

Counting Unimodular Lattices in $\mathbf{R}^{r,s}$

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Abstract. Narain lattices are unimodular lattices in $\mathbf{R}^{r,s}$, subject to certain natural equivalence relation and rationality condition. The problem of describing and counting these rational equivalence classes of Narain lattices in $\mathbf{R}^{2,2}$ has led to an interesting connection to binary forms and their Gauss products, as shown in [HLOYII]. As a sequel, in this paper, we study arbitrary rational Narain lattices and generalize some of our earlier results. In particular in the case of $\mathbf{R}^{2,2}$, a new interpretation of the Gauss product of binary forms brings new light to a number of related objects – rank 4 rational Narain lattices, over-lattices, rank 2 primitive sublattices of an abstract rank 4 even unimodular lattice U^2 , and isomorphisms of discriminant groups of rank 2 lattices.

1. Introduction

The Main Problem. Let E a real quadratic space, i.e. a real vector space equipped with a non-degenerate bilinear form \langle, \rangle of signature (r, s) . Assume that 8 divides $r - s$. Fix a non-degenerate linear subspace $V \subset E$. An even integral unimodular lattice $\Gamma \subset E$ of signature (r, s) is called a Narain lattice in E . If $\text{rank } \Gamma \cap V = \dim V$, we say that Γ is V -rational. The problem is to describe and count V -rational Narain lattices up to the action of the orthogonal subgroup $O(V) \times O(V^\perp) \subset O(E)$. For $r, s > 0$, Narain lattices are parameterized by the set $O(E)/O(\Gamma_0)$, where Γ_0 is a fixed Narain lattice. Geometrically in this case, the problem amounts to describing and counting the “rational points” in the homogeneous space $O(V) \times O(V^\perp) \backslash O(E)/O(\Gamma_0)$, where Γ_0 is a fixed V -rational Narain lattice.

For simplicity, we will concentrate on the case $E = \mathbf{R}^{n,n}$ throughout most of the paper, and return to the general case only at the end. The study of general V -rational Narain lattices in $\mathbf{R}^{n,n}$ leads us to consider the following classes of objects, each of which comes equipped with its own suitable equivalence relation (to be described later):

- i. Rational Narain lattices in $\mathbf{R}^{n,n}$;
- ii. Rank $2n$ even unimodular over-lattices and triples;

- iii. Rank n primitive sublattices of U^n ;
- iv. Isomorphisms of discriminant groups;
- v. Pairs of binary quadratic forms and their Gauss products, when $n = 2$.

Here U is an abstract even integral unimodular hyperbolic rank 2 lattice. Narain lattices in $E = \mathbf{R}^{n,n}$ arise in physics as a way to build modular invariant string theory models [Na][NSW][Po] and conformal field theories [Mo][GV][W]. Here one sets $V = \mathbf{R}^{n,0} \subset E$, and let $E \rightarrow V$, $x \mapsto x_L$, and $E \rightarrow V^\perp$, $x \mapsto x_R$, denote the orthogonal projections. In string theory [Na], to each even unimodular lattice Γ in E , one attaches the real analytic function

$$Z_\Gamma(\tau) = \frac{1}{|\eta(\tau)|^{2n}} \sum_{x \in \Gamma} e^{\pi i \tau \langle x_L, x_L \rangle} e^{\pi i \bar{\tau} \langle x_R, x_R \rangle}$$

known as a partition function, defined on the complex upper half plane. By Poisson summation (cf. [Po][S]), one shows that $Z_\Gamma(\tau)$ is modular invariant. An important problem in string theory is to understand the behavior of the modular functions $Z_\Gamma(\tau)$ as one deforms the “moduli” Γ (see [HLOYII] and references therein).

The theory of over-lattices, also known as gluing theory, came up in coding theory (see [CS] p99 and references therein), and in the study of primitive embeddings of lattices [Ni], in which discriminant groups also play an important role. The latter was a key ingredient in our recent solution to the counting problem of Fourier Mukai partners for K3 surfaces in [HLOYI]. More recently, the same objects (all but iii.) have also arise in the study of rational conformal field theory on real tori [HLOYII]. There, the Gauss product of positive definite coprime binary forms were essential in giving a geometric description of those rational conformal field theories.

The approach taken in this paper is entirely algebraic. Let’s begin with an idea that goes back essentially to Gauss. Given two binary quadratic forms $f(x, y), g(x, y)$ of the same discriminant, Gauss defined his composition law by constructing a third form $h(x, y)$ such that

$$f(x, y)g(x', y') = h(X, Y)$$

where X, Y are certain integral linear forms of x, y, x', y' . This classical construction has, of course, long been superseded by the modern theory of ideal class groups for general Dedekind fields. The key new idea in this paper is this: that Gauss’ original construction of his composition law, in fact, takes on a particularly interesting and potent form, when interpreted *inside* the rank 4 lattice of 2×2 integral matrices with the quadratic form $2 \det X$. This special rank 4 lattice is abstractly isomorphic to U^2 . But because it is also a *ring* with lots of symmetry, all the algebraic structures that come with it can be brought to bear on the study of sublattices. Moreover there is a very interesting *duality*: a binary form can be viewed as a point (an integral symmetric matrix), and a rank 2 primitive sublattice (equipped with a basis) can be view as a quadratic form. It is this duality, together with the algebraic structures of the matrix ring, that makes our algebraic approach work. The details are spelled out in Section 5.

Here is an outline of the paper. Theorem 3.12 gives a 1-1 correspondence between equivalence classes of objects in i. and those in ii. Theorem 4.8 does that same for objects

in ii. and iii. Theorem 5.13 shows precisely how the Gauss products of coprime binary forms enter the description of iii. through a new map Λ when $n = 2$. This is where the quadratic form $2\det X$ on matrices comes in. We also describe briefly the connection between ii. and iv. Finally, Theorems 6.5 and 6.6 use Λ to give a description of iii., hence culminating in a description of V -rational Narain lattices in $\mathbf{R}^{2,2}$, in terms of binary forms. In the last section, we apply this to the positive definite case and recover a result in [HLOYII]. We also discuss the indefinite case as a new application. Finally, we comment on some further interesting generalizations in the last section.

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1.1. General conventions

- Let X, Y be sets equipped with a transformation group G . Let $f : X \rightarrow Y$ be a map of sets. We say that f *descends through* G if f sends each G -orbit into a G -orbit, i.e. $\forall g \in G, \forall x \in X, f(g \cdot x) = g' \cdot f(x)$ for some $g' \in G$. In this case, the composition $X \rightarrow Y \rightarrow Y/G$ descends to an induced map $X/G \rightarrow Y/G$. We say that f is *G -equivariant* if f commutes with the G -action, i.e. $\forall g \in G, \forall x \in X, f(g \cdot x) = g \cdot f(x)$. In this case, f descends through *any* subgroup $K \subset G$. We say that f *factors through* G if f sends each G -orbit to a point, i.e. $\forall g \in G, \forall x \in X, f(g \cdot x) = f(x)$. In this case, f descends to an induced map $X/G \rightarrow Y$.

- If the transformation group G on X is of the form $H \times K$, we can let H acts on the left and K on the right, and denote a G -orbit by $H \cdot x \cdot K$. The orbit space in this case is denoted by the double quotient $H \backslash X / K$. Likewise, we can also speak of the orbit spaces $H \backslash X$ and X / K . If X is a group and H, K are subgroups, then these are the usual left and right coset spaces.

- If $f : X \rightarrow Y$ is an isomorphism of lattices, we denote by $f^* : X^* \rightarrow Y^*$ (the direction is not reversed!) the inverse of the dual isomorphism between their dual lattices. We denote by $f^* : A_X \rightarrow A_Y$ the the induced isometry of discriminant groups.

- We will only consider *even* integral lattices X , i.e. a free abelian group of finite rank equipped with an even *non-degenerate* integral quadratic form \langle, \rangle . The discriminant of a lattice X is defined to be $(-1)^{n-1}$ times the determinant of the matrix $\langle v_i, v_j \rangle$ for a given \mathbf{Z} -base v_i of X . The sign is chosen to make it consistent with the rank 2 case. It is clear that the discriminant is invariant under a change of base, hence it is a well-defined invariant of the lattice. If X is an even lattice equipped with \langle, \rangle , and $r \in \mathbf{Q}$ is a nonzero number such that the scalar multiple $r\langle, \rangle$ remains even integral, then we denote by $X(r)$ the same abelian group but equipped with the form $r\langle, \rangle$. In particular, $X(-1)$ is the lattice with the sign of the form reversed. We say that X is *primitive* if $m = 1$ is the only

positive integer such that $X(\frac{1}{m})$ is even integral. Thus the integral binary quadratic form $[a, b, c] \equiv (\mathbf{Z}^2, \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix})$ is primitive iff $\gcd(a, b, c) = 1$.

- The term primitive is used in one other (somewhat confusing) way. We say that $X \subset Y$ is a *primitive sublattice* of Y if the inclusion is isometric and Y/X is torsion free. Note that this does not mean that Y is primitive, as an abstract lattice.

- The symbol σ is used throughout the paper, but to mean different things in a few different but related contexts. For example, $\sigma : \mathbf{R}^{n,n} \rightarrow \mathbf{R}^{n,n}$ will be an involutive anti-isometry. It induces an involution on the set of rational Narain lattices which we denote by $\sigma : \mathcal{R} \rightarrow \mathcal{R}$. On the set \mathcal{T} of triples (X, Y, φ) , of lattices X, Y and isometry $\varphi : A_Y \rightarrow A_X$, we have an involution $(X, Y, \varphi) \mapsto (Y, X, \varphi^{-1})$. We denote this by $\sigma : \mathcal{T} \rightarrow \mathcal{T}$. The permutation $(P, Q) \mapsto (Q, P)$ acting on the set \mathcal{P} of pairs of quadratic forms. We denote this by $\sigma : \mathcal{P} \rightarrow \mathcal{P}$. All of these will eventually be related.

2. Narain Lattices

- Notations:

$\mathbf{R}^{r,s}$: \mathbf{R}^{r+s} equipped with the quadratic form $\langle x, x \rangle = x_1^2 + \dots + x_r^2 - x_{r+1}^2 - \dots - x_{r+s}^2$.

U : the hyperbolic even unimodular lattice $\mathbf{Z}e \oplus \mathbf{Z}f$ with $\langle e, e \rangle = \langle f, f \rangle = 0$, $\langle e, f \rangle = 1$.

U^n : the direct sum of n copies of U ; we name the i th copy $\mathbf{Z}e_i \oplus \mathbf{Z}f_i$.

$\varepsilon_1, \dots, \varepsilon_{2n}$: the standard basis of \mathbf{R}^{2n} ; $\varepsilon_i = (0, \dots, 0, 1, 0, \dots, 0)$, 1 being at the i th slot.

$E_i^e := \frac{1}{\sqrt{2}}(\varepsilon_i + \varepsilon_{n+i})$, $F_i^e := \frac{1}{\sqrt{2}}(\varepsilon_i - \varepsilon_{n+i})$: called the standard Narain basis of $\mathbf{R}^{n,n}$.

$O(\mathbf{R}^{n,n})$: the group of linear isometries of $\mathbf{R}^{n,n}$.

$O(\Gamma^e)$: the subgroup of $O(\mathbf{R}^{n,n})$ preserving the special even unimodular lattice

$$\Gamma^e := \sum_i (\mathbf{Z}E_i^e + \mathbf{Z}F_i^e) \subset \mathbf{R}^{n,n}.$$

Definition 2.1. *A Narain embedding is an isometric embedding*

$$\nu : U^n \hookrightarrow \mathbf{R}^{n,n}.$$

The set of Narain embeddings is denoted by \mathcal{I}_n . An even unimodular lattice $\Gamma \subset \mathbf{R}^{n,n}$ of signature (n, n) is called a Narain lattice. The set of Narain lattices is denoted by \mathcal{N}_n . The subscript n will be suppressed when there is no confusion.

- Since a Narain embedding ν is an isometry, the images $\nu(e_i), \nu(f_i) \in \mathbf{R}^{n,n}$ form a basis having the same inner products as the basis vectors e_i, f_i of U^n . We shall call such an

ordered \mathbf{R} -basis of $\mathbf{R}^{n,n}$ a Narain basis. Conversely, any Narain basis E_i, F_i of $\mathbf{R}^{n,n}$ defines a Narain embedding ν by declaring

$$\nu(e_i) = E_i, \quad \nu(f_i) = F_i.$$

Thus Narain embeddings correspond 1-1 with Narain bases. Denote by ν_e the Narain embedding corresponding to the special Narain basis E_i^e, F_i^e .

Lemma 2.2. *There is a unique $O(\mathbf{R}^{n,n})$ -equivariant bijection $O(\mathbf{R}^{n,n}) \rightarrow \mathcal{I}$ such that $e \mapsto \nu_e$.*

Proof: An element $g \in O(\mathbf{R}^{n,n})$ acts on Narain embeddings ν by left translations:

$$\nu \mapsto g \circ \nu.$$

The action is transitive because any two Narain bases of $\mathbf{R}^{n,n}$ are related by a *unique* $g \in O(\mathbf{R}^{n,n})$. Conversely, given a Narain basis, its image under a $g \in O(\mathbf{R}^{n,n})$ forms another Narain basis. Thus the map

$$O(\mathbf{R}^{n,n}) \rightarrow \mathcal{I}, \quad g \mapsto \nu_g := g \circ \nu_e.$$

is a bijection. It clearly has the asserted uniqueness and equivariance property. \square

Corollary 2.3. *The composition $O(\mathbf{R}^{n,n}) \rightarrow \mathcal{I} \rightarrow \mathcal{N}$, $g \mapsto \nu_g \mapsto \nu_g(U^n)$, descends to a bijection*

$$O(\mathbf{R}^{n,n})/O(\Gamma^e) \xrightarrow{\sim} \mathcal{N}, \quad g \cdot O(\Gamma^e) \mapsto \nu_g(U^n).$$

Proof: It is clear that the map $O(\mathbf{R}^{n,n}) \rightarrow \mathcal{N}$ factors through $O(\Gamma^e)$, which preserves the lattice $\nu_e(U^n) = \Gamma^e$. By Milnor's theorem every Narain lattice $\Gamma \subset \mathbf{R}^{n,n}$ is in the image of $\mathcal{I} \rightarrow \mathcal{N}$. Thus $O(\mathbf{R}^{n,n})/O(\Gamma^e) \rightarrow \mathcal{N}$ is surjective. Equivalently $O(\mathbf{R}^{n,n})$ acts transitively on \mathcal{N} . With the based point $\nu_e(U^n) \in \mathcal{N}$, the isotropy group is $O(\Gamma^e)$. Thus we have a bijection. \square

2.1. V -equivalence

•Once and for all, we fix a *non-degenerate* linear subspace $V \subset \mathbf{R}^{n,n}$, i.e. $\langle, \rangle|_V$ non-degenerate. The notation V^\perp will always mean the orthogonal complement of V in $\mathbf{R}^{n,n}$. At the end, for $n = 2$, we shall specialize V to the case $V = \mathbf{R}^{2,0} = \{(*, *, 0, 0)\} \subset \mathbf{R}^{2,2}$ or $V = \mathbf{R}^{1,1} = \{(*, 0, *, 0)\} \subset \mathbf{R}^{2,2}$.

•Since $\mathbf{R}^{n,n} = V \oplus V^\perp$ canonically, we have the canonical inclusions of isometry groups

$$O(V), O(V^\perp) \subset O(V) \times O(V^\perp) \subset O(\mathbf{R}^{n,n}).$$

Note that the middle group is the subgroup of $O(\mathbf{R}^{n,n})$ preserving the decomposition $V \oplus V^\perp$, i.e. an element $f \in O(\mathbf{R}^{n,n})$ is in $O(V) \times O(V^\perp)$ iff $fV = V$ and $fV^\perp = V^\perp$.

Definition 2.4. *Two Narain lattices Γ, Γ' are called V -equivalent if $\Gamma' = g\Gamma$ for some $g \in O(V) \times O(V^\perp)$.*

Corollary 2.5. *The map $O(\mathbf{R}^{n,n})/O(\Gamma^e) \xrightarrow{\sim} \mathcal{N}$ above descends to V -equivalence classes, i.e.*

$$\begin{aligned} O(V) \times O(V^\perp) \backslash O(\mathbf{R}^{n,n})/O(\Gamma^e) &\xrightarrow{\sim} O(V) \times O(V^\perp) \backslash \mathcal{N}, \\ O(V) \times O(V^\perp) \cdot g \cdot O(\Gamma^e) &\mapsto O(V) \times O(V^\perp) \cdot \nu_g(U^n). \end{aligned}$$

•Whenever convenient, we shall identify Narain bases with Narain embeddings, and with $O(\mathbf{R}^{n,n})$ via

$$(\nu_g(e_1), \dots, \nu_g(e_n), \nu_g(f_1), \dots, \nu_g(f_n)) \equiv \nu_g \equiv g,$$

and \mathcal{N} with $O(\mathbf{R}^{n,n})/O(\Gamma^e)$ via

$$\nu_g(U^n) \equiv g \cdot O(\Gamma^e).$$

2.2. An involutive anti-isometry σ

Throughout this paper, we assume that V has $\dim V = n$ and has signature $(p, n-p)$. Then V^\perp has signature $(n-p, p)$, hence $V \cong V^\perp(-1)$ as quadratic spaces. We fix an anti-isometry $\sigma : V \rightarrow V^\perp$, and define the linear map

$$\mathbf{R}^{n,n} \rightarrow \mathbf{R}^{n,n}, \quad \mathbf{R}^{n,n} = V \oplus V^\perp \ni (x, y) \mapsto (\sigma^{-1}y, \sigma x).$$

This is clearly an involutive anti-isometry of $\mathbf{R}^{n,n}$ exchanging the two subspaces $V, V^\perp \subset \mathbf{R}^{n,n}$. We also denote this involution by σ . Note that if Γ is a Narain lattice, then its image $\sigma\Gamma \subset \mathbf{R}^{n,n}$ is again an even unimodular lattice as a subset of the quadratic space $\mathbf{R}^{n,n}$, hence $\sigma\Gamma$ is a Narain lattice. So we have an involution $\mathcal{N} \rightarrow \mathcal{N}$, $\Gamma \mapsto \sigma\Gamma$, which we also denote by σ .

Lemma 2.6. *The correspondence $g \mapsto \sigma g \sigma$, is a well-defined involution on $O(\mathbf{R}^{n,n})$, hence on the set of Narain embeddings. This involution also stabilizes the subgroups $O(V) \times O(V^\perp)$, and the identity component $O_0(\mathbf{R}^{n,n})$. If $\sigma\Gamma^e = \Gamma^e$, then σ stabilizes $O(\Gamma^e)$ as well.*

Proof: For $x \in \mathbf{R}^{n,n}$ and $g \in O(\mathbf{R}^{n,n})$, we have

$$\langle \sigma g \sigma x, \sigma g \sigma x \rangle = -\langle g \sigma x, g \sigma x \rangle = -\langle \sigma x, \sigma x \rangle = \langle x, x \rangle.$$

This shows that $\sigma g \sigma \in O(\mathbf{R}^{n,n})$. That conjugation by σ stabilizes $O(V) \times O(V^\perp)$ is obvious from the definition of σ . Since $\sigma \cdot O_0(\mathbf{R}^{n,n}) \cdot \sigma$ is a connected subgroup of $O(\mathbf{R}^{n,n})$ having the same dimension, this subgroup must coincide with $O_0(\mathbf{R}^{n,n})$. The last statement is clear. \square

3. V -Rational Narain Lattices and Triples

Recall that we have fixed a non-degenerate subspace $V \subset \mathbf{R}^{n,n}$.

Definition 3.1. *Given a Narain lattice Γ , the primitive sublattices*

$$\Gamma_L := \Gamma \cap V, \quad \Gamma_R := \Gamma \cap V^\perp$$

are respectively called the left and right V -lattices of Γ . A Narain lattice Γ is called V -rational if Γ_L has maximal rank, i.e. $\text{rk } \Gamma_L = \dim V$. The set of V -rational Narain lattices is denoted by

$$\mathcal{R} \subset \mathcal{N} \equiv O(\mathbf{R}^{n,n})/O(\Gamma^e).$$

The discriminant of a V -rational Narain lattice Γ is defined to be the discriminant of the lattice Γ_L . In this case, we will say that Γ is primitive if Γ_L is primitive as an abstract lattice. Note that we have suppressed V from the notation, even though \mathcal{R} depends on V . We will also drop the mention of V when there is no confusion.

If V has signature (p, q) , and if Γ is V -rational, then Γ_L also has signature (p, q) , by definition. Similarly V^\perp and $\Gamma_R = \Gamma_L^\perp$ in Γ both have signature $(n - p, n - q)$. It follows that a V -rational Narain lattice Γ is an *even unimodular overlattice* of the lattice $\Gamma_L \oplus \Gamma_R \subset \Gamma$.

Lemma 3.2. *If $V, V' \subset \mathbf{R}^{n,n}$ are non-degenerate subspaces with the same signature, there is an isometry $f \in O(\mathbf{R}^{n,n})$ such that $fV = V'$, and that $f\Gamma$ is a V' -rational for every V -rational Narain lattice Γ .*

Proof: Let (p, q) be the signature of V , and let e_1, \dots, e_{2n} be an orthogonal basis for $\mathbf{R}^{n,n}$ such that $e_i \in V$ for $1 \leq i \leq p + q$, that $e_i \in V^\perp$ for $p + q + 1 \leq i \leq 2n$, and that

$$\langle e_i, e_i \rangle = \begin{cases} +1 & 1 \leq i \leq p \\ -1 & p + 1 \leq i \leq p + q \\ +1 & p + q + 1 \leq i \leq n + q \\ -1 & n + q + 1 \leq i \leq 2n. \end{cases}$$

Likewise for e'_1, \dots, e'_{2n} and V' . Then $f : e_i \mapsto e'_i$ defines an element $f \in O(\mathbf{R}^{n,n})$ with $fV = V'$.

If Γ is a V -rational Narain lattice, then $\Gamma \cap V$ has maximal rank, hence $f\Gamma \cap fV = f\Gamma \cap V'$ also has maximal rank, which means that $f\Gamma$ is V' -rational. \square

Lemma 3.3. *For each $\Gamma \in \mathcal{R}$, there exists a unique isometry $\varphi : A_{\Gamma_R(-1)} \rightarrow A_{\Gamma_L}$ such that*

$$(*) \quad \Gamma/(\Gamma_L \oplus \Gamma_R) = \{\varphi(a) \oplus a \mid a \in \Gamma_R(-1)^*/\Gamma_R(-1)\}.$$

In particular, we have

$$\det \Gamma_L = |A_{\Gamma_L}| = |A_{\Gamma_R(-1)}| = \det \Gamma_R(-1).$$

Moreover, when $\dim V = n$, then $\Gamma_L, \Gamma_R(-1)$ have the same signature, and they are in the same genus. In this case, Γ_L is primitive iff Γ_R is primitive.

Proof: The first assertion follows from [Ni]: even unimodular overlattices correspond 1-1 with isometries $\varphi : A_{\Gamma_R(-1)} \xrightarrow{\sim} A_{\Gamma_L}$ such that (*) holds. The second assertion follows the characterization [Ni] of genus that two lattices X, Y are in the same genus iff they have the same signature and have $A_X \cong A_Y$. The last assertion is a result of the next lemma. \square

Lemma 3.4. *Two lattices X, Y in the same genus are either both primitive or both not primitive.*

Proof: Suppose that X, Y are in the same genus, and X is primitive, but Y is not. By definition, we have an isomorphism $\varphi_p : X \otimes \mathbf{Z}_p \rightarrow Y \otimes \mathbf{Z}_p$ of \mathbf{Z}_p integral lattices, for each prime number p . Let q be a prime number such that $Y(\frac{1}{q})$ remains even integral, and let $u \in X$ be a vector such that $\frac{1}{q}\langle u, u \rangle_X \notin 2\mathbf{Z}$. First consider the case $q \neq 2$. Then $\frac{1}{q}\langle u, u \rangle_X \otimes \mathbf{Z}_q \notin 2\mathbf{Z}_q$. (Otherwise, the integer number $\langle u, u \rangle_X$ would have the “zero” to cancel the “pole” $\frac{1}{q}$ in \mathbf{Q}_q .) On the other hand $\frac{1}{q}\langle \cdot, \cdot \rangle_Y$ is an even integral form. In particular, we have $\frac{1}{q}\langle v, v \rangle_Y \otimes \mathbf{Z}_q \in 2\mathbf{Z}_q$ for any $v \in Y \otimes \mathbf{Z}_q$. Since $\varphi_q(u) \in Y \otimes \mathbf{Z}_q$, we have

$$\frac{1}{q}\langle u, u \rangle_X \otimes \mathbf{Z}_q = \frac{1}{q}\langle \varphi_q(u), \varphi_q(u) \rangle_Y \otimes \mathbf{Z}_q \in 2\mathbf{Z}_q$$

which is a contradiction.

For $q = 2$, the same argument works, with the modification that now $\langle u, u \rangle_X$ would have had to have “zero” of at least order 2 to cancel the “pole” $\frac{1}{2}$. \square

Recall that if $\dim V = n$ and has signature $(p, n - p)$, and we fix an anti-isometry $\sigma : V \rightarrow V^\perp$, then we get an involutive anti-isometry $\sigma : \mathbf{R}^{n,n} \rightarrow \mathbf{R}^{n,n}$, $(x, y) \mapsto (\sigma^{-1}y, \sigma x)$, where $(x, y) \in V \oplus V^\perp = \mathbf{R}^{n,n}$.

Lemma 3.5. *The action of $O(V) \times O(V^\perp)$ on the set \mathcal{N} preserves V -rationality, discriminant of the left and right V -lattices, and primitivity of the left and right V -lattices. Likewise the involution $\sigma : \mathcal{N} \rightarrow \mathcal{N}$ preserves all these properties, except for discriminant, which changes by the sign $(-1)^n$. Moreover the involution descends to V -equivalence classes.*

Proof: Let $b \in O(V) \times O(V^\perp)$. Since b preserves the orthogonal decomposition $V \oplus V^\perp$, it follows that

$$\begin{aligned} (b\Gamma)_L &:= (b\Gamma) \cap V = b(\Gamma \cap V) = b\Gamma_L \\ (b\Gamma)_R &:= (b\Gamma) \cap V^\perp = b(\Gamma \cap V^\perp) = b\Gamma_R. \end{aligned}$$

Since

$$rk\ b\Gamma_L = rk\ \Gamma_L, \quad det\ b\Gamma_L = det\ \Gamma_L,$$

it follows that b preserves rationality, discriminant of the left lattices; likewise for the right lattices. By definition, the quadratic form on $(b\Gamma)_L$ is nothing but $\langle b^{-1}-, b^{-1}- \rangle$ where \langle, \rangle is the quadratic form on Γ_L . This shows that Γ_L is primitive iff $(b\Gamma)_L$ is primitive.

Suppose $dim\ V = n$, and Γ is rational. Then

$$\Gamma_L = \Gamma \cap V$$

has maximal rank. Applying σ to this, we get

$$\sigma\Gamma_L = \sigma\Gamma \cap V^\perp = (\sigma\Gamma)_R.$$

This shows that the right (and left) lattice of $\sigma\Gamma$ have rank n . Since σ merely reverses the overall sign of the quadratic form on $\mathbf{R}^{n,n}$, i.e. $\langle \sigma x, \sigma y \rangle = \langle x, y \rangle$, primitivity of left and right lattices are preserved by σ , and the discriminant changes by the sign $(-1)^n$.

By Lemma 2.6, the element $\sigma \in GL(2n, \mathbf{R})$ normalizes the subgroup $O(V) \times O(V^\perp)$. Thus the action of σ on \mathcal{N} descends to the $O(V) \times O(V^\perp)$ orbits of Narain lattices, i.e. to V -equivalence classes. \square

Definition 3.6. *The set of V -equivalence classes of rational Narain lattices is denoted by $\overline{\mathcal{R}} \subset \overline{\mathcal{N}}$.*

Definition 3.7. *A triple (X, Y, φ) consists of even lattices X, Y having the same signature and rank n , and equipped with an isometry $\varphi : A_Y \rightarrow A_X$. The set of triples is denoted by \mathcal{T} . A triple (X, Y, φ) is said to be primitive if X, Y are primitive. Two triples $(X, Y, \varphi), (X', Y', \varphi')$ are said to be properly equivalent if there exist isomorphisms $f : X \rightarrow X', g : Y \rightarrow Y'$ such that*

$$\varphi' = f^* \circ \varphi \circ g^{*-1}.$$

The set of proper equivalence classes of triples is denoted by $\overline{\mathcal{T}}$. The discriminant of a triple (X, Y, φ) is defined to be the discriminant of the lattices X, Y .

Definition 3.8. *Define the involution*

$$\sigma : \mathcal{T} \rightarrow \mathcal{T}, \quad (X, Y, \varphi) \mapsto (Y, X, \varphi^{-1})$$

Likewise on $\overline{\mathcal{T}}$. The triple (X', Y', φ') is said to be (improperly) equivalent to (X, Y, φ) if it is properly equivalent to either (X, Y, φ) or $\sigma(X, Y, \varphi)$.

3.1. The map γ and its invariance properties

Definition 3.9. *Define the map $\gamma : \mathcal{R} \rightarrow \mathcal{T}$, $\Gamma \mapsto (\Gamma_L, \Gamma_R, \varphi)$ where $\varphi : A_{\Gamma_R(-1)} \rightarrow A_{\Gamma_L}$ is the isometry determined by the overlattice $\Gamma \supset \Gamma_L \oplus \Gamma_R$.*

By Lemma 3.4, it follows that $\gamma(\Gamma)$ is primitive iff Γ_L (or Γ_R) is primitive.

We now consider how the map γ interacts with the involution $\sigma : \mathcal{R} \rightarrow \mathcal{R}$ and the group action of $O(V) \times O(V^\perp)$ on \mathcal{R} .

Lemma 3.10.

- i. If $\Gamma \in \mathcal{R}$ and $b \in O(V) \times O(V^\perp)$, then $\gamma(b\Gamma)$ is properly equivalent to $\gamma(\Gamma)$.*
- ii. If $\Gamma \in \mathcal{R}$, then $\gamma(\sigma\Gamma)$ is properly equivalent to $\sigma \cdot \gamma(\Gamma)$.*

Proof: Let b_L, b_R be the restrictions of $b : \Gamma \rightarrow b\Gamma$ to Γ_L, Γ_R respectively. Then, as before, we have isometries $b_L : \Gamma_L \rightarrow b\Gamma_L = (b\Gamma)_L$, $b_R : \Gamma_R \rightarrow b\Gamma_R = (b\Gamma)_R$, and they induce isometries of discriminant groups

$$\begin{aligned} b_R^* : \Gamma_R^*/\Gamma_R &\rightarrow b\Gamma_R^*/b\Gamma_R \\ b_L^* : \Gamma_L^*/\Gamma_L &\rightarrow b\Gamma_L^*/b\Gamma_L. \end{aligned}$$

By Lemma 3.3, we have

$$\Gamma/(\Gamma_L \oplus \Gamma_R) = \{\varphi(a) \oplus a \mid a \in \Gamma_R(-1)^*/\Gamma_R(-1)\}.$$

Applying b to this, we get

$$\begin{aligned} b\Gamma/(b\Gamma_L \oplus b\Gamma_R) &= \{b_L^* \circ \varphi(a) \oplus b_R^*(a) \mid a \in \Gamma_R(-1)^*/\Gamma_R(-1)\} \\ &= \{b_L^* \circ \varphi \circ (b_R^*)^{-1}(c) \oplus c \mid c \in b\Gamma_R(-1)^*/b\Gamma_R(-1)\}. \end{aligned}$$

It follows that $b_L^* \circ \varphi \circ (b_R^*)^{-1} : A_{b\Gamma_R(-1)} \rightarrow A_{b\Gamma_L}$ is the unique isomorphism determined by the overlattice $b\Gamma$ of $b\Gamma_L \oplus b\Gamma_R$. As a result, the triples

$$\gamma(\Gamma) = (\Gamma_L, \Gamma_R(-1), \varphi), \quad \gamma(b\Gamma) = (b\Gamma_L, b\Gamma_R(-1), b_L^* \circ \varphi \circ (b_R^*)^{-1})$$

are properly equivalent. This completes the proof of assertion i.

We now consider the involution $\sigma : \mathcal{R} \rightarrow \mathcal{R}$. For convenience, we put

$$\gamma(\Gamma) = (\Gamma_L, \Gamma_R, \varphi).$$

Recall that

$$(\sigma\Gamma)_L := (\sigma\Gamma) \cap V = \sigma(\Gamma \cap V^\perp) = \sigma\Gamma_R, \quad (\sigma\Gamma)_R := (\sigma\Gamma) \cap V^\perp = \sigma\Gamma_L.$$

Thus σ induces a \mathbf{Z} -module isomorphism

$$\Gamma/(\Gamma_L \oplus \Gamma_R) \xrightarrow{\sim} \sigma\Gamma/((\sigma\Gamma)_R \oplus (\sigma\Gamma)_L), \quad (x, y) \bmod \Gamma_L \oplus \Gamma_R \mapsto (\sigma x, \sigma^{-1}y) \bmod (\sigma\Gamma)_R \oplus (\sigma\Gamma)_L,$$

and lattice isometries defined by

$$\begin{aligned} \iota_L : \Gamma_R(-1) &\rightarrow (\sigma\Gamma)_L, \quad (0, y) \mapsto (\sigma^{-1}y, 0) \\ \iota_R : \Gamma_L &\rightarrow (\sigma\Gamma)_R(-1), \quad (x, 0) \mapsto (0, \sigma x). \end{aligned}$$

In turn, these isometries induce isometries of discriminant groups

$$\begin{aligned}\iota_L^* &: \Gamma_R(-1)^*/\Gamma_R(-1) \rightarrow (\sigma\Gamma)_L^*/(\sigma\Gamma)_L \\ \iota_R^* &: \Gamma_L^*/\Gamma_L \rightarrow (\sigma\Gamma)_R(-1)^*/(\sigma\Gamma)_R(-1).\end{aligned}$$

By Lemma 3.3, we have

$$\Gamma/(\Gamma_L \oplus \Gamma_R) = \{\varphi(a) \oplus a \mid a \in \Gamma_R(-1)^*/\Gamma_R(-1)\}.$$

Applying σ to this, we get

$$\sigma\Gamma/((\sigma\Gamma)_R \oplus (\sigma\Gamma)_L) = \{\iota_L^*(a) \oplus \iota_R^* \circ \varphi(a) \mid a \in \Gamma_R(-1)^*/\Gamma_R(-1)\}.$$

Reparameterizing this set by setting $a' = \iota_R^* \circ \varphi(a) \in (\sigma\Gamma)_R(-1)^*/(\sigma\Gamma)_R(-1)$, we get

$$\begin{aligned}\sigma\Gamma/((\sigma\Gamma)_R \oplus (\sigma\Gamma)_L) &= \{\varphi'(a') \oplus a' \mid a' \in (\sigma\Gamma)_R(-1)^*/(\sigma\Gamma)_R(-1)\} \\ \varphi' &:= \iota_L^* \circ \varphi^{-1} \circ (\iota_R^*)^{-1}.\end{aligned}$$

This shows that the triple $((\sigma\Gamma)_L, (\sigma\Gamma)_R(-1), \varphi')$ coincides with $\gamma(\sigma\Gamma)$, and that it is properly equivalent to $(\Gamma_R(-1), \Gamma_L, \varphi^{-1}) =: \sigma \cdot \gamma(\Gamma)$. This completes the proof of assertion ii. \square

Lemma 3.11. *Let $\Gamma, \Gamma' \in \mathcal{R}$.*

- i. If $\gamma(\Gamma')$ is properly equivalent to $\gamma(\Gamma)$, then $\Gamma' = b\Gamma$ for some $b \in O(V) \times O(V^\perp)$.*
- ii. If $\gamma(\Gamma')$ is properly equivalent to $\sigma \cdot \gamma(\Gamma)$, then $\Gamma' = b\sigma\Gamma$ for some $b \in O(V) \times O(V^\perp)$.*

Proof: Let's prove ii. assuming i. first. By the preceding lemma, $\gamma(\sigma\Gamma)$ is properly equivalent to $\sigma \cdot \gamma(\Gamma)$. Thus by hypothesis of ii., $\gamma(\Gamma')$ is properly equivalent to $\gamma(\sigma\Gamma)$. It follows from i. that $\Gamma' = b\sigma\Gamma$ for some $b \in O(V) \times O(V^\perp)$.

We prove i. now. Write

$$\gamma(\Gamma) = (\Gamma_L, \Gamma_R, \varphi), \quad \gamma(\Gamma') = (\Gamma'_L, \Gamma'_R, \varphi').$$

By hypothesis of i., we have isometries

$$b_L : \Gamma_L \rightarrow \Gamma'_L, \quad b_R : \Gamma_R \rightarrow \Gamma'_R$$

such that

$$\varphi' = b_L^* \circ \varphi \circ (b_R^*)^{-1}.$$

There is a unique linear extension $b : \mathbf{R}^{2n} \rightarrow \mathbf{R}^{2n}$, i.e. $b|_{\Gamma_L} = b_L$ and $b|_{\Gamma_R} = b_R$.

Since b restricted to the rank $2n$ lattice $\Gamma_L \oplus \Gamma_R \subset \mathbf{R}^{n,n}$ is an isometry, it follows that $b \in O(\mathbf{R}^{n,n})$. Since $\Gamma_L \oplus \Gamma_R \subset \Gamma$, it follows that $b(\Gamma_L \oplus \Gamma_R) = \Gamma'_L \oplus \Gamma'_R \subset b\Gamma$,

i.e. $b\Gamma$ is an overlattice of $\Gamma'_L \oplus \Gamma'_R$. By Lemma 3.3, it determines a unique isometry $\varphi_{b\Gamma} : A_{\Gamma'_R(-1)} \rightarrow A_{\Gamma'_L}$. By the same calculation as in the proof of Lemma 3.10i., we see that

$$\varphi_{b\Gamma} = b_L^* \circ \varphi \circ (b_R^*)^{-1}.$$

Hence $\varphi_{b\Gamma} = \varphi'$. Thus $b\Gamma$ must coincide with the unique overlattice corresponding to φ' , i.e.

$$b\Gamma = \Gamma'.$$

Finally, we have

$$\Gamma' \cap V = \Gamma'_L = b\Gamma_L = b(\Gamma \cap V) = b\Gamma \cap bV = \Gamma' \cap bV.$$

This shows that both n dimensional spaces V, bV contains the rank n lattice Γ'_L , hence must be identical: $bV = V$. Likewise $bV^\perp = V^\perp$. It follows that $b \in O(V) \times O(V^\perp)$. This completes the proof of assertion i. \square

Theorem 3.12. *The map $\mathcal{R} \xrightarrow{\gamma} \mathcal{T}$ descends to injections*

$$\overline{\mathcal{R}} \xrightarrow{\gamma} \overline{\mathcal{T}}, \quad \overline{\mathcal{R}}/\sigma \xrightarrow{\gamma} \overline{\mathcal{T}}/\sigma.$$

The image $\gamma(\overline{\mathcal{R}})$ consists of classes of triples (X, Y, φ) where X have the same signature as V .

Proof: Lemma 3.10 says that γ descends to $\overline{\mathcal{R}} \rightarrow \overline{\mathcal{T}}$ and $\overline{\mathcal{R}}/\sigma \rightarrow \overline{\mathcal{T}}/\sigma$. Lemma 3.11 says that the induced maps are injective.

Given a V -rational Narain lattice Γ , then $\Gamma_L, \Gamma_R(-1)$ both have the same signature as V . So, the triple $\gamma(\Gamma)$ has the same signature as V . Conversely, if $(X, Y, \varphi) \in \mathcal{T}$ has the signature of V , then $X, Y(-1)$ can be respectively realized inside the vector spaces V, V^\perp . The isometry $\varphi : A_Y \rightarrow A_X$ corresponds to a unique even unimodular overlattice $\Gamma \supset X \oplus Y(-1)$ in $V \oplus V^\perp = \mathbf{R}^{n,n}$. It is immediate that $\Gamma \in \mathcal{R}$ and $\gamma(\Gamma) = (X, Y, \varphi)$. This proves the second assertion. \square

3.2. Isomorphisms of discriminants vs. triples

Given lattices X, Y , let $Isom(A_X, A_Y)$ denote the set of isometries of their discriminants. Since an isometry of X induces an isometry of A_X , it follows that $O(X)$ acts from the left on $Isom(A_X, A_Y)$. Likewise $O(Y)$ acts on it from the right. The orbit space of this $O(X) \times O(Y)$ action is the double quotient

$$(*) \quad O(X) \backslash Isom(A_X, A_Y) / O(Y).$$

Lemma 3.13. *If the lattices X, Y are not isomorphic, then the double quotient $(*)$ naturally parameterizes the set of improper equivalence classes of triples $[X, Y, \varphi] \in \overline{\mathcal{T}}/\sigma$.*

See Proposition 4.12 in [HLOYII]. The result there is stated for positive definite binary quadratic form, but the proof is valid in general.

Lemma 3.14. *If the lattices X, Y are isomorphic, primitive, and rank 2, then the double quotient $(*)$ naturally parameterizes the set of improper equivalence classes of triples $[X, Y, \varphi] \in \overline{\mathcal{T}}/\sigma$.*

See Proposition 4.12 in [HLOYII] for details.

4. Primitive Sublattices

Definition 4.1. We denote by Π the set of rank n primitive sublattices of the even unimodular lattice U^n . Note that a primitive sublattice need not be primitive as an abstract lattice. We regard $O(U^n)$ as a transformation group of the set Π , and write $\bar{\Pi} = \Pi/O(U^n)$.

In this section we will develop a correspondence between primitive sublattices of U^n and triples, which is parallel to the correspondence we obtained, in the last section, between V -rational Narain lattices and triples.

• *Anti-isometries.* Let $\xi : U^n \rightarrow U^n$ be an anti-isometry, i.e. an automorphism of abelian groups such that

$$\langle \xi(x), \xi(x) \rangle = -\langle x, x \rangle \quad \forall x \in U^n.$$

For example, declaring $\xi : e_i \mapsto -e_i$ and $\xi : f_i \mapsto f_i$, defines an anti-isometry. Obviously if f is any isometry of U^n , then $f \circ \xi$ and $\xi \circ f$ are both anti-isometries. This shows that the set of all anti-isometries is $\xi \cdot O(U^n) = O(U^n) \cdot \xi$, and is independent of the choice of ξ . Likewise if ξ' is any other anti-isometry, then $\xi' \circ \xi \in O(U^n)$, hence ξ always behave like an involution modulo $O(U^n)$. It follows immediately that

$$(*) \quad \xi \cdot O(U^n) \cdot \xi' = O(U^n).$$

Hence the group generated by anti-isometries has index 2 over the group of isometries. It is also straightforward to show that for each $M \in \Pi$, we have

$$(\xi(M))^\perp = \xi(M^\perp).$$

Lemma 4.2. Let ξ be an anti-isometry of U^n . Then as a group of transformations of U^n , the group $\langle O(U^n), \xi \rangle$ is independent of the choice of ξ . Moreover, there is an induced action of this group on the set Π , where $\xi : M \mapsto M^\xi := \xi(M)^\perp$.

Proof: The first assertion follows immediately from the identity (*). The group $O(U^n)$ acts on Π by left translation. To see that the group $\langle O(U^n), \xi \rangle$ acts, it suffices to show that for any $f \in O(U^n)$, and $M \in \Pi$, we have the identities

$$M^{f \circ \xi} = f(M^\xi), \quad M^{\xi \circ f} = (fM)^\xi.$$

But they readily follow from the definition of M^ξ . \square

Corollary 4.3. The set $\bar{\Pi}/\xi$ is independent of the choice of ξ .

From now on, we will fix an anti-isometry ξ of U^n once and for all.

Lemma 4.4. *Let $M \in \Pi$, and $f \in O(U^n)$. Then fM and $M^\perp(-1)$ are isogenous to M , hence they have the same signature, discriminant, and primitivity as M . The lattice $M^\xi \in \Pi$ has the same signature, discriminant, and primitivity as M .*

Proof: Since f is an isometry, fM is isomorphic to M . Now consider $M^\perp(-1)$. If M has signature $(n-p, p)$, then M^\perp has signature $(p, n-p)$, since U^n has signature (n, n) . It follows that $M^\perp(-1)$ has the same signature as M . For discriminant, we have

$$\det M^\perp(-1) = (-1)^n \det M^\perp = (-1)^n (-1)^n \det M = \det M.$$

Since $U^n \supset M \oplus M^\perp(-1)$ is an even unimodular overlattice, it determines an isometry of discriminant groups $\varphi(M) : A_{M^\perp(-1)} \rightarrow A_M$, by Lemma 3.3. It follows that $M^\perp(-1)$ is isogenous to M , hence has the same primitivity as M by Lemma 3.4.

Next we consider M^ξ . That M^ξ and M have the same signature is shown as in the case of $M^\perp(-1)$, and so we omit the details. Since ξ is an anti-isometry, it follows that $\xi(M) \in \Pi$ has the same primitivity as M . Since $\xi(M)^\perp(-1)$ is isogenous to $\xi(M)$, it follows that $M^\xi(-1) = \xi(M)^\perp(-1)$, hence M^ξ has the same primitivity as $\xi(M)$. \square

Definition 4.5. *Define the map*

$$\theta : \Pi \rightarrow \mathcal{T}, \quad M \mapsto (M, M^\perp(-1), \varphi(M)).$$

Here $\varphi = \varphi(M)$ is the isometry of discriminant groups $A_{M^\perp(-1)} \rightarrow A_M$ determined by the even unimodular overlattice $U^n \supset M \oplus M^\perp$.

Now comes the parallels of Lemmas 3.10 and 3.11.

Lemma 4.6.

- i. If $M \in \Pi$ and $f \in O(U^n)$, then $\theta(fM)$ is properly equivalent to $\theta(M)$.*
- ii. If $M \in \Pi$, then $\theta(M^\xi)$ is properly equivalent to $\sigma \cdot \theta(M)$.*

Proof: The proof is word for word the same as the proof of Lemma 3.10. The only slight difference is in ii. Here it suffices to show that the triples

$$\sigma \cdot \theta(M) = (M^\perp(-1), M, \varphi(M)^{-1}), \quad \theta(M^\xi) = (M^\xi, M^{\xi \perp}(-1), \varphi(M^\xi))$$

are equivalent via the isometries

$$\xi|_{M^\perp} : M^\perp(-1) \rightarrow M^\xi, \quad \xi|_M : M \rightarrow M^{\xi \perp}(-1).$$

In other words, we need to show that

$$\xi|_{M^\perp}^* \circ \varphi(M)^{-1} \circ \xi|_M^*{}^{-1} = \varphi(M^\xi).$$

This is a computation very similar to that in Lemma 3.10. We will not repeat it here. \square

Lemma 4.7. *Let $M, M' \in \Pi$.*

- i. If $\theta(M')$ is properly equivalent to $\theta(M)$, then $M' = fM$ for some $f \in O(U^n)$.*
- ii. If $\theta(M')$ is properly equivalent to $\sigma \cdot \theta(M)$, then $M' = fM^\xi$ for some $f \in O(U^n)$.*

Proof: The proof is word for word the same as the proof of Lemma 3.11, with the following minor change in the last part of i. Being given a proper equivalence $\theta(M') \sim \theta(M)$ of triples means that we are given isometries

$$g : M \rightarrow M', \quad g_\perp : M^\perp \rightarrow M'^\perp$$

such that

$$\varphi(M') = g^* \circ \varphi(M) \circ g_\perp^{*-1}.$$

This implies that there is a unique isometry $f \in O(U^n)$ extending g and g_\perp . \square

Theorem 4.8. *The map $\Pi \xrightarrow{\theta} \mathcal{T}$ descends to bijections*

$$\overline{\Pi} \xrightarrow{\theta} \overline{\mathcal{T}}, \quad \overline{\Pi}/\xi \xrightarrow{\theta} \overline{\mathcal{T}}/\sigma.$$

The maps preserves discriminant and primitivity.

Proof: Lemma 4.6 says that θ descends to $\overline{\Pi} \rightarrow \overline{\mathcal{T}}$ and $\overline{\Pi}/\xi \rightarrow \overline{\mathcal{T}}/\sigma$. Lemma 4.7 says that the induced maps are injective. To prove that the maps are surjective, let $(X, Y, \phi) \in \mathcal{T}$. This determines a unique even unimodular overlattice $\Gamma \supset X \oplus Y(-1)$ of signature (n, n) . By Milnor's theorem, there is an isometry $f : \Gamma \rightarrow \Gamma^e$. This induces a triple $\theta(f(X)) = (f(X), f(Y) = f(X)^\perp(-1), \varphi(f(X)))$ which is properly equivalent to (X, Y, ϕ) via f . This proves the asserted surjectivity.

Finally, by Lemma 4.4, the map θ preserves discriminant and (abstract lattice) primitivity. Since the equivalence relations defined on Π and on \mathcal{T} are compatible with primitivity and fixing discriminant, the maps induced by θ must preserve these properties. \square

5. Coprime Pairs and Gauss Product

5.1. Primitive sublattices and concordant pairs

•*Notations.* On the space $M^{2,2}$ of 2×2 real matrices, define the involutions

$$\begin{aligned} t : \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} &\mapsto \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}^t := \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \\ A : \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} &\mapsto \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}^A := \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix} \\ v : \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} &\mapsto \begin{bmatrix} \alpha & -\beta \\ -\gamma & \delta \end{bmatrix} \\ \sigma : \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} &\mapsto \text{diag}(1, -1) \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\delta \end{bmatrix}. \end{aligned}$$

Note that t and A are both anti-involution of the matrix algebra $M^{2,2}$, i.e.

$$(XY)^A = Y^A X^A, \quad (XY)^t = Y^t X^t.$$

But ${}^\vee$ is an isometric involution of algebra, i.e. $(XY)^\vee = X^\vee Y^\vee$, because it is given by a conjugation:

$$X^\vee = \text{diag}(1, -1)X\text{diag}(1, -1).$$

All three are obviously isometries of the quadratic form $2\det$ on $M^{2,2}$. They are also pairwise commuting. The map σ is an anti-isometry:

$$\det \sigma X = -\det X.$$

We will use the fact that $M^{2,2}$ is isometric to $\mathbf{R}^{2,2}$. See Appendix C in [HLOYII].

Recall that an element $(g_1, g_2) \in P(SL(2, \mathbf{R})^2)$ acts on $M^{2,2}$ by isometry via left and right multiplications $X \mapsto g_1 X g_2^{-1}$. Recall also that the subgroup of $P(SL(2, \mathbf{R})^2)$ which stabilizes the lattice $\Gamma^e \equiv \begin{bmatrix} \mathbf{Z} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} \end{bmatrix} \subset M^{2,2}$ is $P(SL(2, \mathbf{Z})^2)$. See Appendix C in [HLOYII] for details.

Lemma 5.1. *The group $O(\Gamma^e)$ is generated by $P(SL(2, \mathbf{Z})^2)$, A , and ${}^\vee$.*

Proof: As shown in Appendix C in [HLOYII], $O(\Gamma^e)$ has the shape $\coprod_{g \in \mathcal{Z}} g \cdot P(SL(2, \mathbf{Z})^2)$ where $\mathcal{Z} \subset O(\Gamma^e)$ is 4-element subgroup with 2 generators, such that

- i. one generator is orientation reversing;
- ii. one generator is orientation preserving, but reversing a positive 2-plane orientation.

Note that $P(SL(2, \mathbf{Z})^2)$ is orientation preserving, and preserving a positive 2-plane orientation as well. Then it is easy to show that $O(\Gamma^e)$ is generated by $P(SL(2, \mathbf{Z})^2)$ plus *any two elements in $O(\Gamma^e)$ with properties i.–ii.* We verify that ${}^A \in O(\Gamma^e)$ has property i., and that ${}^\vee \in O(\Gamma^e)$ has property ii. This completes the proof. \square

All quadratic forms are assumed *even non-degenerate*. We always identify a quadratic form $P = [a, b, c]$ with its matrix $\begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}$, and write $\text{disc } P = b^2 - 4ac$. Recall that $\iota[a, b, c] := [a, -b, c]$. If it is primitive, then it represents the inverse class of the class of $[a, b, c]$ in the class group. Note that under the identification here, we have

$$\iota[a, b, c] \equiv \begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}^\vee.$$

All three operations ${}^t, {}^A, {}^\vee$ operates on symmetric matrices, hence on quadratic forms. We also write $-[a, b, c] = [-a, -b, -c]$. If $P = [a, b, c]$, $Q = [a', b', c']$ are quadratic forms, we sometimes write

$$\text{gcd}(P) := \text{gcd}(a, b, c), \quad \text{gcd}(P, Q) := \text{gcd}(a, b, c, a', b', c').$$

Definition 5.2. Let $P = [a, b, c], Q = [a', b', c']$ be forms with the same discriminant, but not necessarily primitive. Put $\delta := \gcd(a, b, c)$, $\delta' := \gcd(a', b', c')$, $\lambda := \gcd(\delta, \delta')$. We say that P, Q are coprime if $\lambda = 1$. We say that P, Q are concordant if $aa' \neq 0$, $\gcd(a, a') = \lambda$, and $b = b'$. In this case, we define

$$P * Q := [aa', b, \frac{c}{a'}].$$

(cf. Appendix A [HLOYII].) If $d|\delta$, we also write $\frac{1}{d}P = [\frac{a}{d}, \frac{b}{d}, \frac{c}{d}]$. We denote by \mathcal{P} the set of coprime pairs (P, Q) of the same discriminant, and by $\overline{\mathcal{P}}$ the set of coprime pairs $(\overline{P}, \overline{Q})$ of $GL(2, \mathbf{Z})$ equivalence classes $\overline{P}, \overline{Q}$ of quadratic forms. Let $\sigma : \mathcal{P} \rightarrow \mathcal{P}$, $(P, Q) \mapsto (Q, P)$.

Note that P, Q being concordant implies that $ac = a'c'$. Since $\gcd(a, a') = \lambda$, it follows that $\frac{a}{\lambda}, \frac{a'}{\lambda}$ are coprime. Thus we have $\frac{a'}{\lambda} | \frac{c}{\lambda}$ and $\frac{a}{\lambda} | \frac{c'}{\lambda}$. In particular we have $\frac{c}{a'} \in \mathbf{Z}$.

Lemma 5.3. Any pair of quadratic forms of the same discriminant can be $SL(2, \mathbf{Z})$ transformed to a concordant pair.

A proof can be found in Appendix A [HLOYII].

5.2. The map Λ

Recall that Π denotes the set of rank 2 primitive sublattices of lattice $\Gamma^e \equiv \begin{bmatrix} \mathbf{Z} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} \end{bmatrix} \subset M^{2,2}$ equipped with the quadratic form $2\det X$. Fix an anti-isometry $\xi : \Gamma^e \rightarrow \Gamma^e$ once and for all. We denote $\overline{\Pi} := \Pi/O(\Gamma^e)$.

Let P, Q be any (non-degenerate, as always) binary quadratic forms. Introduce the notation

$$\Lambda(P, Q) := \{X \in \Gamma^e | X^t P = Q X^A\}.$$

A priori, $\Lambda(P, Q)$ is just a primitive abelian subgroup of Γ^e . But we will show shortly that it is either zero or a rank 2 sublattice. In fact, it has rank 2 iff P, Q have the same discriminant. The motivation for the defining equation for $\Lambda(P, Q)$ comes from the fact that two quadratic forms P, Q are $SL(2, \mathbf{Z})$ equivalent iff there exists $g \in SL(2, \mathbf{Z})$ such that $g^t P g = Q$, i.e. $g \in \Lambda(P, Q)$ a vector of length 2.

Lemma 5.4. For any quadratic forms P, Q , the abelian group $\Lambda(P, Q)$ is nonzero iff $\text{disc } P = \text{disc } Q$. In this case, $\Lambda(P, Q)$ has rank 2.

Proof: Given $P = [a, b, c], Q = [a', b', c']$, the equation $X^t P = Q X^A$ becomes

$$M\vec{X} = 0, \quad M = \begin{bmatrix} 2a & 0 & b+b' & -2a' \\ b-b' & 2a' & 2c & 0 \\ 0 & 2a & 2c' & b-b' \\ -2c' & b+b' & 0 & 2c \end{bmatrix}.$$

From this we find

$$\det M = \text{disc } Q - \text{disc } P.$$

This proves the first assertion.

We can now compute the 3×3 minors of M . Under the condition $\det M = 0$, it is easy (with some help from Mathematica) to check that all such minors are zero. This shows that M has rank at most 2. To see that it can be no less than 2, observe that for $b = b'$, we have $\text{rk } M = 2$. This shows that when $\det M = 0$, then $\text{rk } M$ is always 2, implying that $\text{rk } \Lambda(P, Q) = 2$ as well. \square

Lemma 5.5. *Let P, Q be any quadratic forms. We have*

- i. $\Lambda(P, Q)^t = \Lambda(Q^A, P^A)$.
- ii. $\Lambda(P, Q)^A = \Lambda(Q, P)$.
- iii. $\Lambda(P, Q)^\vee = \Lambda(P^\vee, Q^\vee)$.
- iv. $\Lambda(g^t P g, Q) = g^{-1} \cdot \Lambda(P, Q)$ for $g \in SL(2, \mathbf{Z})$.
- v. $\Lambda(P, g^t Q g) = \Lambda(P, Q) \cdot g$ for $g \in SL(2, \mathbf{Z})$.
- vi. $\Lambda(P^\vee, Q) = \text{diag}(1, -1) \cdot \Lambda(-P, Q)$.
- vii. $\Lambda(P, Q^\vee) = \Lambda(P, -Q) \cdot \text{diag}(1, -1)$.
- viii. $\Lambda(P, -Q) = \Lambda(-P, Q)$.
- ix. $\Lambda(kP, kQ) = \Lambda(P, Q)$ for nonzero integer k .
- x. $P \stackrel{SL(2, \mathbf{Z})}{\sim} Q$ iff $\exists X \in \Lambda(P, Q)$, $\langle X, X \rangle = 2 \det X = 2$.
- xi. $P \stackrel{SL(2, \mathbf{Z})}{\sim} -Q^\vee$ iff $\exists X \in \Lambda(P, Q)$, $\langle X, X \rangle = 2 \det X = -2$.

Proof: Assertions i.-x. are straightforward. To prove xi., notice that by x., if $P \sim -Q^\vee$, then $\exists Y \in \Lambda(P, -Q^\vee)$ with $\det Y = 2$. But by vii., it follows that $X = Y \cdot \text{diag}(1, -1) \in \Lambda(P, Q)$ has $\det X = -2$. The converse is similar. \square

Lemma 5.6. *For any quadratic forms P, Q of discriminant D , the restriction of \det to $\Lambda(P, Q)$ is nondegenerate. Moreover, $\Lambda(P, Q)$ is indefinite iff P, Q are both indefinite.*

Proof: Since nondegeneracy is invariant under isometries, we are free to replace P, Q by their $SL(2, \mathbf{Z})$ transforms, thanks to Lemma 5.5iv.-v. So let's assume that $P = [a, b, c]$, $Q = [a', b', c']$, are concordant. Then the coefficient matrix M in Lemma 5.4 simplifies, and finding a \mathbf{Q} -basis of $\mathbf{Q} \Lambda(P, Q)$ is easy. The result is

$$\mathbf{Q} \Lambda(P, Q) = \mathbf{Q} \left\{ \begin{bmatrix} \frac{a'}{a} & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{b}{a} & -\frac{c}{a'} \\ 1 & 0 \end{bmatrix} \right\}.$$

Computing the discriminant of the quadratic form on $\mathbf{Q} \Lambda(P, Q)$ using this \mathbf{Q} -basis, we get the result $\frac{D}{a^2}$, where $D = b^2 - 4ac$. This shows that $\Lambda(P, Q)$ is nondegenerate, and that $\Lambda(P, Q)$ is indefinite iff both P, Q are. \square

Definition 5.7. Recall that \mathcal{P} denotes the set of coprime pairs of binary quadratic forms having the same discriminant. Define a map

$$\Lambda : \mathcal{P} \rightarrow \Pi, \quad (P, Q) \mapsto \Lambda(P, Q).$$

Lemma 5.8. Let P, Q be any quadratic forms of the same discriminant D . If P, Q are both positive definite or both negative definite, then $\Lambda(P, Q)$ is positive definite. If P is positive definite and Q is negative definite (or the other way around), then $\Lambda(P, Q)$ is negative definite.

Proof: By the preceding lemma, if P, Q are definite, then $\Lambda(P, Q)$ is also definite. So to determine the sign, it suffices to compute the length of a single nonzero vector $X \in \Lambda(P, Q)$. We have

$$X^t P = Q X^A.$$

Since $X^A = (\det X) X^{-1}$, we get an equivalent equation

$$X^t P X = (\det X) Q.$$

Note that $X^t P X$ have the same signature as P . So if P, Q are both positive definite or both negative definite, then $\det X$ must be positive, and hence $\Lambda(P, Q)$ is positive definite. If P is positive definite and Q is negative definite, then $\det X$ is negative and $\Lambda(P, Q)$ is negative definite. \square

Lemma 5.9. For any quadratic forms P, Q of the same discriminant, we have

$$\Lambda(P, Q)^\perp = \Lambda(P, -Q) = ({}^t \circ \sigma \circ {}^t) \cdot \Lambda(P, Q^\vee)$$

Proof: The second equality follows from that

$$({}^t \circ \sigma \circ {}^t) Z = Z \operatorname{diag}(1, -1), \quad \forall Z \in M^{2,2},$$

and Lemma 5.5vii. that $\Lambda(P, Q^\vee) \cdot \operatorname{diag}(1, -1) = \Lambda(P, -Q)$.

Let's consider the first equality. Since both $\Lambda(P, -Q), \Lambda(P, Q)$ are rank 2 primitive sublattices in Γ^e , it suffices to show that if $X \in \Lambda(P, -Q)$ and $Y \in \Lambda(P, Q)$, then $\langle X, Y \rangle = 0$. So suppose that

$$X^t P = -Q X^A, \quad Y^t P = Q Y^A.$$

Then we have

$$(X + Y)^t P = -Q(X - Y)^A.$$

Taking the determinant on both sides, and noting that $\det P = \det Q \neq 0$, we see that $\det(X + Y) = \det(X - Y)$. This shows that

$$\langle X, Y \rangle = \langle X, -Y \rangle = -\langle X, Y \rangle.$$

Here we've used the fact that X, Y are 2×2 . This completes the proof. \square

Lemma 5.10. *Suppose that $P = [a, b, c]$, $Q^\vee = [a', b, c']$ are concordant forms with $\gcd(a, a') = 1$. Then*

$$\Lambda(P, Q) = \mathbf{Z} \left\{ \begin{bmatrix} a' & -b \\ 0 & a \end{bmatrix}, \begin{bmatrix} 0 & -\frac{c}{a'} \\ 1 & 0 \end{bmatrix} \right\}.$$

Proof: Name the two given generators X_1, X_2 . It is trivial to check that the X_i solve the linear equation $X^t P - Q X^A = 0$. So the X_i form a \mathbf{Q} -basis of $\Lambda(P, Q)$. To show that the X_i form a \mathbf{Z} -basis, it suffices to show that the vectors $X_i \in \mathbf{Z}^4$ generate a primitive sublattice of \mathbf{Z}^4 . Since $X_1 = (a', -b, a, 0)$, $X_2 = (0, -\frac{c}{a'}, 0, 1)$, we can find two additional vectors $Y_1, Y_2 \in \mathbf{Z}^4$ of the shape $(*, *, *, 0)$ such that the four vectors X_1, X_2, Y_1, Y_2 form a unimodular matrix. This shows that the first two vectors generates a primitive sublattice of \mathbf{Z}^4 . \square

Corollary 5.11. *Let $P = [a, b, c]$, $Q^\vee = [a', b', c']$ be quadratic forms of discriminant D . Put $\delta := \gcd(P)$, $\delta' := \gcd(Q)$, $\lambda := \gcd(\delta, \delta')$, as before. If P, Q^\vee are concordant, then*

$$\Lambda(P, Q) = \mathbf{Z} \left\{ \frac{1}{\lambda} \begin{bmatrix} a' & -b \\ 0 & a \end{bmatrix}, \begin{bmatrix} 0 & -\frac{c}{a'} \\ 1 & 0 \end{bmatrix} \right\}.$$

In particular, it has discriminant $\frac{1}{\lambda^2}(b^2 - 4ac)$. More generally if P, Q are arbitrary, then $\Lambda(P, Q)$ has discriminant $\frac{D}{\lambda^2}$.

Proof: By definition, that P, Q^\vee are concordant means that $aa' \neq 0$, $\gcd(a, a') = \lambda$, and $b = b'$. It follows that $\frac{1}{\lambda}P, \frac{1}{\lambda}Q^\vee$ are also concordant, but with coprime and nonzero leading coefficients $\frac{a}{\lambda}, \frac{a'}{\lambda}$. By Lemma 5.5ix., we have

$$\Lambda(P, Q) = \Lambda\left(\frac{1}{\lambda}P, \frac{1}{\lambda}Q\right).$$

Now applying Lemma 5.10 to compute the right hand side, we get our first assertion. Computing the discriminant using the \mathbf{Z} -base we found is straightforward, and we get $\frac{1}{\lambda^2}(b^2 - 4ac)$.

Finally, if P, Q^\vee are arbitrary forms of discriminant D , we can replace them with their $SL(2, \mathbf{Z})$ transforms without changing the values of δ, δ', λ , or the discriminant of $\Lambda(P, Q)$, thanks to Lemma 5.5iv.-v. We can choose the $SL(2, \mathbf{Z})$ transforms to be a concordant pair, by Lemma 5.3. Now our third assertion follows from the second assertion. \square

Corollary 5.12. $\Lambda(P, Q)$ is primitive as an abstract lattice iff $\frac{1}{\lambda}P, \frac{1}{\lambda}Q$ are both primitive as quadratic forms.

Proof: Again, isometries preserve primitivity. Thus, we may as well assume that $P = [a, b, c], Q^\vee = [a', b, c']$ are concordant, thanks to Lemmas 5.5iv.-v. and the existence of concordant forms. So we can use the \mathbf{Z} -base found in the preceding corollary to compute the quadratic form of $\Lambda(P, Q)$. Since

$$\Lambda(P, Q) = \Lambda\left(\frac{1}{\lambda}P, \frac{1}{\lambda}Q\right),$$

we may as well assume that $\lambda = 1$. Then the resulting quadratic form is $[aa', b, \frac{c}{a'}]$. We put

$$m := \gcd(aa', b, \frac{c}{a'}).$$

Suppose P, Q are both primitive. Then we have $m|aa', m|b$, and $m|\frac{c}{a'}$, hence $m|c$. Since $[a, b, c]$ is assumed primitive i.e. $\gcd(a, b, c) = 1$, it follows that $m|a'$. Since $a'c' = ac$, and $m|\frac{c}{a'}$, it follows that $m|\frac{c'}{a}$, hence $m|c'$. So we find that $m|a', b, c'$. But $[a', b, c']$ is also assumed primitive. It follows that $m = 1$, and so the quadratic form of $\Lambda(P, Q)$ is primitive.

Conversely suppose that one of P, Q is not primitive. By Lemma 5.5ii., we may assume that P is not primitive. So $\delta := \gcd(a, b, c) > 1$. Since P, Q are assumed concordant, we have $\gcd(a, a') = 1$, hence $\gcd(\delta, a') = 1$. It follows that $\delta|\frac{c}{a'}$. But we also have $\delta|a, b$. So we have $\delta|m = \gcd(aa', b, \frac{c}{a'})$. This shows that $m > 1$, hence $\Lambda(P, Q)$ is not primitive. \square

Theorem 5.13. If P, Q^\vee are coprime concordant forms, then the lattice $\Lambda(P, Q)$ equipped with the \mathbf{Z} -base as in Lemma 5.10, coincides with the quadratic form $[aa', b, \frac{c}{a'}] = P * Q^\vee$. More generally, for any coprime forms P, Q , the lattices $\Lambda(P, Q), \Lambda(P, Q)^\perp(-1)$, are respectively isomorphic to $P * Q^\vee, P * Q$ as lattices.

Proof: The first assertion follows from a straightforward computation using the given explicit \mathbf{Z} -base. That $\Lambda(P, Q)$ is isomorphic to $P * Q^\vee$ follows from the first assertion and Lemma 5.5iv.-v.

Note that the transformation $\xi := {}^t \circ \sigma \circ {}^t : \Gamma^e \rightarrow \Gamma^e$ is an anti-isometry. In particular for any sublattice $M \subset \Gamma^e$, we have $\xi(M)(-1) \cong M$ as lattices. By Lemma 5.9, it follows that $\Lambda(P, Q)^\perp(-1)$ is isomorphic to $\Lambda(P, Q^\vee)$. The latter is isomorphic to $P * Q$, by the first assertion. \square

Corollary 5.14. When restricted to primitive forms of discriminant D , the map Λ descends to

$$\Lambda : Cl_D \times Cl_D \rightarrow Cl_D/\iota, \quad (P, Q) \mapsto [\Lambda(P, Q)] = [P * Q^\vee].$$

Here Cl_D is the group of proper equivalence classes of primitive forms (cf. p337 [Ca]), and $\iota : C \mapsto C^{-1}$.

Proof: By Lemma 5.5iv.-v., the isomorphism class $[\Lambda(P, Q)]$ depends only on the $SL(2, \mathbf{Z})$ equivalence classes of P, Q , i.e. Λ is a class function. We evaluate $\Lambda(P, Q)$ by choosing P, Q^\vee to be concordant. Now the preceding corollary completes the proof. \square

If we restrict ourselves to positive definite primitive forms, then it can be shown that there is a natural lifting to

$$\Lambda : Cl_D \times Cl_D \rightarrow Cl_D, \quad (C_1, C_2) \mapsto (C_1 * C_2^{-1}).$$

The point is that because the Grassmannian of *positive* 2-planes is contractible, we can choose an orientation for every positive 2-plane so that they are all compatible under deformation. In particular, we can assign compatible orientations to all $\Lambda(P, Q)$, so that these lattices become quadratic forms. There are obviously two ways to do so. The observation here is that one of them yields the map $(C_1, C_2) \mapsto (C_1 * C_2^{-1})$, and the other choice yields $(C_1, C_2) \mapsto (C_1^{-1} * C_2)$.

Define the following transformations on the set \mathcal{P} :

$$\begin{aligned} P(SL(2, \mathbf{Z})^2) \ni (g_1, g_2) : (P, Q) &\mapsto (g_1^t \ ^{-1} P g_1^{-1}, g_2^t \ ^{-1} Q g_2^{-1}) \\ \sigma = \iota_1 : (P, Q) &\mapsto (Q, P) \\ \iota_2 : (P, Q) &\mapsto (P^\vee, Q^\vee) \\ \iota_3 : (P, Q) &\mapsto (P, Q^\vee). \end{aligned}$$

It is clear that each of these transformations preserves discriminant, coprimeness, and primitivity (but not necessary signature). We denote the group generated by these transformations by $\mathbf{G} = \langle P(SL(2, \mathbf{Z})^2), \iota_1, \iota_2, \iota_3 \rangle$. It is straightforward to check that $\overline{\mathcal{P}} = \mathcal{P}/K$ (see Definition 5.2), where $K \subset \mathbf{G}$ is the subgroup generated by $P(SL(2, \mathbf{Z})^2)$ and ι_2, ι_3 . Note that among the generators of the group \mathbf{G} presented above, ι_1 is the only one which does not preserve the signatures of a pair, since $(P, Q), (Q, P) \in \mathcal{P}$ have different signatures when P is positive definite and Q is negative definite.

By Lemma 5.1, the group $O(\Gamma^e)$ is generated by $P(SL(2, \mathbf{Z})^2), ^A, ^\vee$. Together with the anti-isometry ξ , they act on primitive sublattices $M \in \Pi$ by

$$\begin{aligned} P(SL(2, \mathbf{Z})^2) \ni (g_1, g_2) : M &\mapsto g_1 M g_2^{-1} \\ ^A : M &\mapsto M^A \\ ^\vee : M &\mapsto M^\vee \\ \xi : M &\mapsto M^\xi. \end{aligned}$$

By Lemma 4.4, each of these transformations preserves discriminant, and (abstract lattice) primitivity, and signature.

Theorem 5.15. *The map $\Lambda : \mathcal{P} \rightarrow \Pi$ preserves discriminant, primitivity, and is equivariant with respect to the group action of \mathbf{G} on \mathcal{P} , and $\langle O(\Gamma^e), \xi \rangle$ on Π . Thus Λ descends to $\overline{\mathcal{P}}/\sigma \rightarrow \overline{\Pi}/\xi$.*

Proof: If (P, Q) is a coprime pair of discriminant D , then $\Lambda(P, Q)$ has discriminant D , by Corollary 5.11, and so Λ preserves discriminant. If (P, Q) is a primitive pair of discriminant D , then $\Lambda(P, Q)$ is primitive, by Lemma 5.12, and so Λ preserves primitivity.

Moreover, by Lemmas 5.5iv.-v., ii.-iii., the map Λ is equivariant with respect to $P(SL(2, \mathbf{Z})^2)$ and the involutions $\sigma = \iota_1 \leftrightarrow^A, \iota_2 \leftrightarrow^V$. Since $\overline{\Pi}/\xi = \Pi/\langle O(\Gamma^e), \xi \rangle$ is independent of the choice of the anti-isometry ξ , by Lemma 4.2, we can choose ξ to be

$$\xi = {}^t \circ \sigma \circ {}^t.$$

Then Lemmas 5.9 and 4.2 show that the map Λ is also equivariant with respect to the involutions $\iota_3 \leftrightarrow^\xi$, i.e. $\Lambda(\iota_3(P, Q)) = \Lambda(P, Q)^\xi$. Thus we have shown that the map Λ is equivariant with respect to the group action of \mathbf{G} on \mathcal{P} , and $\langle O(\Gamma^e), \xi \rangle$ on Π . \square

6. Surjectivity of Λ

Lemma 6.1. *Let $A \in GL(2, \mathbf{Q})$ which is not a multiple of the identity. Let $M_{\mathbf{Q}}^{2,2} = \begin{bmatrix} \mathbf{Q} & \mathbf{Q} \\ \mathbf{Q} & \mathbf{Q} \end{bmatrix}$. Then the linear map $L_A : M_{\mathbf{Q}}^{2,2} \rightarrow M_{\mathbf{Q}}^{2,2}, P \mapsto A^t P A - (\det A)P$ has rank 2, and we have*

$$\ker L_A = \mathbf{Q} \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -c & a-d \\ 0 & b \end{bmatrix} \right\}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Proof: This is straightforward linear algebra. \square

Corollary 6.2. *Under the same assumption as in the preceding lemma, there is a unique, up to sign, primitive quadratic form in $\ker L_A$.*

Proof: It is obvious that the 2-dimensional space $\ker L_A$ contains nonsymmetric matrices. So the subspace of symmetric matrices is at most 1-dimensional. But $\ker L_A$ is closed under transpose. It follows that the subspace of symmetric matrices in $\ker L_A$ is exactly 1-dimensional. Since this is over \mathbf{Q} , there is a unique, up to sign, primitive vector

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \begin{bmatrix} \mathbf{Z} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} \end{bmatrix}$$

in this subspace, i.e. $\gcd(x, y, z) = 1$. If both x, z are even, then $\pm \begin{bmatrix} x & y \\ y & z \end{bmatrix}$ are the quadratic forms we seek. If not, then $\pm 2 \begin{bmatrix} x & y \\ y & z \end{bmatrix}$ are. \square

Corollary 6.3. *Let $X_1, X_2 \in \Gamma^e$, be any linearly independent vectors with nonzero lengths. Then there is a unique, up to overall sign, pair of coprime forms P, Q of the same discriminant such that $X_1, X_2 \in \Lambda(P, Q)$.*

Proof: Given the X_i , we want to solve the equations

$$X_1^t P - Q X_1^A = 0, \quad X_2^t P - Q X_2^A = 0.$$

Since $\det X_i \neq 0$ by assumption, it follows that $X_i^A = (\det X_i) X_i^{-1}$. Moreover any solution P, Q necessarily have $(\det X_i)(\det P) = (\det Q)(\det X_i)$, i.e. P, Q have the same discriminant. We also get equivalent equations

$$(*) \quad X_1^t P X_1 - (\det X_1) Q = 0, \quad X_2^t P X_2 - (\det X_2) Q = 0.$$

Eliminating Q , we get

$$A^t P A - (\det A) P = 0, \quad A := X_1 X_2^{-1}.$$

Note that $A \in GL(2, \mathbf{Q})$, and that A is not a scalar multiple of the identity, because X_1, X_2 are assumed linearly independent. So there is a unique, up to sign, primitive quadratic form P solving the last equation. Plugging P back into either equation in $(*)$ and solve for Q , we get

$$Q = \frac{1}{\det X_1} X_1^t P X_1 = \frac{1}{\det X_2} X_2^t P X_2.$$

It is symmetric and nondegenerate because P is. Its entries are rational numbers. So by a suitable scaling of P by an integer, we get an even integral quadratic form Q .

From the construction, it is clear that P, Q can be made coprime. Uniqueness is also clear. \square

Corollary 6.4. *Any rank 2 primitive sublattice $M \subset \Gamma^e$ can be realized as $\Lambda(P, Q)$ for a pair of coprime quadratic forms P, Q of the same discriminant. Moreover P, Q are unique up to an overall sign.*

Proof: By assumption, we can find two linearly independent vectors $X_1, X_2 \in M$ with nonzero lengths (nonzero lengths because \det restricted to M is assumed nondegenerate). By the preceding corollary, we can find P, Q such that $X_1, X_2 \in \Lambda(P, Q)$. Since M and $\Lambda(P, Q)$ are both primitive sublattices of Γ^e containing X_1, X_2 , it follows that $M = \Lambda(P, Q)$.

Now a priori, P, Q may depend on the choice of X_1, X_2 . But if $\Lambda(P, Q) = \Lambda(P', Q')$ for two coprime pairs $(P, Q), (P', Q')$, then $X_1, X_2 \in \Lambda(P', Q')$ also determine P', Q' in terms of X_1, X_2 up to an overall sign. It follows that $(P', Q') = \pm(P, Q)$. \square

Theorem 6.5. *The map $\Lambda : \mathcal{P} \rightarrow \Pi$ is unramified 2:1 surjective, with fiber $\Lambda^{-1}(\Lambda(P, Q)) = \{\pm(P, Q)\}$.*

Proof: By Lemma 5.5viii., we have $\Lambda(-P, -Q) = \Lambda(P, Q) =: M$. The preceding corollary says Λ is surjective with fiber $\Lambda^{-1}(M) = \{\pm(P, Q)\}$. Since $(P, Q), (-P, -Q)$ are never equal in \mathcal{P} , it follows that the map Λ is 2:1 everywhere. \square

Theorem 6.6. *The map $\Lambda : \mathcal{P} \rightarrow \Pi$ descends to a bijection*

$$\Lambda : \overline{\mathcal{P}}/\langle\sigma, -\rangle \rightarrow \overline{\Pi}/\xi$$

where $\sigma, -$ are the respective involutions on pairs $\sigma : (P, Q) \mapsto (Q, P)$, $- : (P, Q) \mapsto (-P, -Q)$. Moreover, Λ preserves discriminant and primitivity.

Proof: By Theorem 5.15, $\Lambda : \mathcal{P} \rightarrow \Pi$ is equivariant with respect to \mathbf{G} acting on \mathcal{P} and $K := \langle O(\Gamma^e), \xi \rangle$ acting on Π . By Theorem 6.5, this map descends to an equivariant bijection

$$\Lambda : \mathcal{P}/\langle-\rangle \rightarrow \Pi.$$

(Note that the \mathbf{G} action commutes with $-$.) Now passing to the orbit spaces of \mathbf{G} and K , we get the asserted induced bijection.

That Λ preserves discriminant follows from Theorem 5.13. That Λ preserves primitivity follows from Corollary 5.12. The same holds for the induced Λ , since the equivalence relations on \mathcal{P} and Π are compatible with primitivity and fixing discriminant. \square

7. Applications and Conclusions

7.1. Binary forms

Throughout this section, we set $n = 2$.

Fix a negative integer D , and let $\mathcal{P}_D^+ \subset \mathcal{P}$ be the set of coprime pairs of positive definite forms in \mathcal{P} of discriminant D . Let $\Pi_D^+ \subset \Pi$ be the positive definite lattices in Π of discriminant D . Then Theorem 6.6 says that

$$\overline{\mathcal{P}}_D^+/\sigma \xrightarrow{\Lambda} \overline{\Pi}_D^+/\xi. \tag{7.1}$$

The correspondence (7.1) also preserves primitivity. Note that this correspondence can be precisely described by the composition law $*$, as in Theorem 5.13.

Let \mathcal{T}_D^+ be the set of positive definite triples in \mathcal{T} of discriminant D . Then Theorem 4.8 says that

$$\overline{\Pi}_D^+ / \xi \xleftrightarrow{\theta \ 1:1} \overline{\mathcal{T}}_D^+ / \sigma. \quad (7.2)$$

This correspondence also preserves primitivity.

Now put $V = \mathbf{R}^{2,0}$, and let \mathcal{R}_D^+ be the set of V -rational Narain lattices in \mathcal{R} of discriminant D . Then Theorem 3.12 says that

$$\overline{\mathcal{R}}_D^+ / \sigma \xleftrightarrow{\gamma \ 1:1} \overline{\mathcal{T}}_D^+ / \sigma. \quad (7.3)$$

Note that the same is true if V is replaced by any other positive definite two-plane in $\mathbf{R}^{2,2}$. The three correspondences above together recover a result in [HLOYII] (cf. Theorem 5.8 there).

Let's consider now the indefinite case, i.e. D a positive integer, and $V = \mathbf{R}^{1,1} = \{(*, 0, *, 0)\} \subset \mathbf{R}^{2,2}$. Let \mathcal{P}_D^{+-} , Π_D^{+-} , \mathcal{T}_D^{+-} , \mathcal{R}_D^{+-} , be respectively the set of coprime pairs of indefinite forms in \mathcal{P} , the set of indefinite lattices in Π , the set of indefinite triples in \mathcal{T} , and finally, the set of V -rational Narain lattices in \mathcal{R} , all having discriminant D . Then the correspondences analogous to (7.2) and (7.3) hold. But (7.1) must be replaced by

$$\overline{\mathcal{P}}_D^{+-} / \langle \sigma, - \rangle \xleftrightarrow{\Lambda \ 1:1} \overline{\Pi}_D^{+-} / \xi. \quad (7.4)$$

While in the definite case, the involution $- : (P, Q) \mapsto (-P, -Q)$ always identifies two distinct pairs in $\overline{\mathcal{P}}$, it is no longer so in the indefinite case. The reason is that a given indefinite form P may or may not be $GL(2, \mathbf{Z})$ equivalent to $-P$. Both possibilities can occur.

For simplicity, let's consider just the primitive indefinite forms. There are two kinds of $GL(2, \mathbf{Z})$ classes. There are classes with $\bar{P} = -\bar{P}$, and those with $\bar{Q} \neq -\bar{Q}$. Clearly the latter kind comes in pairs. Thus the complete list of pairwise distinct $GL(2, \mathbf{Z})$ classes has the shape

$$\bar{P}_1, \dots, \bar{P}_u, \bar{Q}_1, \dots, \bar{Q}_v, -\bar{Q}_1, \dots, -\bar{Q}_v.$$

The numbers u, v depend on D .

To describe the primitive objects in the set $\overline{\mathcal{P}}_D^{+-} / \langle \sigma, - \rangle$, it suffices to list all pairs which are not congruent modulo $\langle \sigma, - \rangle$. We get

$$\begin{aligned} &(\bar{P}_i, \bar{P}_j), \quad i \leq j \\ &(\bar{P}_i, \bar{Q}_j), \quad \forall i, j \\ &(\bar{Q}_i, \bar{Q}_j), \quad i \leq j \\ &(\bar{Q}_i, -\bar{Q}_j), \quad \forall i, j. \end{aligned}$$

This shows that

$$\# \text{primitive objects in } \overline{\mathcal{P}}_D^{+-} / \langle \sigma, - \rangle = \frac{1}{2}u(u+1) + uv + \frac{1}{2}v(v+1) + v^2 = \frac{1}{2}(u+v)(u+v+1) + v^2.$$

It follows that this is also the number of primitive objects in each of $\overline{\Pi}_D^{+-} / \xi$, $\overline{\mathcal{T}}_D^{+-} / \sigma$, $\overline{\mathcal{R}}_D^{+-} / \sigma$.

It is known that for $D = p$ a prime number, every quadratic form P of discriminant D is $GL(2, \mathbf{Z})$ equivalent to $-P$. Thus, in this case, we have $v = 0$.

7.2. Concluding remarks

We now return to the general setting of *The Main Problem* in the Introduction. Here $\mathbf{R}^{n,n}$ is replaced by an arbitrary quadratic space E of signature (r, s) with $8|(r-s)$ (which can be taken to be $\mathbf{R}^{r,s}$ without loss of generality), and $V \subset E$ an arbitrary non-degenerate subspace (which can be taken to be $\mathbf{R}^{p,q} := \{(*^p, 0^{r-p}, *^q, 0^{s-q})\} \subset \mathbf{R}^{r,s}$). It is further assumed that E is indefinite. (Note that Milnor's theorem for uniqueness of indefinite unimodular lattices is used in Theorem 4.8.) Then Theorems 3.12 and 4.8 can be readily generalized to this case. The notion of a triple must be weakened as follows: *A triple (X, Y, φ) consists of a pair of even lattices X, Y , and an isometry $\varphi : A_Y \rightarrow A_X$, such that $X \oplus Y(-1)$ have signature (r, s) .* The notions of V -equivalence on V -rational Narain lattices and proper equivalence on triples remain the same as before. Thus \mathcal{R} is now the set of V -rational Narain lattices, \mathcal{T} the class of triples, and $\overline{\mathcal{R}}, \overline{\mathcal{T}}$ the respective sets of equivalence classes. The set Π now becomes the set of primitive sublattices of a given abstract even unimodular lattice \mathbf{U} of signature (r, s) , and $\overline{\Pi}$ the set of $O(\mathbf{U})$ equivalence classes in Π . The lattice \mathbf{U} now plays the role of the lattice U^n .

In this generality, however, we no longer have an anti-isometry σ of E which exchanges V and V^\perp . Likewise, for $M \in \Pi$, the lattice $M^\perp(-1)$ is no longer isogenous to M in general. In any case, the set of proper equivalence classes of triples (X, Y, φ) with $\varphi \in \text{Isom}(A_Y, A_X)$, and X, Y are lattices of fixed discriminant D with $\text{sign } X \oplus Y(-1) = (r, s)$, is clearly finite. Therefore, the problem of counting V -rational Narain lattices, triples, and primitive sublattices of \mathbf{U} , still makes sense. In fact, Theorems 3.12 and 4.8 generalize to

Theorem 7.1. *The map $\mathcal{R} \xrightarrow{\gamma} \mathcal{T}$, $\Gamma \mapsto (\Gamma \cap V, (\Gamma \cap V^\perp)(-1), \varphi_\Gamma)$, descends to an injection $\overline{\mathcal{R}} \xrightarrow{\gamma} \overline{\mathcal{T}}$. The image $\gamma(\overline{\mathcal{R}})$ consists of classes of triples (X, Y, φ) where X has the same signature as V .*

Theorem 7.2. *The map $\Pi \xrightarrow{\theta} \mathcal{T}$, $M \mapsto (M, M^\perp(-1), \varphi(M))$, descends to a bijection $\overline{\Pi} \xrightarrow{\theta} \overline{\mathcal{T}}$.*

The proofs of Theorems 3.12 and 4.8 are valid here verbatim. We give two applications of these theorems.

Let $E = \mathbf{R}^{1,1}$, and $V \subset E$ a 1 dimensional positive definite subspace. We want to count V -rational Narain lattices Γ with $\Gamma \cap V$ rank 1 of discriminant $D > 0$. This amounts to counting rank 1 primitive sublattices $M \subset U$ of discriminant D modulo $O(U)$. It is not hard to show that the answer is 1 if $D = 2$, and exactly $2^{\tau(\frac{D}{2})-1}$ if $D > 2$. Here $\tau(k)$ is the number of distinct prime factors of k .

We now consider the general case $E = \mathbf{R}^{r,s}$, and $V = \mathbf{R}^{p,q} \subset E$. As before, denote by \mathcal{R}_D the set of V -rational Narain lattices Γ such that $\Gamma \cap V$ has discriminant D , and by $\overline{\mathcal{R}}_D$ the set of V -equivalence classes in \mathcal{R}_D . Let \mathcal{L}_D denotes the class of abstract even lattices L of discriminant D , and $\overline{\mathcal{L}}_D$ the set of isomorphism classes $[L]$ in \mathcal{L}_D . Note that

$\overline{\mathcal{L}}_D$ decomposes into finite subsets (see [Ca]) $\overline{\mathcal{L}}_D(p, q)$ consisting of isomorphism classes of lattices of discriminant D and signature (p, q) .

Proposition 7.3. *If $r > p$, $s > q$, and $\dim V \neq \frac{1}{2}\dim E$, then $|\overline{\mathcal{R}}_D| = |\overline{\mathcal{L}}_D(p, q)|$.*

Proof: Note that a Narain lattice $\Gamma \subset E$ is V -rational iff it is also V^\perp -rational. Thus it suffices to consider the case $\dim V < \frac{1}{2}\dim E$, since $\dim V + \dim V^\perp = \dim E$. By the preceding theorems, it suffices to show that the map

$$(*) \quad \overline{\Pi}_D(p, q) \rightarrow \overline{\mathcal{L}}_D(p, q), \quad M \bmod O(\mathbf{U}) \mapsto [M],$$

is a bijection, where $\overline{\Pi}_D(p, q)$ is the set consisting of the objects of discriminant D and signature (p, q) in $\overline{\Pi}$.

Let $[M] = [M']$ for some primitive sublattices $M, M' \in \Pi$ of \mathbf{U} with discriminant D , signature (p, q) , and let $i : M \hookrightarrow \mathbf{U}$, $i' : M' \hookrightarrow \mathbf{U}$ be their respective inclusions. Then we have a lattice isomorphism $M \xrightarrow{g} M'$, and two primitive embeddings $i, i' \circ g : M \rightarrow \mathbf{U}$. Since $p + q = \text{rank } M = \dim V \leq \frac{1}{2}\dim E - 1 = \frac{1}{2}(r + s) - 1$, it follows that

$$r + s - p - q \geq 2 + \text{rank } M \geq 2 + l(A_M)$$

where $l(A_M)$ is the minimal number of generators of A_M . Moreover (r, s) is the signature of \mathbf{U} , and we assume that $r > p$ and $s > q$. By Theorem 1.14.4 (analog of Witt's theorem) [Ni], it follows that there exists a unique primitive embedding of M into \mathbf{U} , up to $O(\mathbf{U})$. In particular, we have $f \circ i = i' \circ g$ for some $f \in O(\mathbf{U})$ by uniqueness. Applying this to M , we find that $fM = M'$, i.e. $M = M' \bmod O(\mathbf{U})$. This proves that $(*)$ is injective. Likewise, every $[L] \in \overline{\mathcal{L}}_D(p, q)$ is represented by some $M \in \Pi$, by the existence of primitive embedding $L \hookrightarrow \mathbf{U}$. This completes the proof. \square

Note that the map $(*)$ in the preceding proof is well-defined without any of the hypotheses in the proposition. However the example above, where $E = \mathbf{R}^{1,1}$ and $V \subset E$ a 1 dimensional positive definite, shows that $(*)$ need not be a bijection without those hypotheses.

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