m^{k+1}}\lambda, -\Delta^D_{\mu|_{K_\ell}}\Big) + \boldsymbol{z}(n, \lambda) \\
&= \sum_{\ell \in \mathscr{T}_1} \sum_{k=0}^{\infty} N\Big(\big(\frac{p_0}{m}\big)^{k+1}\lambda, -\Delta^D_{\mu|_{K_\ell}}\Big) + \sum_{\ell \in \mathscr{T}_1} \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} \sum_{J \in \mathscr{T}_0^{k-i-1}} N\Big(\frac{p_0^i c_{0,J}}{m^{k+1}}\lambda, -\Delta^D_{\mu|_{K_\ell}}\Big) \\
&\quad - \sum_{\ell \in \mathscr{T}_1} \sum_{k=n}^{\infty} N\Big(\big(\frac{p_0}{m}\big)^{k+1}\lambda, -\Delta^D_{\mu|_{K_\ell}}\Big) - \sum_{\ell \in \mathscr{T}_1} \sum_{k=n}^{\infty} \sum_{i=0}^{k-1} \sum_{J \in \mathscr{T}_0^{k-i-1}} N\Big(\frac{p_0^i c_{0,J}}{m^{k+1}}\lambda, -\Delta^D_{\mu|_{K_\ell}}\Big) + \boldsymbol{z}(n, \lambda)
\end{aligned} \tag{5.15}$$

for all $n \geq 1$ and $\lambda > 0$, where $0 \leq \epsilon(\mathbf{P}_n, \lambda) \leq 2\#\mathbf{P}_n$ and

$$\boldsymbol{z}(n, \lambda) := \sum_{J \in \mathscr{T}_0^n} N(\lambda, -\Delta^D_{\mu|_{K_{0J}}}) + \epsilon(\mathbf{P}_n, \lambda).$$

Since the first eigenvalue of $-\Delta^D_{\mu|_{K_\ell}}$ is positive for all $\ell \in \mathscr{T}_1$, there exists $\lambda_0 > 0$ such that $N(\lambda, -\Delta^D_{\mu|_{K_\ell}})$ for all $\lambda < \lambda_0$ and all $\ell \in \mathscr{T}_1$. For $\lambda > 0$, n_λ is the smallest integer such that

$$\lambda \cdot \max\Big\{\frac{p_0^i c_{0,J}}{m^{n_\lambda+1}} : 0 \leq i \leq n_\lambda - 1, J \in \mathscr{T}_0^{n_\lambda - i - 1}\Big\} < \lambda_0.$$

Letting $n = n_\lambda$ in (5.15). Then the third and fourth summations in (5.15) vanish, and thus we get

$$\begin{aligned}
N(\lambda, -\Delta^D_{\mu|_K}) &= \sum_{\ell \in \mathscr{T}_1} \sum_{k=0}^{\infty} N\Big(\big(\frac{p_0}{m}\big)^{k+1}\lambda, -\Delta^D_{\mu|_{K_\ell}}\Big) \\
&\quad + \sum_{\ell \in \mathscr{T}_1} \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} \sum_{J \in \mathscr{T}_0^{k-i-1}} N\Big(\frac{p_0^i c_{0,J}}{m^{k+1}}\lambda, -\Delta^D_{\mu|_{K_\ell}}\Big) + \boldsymbol{z}(n_\lambda, \lambda).
\end{aligned} \tag{5.16}$$

Similar to that proof of [32, Section 6], we can check that $\boldsymbol{z}(n_\lambda, \lambda) = o(\lambda^{d_s/2})$ as $\lambda \to \infty$. Thus by applying Lemma 5.3 and (5.16), we have as $\lambda \to \infty$,

$$\begin{aligned} N(\lambda,-\Delta^D_{\mu|_{K_0}}) &= \sum_{\ell\in\mathscr{T}_1}\Big(\sum_{k=0}^{\infty}\Big(\frac{p_0}{m}\Big)^{(k+1)d_s/2}+\sum_{k=1}^{\infty}\sum_{i=0}^{k-1}\sum_{J\in\mathscr{T}_0^{k-i-1}}\Big(\frac{p_0^i c_{0,J}}{m^{k+1}}\Big)^{d_s/2}\Big)\nu(K_\ell)\lambda^{d_s/2}(1+o(1)) \\ &= \sum_{\ell\in\mathscr{T}_1}\Big(\sum_{k=0}^{\infty}\nu(K_{0^{k+1}\ell})+\sum_{k=1}^{\infty}\sum_{i=0}^{k-1}\sum_{J\in\mathscr{T}_0^{k-i-1}}\nu(K_{0^{i+1}(m-1)J\ell})\Big)\lambda^{d_s/2}(1+o(1)) \\ &= \nu(K_0)\lambda^{d_s/2}(1+o(1)), \end{aligned}$$

which completes the proof. □

Let $J\in\mathscr{T}_0^*$ be a finite word. Similar to that proof of Lemma 5.5, we can show that

$$\begin{aligned} N(\lambda,-\Delta^D_{\mu|_{K_{0(m-1)J}}}) &= \sum_{\ell\in\mathscr{T}_1}\sum_{k=0}^{\infty}\sum_{I\in\mathscr{T}_0^k}N\Big(\frac{c_{0,JI}}{m^{|JI|+2}}\lambda,-\Delta^D_{\mu|_{K_\ell}}\Big)+o(\lambda^{d_s/2}) \\ &= \sum_{\ell\in\mathscr{T}_1}\sum_{k=0}^{\infty}\sum_{I\in\mathscr{T}_0^k}\Big(\frac{c_{0,JI}}{m^{|JI|+2}}\Big)^{d_s/2}\nu(K_\ell)\lambda^{d_s/2}(1+o(1)) \\ &= \sum_{\ell\in\mathscr{T}_1}\sum_{k=0}^{\infty}\sum_{I\in\mathscr{T}_0^k}\nu(K_{0(m-1)JI\ell})\lambda^{d_s/2}(1+o(1)) \\ &= \nu(K_{0(m-1)J})\lambda^{d_s/2}(1+o(1)) \qquad \text{as } \lambda\to\infty, \end{aligned} \tag{5.17}$$

where equation (5.9) has been used in second and third equalities.

Lemma 5.6. *Use the notation as above. Let ν be given by* (5.10)*. Then*

$$N(\lambda,-\Delta^D_{\mu|_K}) = \nu(K)\lambda^{d_s/2}(1+o(1)) \qquad \text{as } \lambda\to\infty.$$

Proof. Since $\mathbf{P}:=\{K_i: i\in\mathscr{T}\}$ is a partition of K, Propositions 2.5 implies

$$N(\lambda,-\Delta^D_{\mu|_K}) = \sum_{\ell\in\mathscr{T}} N(\lambda,-\Delta^D_{\mu|_{K_\ell}})+\epsilon(\mathbf{P},\lambda),$$

where $0\le\epsilon(\mathbf{P},\lambda)\le 2m$. Thus the assertion follows from Lemmas 5.3 and 5.5. □

Lemma 5.7. *Use the notation as above. Let ν be given by* (5.10)*. Then as $\lambda\to\infty$, we have*

$$N(\lambda,-\Delta^D_{\mu|_{K_{i\boldsymbol{\omega}}}}) = \nu(K_{i\boldsymbol{\omega}})\lambda^{d_s/2}(1+o(1)) \qquad \textit{for all } i\in\mathscr{T}_0 \textit{ and finite words } \boldsymbol{\omega}\in\mathscr{T}^*.$$

Proof. Similarly, it suffices to show that as $\lambda\to\infty$,

$$N(\lambda,-\Delta^D_{\mu|_{K_{0\boldsymbol{\omega}}}}) = \nu(K_{0\boldsymbol{\omega}})\lambda^{d_s/2}(1+o(1)) \tag{5.18}$$

for all finite words $\boldsymbol{\omega}\in\mathscr{T}^*$. Lemma 5.5 tells us that (5.18) holds for all $\boldsymbol{\omega}\in\mathscr{T}^0$. Assume that as $\lambda\to\infty$, we have

$$N(\lambda,-\Delta^D_{\mu|_{K_{0\boldsymbol{\omega}}}}) = \nu(K_{0\boldsymbol{\omega}})\lambda^{d_s/2}(1+o(1)) \qquad \text{for all } \boldsymbol{\omega}\in\bigcup_{k=0}^{n}\mathscr{T}^k, \tag{5.19}$$

where $n \geq 0$. Let $\boldsymbol{\omega} \in \mathscr{T}^n$. Proposition 5.2, Lemma 5.3 and (5.19) give as $\lambda \to \infty$

$$\begin{aligned} N(\lambda, -\Delta^D_{\mu|_{K_{0j\boldsymbol{\omega}}}}) &= N\Big(\frac{p_0}{m}\lambda, -\Delta^D_{\mu|_{K_{j\boldsymbol{\omega}}}}\Big) = \Big(\frac{p_0}{m}\Big)^{d_s/2}\nu(K_{j\boldsymbol{\omega}})\lambda^{d_s/2}(1+o(1)) \\ &= \nu(K_{0j\boldsymbol{\omega}})\lambda^{d_s/2}(1+o(1)) \qquad \text{for all } 0 \leq j \leq m-2, \end{aligned} \tag{5.20}$$

where equation (5.9) was used in the last equality. If $\boldsymbol{\omega} \in \mathscr{T}_0^n$, then it follows from (5.17) that

$$N(\lambda, -\Delta^D_{\mu|_{K_{0(m-1)\boldsymbol{\omega}}}}) = \nu(K_{0(m-1)\boldsymbol{\omega}})\lambda^{d_s/2}(1+o(1)) \qquad \text{as } \lambda \to \infty. \tag{5.21}$$

On the other hand, if $\boldsymbol{\omega} \in \mathscr{T}^n \setminus \mathscr{T}_0^n$, then there exist finite words $J \in \mathscr{T}_0^*$, $\boldsymbol{\tau} \in \mathscr{T}^*$ and $j \in \mathscr{T}_1$ such that $\boldsymbol{\omega} = Jj\boldsymbol{\tau}$. Thus Proposition 5.2 and Lemmas 5.3 imply that for any $\boldsymbol{\omega} \in \mathscr{T}^n \setminus \mathscr{T}_0^n$,

$$\begin{aligned} N(\lambda, -\Delta^D_{\mu|_{K_{0(m-1)\boldsymbol{\omega}}}}) &= N\big(\lambda, -\Delta^D_{\mu|_{K_{0(m-1)Jj\boldsymbol{\tau}}}}\big) = N\Big(\frac{c_{0,J}}{m^{|J|+2}}\lambda, -\Delta^D_{\mu|_{K_{j\boldsymbol{\tau}}}}\Big) \\ &= \Big(\frac{c_{0,J}}{m^{|J|+2}}\Big)^{d_s/2}\nu(K_{j\boldsymbol{\tau}})\lambda^{d_s/2}(1+o(1)) \\ &= \nu(K_{0(m-1)\boldsymbol{\omega}})\lambda^{d_s/2}(1+o(1)) \qquad \text{as } \lambda \to \infty, \end{aligned} \tag{5.22}$$

where equation (5.9) was used in the last equality. In view of (5.20), (5.21) and (5.22), we have (5.18) holds for all $\boldsymbol{\omega} \in \mathscr{T}^{n+1}$. By induction, the lemma holds. □

Using Proposition 2.5, Lemmas 5.3, 5.6 and 5.7, we can deduce that ν is additive (see Proposition 5.8(a) below).

Proposition 5.8. *Use the notation as above. Let ν be given by* (5.10)*. Then*

(a) $\nu(K_{\boldsymbol{\omega}}) = \sum_{\ell \in \mathscr{T}} \nu(K_{\boldsymbol{\omega}\ell})$ *for all finite words* $\boldsymbol{\omega} \in \mathscr{T}^*$.
(b) $\max\big\{\nu(K_{\boldsymbol{\omega}}) : \boldsymbol{\omega} \in \mathscr{T}^n\big\} \to 0$ *as* $n \to \infty$.

Consequently, ν is a well-defined measure on K.

Proof. (a) Let $\boldsymbol{\omega} \in \mathscr{T}^*$ be a finite word. Since $\mathbf{P} := \{K_{\boldsymbol{\omega}\ell} : \ell \in \mathscr{T}\}$ is a partition of $K_{\boldsymbol{\omega}}$, Proposition 2.5 implies

$$N(\lambda, -\Delta^D_{\mu|_{K_{\boldsymbol{\omega}}}}) = \sum_{\ell \in \mathscr{T}} N(\lambda, -\Delta^D_{\mu|_{K_{\boldsymbol{\omega}\ell}}}) + \epsilon(\mathbf{P}, \lambda)$$

for all $\lambda \geq 0$, where $0 \leq \epsilon(\mathbf{P}, \lambda) \leq 2(m-1)$. Letting $\lambda \to \infty$. Then Lemmas 5.3,5.6 and 5.7 imply $\nu(K_{\boldsymbol{\omega}}) = \sum_{\ell \in \mathscr{T}} \nu(K_{\boldsymbol{\omega}\ell})$, which completes the proof.

(b) By the definition of $\nu(K_{\boldsymbol{\omega}})$, it suffices to show that

$$\max\Big\{\nu(K_{ii'J}) : i \in \mathscr{T}_1, J \in \mathscr{T}_0^n\Big\} \to 0, \quad \text{as } n \to \infty. \tag{5.23}$$

It follows from (5.5) that the series

$$\sum_{i \in \mathscr{T}_1} \sum_{k=0}^{\infty} \sum_{J \in \mathscr{T}_0^k} \Big(\frac{c_{i,J}}{m^{|J|+2}}\Big)^{d_s/2}$$

is convergent, which implies, using (5.8) again, that

$$\begin{aligned}\sum_{i\in\mathscr{T}_1}\sum_{J\in\mathscr{T}_0^n}\nu(K_{ii'J}) &= \sum_{i\in\mathscr{T}_1}\sum_{J\in\mathscr{T}_0^n}\Big(\sum_{\ell\in\mathscr{T}_1}\sum_{k=0}^{\infty}\sum_{I\in\mathscr{T}_0^k}\Big(\frac{c_{i,JI}}{m^{|JI|+2}}\Big)^{d_s/2}\nu(K_\ell)\Big)\\ &= \sum_{i\in\mathscr{T}_1}\sum_{\ell\in\mathscr{T}_1}\sum_{k=n}^{\infty}\sum_{I\in\mathscr{T}_0^k}\Big(\frac{c_{i,I}}{m^{|I|+2}}\Big)^{d_s/2}\nu(K_\ell)\\ &\le C\sum_{i\in\mathscr{T}_1}\sum_{k=n}^{\infty}\sum_{I\in\mathscr{T}_0^k}\Big(\frac{c_{i,I}}{m^{|I|+2}}\Big)^{d_s/2}\to 0 \quad \text{as } n\to\infty,\end{aligned}$$

where $C = m\cdot\max\{C_i, i\in\mathscr{T}_1\}$, and $(C_i)_{i\in\mathscr{T}_1}$ comes from (5.6). Hence, (5.23) holds. This proves part (b). □

Now we prove Theorem 1.7.

Proof of Theorem 1.7. (a) Combine Lemmas 5.3, 5.6, 5.7 and Proposition 5.8.

(b) Let $\mathbf{P}_k = \{K_{\boldsymbol{\omega}} : \boldsymbol{\omega}\in\mathscr{T}^k\}$ for $k\ge 0$ and ν be defined as in (5.10). Using Proposition 5.8(b), we can deduce that $(\mathbf{P}_k)_{k\ge 0}$ is refining with respect to ν. Combining it with part (a) and Remark 2.4, we have Assumption 1.2 holds. Thus the assertion follows from Theorem 1.3(a). □

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