CY PRINCIPAL BUNDLES OVER COMPACT KÄHLER MANIFOLDS

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Abstract. A CY bundle on a compact complex manifold X was a crucial ingredient in constructing differential systems for period integrals in [LY], by lifting line bundles from the base X to the total space. A question was therefore raised as to whether there exists such a bundle that supports the liftings of *all* line bundles from X, simultaneously. This was a key step for giving a uniform construction of differential systems for arbitrary complete intersections in X. In this paper, we answer the existence question in the affirmative if X is assumed to be Kähler, and also in general if the Picard group of X is assumed to be free. Furthermore, we prove a rigidity property of CY bundles if the principal group is an algebraic torus, showing that such a CY bundle is essentially determined by its character map.

Contents

1. Introduction

1.1. **Background.** Let G, H be complex Lie groups and X a compact complex manifold. A G-equivariant principal H-bundle, denoted by $M \equiv (G, H -$

$M\xrightarrow{\pi}X)$, is a holomorphic principal H-bundle M over X equipped with an action

$$G \times H \times M \to M,\ (g,h,m) \mapsto gmh^{-1}.$$

Since the H-action is assumed to be principal, the projection π induces an isomorphism $M/H \simeq X$.

Given such an M, it is easy to show that there is an equivalence of categories

$$\{G\text{-equiv. vector bundles on } X\} \leftrightarrow \{G \times H\text{-equiv. vector bundles on } M\}.$$

Restricting this to line bundles, we get a natural isomorphism

$$\mathrm{Pic}_G(X) \simeq \mathrm{Pic}_{G\times H}(M).$$

Let $\chi \in \mathrm{Hom}(H, \mathbb{C}^\times) \equiv \widehat{H}$ be a holomorphic character of H, and $\mathbb{C}_\chi$ be the corresponding 1-dimensional representation. Then $G \times H$ acts on $M \times \mathbb{C}_\chi$ as a G-equivariantly trivial bundle which is not necessarily H-equivariantly trivial. Composing with the Picard group isomorphism above one gets a canonical homomorphism

$$\begin{array}{rcccl} \lambda_M : \widehat{H} & \to & \mathrm{Pic}_{G\times H}(M) & \to & \mathrm{Pic}_G(X) \\ \chi & \mapsto & M \times \mathbb{C}_\chi & \mapsto & L_\chi := (M \times \mathbb{C}_\chi)/H. \end{array}$$

We call this *the character map* of M.

This map allows us to describe the line bundles in the image of λ_M together with their sections purely in terms of 1-dimensional representations of H. For example, one can show that there is a canonical G-equivariant isomorphism [LY]

$$\Gamma(X, L_\chi) \simeq \mathcal{O}_\chi(M) := \{f \in \mathcal{O}(M) | f(mh^{-1}) = \chi(h) f(m)\, \forall m \in M\}.$$

Definition 1.1. [LY] *We say that M is a CY bundle if it admits a CY structure $(\mathbb{C}\omega, \chi)$. Namely, $\chi \in \widehat{H}$ is a holomorphic character of H, and ω is a G-invariant nonvanishing holomorphic top form on M such that*

$$\Gamma_h \omega = \chi(h)\omega,\ h \in H.$$

A prototype example of this definition is given by the following example due to Calabi.

Example 1.2. ([Ca], 1979) Let $M := K_X^\times$ be the complement of the zero section of K_X, and let $\omega = dz_w \wedge dw_1 \wedge \cdots \wedge dw_d$, where w is a local coordinate chart and z_w is the coordinate induced by w along the local fibers of K_X. Then ω is a globally defined CY structure on the bundle $(\mathrm{Aut}\ X, \mathbb{C}^\times - M \to X)$ with $\chi = \mathrm{id}_{\mathbb{C}^\times}$.

Let us mention a number of important applications of this notion. First, we note that CY structures, if exist, can be classified by a coset of the kernel of the character map.

Theorem 1.3 (Classification of CY structures [LY]). *Given a principal H-bundle over a compact complex manifold X, there is a bijection*

$$\{CY \text{ structures on } M\} \longleftrightarrow \lambda_M^{-1}([K_X]), \ (\mathbb{C}\omega, \chi) \leftrightarrow \chi\chi_{\mathfrak{h}}$$

where $\chi_{\mathfrak{h}}$ is the 1-dimensional representation $\wedge^{\dim \mathfrak{h}}\mathfrak{h}$ induced by the adjoint representation of H.

Next, if such a structure exists, one gets a bundle version of the adjunction formula, and it allows us to describe K_X in purely functional terms.

Theorem 1.4 (Adjunction for bundle [LY]). *Let $(\mathbb{C}\omega, \chi)$ be a CY structure on M. Then there is a canonical isomorphism*

$$K_X \simeq L_{\chi\chi_{\mathfrak{h}}}.$$

As a consequence, we have the following corollary:

Corollary 1.5. *There is a canonical embedding of the pluri-(anti)canonical ring of X as a subring of the ring of holomorphic functions $\mathcal{O}(M)$.*

Let us mention two other important recent applications of the theory of CY bundles.

(1) One can use the functional description of K_X above to give an explicit formula for the family version of the Poincaré residue map for complete intersections. This is a powerful tool for studying Picard-Fuchs systems for period integrals. This technique can reconstruct virtually all known PF systems, plus a large class of new ones (tautological systems). See recent papers [LSY],[LY] and [HLZ].

(2) CY structures are also very useful for studying the D-modules associated with the PF systems. A CY structure allows us to describe the D-modules in terms of Lie algebra homology. In some important cases, this can be recast as the de Rham cohomology of an affine algebraic variety via the Riemann-Hilbert correspondence. See recent papers [BHLSY] and [HLZ].

1.2. **Existence and uniqueness problems for CY bundles.** The aim of this paper is to study the existence and uniqueness questions for CY bundles. We are particularly interested in those CY bundles whose character map λ_M is surjective. In this case, *all line bundles on X can be simultaneously realized by H-characters.*

This was an important open question raised in [LY]. Such a structure would fill a crucial step in the construction of tautological systems for period integrals of arbitrary complete intersections in X, as mentioned in (1)-(2) before. In addition, it provides a uniform treatment for all line bundles on X.

We begin with the special case $H \simeq (\mathbb{C}^\times)^p$ for a positive integer p, so that $\chi_{\mathfrak{h}}$ is trivial. For simplicity, we will also assume that $G = 1$. Every result we discuss has a G-equivariant version, where G is a suitable lifting of any closed subgroup of Aut X.

The following result says that a CY bundle is uniquely determined by its character map.

Theorem 1.6 (Rigidity of CY bundle)**.** *Let $M_i \equiv (H - M_i \to X)$, $i = 1, 2$, be CY bundles such that the following diagram commutes*

$$\begin{array}{ccc} \widehat{H} & \xrightarrow{\lambda_{M_1}} & Pic(X) \\ \xi \downarrow \simeq & \nearrow_{\lambda_{M_2}} & \\ \widehat{H} & & \end{array}$$

for some $\xi \in Aut\ \widehat{H}$. Then there is an isomorphism $M_1 \simeq M_2$, canonical up to a twist by the induced automorphism $\xi^\vee \in Aut\ H$.

This result will be proved in Section 2.2.

We now consider the existence question. The first crucial step is the following criterion, which can also be seen as a consequence of Theorem 1.3.

Theorem 1.7 (Obstruction criterion[LY])**.** *An H-principal bundle M admits a CY structure iff K_X is in the image of the character map λ_M. (By adjunction for bundles, we necessarily get $K_X = \lambda_M(\chi\chi_{\mathfrak{h}}) = L_{\chi\chi_{\mathfrak{h}}}$.)*

This gives a classification for rank 1 CY bundles.

Corollary 1.8. *The bundle $M \equiv (\mathbb{C}^\times - M \to X)$ admits a CY structure iff M is the complement of zero section of a line bundle L which is a root of K_X, i.e. $K_X \simeq kL$ for some integer k.*

Example 1.9. Take $X = \mathbb{P}^d$, $H = \mathbb{C}^\times$. Then rank 1 CY bundles on X are exactly those of the form $M' \simeq \mathcal{O}(k)^\times$ for some $k|(d+1)$. Its character map is then

$$\lambda_{M'} : \widehat{H} = \mathbb{Z} \to \mathrm{Pic}(X) \simeq \mathbb{Z},\ 1 \mapsto k.$$

Thus $\lambda_{M'}$ is isomorphic iff $k = \pm 1$. Let $M_1 := \mathcal{O}(-1)^\times, M_2 := \mathcal{O}(1)^\times$, then $\xi : \widehat{H} \to \widehat{H}, 1 \mapsto -1$ is an isomorphism and $\lambda_{M_1} = \lambda_{M_2} \circ \xi$. So, by the rigidity theorem above, we have $\mathcal{O}(-1)^\times \simeq \mathcal{O}(1)^\times$ up to a twist by $\xi^\vee : H \to H,\ h \mapsto h^{-1}$.

We can describe this isomorphism explicitly as follows. Let $M := \mathbb{C}^{d+1} \setminus \{0\}$, since $\mathcal{O}(-1) \subset \mathbb{P}^d \times \mathbb{C}^{d+1}$, we have an isomorphism $\alpha : M \to \mathcal{O}(-1)^\times, m \mapsto ([m], m)$. Define a linear function $m^{-1} : \mathcal{O}(-1)_{[m]} \to \mathbb{C}, ([m], cm) \mapsto c$. Then $m^{-1} \in (\mathcal{O}(-1)_{[m]})^\vee \simeq \mathcal{O}(1)_{[m]}$. Let $\beta : M \to \mathcal{O}(1)^\times, m \mapsto ([m], m^{-1})$. Then β is an isomorphism as well. Thus we can conclude that $\mathcal{O}(-1)^\times \simeq \mathcal{O}(1)^\times$, but the H-actions on them are different. H acts on M by $h \cdot m = mh^{-1}$. H acts on M_1 by $h \cdot ([m], m) = ([m], mh^{-1})$, and H acts on M_2 by $h \cdot ([m], m^{-1}) = ([m], m^{-1}h)$. With these actions M_1 and M_2 are isomorphic to M as principal H-bundles. The H-actions on M_1 and M_2 are twisted by $\xi^\vee$.

This example important generalization to any smooth toric variety. As an application, we can give a simple characterization of a special toric variety constructed by Audin and Cox in the early 90's. Namely, let $T = (\mathbb{C}^\times)^d$ and X be a smooth complete toric variety with respect to the group T.

Theorem 1.10 (Audin-Cox variety)**.** *Let t be the number of T-orbits in X. Then there is a canonical $(\mathbb{C}^\times)^t$-invariant Zariski open subset $M \subset \mathbb{C}^t$ and a $(t-d)$-dimensional closed algebraic subgroup $H \subset (\mathbb{C}^\times)^t$ such that the geometric quotient M/H is isomorphic to X.*

It can be shown that the character map λ_M for the Audin-Cox variety is isomorphic. In particular M is a CY bundle over X. The $\mathbb{P}^d$ example above is a special case of this construction. Now as a consequence of our rigidity theorem, we have the following characterization of M:

Corollary 1.11. *The Audin-Cox variety M is the unique CY bundle over X with the property that λ_M is a group isomorphism.*

Next, we will see that this allows us to reconstruct the same space M in many different ways as an algebraic variety. The characterization also shows that a CY bundle over a general complex manifold can be viewed as a generalization of the Audin-Cox construction for toric variety.

Here is one of our main results.

Theorem 1.12. *Let X be a compact complex manifold. If Pic(X) is free, then X admits a unique CY H-bundle whose character map is isomorphic. If X is Kähler, then it admits a CY bundle whose character map is onto.*

These results will be proved in Section 2 and Section 4.

When Pic(X) is free, the proof uses the obstruction criterion of [LY], and the rigidity theorem above. When Pic(X) is not free, then by using Kähler condition, we can lift the construction to the universal cover $\tilde{X}$ where Pic($\tilde{X}$) is finitely generated, and then apply a construction similar to the first case.

Moreover, making use of the Remmert-Morimoto decomposition for connected abelian complex Lie groups, we have:

Theorem 1.13. *If X is Kähler and H is a sufficiently large connected abelian group, then there exists a CY H-bundle whose character map is onto.*

This result will be proved in Section 5.

2. Existence and uniqueness of $(\mathbb{C}^\times)^p$-principal bundles

2.1. **Existence of $(\mathbb{C}^\times)^p$-principal bundles.** Recall that a holomorphic principal H-bundle, where H denotes a complex Lie group, is a holomorphic bundle $\pi : M \to X$ equipped with a holomorphic right action $M \times H \to M$, such that H acts on each fiber of π freely and transitively. We denote the principal bundle as $H - M \to X$. If we have two principal bundles $H_1 - M_1 \to X$ and $H_2 - M_2 \to X$, then its Whitney sum $M_1 \oplus M_2$ over X is a principal $(H_1 \times H_2)$-bundle over X.

Let X be a complex manifold, and L a holomorphic line bundle over X. Let $L^\times$ denote the complement of the zero section in L. Then we have a natural $\mathbb{C}^\times$-action on $L^\times$:

$$\mathbb{C}^\times \times L^\times \to L^\times, (h, l) \mapsto lh^{-1}$$

where $h \in \mathbb{C}^\times, l \in L^\times$.

It is clear that this action preserves each fiber of $L^\times \to X$ and it is free and transitive, which means that the projection $L^\times \to X$ defines a principal $\mathbb{C}^\times$-bundle over X.

Now if we have two holomorphic line bundles L_1, L_2 on X, then $L_1^\times \oplus L_2^\times \subset L_1 \oplus L_2$ is a principal $(\mathbb{C}^\times)^2$-bundle over X with the $(\mathbb{C}^\times)^2$-action given by

$$(h_1, h_2) \cdot (l_1, l_2) = (l_1 h_1^{-1}, l_2 h_2^{-1}).$$

We can represent any given $(\mathbb{C}^\times)^2$-character $\chi \in \widehat{(\mathbb{C}^\times)^2} = \mathbb{Z}^2$ uniquely as a product

$$\chi_k \chi_l : (\mathbb{C}^\times)^2 \to \mathbb{C}^\times, (h_1, h_2) \mapsto h_1^k h_2^l$$

for some $k, l \in \mathbb{Z}$, where $\chi_k \in \widehat{\mathbb{C}^\times}$.

Proposition 2.1. *Given holomorphic line bundles L_1, L_2 on X, we have an isomorphism of holomorphic line bundles:*

$$(L_1^\times \oplus L_2^\times) \times_{(\mathbb{C}^\times)^2} \mathbb{C}_{\chi_k \chi_l} \simeq kL_1 + lL_2.$$

Proof. Define a map

$$\begin{aligned} \rho : (L_1^\times \oplus L_2^\times) \times \mathbb{C}_{\chi_k \chi_l} &\to kL_1 + lL_2, \\ ((l_1, l_2), c) &\mapsto c l_1^{\otimes k} \otimes l_2^{\otimes l}. \end{aligned}$$

Since $(\mathbb{C}^\times)^2$ acts on $(L_1^\times \oplus L_2^\times) \times \mathbb{C}_{\chi_k \chi_l}$ by:

$$(h_1, h_2) \cdot ((l_1, l_2), c) = ((l_1 h_1^{-1}, l_2 h_2^{-1}), c h_1^k h_2^l),$$

we have

$$\rho((h_1, h_2) \cdot ((l_1, l_2), c)) = c h_1^k h_2^l (l_1 h_1^{-1})^{\otimes k} \otimes (l_2 h_2^{-1})^{\otimes l} = \rho(((l_1, l_2), c))$$

for any $(h_1, h_2) \in (\mathbb{C}^\times)^2$. It follows that ρ descends to

$$\tilde{\rho} : (L_1^\times \oplus L_2^\times) \times_{(\mathbb{C}^\times)^2} \mathbb{C}_{\chi_k \chi_l} \to kL_1 + lL_2.$$

It is clear that $\tilde{\rho}$ induces a linear isomorphism on fibers and it commutes with quotients to X, it is an isomorphism. □

Corollary 2.2. *Let L_1, L_2 be line bundles and $k, l \in \mathbb{Z}$ such that $K_X \simeq kL_1 + lL_2$. Then $L_1^\times \oplus L_2^\times$ admits a CY $(\mathbb{C}^\times)^2$-bundle structure with character $\chi_k \chi_l$.*

Proof. Put $H := (\mathbb{C}^\times)^2, M := L_1^\times \oplus L_2^\times, \chi := \chi_k \chi_l$. Proposition 2.1 tells us that

$$K_X \simeq kL_1 + lL_2 \simeq M \times_H \mathbb{C}_\chi = \lambda_M(\chi).$$

By the obstruction criterion, i.e. Theorem 1.7, $L_1^\times \oplus L_2^\times$ admits a CY H-bundle structure $(\mathbb{C}\omega_M, \chi_M)$. Moreover, since the H-character $\chi_{\mathfrak{h}}$ is trivial, $\chi_M = \chi \chi_{\mathfrak{h}}^{-1} = \chi$ by Theorem 1.4. □

Corollary 2.3. *For every line bundle L, there exists a CY bundle $(\mathbb{C}^\times)^2 - M \to X$ and a character $\chi \in \widehat{(\mathbb{C}^\times)^2}$ such that $L \simeq M \times_{(\mathbb{C}^\times)^2} \mathbb{C}_\chi$.*

Proof. Let $M := L^\times \oplus K_X^\times$. Since $K_X = 0 \cdot L + 1 \cdot K_X$, Corollary 2.2 tells us that M is a CY bundle with character $\chi_0\chi_1$. Now for $\chi = \chi_1\chi_0$, we have $M \times_{(\mathbb{C}^\times)^2} \mathbb{C}_\chi \simeq 1 \cdot L + 0 \cdot K_X = L$.

□

Example 2.4. Since $K_{\mathbb{P}^n} = \mathcal{O}(-n-1)$, $K_{\mathbb{P}^n} \simeq k\mathcal{O}(-1) + l\mathcal{O}(-1)$ whenever $k+l = n+1$. Corollary 2.2 shows that $M := \mathcal{O}(-1)^\times \oplus \mathcal{O}(-1)^\times$ is a CY bundle over $\mathbb{P}^n$. We now give an explicit description of the CY structure $(\mathbb{C}\omega_M, \chi_M)$. We have just seen that $\chi_M = \chi_k\chi_l$.

We can take

$$M = \{([m], m, m') | m \in \mathbb{C}^{n+1} \setminus \{0\},\ \mathbb{C}m = \mathbb{C}m'\} \subset \mathbb{P}^n \times \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}.$$

Then we have a canonical isomorphism

$$M \simeq (\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C}^\times,\ ([m], m, cm) \leftrightarrow (m, c).$$

The H-action then becomes $h(m, c) = (mh_1^{-1}, ch_1h_2^{-1})$ for $h = (h_1, h_2) \in H$. On the right hand side, we have the global coordinates $(z, \zeta) \equiv (z_0, .., z_n, \zeta) : (m, c) \mapsto (m, c)$. We then find that the CY structures on M are just $(\mathbb{C}\omega_l, \chi_{n+1-l}\chi_l)$, $l \in \mathbb{Z}$, where

$$\omega_l := \zeta^{l-1} dz_0 \wedge \cdots \wedge dz_n \wedge d\zeta, \quad \chi_k\chi_l(h) = h_1^k h_2^l.$$

We can generalize Proposition 2.1 and Corollary 2.2 to cases involving finitely many line bundles:

Theorem 2.5. *Given holomorphic line bundles $L_1, \cdots, L_p$ on X, we have an isomorphism of holomorphic line bundles:*

$$(L_1^\times \oplus L_2^\times \oplus \cdots \oplus L_p^\times) \times_{(\mathbb{C}^\times)^p} \mathbb{C}_{\chi_{k_1}\cdots\chi_{k_p}} \simeq k_1L_1 + \cdots + k_pL_p$$

where $\chi_{k_1} \cdots \chi_{k_p} : (\mathbb{C}^\times)^p \to \mathbb{C}^\times, (h_1, \cdots, h_p) \mapsto h_1^{k_1} \cdots h_p^{k_p}$.

Corollary 2.6. *If there exist line bundles $L_1, \cdots, L_p$ and integers $k_1, \cdots, k_p$ such that $k_1L_1 + k_2L_2 + \cdots + k_pL_p \simeq K_X$, then $L_1^\times \oplus L_2^\times \oplus \cdots \oplus L_p^\times$ is a CY $(\mathbb{C}^\times)^p$-bundle over X with character $\chi_{k_1}\chi_{k_2}\cdots\chi_{k_p}$.*

2.2. **Rigidity of H-principal bundles.** In this section we set $H := (\mathbb{C}^\times)^p$. Assume M, N are two H-principal bundles over a complex manifold X. Then we have character maps:

$$\lambda_M : \widehat{H} \to \mathrm{Pic}(X)$$
$$\chi \mapsto L_\chi^M := M \times_H \mathbb{C}_\chi$$

and

$$\lambda_N : \widehat{H} \to \mathrm{Pic}(X)$$
$$\rho \mapsto L_\rho^N := N \times_H \mathbb{C}_\rho$$

where $\chi, \rho \in \widehat{H}$.

We assume that there is an automorphism $\xi \in \text{Aut}\,(\widehat{H})$ such that the following diagram

$$\begin{array}{ccc} \widehat{H} & \xrightarrow{\lambda_M} & \text{Pic}(X) \\ \xi \downarrow \simeq & \nearrow_{\lambda_N} & \\ \widehat{H} & & \end{array}$$

is commutative.

Since

$$\widehat{H} = \widehat{(\mathbb{C}^\times)^p} = (\widehat{\mathbb{C}^\times})^p \simeq \mathbb{Z}^p,$$

we can pick a $\mathbb{Z}$-basis $\{\chi_1, \ldots, \chi_p\}$ of $\widehat{H}$. Let $\rho_i := \xi(\chi_i)$, since ξ is an automorphism, $\{\rho_1, \ldots, \rho_p\}$ is also an $\mathbb{Z}$-basis of $\widehat{H}$. Moreover, the commutative diagram tells us that there exists a holomorphic line bundle isomorphism $\eta_i : L^M_{\chi_i} \to L^N_{\rho_i}$.

Let $[m, c]^M_{\chi_i}$ denote the class of $(m, c) \in M \times \mathbb{C}$ in $L^M_{\chi_i}$.

Lemma 2.7. *The bundle* $(L^M_{\chi_1})^\times \oplus \cdots \oplus (L^M_{\chi_p})^\times$ *with* H*-action*

$$h \cdot ([m_1, c_1]^M_{\chi_1}, \ldots, [m_p, c_p]^M_{\chi_p}) = ([m_1 h^{-1}, c_1]^M_{\chi_1}, \ldots, [m_p h^{-1}, c_p]^M_{\chi_p})$$

is a principal H*-bundle over* X*.*

Proof. It is clear that $(L^M_{\chi_1})^\times \oplus \cdots \oplus (L^M_{\chi_p})^\times$ is a bundle over X with fiber $(\mathbb{C}^\times)^p = H$ and the H-action preserves the fiber. Since $[m_i h^{-1}, \chi(h) c_i] = [m_i, c_i]$, we have $[m_i h^{-1}, c_i] = [m_i, \chi(h^{-1}) c_i]$. We can rewrite the H-action as

$$h \cdot ([m_1, c_1]^M_{\chi_1}, \ldots, [m_p, c_p]^M_{\chi_p}) = ([m_1, \chi_1(h^{-1}) c_1]^M_{\chi_1}, \ldots, [m_p, \chi_p(h^{-1}) c_p]^M_{\chi_p}).$$

Claim 2.8. $\hat{\chi} := (\chi_1, \ldots, \chi_p)$ *gives an automorphism of* H*.*

Proof of claim. It is clear that $\hat{\chi}$ is a homomorphism of H. Note that $\text{Aut}\,(H) \simeq \text{GL}_p(\mathbb{Z})$. Since $\chi_i \in \widehat{H} \simeq \mathbb{Z}^p$, there exists $a^i_1, \ldots, a^i_p \in \mathbb{Z}$ such that

$$\chi_i : H \to \mathbb{C}^\times$$
$$(h_1, \ldots, h_p) \mapsto h_1^{a^i_1} \cdots h_p^{a^i_p}.$$

In this way we may represent χ_i by $(a^i_1, \ldots, a^i_p)$, then $\hat{\chi}$ can be represented by the matrix (a^i_j). Since $\{\chi_1, \ldots, \chi_q\}$ is a $\mathbb{Z}$-basis of $\widehat{H}$, we have $\det(a^i_j) = 1$ and $(a^i_j) \in \text{GL}_p(\mathbb{Z})$. Thus $\hat{\chi} \in \text{Aut}\,(H)$. □

Thus H acts freely and transitively on the fiber of $(L^M_{\chi_1})^\times \oplus \cdots \oplus (L^M_{\chi_p})^\times$, i.e., it is a principal H-bundle. □

Similarly, $(L^N_{\rho_1})^\times \oplus \cdots \oplus (L^N_{\rho_p})^\times$ is a principal H-bundle with the action

$$h \cdot ([n_1, d_1]^M_{\rho_1}, \ldots, [n_p, d_p]^M_{\rho_p}) = ([n_1 h^{-1}, d_1]^M_{\rho_1}, \ldots, [n_p h^{-1}, d_p]^M_{\rho_p}).$$

Claim 2.9. *The map*

$$\begin{aligned} \alpha : M &\to (L^M_{\chi_1})^\times \oplus \cdots \oplus (L^M_{\chi_p})^\times \\ m &\mapsto ([m,1]^M_{\chi_1}, \ldots, [m,1]^M_{\chi_p}) \end{aligned}$$

is an isomorphism of principle H-bundles.

Proof. Since

$$\alpha(h \cdot m) = ([m_1 h^{-1}, 1]^M_{\chi_1}, \ldots, [m_p h^{-1}, 1]^M_{\chi_p}) = h \cdot \alpha(m),$$

α is H-equivariant.

Let $p : M \to X$ and $p' : (L^M_{\chi_1})^\times \oplus \cdots \oplus (L^M_{\chi_p})^\times \to X$ denote the projections respectively. Then $p' \circ \alpha(m) = p'(([m,1]^M_{\chi_1}, \ldots, [m,1]^M_{\chi_p})) = p(m)$. Thus α is a principal morphism over X. Since every principal morphism over X is an isomorphism [Hu, p. 43], α is an isomorphism of principal H-bundles over X. □

Similarly, let $\hat\rho := (\rho_1, \ldots, \rho_p)$, then $\hat\rho \in \text{Aut}\,(H)$, and we have an isomorphism of principal H-bundles:

$$\beta : N \to (L^N_{\rho_1})^\times \oplus \cdots \oplus (L^N_{\rho_p})^\times.$$

Now we can see that by definition $(L^M_{\chi_1})^\times \oplus \cdots \oplus (L^M_{\chi_p})^\times \simeq (L^N_{\rho_1})^\times \oplus \cdots \oplus (L^N_{\rho_p})^\times$ as holomorphic H-bundles, but the H-actions are different.

Definition 2.10. *Given $\sigma \in Aut\,(H)$ and a principal H-bundle $H - M \to X$. Define M^σ to be the bundle twisted by σ as follows. The total space of M^σ is M as a complex manifold, but the H-action on M^σ is defined to be*

$$\begin{aligned} H \times M^\sigma &\to M^\sigma \\ (h, m) &\mapsto m(\sigma(h)^{-1}). \end{aligned}$$

It is clear from the definition that the H-action on M^σ preserves the fiber and acts freely and transitively on the fiber, so it is a principal H-bundle as well.

Definition 2.11. *Given principal bundles $H - M_i \to X, i = 1, 2$, we call a holomorphic bundle map $\varphi : M_1 \to M_2$ a twisted principal H-bundle isomorphism if there exists $\sigma \in Aut\,(H)$ such that $\varphi : M_1 \to M_2^\sigma$ is a principal H-bundle isomorphism. In this case, we say that M_1 is isomorphic to M_2 up to a twist by σ.*

Theorem 2.12 (Rigidity of CY bundle)**.** *Assume M, N are two H-principal bundles over a complex manifold X and there exists an automorphism $\xi \in Aut\,(\widehat{H})$ such that the following diagram*

$$\begin{array}{ccc} \widehat{H} & \xrightarrow{\lambda_M} & Pic(X) \\ {\scriptstyle \xi}\downarrow\simeq & \nearrow_{\lambda_N} & \\ \widehat{H} & & \end{array}$$

commutes. Then M is isomorphic to N up to a twist by the induced automorphism $\xi^\vee \in Aut\ H$.

Proof. First we describe the induced automorphism $\xi^\vee$. It is clear that there is a natural homomorphism

$$\begin{aligned}\theta : \text{Aut } H &\to \text{Aut } (\widehat{H}) \\ f &\mapsto f^\vee := (\gamma \mapsto \gamma \circ f^{-1})\end{aligned}$$

for $f \in \text{Aut } H$ and $\gamma \in \widehat{H}$. The inverse of this map does not necessarily exist for general H. But in the case $H = (\mathbb{C}^\times)^p$,

$$\text{Aut } H \simeq \text{GL}_p(\mathbb{Z}) \simeq \text{Aut } (\widehat{H}),$$

we can construct an inverse of θ.

Claim 2.13. *Given a basis $\{\chi_1, \ldots, \chi_p\}$ of $\widehat{H}$, let $\hat{\chi} := (\chi_1, \ldots, \chi_p)$ and $\hat{\rho} := (\xi(\chi_1), \ldots, \xi(\chi_p))$. Then*

$$\begin{aligned}\tau : Aut\ (\widehat{H}) &\to Aut\ H \\ \xi &\mapsto \xi^\vee := \hat{\rho}^{-1} \circ \hat{\chi}\end{aligned}$$

is an inverse of θ.

Proof of claim: It is clear that $\xi^\vee \in \text{Aut } H$. We want to show $(\theta \circ \tau)(\xi) = \xi$, it suffices to show that on the $\mathbb{Z}$-basis we have: $((\theta \circ \tau)(\xi))(\chi_i) = \xi(\chi_i) = \rho_i$.

Let $\text{pr}_i : H \to \mathbb{C}^\times$ be the i-th projection, then $\chi_i = \text{pr}_i \circ \hat{\chi}$ and $\rho_i = \text{pr}_i \circ \hat{\rho}$. Thus

$$((\theta \circ \tau)(\xi))(\chi_i) = \chi_i \circ (\xi^\vee)^{-1} = \text{pr}_i \circ \hat{\rho} = \rho_i$$

as we expected. □

Consider $M' := (L^M_{\chi_1})^\times \oplus \cdots \oplus (L^M_{\chi_p})^\times$ and $N' := (L^N_{\rho_1})^\times \oplus \cdots \oplus (L^N_{\rho_p})^\times$ with H-actions described as in Lemma 2.7. Let $\eta := \eta_1 \oplus \cdots \oplus \eta_p$, then $\eta : M' \to N'$ is a holomorphic H-bundle isomorphism. Given any $[m_i, c_i]^M_{\chi_i} \in L^M_{\chi_i}$, let $[n_i, d_i]^N_{\rho_i} := \eta_i([m_i, c_i]^M_{\chi_i})$. Since isomorphism of line bundles preserves scaling on fibers, given any $a_i \in \mathbb{C}^\times$, $\eta_i([m_i, c_i \cdot a_i]^M_{\chi_i}) = [n_i, d_i \cdot a_i]^N_{\rho_i}$.

The H-action on $(N')^{\xi^\vee}$ becomes:

$$\begin{aligned}h \cdot_{\xi^\vee} ([n_1, d_1]^N_{\rho_1}, \ldots, [n_p, d_p]^N_{\rho_p}) &= ([n_1\xi^\vee(h^{-1}), d_1]^N_{\rho_1}, \ldots, [n_p\xi^\vee(h^{-1}), d_p]^N_{\rho_p}) \\ &= ([n_1, d_1\rho_1(\xi^\vee(h^{-1}))]^N_{\rho_1}, \ldots, [n_p, d_p\rho_p(\xi^\vee(h^{-1}))]^N_{\rho_p}) \\ &= ([n_1, d_1\chi_1(h^{-1})]^N_{\rho_1}, \ldots, [n_p, d_p\chi_p(h^{-1})]^N_{\rho_p})\end{aligned}$$

since $\rho_i \circ \xi^\vee = \text{pr}_i \circ \hat{\rho} \circ \hat{\rho}^{-1} \circ \hat{\chi} = \text{pr}_i \circ \hat{\chi} = \chi_i$.

On the other hand,

$$\begin{aligned}\eta(h \cdot ([m_1, c_1]^M_{\chi_1}, \ldots, [m_p, c_p]^M_{\chi_p})) &= \eta(([m_1, \chi_1(h^{-1})c_1]^M_{\chi_1}, \ldots, [m_p, \chi_p(h^{-1})c_p]^M_{\chi_p})) \\ &= ([n_1, d_1\chi_1(h^{-1})]^N_{\chi_1}, \ldots, [n_p, d_p\chi_p(h^{-1})]^N_{\chi_p})\end{aligned}$$

Therefore η is H-equivariant and thus a principal H-bundle isomorphism between M' and $(N')^{\xi^\vee}$. Applying Claim 2.9 we can conclude that M and N are isomorphic up to a twist by $\xi^\vee$. □

In fact, we can further prove:

Theorem 2.14. *Given $H = (\mathbb{C}^\times)^p$ and any group homomorphism*

$$\lambda : \widehat{H} \to Pic(X)$$

with K_X in the image of λ. Then there exists a unique CY H-bundle M such that $\lambda = \lambda_M$.

Proof. Since $\widehat{H} = \{\chi_{k_1} \cdots \chi_{k_p} \mid k_i \in \mathbb{Z}\}$, let $e_i := \chi_0 \cdots \chi_1 \cdots \chi_0$, where χ_1 exists only on the i-th component, be the standard basis. Let $M := (\lambda(e_1))^\times \oplus \cdots \oplus (\lambda(e_p))^\times$. Let H act on M by

$$(h_1, \ldots, h_p) \cdot (l_1, \ldots, l_p) = (l_1 h_1^{-1}, \ldots, l_p h_p^{-1}).$$

Then M is a principal H-bundle on X. From Theorem 2.5 we have

$$\lambda_M(\chi_{k_1} \cdots \chi_{k_p}) = k_1 \lambda(e_1) + \cdots + k_p \lambda(e_p) = \lambda(\chi_{k_1} \cdots \chi_{k_p}),$$

i.e. $\lambda_M = \lambda$.

Now we know further that $K_X \in \operatorname{Im} \lambda$, then we have $K_X \in \operatorname{Im} \lambda_M$ and thus by Theorem 1.7 M is a CY H-bundle on X.

Now suppose there exists another N such that $\lambda_N = \lambda$. Then we have a commutative diagram

$$\begin{array}{ccc} \widehat{H} & \xrightarrow{\lambda_M} & \operatorname{Pic}(X) \\ \xi \downarrow = & \nearrow_{\lambda_N} & \\ \widehat{H} & & \end{array}$$

where ξ is the identity map. Then apply Theorem 2.12 we can conclude that M and N are isomorphic as principal H-bundles. □

Remark 2.15. *If we drop the condition $K_X \in \operatorname{Im} \lambda$, then the theorem holds with "CY H-bundle" replaced by "principal H-bundle".*

Now we can prove the first part of our main theorem:

Theorem 2.16. *Let X be a compact complex manifold. If $Pic(X)$ is free, then X admits a CY $(\mathbb{C}^\times)^p$-bundle whose character map is isomorphic, and the bundle is unique up to a twist by an automorphism of $(\mathbb{C}^\times)^p$.*

Proof. Since X is compact, $\operatorname{Pic}(X)$ is finitely generated. Assume $\operatorname{Pic}(X) \simeq \mathbb{Z}^p$. Then there exists line bundles $L_1, \ldots, L_p$ such that

$$\operatorname{Pic}(X) \simeq \mathbb{Z}L_1 + \cdots + \mathbb{Z}L_p.$$

Let $M := L_1^\times \oplus L_2^\times \oplus \cdots \oplus L_p^\times$ and $H := (\mathbb{C}^\times)^p$, then M is a principal H-bundle over X. Then by Theorem 2.5

$$\lambda_M(\widehat{H}) = \mathbb{Z}L_1 + \cdots + \mathbb{Z}L_p \simeq \operatorname{Pic}(X),$$

meaning that the character map is an isomorphism.

If we have another bundle $H - N \to X$ such that

$$\lambda_N(\widehat{H}) \simeq \mathrm{Pic}(X),$$

then there exists $\xi_1, \ldots, \xi_p \in \widehat{H}$ such that

$$\lambda_N(\xi_i) \simeq L_i, \quad \forall i.$$

Since $L_1, \ldots, L_p$ is a $\mathbb{Z}$-basis of $\mathrm{Pic}(X)$, $\{\xi_1, \ldots, \xi_p\}$ is a $\mathbb{Z}$-basis of $\widehat{H}$.

Let $\mathrm{pr}_i : H \to \mathbb{C}^\times$ be the i-th projection. Then $\{\mathrm{pr}_1, \ldots, \mathrm{pr}_p\}$ is a $\mathbb{Z}$-basis of $\widehat{H}$, and $\lambda_M(\mathrm{pr}_i) \simeq L_i$.

Define a homomorphism $\xi : \widehat{H} \to \widehat{H}$ where on generators $\xi(\mathrm{pr}_i) := \xi_i$ for all i. Since ξ maps a $\mathbb{Z}$-basis to a $\mathbb{Z}$-basis, it is an isomorphism. Moreover,

$$\lambda_N \circ \xi(\mathrm{pr}_i) = \lambda_N(\xi_i) \simeq L_i \simeq \lambda_M(\mathrm{pr}_i),$$

for all i. Thus $\lambda_N \circ \xi = \lambda_M$ and by Theorem 2.12 M and N are isomorphic up to a twist by the automorphism $\xi^\vee = (\xi_1, \ldots, \xi_p)^{-1}$. □

3. Character map of the universal cover of X

3.1. **Realizing Pic_0 for Kähler manifolds.** In this section, we describe the connected component $\mathrm{Pic}_0(X)$ of the Picard group of an arbitrary connected compact Kähler manifold X. Aside from a few special cases (such as curves and tori), part of the description seems to be folklore in complex geometry, and the authors have been unable to locate a source for the general case in the literature.

Let $\tilde{X}$ be the universal cover of X. Then $\tilde{X}$ is a principal bundle $\pi_1(X)$-bundle over X and we have a character map

$$\lambda_{\tilde{X}} : \widehat{\pi_1(X)} \to \mathrm{Pic}(X), \quad \gamma \mapsto L_\gamma := \tilde{X} \times_{\pi_1(X)} \mathbb{C}_\gamma.$$

First we want to describe the action of $\pi_1(X)$ on $\tilde{X} \times \mathbb{C}_\gamma$. We have a left action of $\pi_1(X)$ on $\tilde{X}$ by deck transformation $\rho \cdot \tilde{z}$ for $\rho \in \pi_1(X)$, which we shall regard as a right action where $\tilde{z}\rho^{-1} := \rho \cdot \tilde{z}$. Then $\pi_1(X)$ acts on $\tilde{X} \times \mathbb{C}_\gamma$ by

$$\rho \cdot (\tilde{z}, c) := (\tilde{z}\rho^{-1}, \gamma(\rho)c) = (\rho \cdot \tilde{z}, \gamma(\rho)c)$$

for $\rho \in \pi_1(X), \tilde{z} \in \tilde{X}, c \in \mathbb{C}$.

In this section we are going to prove that $\mathrm{Pic}_0(X)$ is contained in the image of $\lambda_{\tilde{X}}$.

Lemma 3.1. [GH, p313] *Any line bundle on X with Chern class 0 can be given by constant transition functions.*

Since we will need a description of the constant transition functions, we sketch a proof here.

Proof. Consider the inclusion of exact sheaf sequences on X:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{O} & \xrightarrow{\exp} & \mathcal{O}^\times & \longrightarrow & 0 \\ & & \| & & \uparrow{\scriptstyle \iota_1} & & \uparrow{\scriptstyle \iota_2} & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{C} & \longrightarrow & \mathbb{C}^\times & \longrightarrow & 0. \end{array}$$

It induces a commutative diagram

$$\begin{array}{ccccc} H^1(X,\mathcal{O}) & \xrightarrow{\alpha} & H^1(X,\mathcal{O}^\times) & \xrightarrow{c_1} & H^2(X,\mathbb{Z}) \\ \uparrow{\scriptstyle \iota_1^*} & & \uparrow{\scriptstyle \iota_2^*} & & \| \\ H^1(X,\mathbb{C}) & \xrightarrow{\beta} & H^1(X,\mathbb{C}^\times) & \longrightarrow & H^2(X,\mathbb{Z}). \end{array}$$

The map ι_1^* represents projection of $H^1(X,\mathbb{C}) \simeq H^{1,0}(X) \oplus H^{0,1}(X) = H^0(X,\Omega) \oplus H^1(X,\mathcal{O})$ onto the second factor, and so is surjective. Then

$$(*) \qquad \mathrm{Pic}_0(X) = \ker\, c_1 = \mathrm{Im}\,\alpha = \mathrm{Im}\,\alpha \circ \iota_1^* = \mathrm{Im}\,\iota_2^* \circ \beta \subseteq \mathrm{Im}\,\iota_2^*.$$

It follows that any cocycle $\gamma \in H^1(X,\mathcal{O}^\times)$ in the kernel of c_1 is in the image of ι_2^*, i.e., is cohomologous to a cocycle with constant coefficients. Thus any line bundle on X with Chern class 0 can be given by constant transition functions. □

Equation $(*)$ says that the image of the map

$$\iota_2^* : H^1(X,\mathbb{C}^\times) \to H^1(X,\mathcal{O}^\times)$$

contains $\mathrm{Pic}_0(X)$. We want to compare it with the character map $\lambda_{\tilde{X}}$:

Proposition 3.2. *If X is a complex manifold (not necessarily Kähler), there exists an isomorphisms ψ such that the following diagram*

$$\begin{array}{ccc} \widehat{\pi_1(X)} & \xrightarrow{\lambda_{\tilde{X}}} & Pic(X) \\ \downarrow{\scriptstyle \psi} & & \| \\ H^1(X,\mathbb{C}^\times) & \xrightarrow{\iota_2^*} & H^1(X,\mathcal{O}^\times) \end{array}$$

commutes.

Proof. The universal coefficient theorem tells us that

$$H^1(X,\mathbb{C}^\times) \simeq \mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{C}^\times) \oplus \mathrm{Ext}(H_0(X,\mathbb{Z}),\mathbb{C}^\times).$$

Since $\mathbb{C}^\times$ is abelian and $H_1(X,\mathbb{Z})$ is the abelianization of $\pi_1(X)$,

$$\widehat{\pi_1(X)} = \mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{C}^\times).$$

Since $H_0(X,\mathbb{Z})$ is free, $\mathrm{Ext}(H_0(X,\mathbb{Z}),\mathbb{C}^\times) = 0$. Therefore we have

$$H^1(X,\mathbb{C}^\times) \simeq \widehat{\pi_1(X)}.$$

Next, we want to describe this isomorphism ψ in detail. We will use the notion of G-coverings which can be found in [Fu, p.160]. Let G be a group.

A covering $p : Y \to X$ is called a G-covering if it arises from a properly discontinuous action of G on Y. Since X is a manifold, we can find an open cover $\mathcal{U} = \{U_\alpha\}$ where U_α are connected and contractible. If G is abelian, there are isomorphism of abelian groups:

$$\mathrm{Hom}(\pi_1(X), G) \simeq \{G\text{-coverings of } X\}/\text{isomorphism} \simeq H^1(\mathcal{U}, G). \tag{3.1}$$

The first isomorphism is given by $\rho \mapsto [Y_\rho]$ where $Y_\rho := \tilde{X} \times_{\pi_1(X)} G$. Here $\pi_1(X)$ acts on $\tilde{X} \times G$ by

$$\sigma(\tilde{z}, g) = (\tilde{z}\sigma^{-1}, \rho(\sigma) \cdot g) \text{ for } \rho \in \pi_1(X),$$

and G acts on Y_ρ by

$$g' \cdot [(\tilde{z}, g)] = [(\tilde{z}, g \cdot (g')^{-1})].$$

Note that our convention differs from that of [Fu] in that the left and right group actions are switched. But the results there carry over. We now specialize to $G = \mathbb{C}^\times$. Let $\rho \in \widehat{\pi_1(X)}$. Since U_α is contractible, $Y_\rho|_{U_\alpha}$ is a trivial $\mathbb{C}^\times$-covering. Then under (3.1), the image of $[Y_\rho]$ in $H^1(\mathcal{U}, \mathbb{C}^\times)$ must be a collection of associated transition functions $\{g_{\alpha\beta}\}_{\alpha,\beta}$, where the $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^\times$ are locally constant functions.

Since each U_α is a contractible open set, we have $H^1(U_\alpha, \mathbb{C}^\times) = 0$. Therefore, we have a canonical isomorphism $H^1(X, \mathbb{C}^\times) \simeq H^1(\mathcal{U}, \mathbb{C}^\times)$ and we can regard $\{g_{\alpha\beta}\}_{\alpha,\beta}$ as an element in $H^1(X, \mathbb{C}^\times)$. Define

$$\psi : \widehat{\pi_1(X)} \to H^1(X, \mathbb{C}^\times), \rho \mapsto \{g_{\alpha\beta}\}_{\alpha,\beta},$$

then ψ is an isomorphism.

Moreover, since ι_2 is an inclusion, $\iota_2^*(\{g_{\alpha\beta}\}_{\alpha,\beta}) = \{\iota_2(g_{\alpha\beta})\}_{\alpha,\beta} = \{g_{\alpha\beta}\}_{\alpha,\beta}$. Therefore

$$\iota_2^* \circ \psi(\rho) = \{g_{\alpha\beta}\}_{\alpha,\beta}.$$

Now we consider

$$\lambda_{\tilde{X}}(\rho) = L_\rho := \tilde{X} \times_{\pi_1(X)} \mathbb{C}_\rho$$

where $[\tilde{z}, c] = [\sigma \cdot \tilde{z}, \rho(\sigma)c]$. $\mathcal{U}$ defined above is a local trivialization of L_ρ, and from the way we define Y_ρ and L_ρ we can see that they have the same transition functions. Therefore

$$\lambda_{\tilde{X}}(\rho) = \{g_{\alpha\beta}\}_{\alpha,\beta} = \iota_2^* \circ \psi(\rho),$$

which means that the diagram in the proposition commutes, as desired. □

An immediate result of this proposition is the following:

Corollary 3.3. *$Pic_0(X)$ is contained in the image of $\lambda_{\tilde{X}}$. I.e., given a line bundle $L \in Pic_0(X)$, there exists a character $\gamma \in \widehat{\pi_1(X)}$ such that $L \simeq \tilde{X} \times_{\pi_1(X)} \mathbb{C}_\gamma$.*

Example 3.4. Let X be a complex torus of dimension g. Then $\pi_1(X) \simeq \mathbb{Z}^{2g}$. Corollary 3.3 tells us that the character map

$$\lambda_{\tilde{X}} : \widehat{\pi_1(X)} \simeq (\mathbb{C}^\times)^{2g} \to \mathrm{Pic}(X)$$

contains $\mathrm{Pic}_0(X) \simeq \mathbb{C}^g/\mathbb{Z}^{2g}$ in its image. It is clear from a dimension argument that this map has a kernel. We will describe the kernel in Section 3.3.

We now give an explicit description of the character γ under the assumption that X is Kähler. This generalizes the case of complex torus given in [GH, p313,308].

Given a line bundle L in $\mathrm{Pic}_0(X)$, we can find an open cover $U = \{U_\alpha\}$ of $\tilde{X}$ such that for each α, $\pi^{-1}(U_\alpha) = \cup_j U_{\alpha,j}$ is a disjoint union of open sets $U_{\alpha,j}$ isomorphic via π to U_α, and a collection of trivializations $\varphi_\alpha : L|_{U_\alpha} \to U_\alpha \times \mathbb{C}$ having constant transition functions $\{g_{\alpha\alpha'}\}$ where $g_{\alpha\alpha'}(z) = (\varphi_\alpha \circ \varphi_{\alpha'}^{-1})|_{\{z\}\times\mathbb{C}}$, $z \in U_\alpha \cap U_{\alpha'}$.

For each α, pick some $U_{\alpha,j}$ and denote it by $U_{\alpha,1}$. Since $\pi^{-1}(U_\alpha) \simeq U_\alpha \times \pi_1(X)$, if we set $U_{\alpha,\lambda} := \lambda \cdot U_{\alpha,1}$ for $\lambda \in \pi_1(X)$ and the action is given by deck transformations, then $\pi^{-1}(U_\alpha) = \amalg_{\lambda\in\pi_1(X)} U_{\alpha,\lambda}$.

We can define a collection of nonzero complex numbers $\{h_{\alpha,\lambda}\}_{\alpha,\lambda}$ by taking $h_{\alpha_0,1} \equiv 1$ for some α_0 and setting

$$h_{\alpha,\lambda} = h_{\alpha',\lambda'} \cdot g_{\alpha'\alpha} \text{ for } \alpha, \lambda, \alpha', \lambda' \text{ whenever } U_{\alpha,\lambda} \cap U_{\alpha',\lambda'} \neq \emptyset.$$

By a straightforward computation we can see that by the cocycle rule on $g_{\alpha\alpha'}$ this is well-defined.

Now consider the pull back bundle π^*L on $\tilde{X}$. We can define maps:

$$\varphi_{\alpha,\lambda} : (\pi^*L)|_{U_{\alpha,\lambda}} \to U_{\alpha,\lambda} \times \mathbb{C} \text{ where } \varphi_{\alpha,\lambda} = h_{\alpha,\lambda} \cdot \pi^*\varphi_\alpha$$

where $\pi^*\varphi_\alpha$ is the pull back of φ_α via $\pi \times \mathrm{id}_\mathbb{C}$. We can see that these $\{\varphi_{\alpha,\lambda}\}$ give a local trivialization of π^*L.

Since $\varphi_{\alpha',\lambda'}^{-1} = h_{\alpha',\lambda'}^{-1} \cdot (\pi^*\varphi_{\alpha'})^{-1}$, if $U_{\alpha,\lambda} \cap U_{\alpha',\lambda'} \neq \emptyset$ then

$$g_{\alpha,\lambda;\alpha',\lambda'}(\tilde{z}) = h_{\alpha,\lambda} h_{\alpha',\lambda'}^{-1} \pi^*((\varphi_\alpha \circ \varphi_{\alpha'}^{-1})|_{\{z\}\times\mathbb{C}}) = (h_{\alpha',\lambda'} \cdot g_{\alpha'\alpha}) h_{\alpha',\lambda'}^{-1} g_{\alpha\alpha'} = 1$$

for $\tilde{z} \in U_{\alpha,\lambda} \cap U_{\alpha',\lambda'}$ and $z := \pi(\tilde{z})$.

This means that $\{\varphi_{\alpha,\lambda}\}$ matches on overlaps and give a global trivialization of π^*L, which we denote by $\varphi : \pi^*L \to \tilde{X} \times \mathbb{C}$. To summarize, we have:

Proposition 3.5. *For any $L \in Pic_0(X)$, π^*L is a trivial line bundle on $\tilde{X}$.*

Now for $\tilde{z} \in \tilde{X}$ and $\rho \cdot \tilde{z}$ for $\rho \in \pi_1(X)$, the fibers of π^*L at $\tilde{z}$ and $\rho \cdot \tilde{z}$ are by definition both identified with the fiber of L at z, and comparing the trivialization φ at $\tilde{z}$ and $\rho \cdot \tilde{z}$ yields a linear automorphism of $\mathbb{C}$:

$$\mathbb{C} \xleftarrow{\varphi|_{\{\tilde{z}\}\times\mathbb{C}}} (\pi^*L)|_{\tilde{z}} = L|_z = (\pi^*L)|_{\rho\cdot\tilde{z}} \xrightarrow{\varphi|_{\{\rho\cdot\tilde{z}\}\times\mathbb{C}}} \mathbb{C}.$$

Such an automorphism is given as multiplication by a nonzero complex number, which we denote by $e_\rho(\tilde{z})$. It is clearly holomorphic in $\tilde{z}$, hence we obtain a collection of functions

$$\{e_\rho \in \mathcal{O}^\times(\tilde{X})\}_{\rho\in\pi_1(X)}$$

satisfying the compatibility relation

$$e_{\rho'}(\rho \cdot \tilde{z})e_\rho(\tilde{z}) = e_\rho(\rho' \cdot \tilde{z})e_{\rho'}(\tilde{z}) = e_{\rho\rho'}(\tilde{z})$$

for all $\rho, \rho' \in \pi_1(X)$. This collection is called a set of *multipliers* for L. (It is also called *factors of automorphy*.)

Suppose $\tilde{z} \in U_{\alpha,\lambda}$ for some α, λ, then $\rho \cdot \tilde{z} \in U_{\alpha,\rho\cdot\lambda}$. Then we have

$$\varphi|_{\{\tilde{z}\}\times\mathbb{C}} = h_{\alpha,\lambda} \cdot (\pi^*\varphi_\alpha)|_{\{\tilde{z}\}\times\mathbb{C}} = h_{\alpha,\lambda} \cdot \pi^*(\varphi_\alpha|_{\{z\}\times\mathbb{C}})$$
$$\varphi|_{\{\rho\cdot\tilde{z}\}\times\mathbb{C}} = h_{\alpha,\rho\lambda} \cdot (\pi^*\varphi_\alpha)|_{\{\rho\cdot\tilde{z}\}\times\mathbb{C}} = h_{\alpha,\rho\lambda} \cdot \pi^*(\varphi_\alpha|_{\{z\}\times\mathbb{C}}).$$

Thus the linear automorphism is given by

$$e_\rho(\tilde{z}) = \frac{\varphi|_{\{\rho\cdot\tilde{z}\}\times\mathbb{C}}}{\varphi|_{\{\tilde{z}\}\times\mathbb{C}}} = \frac{h_{\alpha,\rho\lambda}}{h_{\alpha,\lambda}}.$$

which is a nonzero constant on $U_{\alpha,\lambda}$.

Moreover, if $U_{\alpha,\lambda} \cap U_{\alpha',\lambda'} \neq \emptyset$, then $\rho \cdot (U_{\alpha,\lambda} \cap U_{\alpha',\lambda'}) = U_{\alpha,\rho\lambda} \cap U_{\alpha',\rho\lambda'} \neq \emptyset$. Thus we have

$$h_{\alpha,\lambda} = h_{\alpha',\lambda'} \cdot g_{\alpha'\alpha}$$
$$h_{\alpha,\rho\lambda} = h_{\alpha',\rho\lambda'} \cdot g_{\alpha'\alpha},$$

which implies

$$e_\rho(\tilde{z}') = \frac{\varphi|_{\{\rho\cdot\tilde{z}'\}\times\mathbb{C}}}{\varphi|_{\{\tilde{z}'\}\times\mathbb{C}}} = \frac{h_{\alpha',\rho\lambda'} \cdot g_{\alpha'\alpha}}{h_{\alpha',\lambda'} \cdot g_{\alpha'\alpha}} = \frac{h_{\alpha,\rho\lambda}}{h_{\alpha,\lambda}} = e_\rho(\tilde{z}),$$

for any $\tilde{z}' \in U_{\alpha',\lambda'}$.

Since $\tilde{X}$ is connected, this tells us that e_ρ is a constant on $\tilde{X}$, i.e. L has constant multipliers.

Now we have a map

$$\gamma : \pi_1(X) \to \mathbb{C}^\times$$
$$\rho \mapsto e_\rho$$

The compatibility relation becomes

$$e_{\rho'}e_\rho = e_\rho e_{\rho'} = e_{\rho\rho'}$$

for all $\rho, \rho' \in \pi_1(X)$. Thus γ is a group homomorphism, i.e., $\gamma \in \widehat{\pi_1(X)}$.

Claim 3.6. *There is an isomorphism of line bundles* $L \simeq L_\gamma := \tilde{X} \times_{\pi_1(X)} \mathbb{C}_\gamma$.

Proof. Since $\{U_\alpha\}$ defined above gives a local trivialization of $\tilde{X}$ regarded as a principal bundle on X, it induces a trivialization of L_γ on X, which will be described below.

We have a $\pi_1(X)$-bundle isomorphism:

$$\tilde{X}|_{U_\alpha} \times \mathbb{C}_\gamma \xrightarrow{\sim} (\coprod_{\lambda\in\pi_1(X)} U_{\alpha,\lambda}) \times \mathbb{C}_\gamma)$$
$$(\tilde{z}, c) \mapsto ((z, \lambda), c)$$

where $\pi(\tilde{z}) := z \in U_\alpha$. And the $\pi_1(X)$-action carries to the image as well:

$$\rho \cdot ((z,\lambda), c) = ((z, \rho\lambda), \gamma(\rho)c)$$

where $\rho \in \pi_1(X)$. We identify the two isomorphic bundles and use them interchangeably.

Define a map

$$\begin{aligned} \tilde{X}|_{U_\alpha} \times \mathbb{C}_\gamma &\to U_\alpha \times \mathbb{C} \\ ((z,\lambda), c) &\mapsto (z, \gamma^{-1}(\lambda)c). \end{aligned}$$

Then

$$((z,\rho\lambda), \gamma(\rho)c) \mapsto (z, \gamma^{-1}(\rho\lambda)\gamma(\rho)c) = (z, \gamma^{-1}(\lambda)c),$$

which means that this map is constant under $\pi_1(X)$-action. It thus induces a trivialization of $L_\gamma|_{U_\alpha}$, denote it by φ'_α:

$$\begin{aligned} \varphi'_\alpha : L_\gamma|_{U_\alpha} = \tilde{X}|_{U_\alpha} \times_{\pi_1(X)} \mathbb{C}_\gamma &\to U_\alpha \times \mathbb{C} \\ [((z,\lambda), c)] &\mapsto (z, \gamma^{-1}(\lambda)c). \end{aligned}$$

Suppose $U_\alpha \cap U_{\alpha'} \neq \emptyset$, and we consider $\varphi'_\alpha \circ (\varphi'_{\alpha'})^{-1}$. Given $z \in U_\alpha \cap U_{\alpha'}$, then there exists some $\lambda \in \pi_1(X)$ such that $U_{\alpha,\lambda} \cap U_{\alpha',1} \neq 0$ and the pre-image of z are $(z,1) \in U_{\alpha',1}$ and $(z,\lambda) \in U_{\alpha,\lambda}$. Thus we have:

$$\begin{array}{ccccccc} U_{\alpha'} \times \mathbb{C} & \xrightarrow{(\varphi'_{\alpha'})^{-1}} & L_\gamma|_{U_\alpha \cap U_{\alpha'}} & \xrightarrow{\sim} & L_\gamma|_{U_\alpha \cap U_{\alpha'}} & \xrightarrow{\varphi'_\alpha} & U_\alpha \times \mathbb{C} \\ (z,c) & \mapsto & [(z,1), c] & \mapsto & [(z,\lambda), c] & \mapsto & (z, \gamma^{-1}(\lambda)c). \end{array}$$

Therefore the transition function $g'_{\alpha\alpha'} = \gamma^{-1}(\lambda)$.

$$g'_{\alpha\alpha'} = \gamma(\lambda^{-1}) = e_{\lambda^{-1}}((z,\lambda)) = h_{\alpha,1}(h_{\alpha,\lambda})^{-1} = h_{\alpha,1} g_{\alpha\alpha'} (h_{\alpha',1})^{-1}.$$

Thus the transition functions of L and of L_γ differ by an element in the coboundary, which means L and L_γ are isomorphic as line bundles.

It is clear that the above isomorphisms are holomorphic, therefore the L is isomorphic to L_γ as holomorphic line bundles. □

3.2. **More about the character map $\lambda_{\tilde{X}}$.** In this section we are going to discuss two important facts about the character map.

First we want to show:

Proposition 3.7. *If X is a compact complex manifold (not necessarily Kähler), $c_1 \circ \lambda_{\tilde{X}}$ maps onto the torsion of $c_1(Pic(X))$.*

Proof. Consider the following commutative diagram induced by inclusion of the exponential sheaf sequences:

$$\begin{array}{ccccccc} H^1(X,\mathcal{O}) & \xrightarrow{\alpha} & H^1(X,\mathcal{O}^\times) & \xrightarrow{c_1} & H^2(X,\mathbb{Z}) & \longrightarrow & H^2(X,\mathcal{O}) \\ \uparrow & & \uparrow \iota_2^* & & \| & & \uparrow \\ H^1(X,\mathbb{C}) & \xrightarrow{\beta} & H^1(X,\mathbb{C}^\times) & \xrightarrow{\gamma} & H^2(X,\mathbb{Z}) & \xrightarrow{\epsilon} & H^2(X,\mathbb{C}). \end{array}$$

Since X is compact, $H_i(X) := H_i(X,\mathbb{Z})$ is a finitely generated abelian group for all i. In this subsection as assume:

$$\mathbb{Z}^a \simeq H_1(X)_{\mathrm{free}}, \quad T_1 := H_1(X)_{\mathrm{tor}}, \quad \mathbb{Z}^b \simeq H_2(X)_{\mathrm{free}}, \quad T_2 := H_2(X)_{\mathrm{tor}}$$

where $a, b \in \mathbb{Z}$ and T_1, T_2 are finite groups.

The universal coefficient theorem says

$$0 \to \mathrm{Ext}(H_1(X),\mathbb{Z}) \to H^2(X,\mathbb{Z}) \to \mathrm{Hom}(H_2(X),\mathbb{Z}) \to 0,$$
$$0 \to \mathrm{Ext}(H_1(X),\mathbb{C}) \to H^2(X,\mathbb{C}) \to \mathrm{Hom}(H_2(X),\mathbb{C}) \to 0,$$

and these sequences split.

We know that $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z}$. Since finite abelian groups can be written as a direct sum of finite cyclic groups, $\mathrm{Ext}(T_1,\mathbb{Z}) \simeq T_1$ and

$$\mathrm{Ext}(H_1(X),\mathbb{Z}) \simeq \mathrm{Ext}(\mathbb{Z}^a \oplus T_1,\mathbb{Z}) \simeq T_1$$

Similarly, $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z},\mathbb{C}) \simeq \mathbb{C}/n\mathbb{C} = 0$ and thus $\mathrm{Ext}(T_1,\mathbb{C}) = 0$ and

$$\mathrm{Ext}(H_1(X),\mathbb{C}) = 0.$$

Since $\mathrm{Hom}(H_2(X),\mathbb{Z}) \simeq \mathbb{Z}^b$ and $\mathrm{Hom}(H_2(X),\mathbb{C}) \simeq \mathbb{C}^b$, we have

$$H^2(X,\mathbb{Z}) \simeq T_1 \oplus \mathbb{Z}^b, \quad H^2(X,\mathbb{C}) \simeq \mathbb{C}^b.$$

From the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^\times \to 0$$

we get a left exact sequence

$$0 \to \mathrm{Hom}(G,\mathbb{Z}) \to \mathrm{Hom}(G,\mathbb{C}) \to \mathrm{Hom}(G,\mathbb{C}^\times).$$

Thus $\mathrm{Hom}(H_2(X),\mathbb{Z}) \simeq \mathbb{Z}^b \to \mathrm{Hom}(H_2(X),\mathbb{C}) \simeq \mathbb{C}^b$ is injective.

Consider $H^2(X,\mathbb{Z}) \simeq T_1 \oplus \mathbb{Z}^b \xrightarrow{\epsilon} H^2(X,\mathbb{C}) \simeq \mathbb{C}^b$, since T_1 is a finite group, it has to be mapped to $0 \in \mathbb{C}^b$. Then

$$T_1 \simeq \ker \epsilon = \mathrm{Im}\,\gamma = \mathrm{Im}\, c_1 \circ \iota_2^*,$$

which implies that $\mathrm{Im}\, c_1 \circ \iota_2^* = H^2(X,\mathbb{Z})_{\mathrm{tor}}$.

Proposition 3.2 tells us that $\lambda_{\tilde{X}} = \psi \circ \iota_2^*$ and ψ is an isomorphism. Thus

$$\mathrm{Im}\, c_1 \circ \lambda_{\tilde{X}} = \mathrm{Im}\, c_1 \circ \iota_2^* = H^2(X,\mathbb{Z})_{\mathrm{tor}}.$$

It further implies that $c_1(\mathrm{Pic}(X))_{\mathrm{tor}} = H^2(X,\mathbb{Z})_{\mathrm{tor}}$ and $c_1 \circ \lambda_{\tilde{X}}$ maps onto the torsion of $c_1(\mathrm{Pic}(X))$. □

Next, we will assume X to be Kähler and want to look further into the first square of the commutative diagram:

(3.2)

$$\begin{array}{ccccccccc}
 & & H^1(X,\mathcal{O}) & \xrightarrow{\alpha} & H^1(X,\mathcal{O}^\times) & \xrightarrow{c_1} & H^2(X,\mathbb{Z}) & \longrightarrow & H^2(X,\mathcal{O}) \\
 & & \uparrow{\scriptstyle \iota_1^*} & & \uparrow{\scriptstyle \iota_2^*} & & \| & & \uparrow \\
H^1(X,\mathbb{Z}) & \xrightarrow{\sigma} & H^1(X,\mathbb{C}) & \xrightarrow{\beta} & H^1(X,\mathbb{C}^\times) & \xrightarrow{\gamma} & H^2(X,\mathbb{Z}) & \xrightarrow{\epsilon} & H^2(X,\mathbb{C}).
\end{array}$$

The universal coefficient theorem tells us that

$$\begin{aligned}
&H^1(X,\mathbb{Z}) \simeq \mathrm{Ext}(H_0(X),\mathbb{Z}) \oplus \mathrm{Hom}(H_1(X),\mathbb{Z}) \simeq \mathrm{Hom}(\mathbb{Z}^a,\mathbb{Z});\\
&H^1(X,\mathbb{C}) \simeq \mathrm{Ext}(H_0(X),\mathbb{C}) \oplus \mathrm{Hom}(H_1(X),\mathbb{C}) \simeq \mathrm{Hom}(\mathbb{Z}^a,\mathbb{C});\\
&H^1(X,\mathbb{C}^\times) \simeq \mathrm{Ext}(H_0(X),\mathbb{C}^\times) \oplus \mathrm{Hom}(H_1(X),\mathbb{C}^\times) \simeq \mathrm{Hom}(\mathbb{Z}^a,\mathbb{C}^\times) \oplus T_1.
\end{aligned}$$

Now we can rewrite the bottom exact sequence in (3.2) as:

$$\mathrm{Hom}(\mathbb{Z}^a,\mathbb{Z}) \xrightarrow{\sigma} \mathrm{Hom}(\mathbb{Z}^a,\mathbb{C}) \xrightarrow{\beta} \mathrm{Hom}(\mathbb{Z}^a,\mathbb{C}^\times) \oplus T_1.$$

Since $\mathrm{Hom}(\mathbb{Z}^a,-)$ is an exact functor,

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^\times \to 0$$

implies that $\mathrm{Im}\,\beta \simeq \mathrm{Hom}(\mathbb{Z}^a,\mathbb{C}^\times)$.

Therefore

$$\iota_2^*(\mathrm{Hom}(\mathbb{Z}^a,\mathbb{C}^\times)) \simeq \mathrm{Im}\,\iota_2^* \circ \beta = \mathrm{Im}\,\alpha \circ \iota_1^* = \mathrm{Pic}_0(X),$$

where the last equality comes from $(*)$ in Section 3.1.

Since $\mathbb{Z}^a \simeq H_1(X)_{\mathrm{free}} \simeq \overline{\pi_1(X)}_{\mathrm{free}}$ where $\overline{\pi_1(X)}$ means the abelianization of $\pi_1(X)$, using the isomorphism between $\widehat{\pi_1(X)}$ and $H^1(X,\mathbb{C}^\times)$ again we can conclude that

$$\lambda_{\tilde{X}}(\mathrm{Hom}(\overline{\pi_1(X)}_{\mathrm{free}},\mathbb{C}^\times)) = \mathrm{Pic}_0(X).$$

To sum up, we have:

Proposition 3.8. *Let X be a compact Kähler complex manifold, then the character map $\lambda_{\tilde{X}}$ restricted to Hom($\overline{\pi_1(X)}_{free}$, $\mathbb{C}^\times$) maps exactly onto Pic$_0(X)$.*

We will use these two results later.

3.3. **Further description of $\mathrm{Pic}_0(X)$ and kernel of $\lambda_{\tilde{X}}$.** The character map $\lambda_{\tilde{X}}$ is usually not injective.

First we want to give a more detailed description of $\mathrm{Pic}_0(X)$. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} & \xrightarrow{\exp} & S^1 & \longrightarrow & 0\\
 & & \| & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^\times & \longrightarrow & 0\\
 & & \| & & \downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle \iota_2} & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{O} & \xrightarrow{\exp} & \mathcal{O}^\times & \longrightarrow & 0
\end{array}$$

where S^1 denotes the unit circle. The horizontal lines are exact (but vertical lines are not.)

It induces a diagram of long exact sequences:

$$\begin{array}{ccccccccc}
H^0(X,S^1) & \xrightarrow{\delta_1} & H^1(X,\mathbb{Z}) & \longrightarrow & H^1(X,\mathbb{R}) & \xrightarrow{\exp_1^*} & H^1(X,S^1) & \longrightarrow & H^2(X,\mathbb{Z}) \\
\downarrow & & \| & & \downarrow{\scriptstyle i_1^*} & & \downarrow{\scriptstyle i_2^*} & & \| \\
H^0(X,\mathbb{C}^\times) & \xrightarrow{\delta_2} & H^1(X,\mathbb{Z}) & \longrightarrow & H^1(X,\mathbb{C}) & \xrightarrow{\exp_2^*} & H^1(X,\mathbb{C}^\times) & \longrightarrow & H^2(X,\mathbb{Z}) \\
\downarrow & & \| & & \downarrow{\scriptstyle \iota_1^*} & & \downarrow{\scriptstyle \iota_2^*} & & \| \\
H^0(X,\mathcal{O}^\times) & \xrightarrow{\delta_3} & H^1(X,\mathbb{Z}) & \longrightarrow & H^1(X,\mathcal{O}) & \xrightarrow{\exp_3^*} & H^1(X,\mathcal{O}^\times) & \longrightarrow & H^2(X,\mathbb{Z})
\end{array}$$

where the rows are exact but not necessarily the columns. By universal coefficient theorem and that $\mathrm{Hom}(\mathbb{Z},-)$ is exact, we see that $\delta_1,\delta_2,\delta_3$ are inclusions.

From the universal coefficient theorem it is also clear that

$$\mathrm{Im}(\exp_1^*) \simeq \mathrm{Hom}(\overline{\pi_1(X)}_{\mathrm{free}}, S^1), \quad \mathrm{Im}(\exp_2^*) \simeq \mathrm{Hom}(\overline{\pi_1(X)}_{\mathrm{free}}, \mathbb{C}^\times)$$

where $\overline{\pi_1(X)}_{\mathrm{free}}$ denotes the free part of the abelianization of $\pi_1(X)$. Since $\mathrm{Im}(\exp_3^*) = \mathrm{Pic}_0(X)$, we can rewrite the middle part of the sequences as

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(X,\mathbb{Z}) & \longrightarrow & H^1(X,\mathbb{R}) & \xrightarrow{\exp_1^*} & \mathrm{Hom}(\overline{\pi_1(X)}_{\mathrm{free}}, S^1) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle i_1^*} & & \downarrow{\scriptstyle i_2^*} & & \\
0 & \longrightarrow & H^1(X,\mathbb{Z}) & \longrightarrow & H^1(X,\mathbb{C}) & \xrightarrow{\exp_2^*} & \mathrm{Hom}(\overline{\pi_1(X)}_{\mathrm{free}}, \mathbb{C}^\times) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \iota_1^*} & & \downarrow{\scriptstyle \lambda_{\tilde{X}}} & & \\
0 & \longrightarrow & H^1(X,\mathbb{Z}) & \longrightarrow & H^1(X,\mathcal{O}) & \xrightarrow{\exp_3^*} & \mathrm{Pic}_0(X) & \longrightarrow & 0
\end{array}$$

Claim 3.9. *$\iota_1^* \circ i_1^*$ is an isomorphism between $H^1(X,\mathbb{R})$ and $H^1(X,\mathcal{O})$ as real vector spaces.*

Proof. Since we assume X to be Kähler, $H^1(X,\mathbb{C}) = H^1(X,\mathcal{O}) \oplus \overline{H^1(X,\mathcal{O})}$. For $H^1(X,\mathbb{R}) \subset H^1(X,\mathbb{C})$ is real,

$$H^1(X,\mathbb{R}) \cap H^1(X,\mathcal{O}) = H^1(X,\mathbb{R}) \cap \overline{H^1(X,\mathcal{O})} = 0.$$

Since ι_1^* is the projection from the Hodge decomposition, ker $(\iota_1^*) = \overline{H^1(X,\mathcal{O})}$. It follows that ker $(\iota_1^*) \cap H^1(X,\mathbb{R}) = 0$. Since i_1^* is an inclusion, ker $(\iota_1^* \circ i_1^*) \cap H^1(X,\mathbb{R}) = 0$. Thus $\iota_1^* \circ i_1^*$ is injective. Since $H^1(X,\mathbb{R})$ and $H^1(X,\mathcal{O})$ have the same dimension as real vector spaces, $\iota_1^* \circ i_1^*$ is an isomorphism. □

From the universal coefficient theorem it is clear that i_2^* is the natural inclusion. Now the first and third rows become

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(X,\mathbb{Z}) & \longrightarrow & H^1(X,\mathbb{R}) & \xrightarrow{\exp_1^*} & \mathrm{Hom}(\overline{\pi_1(X)}_{\mathrm{free}}, S^1) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \iota_1^* \circ i_1^*} & & \downarrow{\scriptstyle \lambda_{\tilde{X}} \circ i_2^*} & & \\
0 & \longrightarrow & H^1(X,\mathbb{Z}) & \longrightarrow & H^1(X,\mathcal{O}) & \xrightarrow{\exp_3^*} & \mathrm{Pic}_0(X) & \longrightarrow & 0.
\end{array}$$

Finally, the five-lemma yields

Proposition 3.10. *$\lambda_{\tilde{X}} \circ i_2^*$ gives an isomorphism of $Hom(\overline{\pi_1(X)}_{free}, S^1)$ and $Pic_0(X)$ as real tori.*

Now we want to describe the kernel of $\lambda_{\tilde{X}}$. Proposition 3.2 tells us that $\ker(\lambda_{\tilde{X}}) \simeq \ker(\iota_2^*)$. We consider the following exact sheaf sequences[1]:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & & \\
 & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow & & \\
0 & \longrightarrow & \mathbb{C} & \xrightarrow{\iota_1} & \mathcal{O} & \xrightarrow{d} & \Omega^1_c & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} & & \downarrow & & \\
0 & \longrightarrow & \mathbb{C}^\times & \xrightarrow{\iota_2} & \mathcal{O}^\times & \xrightarrow{d\log} & \Omega^1_c & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where Ω^1_c is the sheaf of holomorphic closed 1-forms on X. Then we have a long exact sequence:

$$0 \to H^0(X, \mathbb{C}^\times) \to H^0(X, \mathcal{O}^\times) \to H^0(X, \Omega^1_c) \xrightarrow{\delta} H^1(X, \mathbb{C}^\times) \xrightarrow{\iota_2^*} H^1(X, \mathcal{O}^\times) \to \cdots$$

Since X is compact, $H^0(X, \mathcal{O}^\times) \simeq \mathbb{C}^\times \simeq H^0(X, \mathbb{C}^\times)$. Thus δ is an injection. Thus $\ker(\iota_2^*) = \delta(H^0(X, \Omega^1_c)) \simeq H^0(X, \Omega^1_c)$. Therefore we have:

Proposition 3.11. *$ker(\lambda_{\tilde{X}}) \simeq H^0(X, \Omega^1_c)$.*

Proposition 3.7 and Proposition 3.8 tells us that there is a short exact sequence

$$0 \to \ker \lambda_{\tilde{X}} \to \mathrm{Hom}(\overline{\pi_1(X)}_{\mathrm{free}}, \mathbb{C}^\times) \xrightarrow{\lambda_{\tilde{X}}} \mathrm{Pic}_0(X) \to 0.$$

Proposition 3.10 tells us that $\lambda_{\tilde{X}}$ restricted to $\mathrm{Hom}(\overline{\pi_1(X)}_{\mathrm{free}}, S^1)$ is an isomorphism to $\mathrm{Pic}_0(X)$. Therefore the inverse of $\lambda_{\tilde{X}} \circ i_2^*$ splits this sequence of groups, and so we have

$$\mathrm{Hom}(\overline{\pi_1(X)}_{\mathrm{free}}, \mathbb{C}^\times) \simeq \ker \lambda_{\tilde{X}} \oplus \mathrm{Pic}_0(X)$$

(Note that this splitting is not holomorphic.)

Example 3.12. Let X be a compact Riemann surface with genus g. Then every homomorphic 1-form on X is closed, i.e. $\Omega^1_c = \Omega^1$. We know that $H^0(X, \Omega^1) \simeq \mathbb{C}^g$, thus in this case $\mathrm{Hom}(\pi_1(X), \mathbb{C}^\times) \simeq (\mathbb{C}^\times)^{2g}$, $\mathrm{Pic}_0(X) \simeq \mathbb{C}^g/\mathbb{Z}^{2g}$ and $\ker(\lambda_{\tilde{X}}) \simeq \mathbb{C}^g$.

[1] We thank Prof. Alan Mayer for pointing out this diagram to us.

4. Existence of CY bundles on Kähler manifolds

Let X be a connected compact Kähler manifold. Since we assume X to be compact, $H^2(X,\mathbb{Z})$ is a finitely generated abelian group. Then as a subgroup of $H^2(X,\mathbb{Z})$, $c_1(\mathrm{Pic}(X))$ is a finitely generated abelian group as well.

Let p be the rank of $c_1(\mathrm{Pic}(X))$ and $L_1,\cdots,L_p$ be holomorphic line bundles on X such that $\{c_1(L_1),\cdots,c_1(L_p)\}$ generate the free part of $c_1(\mathrm{Pic}(X))$. Let $M := L_1^\times \oplus L_2^\times \oplus \cdots \oplus L_p^\times$, then M is a principal $(\mathbb{C}^\times)^p$-bundle over X. Let $\tilde{M} := \tilde{X} \oplus M$ be the Whitney sum of $\tilde{X}$ and M over X. Let $\tilde{H} := \pi_1(X) \times (\mathbb{C}^\times)^p$. Then $\tilde{M}$ is a principal $\tilde{H}$-bundle over X.

Theorem 4.1. *The character map*

$$\lambda_{\tilde{M}} : \hat{\tilde{H}} \to Pic(X), \quad \chi \mapsto \tilde{M} \times_{\tilde{H}} \mathbb{C}_\chi$$

is surjective.

Proof. We are going to show that given any line bundle L on X, there exists an element χ in $\hat{\tilde{H}}$ such that $\tilde{M} \times_{\tilde{H}} \mathbb{C}_\chi = L$.

First note that

$$\hat{\tilde{H}} \simeq \widehat{\pi_1(X)} \times \widehat{(\mathbb{C}^\times)^p}$$

By assumption on $L_1,\ldots,L_p$, there exist integers $k_1,\ldots,k_p$ such that

$$\tilde{\sigma} := c_1(L) - (k_1 c_1(L_1) + \cdots + k_p c_1(L_p)) \in c_1(\mathrm{Pic}(X))_{\mathrm{tor}}.$$

By Proposition 3.7 there exists $\rho \in \widehat{\pi_1(X)}$ such that

$$c_1(\tilde{X} \times_{\pi_1(X)} \mathbb{C}_\rho) = \tilde{\sigma},$$

i.e.,

$$c_1(\tilde{X} \times_{\pi_1(X)} \mathbb{C}_\rho) = c_1(L - (k_1 L_1 + \cdots + k_p L_p))$$

Then

$$L_0 := L - (k_1 L_1 + \cdots + k_p L_p) - \tilde{X} \times_{\pi_1(X)} \mathbb{C}_\rho \in \mathrm{Pic}_0(X).$$

By Corollary 3.3 there exists $\gamma \in \widehat{\pi_1(X)}$ such that

$$L_0 \simeq \tilde{X} \times_{\pi_1(X)} \mathbb{C}_\gamma.$$

Then we have

$$\tilde{M} \times_{\tilde{H}} \mathbb{C}_{(\gamma\cdot\rho,\chi_0\cdots\chi_0)} \simeq L_0 + \tilde{X} \times_{\pi_1(X)} \mathbb{C}_\rho$$

where $\chi_0 \cdots \chi_0$ is the trivial character in $\widehat{(\mathbb{C}^\times)^p}$.

Theorem 2.5 tells us that

$$M \times_{(\mathbb{C}^\times)^p} \mathbb{C}_{\chi_{k_1}\cdots\chi_{k_p}} \simeq k_1 L_1 + \cdots + k_p L_p,$$

then

$$\tilde{M} \times_{\tilde{H}} \mathbb{C}_{(e,\chi_{k_1}\cdots\chi_{k_p})} \simeq k_1 L_1 + \cdots + k_p L_p$$

where e denotes the trivial character $e : \pi_1(X) \to \mathbb{C}^\times, \rho \to 1, \forall \rho \in \pi_1(X)$.

Since

$$\hat{\tilde{H}} \to Pic(X), \quad \chi \mapsto \tilde{M} \times_{\tilde{H}} \mathbb{C}_\chi$$

is a group homomorphism, and

$$(e, \chi_{k_1} \cdots \chi_{k_p}) \cdot (\gamma \cdot \rho, \chi_0 \cdots \chi_0) = (\gamma \cdot \rho, \chi_{k_1} \cdots \chi_{k_p}),$$

we have

$$\tilde{M} \times_{\tilde{H}} \mathbb{C}_{(\gamma\cdot\rho, \chi_{k_1} \cdots \chi_{k_p})} \simeq L_0 + \tilde{X} \times_{\pi_1(X)} \mathbb{C}_\rho + k_1 L_1 + \cdots + k_p L_p = L.$$

I.e., $\tilde{M} \times_{\tilde{H}} \mathbb{C}_\chi \simeq L$ where $\chi = (\gamma \cdot \rho, \chi_{k_1} \cdots \chi_{k_p})$. □

In particular, since $K_X \in \mathrm{Pic}(X)$, we conclude that:

Corollary 4.2. *$\tilde{H} - \tilde{M} \to X$ is a CY bundle over X whose character map is onto.*

Unlike the case where $\mathrm{Pic}(X)$ is free, the character map $\lambda_{\tilde{M}}$ is usually not injective.

We have $\mathrm{Hom}(\tilde{H}, \mathbb{C}^\times) = \widehat{\pi_1(X)} \times \widehat{H}$. Consider the character map $\lambda_{\tilde{M}}$ restricted to $\widehat{H}$:

$$\mathbb{Z}^p \simeq \widehat{H} \to \mathrm{Pic}(X),\ \chi := (e, \chi_{k_1} \cdots \chi_{k_p}) \mapsto \tilde{M} \times_{\tilde{H}} \mathbb{C}_\chi = k_1 L_1 + \cdots + k_p L_p.$$

From the choice of $L_1, \ldots, L_p$ we know that $c_1(L_1), \ldots, c_1(L_p)$ are $\mathbb{Z}$-independent, which implies that $L_1, \ldots, L_p$ are also $\mathbb{Z}$-independent, meaning that the above map is an injection. Thus we can conclude that

$$\ker \lambda_{\tilde{M}} \simeq \ker \lambda_{\tilde{X}} \simeq \ker \iota_2^*.$$

Therefore we have:

Proposition 4.3. *$\ker \lambda_{\tilde{M}} \simeq H^0(X, \Omega_c^1)$.*

5. Existence of CY bundles for abelian structure groups

In this section, we assume that H is a connected abelian complex Lie group. We will give a sufficient condition for the existence of a principal CY H-bundle whose character map is onto.

5.1. Some preparation.

• Connected abelian complex Lie groups.

Definition 5.1. *A connected abelian complex Lie group G having no non-constant holomorphic functions is called a Cousin group.*

There is a very nice result on the classification of connected abelian complex groups that we will need:

Theorem 5.2 (Remmert-Morimoto decomposition)**.** *Any connected abelian complex Lie group G is holomorphically isomorphic to a group of the form*

$$(\mathbb{C}^\times)^a \times \mathbb{C}^b \times G_0,$$

where G_0 is a Cousin group. Moreover, $a, b \in \mathbb{Z}$ and G_0 (up to isomorphism) are uniquely determined by G.

• **Principal bundles.** We want to make use of our previous construction of CY bundles, where $\tilde{H} = \pi_1(X) \times (\mathbb{C}^\times)^a$ is neither abelian nor connected in general. The idea is to construct new principal bundles out of the old one.

Given a holomorphic Lie group homomorphism $f : H \to K$, it induces a homomorphism $\circ f : \widehat{H} \to \widehat{K},\ \chi \mapsto \chi \circ f$. f induces a holomorphic action of H on K as well: $H \times K \to K,\ (h, k) \mapsto f(h) \cdot k$.

If we are given a principal H bundle M over X, let $N := M \times_H K$ where the action of H on K is induced by f. Then N is a holomorphic K-bundle over X. Define a K-action on N by right translation on fibers:

$$K \times N \to N,\ (k, [m, l]) \mapsto [m, l \cdot k^{-1}].$$

It is clear that this action preserves the fibers and acts freely and transitively on them. Thus with this action N is a principal K-bundle over X, we call it *the principal K-bundle induced by f and M.*

Proposition 5.3. *If N is the principal K-bundle induced by f and M, then the diagram of character maps*

$$\begin{array}{ccc} \widehat{K} & \xrightarrow{\lambda_N} & Pic(X) \\ {\scriptstyle \circ f}\downarrow & \nearrow{\scriptstyle \lambda_M} & \\ \widehat{H} & & \end{array}$$

commutes.

Proof. It suffices to show that given a character $\chi \in \widehat{K}$, there is a holomorphic line bundle isomorphism $N \times_K \mathbb{C}_\chi \simeq M \times_H \mathbb{C}_{\chi\circ f}$. Define a map

$$\begin{aligned} \xi : (M \times_H K) \times_K \mathbb{C}_\chi &\to M \times_H \mathbb{C}_{\chi\circ f} \\ [[m, l], c] &\mapsto [m, \chi(l)c] \end{aligned}$$

Since

$$\begin{aligned} &\xi([[mh^{-1}, f(h)l], c]) = [mh^{-1}, \chi(f(h)l)c] = [mh^{-1}, \chi(f(h))\chi(l)c] = [m, \chi(l)c] \\ &\xi([[m, lk^{-1}], \chi(k)c]) = [m, \chi(lk^{-1})\chi(k)c] = [m, \chi(l)c], \end{aligned}$$

ξ is well defined. It is clear that ξ induces a linear isomorphism on fibers and commutes with quotients to X, thus it is an isomorphism between holomorphic line bundles. □

5.2. **Principal $\mathbb{C}^q$-bundles.** In this subsection let X be a compact Kähler complex manifold.

The abelianization of $\pi_1(X)$ gives a Lie group homomorphism $\mathrm{Ab} : \pi_1(X) \to \overline{(\pi_1(X))}$. Let $\bar{X} := \tilde{X} \times_{\pi_1(X)} \overline{\pi_1(X)}$ be the principal $\overline{\pi_1(X)}$-bundle induced by Ab and $\tilde{X}$. Then $\bar{X}$ is a cover of X whose deck transformation group is $\overline{\pi_1(X)}$.

Proposition 5.4. *The image of the character map of $\bar{X}$ contains $Pic_0(X)$. Moreover, $\lambda_{\bar{X}}(Hom(\overline{\pi_1(X)}_{free}, \mathbb{C}^\times)) = Pic_0(X)$.*

Proof. Proposition 5.3 implies the following diagram

$$\begin{array}{ccc} \widehat{\overline{\pi_1(X)}} & \xrightarrow{\lambda_{\bar{X}}} & \mathrm{Pic}(X) \\ {\scriptstyle \circ\mathrm{Ab}}\downarrow & \nearrow{\scriptstyle \lambda_{\tilde{X}}} & \\ \widehat{\pi_1(X)} & & \end{array}$$

commutes. Since $\mathbb{C}^\times$ is abelian, $\circ\mathrm{Ab}$ is an isomorphism. From Corollary 3.3 we know that $\mathrm{Pic}_0(X)$ is contained in the image of $\lambda_{\tilde{X}}$, thus the image of $\lambda_{\bar{X}}$ contains $\mathrm{Pic}_0(X)$ as well.

Proposition 3.8 shows that $\lambda_{\tilde{X}}(\mathrm{Hom}(\overline{\pi_1(X)}_{\mathrm{free}}, \mathbb{C}^\times)) = \mathrm{Pic}_0(X)$, composing with the isomorphism $\circ\mathrm{Ab}$ we have $\lambda_{\bar{X}}(\mathrm{Hom}(\overline{\pi_1(X)}_{\mathrm{free}}, \mathbb{C}^\times)) = \mathrm{Pic}_0(X)$. $\square$

Let q be the rank of $\overline{\pi_1(X)}$. The goal is to construct a principal $\mathbb{C}^q$-bundle whose character map contains $\mathrm{Pic}_0(X)$ in its image.

Since $\overline{\pi_1(X)}$ is finitely generated, there is an isomorphism $\phi : \overline{\pi_1(X)} \xrightarrow{\simeq} \mathbb{Z}^q \oplus T$ for some finite group T. Let $\mathrm{pr} : \mathbb{Z}^q \oplus T \to \mathbb{Z}^q$ be the projection to the first factor. Then we have a homomorphism $\mathrm{pr} \circ \phi : \overline{\pi_1(X)} \to \mathbb{Z}^q$. Let $N := \bar{X} \times_{\overline{\pi_1(X)}} \mathbb{Z}^q$ be the principal $\mathbb{Z}^q$-bundle induced by $\mathrm{pr} \circ \phi$ and $\bar{X}$. Then it follows from Proposition 5.3 that there is a commutative diagram

$$\begin{array}{ccc} \widehat{\mathbb{Z}^q} & \xrightarrow{\lambda_N} & \mathrm{Pic}(X) \\ {\scriptstyle \circ\mathrm{pr}\circ\phi}\downarrow & \nearrow{\scriptstyle \lambda_{\bar{X}}} & \\ \widehat{\overline{\pi_1(X)}}. & & \end{array}$$

Since $\widehat{\overline{\pi_1(X)}} \simeq \widehat{\overline{\pi_1(X)}_{\mathrm{free}}} \times \widehat{\overline{\pi_1(X)}_{\mathrm{tor}}}$ and it is clear that $\mathrm{pr} \circ \phi$ induces an isomorphism from $\widehat{\mathbb{Z}^q}$ to $\widehat{\overline{\pi_1(X)}_{\mathrm{free}}}$. Then by Proposition 5.4 $\mathrm{Pic}_0(X)$ is contained in the image of the composition map λ_N.

Next we consider the inclusion $\iota : \mathbb{Z}^q \to \mathbb{C}^q$. Let $Q := N \times_{\mathbb{Z}^q} \mathbb{C}^q$ be the principal $\mathbb{C}^q$-bundle induced by ι and N. Then from Proposition 5.3 we have a commutative diagram

$$\begin{array}{ccc} \widehat{\mathbb{C}^q} & \xrightarrow{\lambda_Q} & \mathrm{Pic}(X) \\ {\scriptstyle \circ\iota}\downarrow & \nearrow{\scriptstyle \lambda_N} & \\ \widehat{\mathbb{Z}^q}. & & \end{array}$$

Since $\mathbb{C}^\times$ is a divisible group, $\mathrm{Hom}(-, \mathbb{C}^\times)$ is an exact contravariant functor. So the following sequence

$$\mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^\times) \xrightarrow{\circ\iota} \mathrm{Hom}(\mathbb{Z}^q, \mathbb{C}^\times) \to 0$$

is exact, meaning that $\circ\iota$ is surjective. Thus $\mathrm{Pic}_0(X)$ is contained in the image of the character map λ_Q.

To sum up, we have:

Theorem 5.5. *Let X be a compact Kähler manifold and q be the rank of $\overline{\pi_1(X)}$. Then there exists a principal $\mathbb{C}^q$-bundle over X such that the image of its character map contains $Pic_0(X)$.*

5.3. **General case.** In this subsection, assume X to be a connected compact Kähler manifold. We now assemble results of the preceding sections to construct CY bundles over X with arbitrary abelian structure groups.

Since $\mathrm{Pic}(X)/\mathrm{Pic}_0(X) \simeq c_1(\mathrm{Pic}_(X)) \subset H^2(X)$, it is finitely generated. Let $p \in \mathbb{Z}$ be the minimum number of generators of $c_1(\mathrm{Pic}_(X))$ and let $\{c_1(L_1), \ldots, c_1(L_p)\}$ be a set of generators. Let q be the rank of $\overline{\pi_1(X)}$ and Q be the principal $\mathbb{C}^q$-bundle over X such that $\mathrm{Pic}_0(X) \subset \mathrm{Im}\,\lambda_Q$, as described in the previous subsection. Let $P := L_1^\times \oplus L_2^\times \oplus \cdots \oplus L_p^\times \oplus Q$, then it is a principal $((\mathbb{C}^\times)^p \times \mathbb{C}^q)$-bundle over X.

Proposition 5.6. *P is a CY $((\mathbb{C}^\times)^p \times \mathbb{C}^q)$-bundle whose character map is onto.*

Proof. First, we can identify $\widehat{(\mathbb{C}^\times)^p \times \mathbb{C}^q} \equiv \widehat{(\mathbb{C}^\times)^p} \times \widehat{\mathbb{C}^q}$.

By assumption on $L_1, \ldots, L_p$, there exist integers $k_1, \ldots, k_p$ such that $c_1(L) = k_1 c_1(L_1) + \cdots + k_p c_1(L_p)$. Thus

$$L_0 := L - (k_1 L_1 + \cdots + k_p L_p) \in \mathrm{Pic}_0(X).$$

By Theorem 2.5 we know that

$$P \times_{(\mathbb{C}^\times)^p \times \mathbb{C}^q} \mathbb{C}_{(\chi_{k_1} \cdots \chi_{k_p}, e)} \simeq k_1 L_1 + \cdots + k_p L_p$$

where e denotes the trivial character $e : \mathbb{C}^q \to \mathbb{C}^\times, a \mapsto 1, \forall a \in \mathbb{C}^q$.

By Theorem 5.5, there exists $\gamma \in \widehat{\mathbb{C}^q}$ such that

$$L_0 \simeq Q \times_{\mathbb{C}^q} \mathbb{C}_\gamma.$$

Thus we have

$$P \times_{(\mathbb{C}^\times)^p \times \mathbb{C}^q} \mathbb{C}_{(\chi_0 \cdots \chi_0, \gamma)} \simeq L_0$$

where $\chi_0 \cdots \chi_0$ is the trivial character in $\widehat{(\mathbb{C}^\times)^p}$. Hence

$$P \times_{(\mathbb{C}^\times)^p \times \mathbb{C}^q} \mathbb{C}_{(\chi_{k_1} \cdots \chi_{k_p}, \gamma)} \simeq k_1 L_1 + \cdots + k_p L_p + L_0 = L.$$

I.e., $\lambda_P(\chi) = L$ where $\chi = (\chi_{k_1} \cdots \chi_{k_p}, \gamma)$.

Thus λ_P is onto and therefore P is a CY bundle over X. □

Theorem 5.7. *If H is holomorphically isomorphic to $(\mathbb{C}^\times)^a \times \mathbb{C}^b \times G_0$ for some $a \geq p$ and $b \geq q$, where p, q are defined as above. Then there exists a CY H-bundle whose character map is onto.*

Proof. The preceding proposition tells us that P defined above is a CY $((\mathbb{C}^\times)^p \times \mathbb{C}^q)$-bundle whose character map is onto. Let

$$N := P \oplus (X \times ((\mathbb{C}^\times)^{a-p} \times \mathbb{C}^{b-q} \times G_0)).$$

Then N is a principal $((\mathbb{C}^\times)^a \times \mathbb{C}^b \times G_0)$-bundle.

Since $X \times ((\mathbb{C}^\times)^{a-p} \times \mathbb{C}^{b-q} \times G_0)$ is the trivial $((\mathbb{C}^\times)^{a-p} \times \mathbb{C}^{b-q} \times G_0)$-bundle over X, its character map is trivial. It follows that the image of the character map of N is the same as that of P, thus N is a CY $((\mathbb{C}^\times)^a \times \mathbb{C}^b \times G_0)$-bundle whose character map is onto.

Let $\psi : (\mathbb{C}^\times)^a \times \mathbb{C}^b \times G_0 \to H$ be a holomorphic isomorphism. Let $M := N \times_{(\mathbb{C}^\times)^a \times \mathbb{C}^b \times G_0} H$ be the principal H-bundle induced by ψ and N. The by Proposition 5.3 the following diagram is commutative

$$\begin{array}{ccc} \widehat{H} & \xrightarrow{\lambda_M} & Pic(X) \\ \downarrow{\scriptstyle \circ\psi} & \nearrow{\scriptstyle \lambda_N} & \\ \widehat{(\mathbb{C}^\times)^a \times \mathbb{C}^b} \times G_0. & & \end{array}$$

Since ψ is an isomorphism, $\circ\psi$ is an isomorphism as well. Thus the image of λ_M is the same as that of λ_N, i.e. λ_M is onto. □

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