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Jürgen Jost

Riemannian Geometry and Geometric Analysis

Fourth Edition

With 14 Figures

 Springer

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**Dedicated to Shing-Tung Yau,
for so many discussions about
mathematics and Chinese culture**

Preface

Riemannian geometry is characterized, and research is oriented towards and shaped by concepts (geodesics, connections, curvature, ...) and objectives, in particular to understand certain classes of (compact) Riemannian manifolds defined by curvature conditions (constant or positive or negative curvature, ...). By way of contrast, geometric analysis is a perhaps somewhat less systematic collection of techniques, for solving extremal problems naturally arising in geometry and for investigating and characterizing their solutions. It turns out that the two fields complement each other very well; geometric analysis offers tools for solving difficult problems in geometry, and Riemannian geometry stimulates progress in geometric analysis by setting ambitious goals.

It is the aim of this book to be a systematic and comprehensive introduction to Riemannian geometry and a representative introduction to the methods of geometric analysis. It attempts a synthesis of geometric and analytic methods in the study of Riemannian manifolds.

The present work is the fourth edition of my textbook on Riemannian geometry and geometric analysis. It has developed on the basis of several graduate courses I taught at the Ruhr-University Bochum and the University of Leipzig. Besides several smaller additions, reorganizations, corrections (I am grateful to J.Weber and P.Hinow for useful comments), and a systematic bibliography, the main new features of the present edition are a systematic introduction to Kähler geometry and the presentation of additional techniques from geometric analysis.

Let me now briefly describe the contents:

In the first chapter, we introduce the basic geometric concepts, like differentiable manifolds, tangent spaces, vector bundles, vector fields and one-parameter groups of diffeomorphisms, Lie algebras and groups and in particular Riemannian metrics. We also derive some elementary results about geodesics.

The second chapter introduces de Rham cohomology groups and the essential tools from elliptic PDE for treating these groups. In later chapters, we shall encounter nonlinear versions of the methods presented here.

The third chapter treats the general theory of connections and curvature.

In the fourth chapter, we introduce Jacobi fields, prove the Rauch comparison theorems for Jacobi fields and apply these results to geodesics.

These first four chapters treat the more elementary and basic aspects of the subject. Their results will be used in the remaining, more advanced chapters that are essentially independent of each other.

The fifth chapter treats symmetric spaces as important examples of Riemannian manifolds in detail.

The sixth chapter is devoted to Morse theory and Floer homology.

The seventh chapter treats variational problems from quantum field theory, in particular the Ginzburg-Landau and Seiberg-Witten equations. The background material on spin geometry and Dirac operators is already developed in earlier chapters.

In the eighth chapter, we treat harmonic maps between Riemannian manifolds. We prove several existence theorems and apply them to Riemannian geometry. The treatment uses an abstract approach based on convexity that should bring out the fundamental structures. We also display a representative sample of techniques from geometric analysis.

A guiding principle for this textbook was that the material in the main body should be self contained. The essential exception is that we use material about Sobolev spaces and linear elliptic PDEs without giving proofs. This material is collected in Appendix A. Appendix B collects some elementary topological results about fundamental groups and covering spaces.

Also, in certain places in Chapter 6, we do not present all technical details, but rather explain some points in a more informal manner, in order to keep the size of that chapter within reasonable limits and not to lose the patience of the readers.

We employ both coordinate-free intrinsic notations and tensor notations depending on local coordinates. We usually develop a concept in both notations while we sometimes alternate in the proofs. Besides my not being a methodological purist, the reasons for often preferring the tensor calculus to the more elegant and concise intrinsic one are the following. For the analytic aspects, one often has to employ results about (elliptic) partial differential equations (PDEs), and in order to check that the relevant assumptions like ellipticity hold and in order to make contact with the notations usually employed in PDE theory, one has to write down the differential equation in local coordinates. Also, recently, manifold and important connections have been established between theoretical physics and our subject. In the physical literature, tensor notation is usually employed, and therefore familiarity with that notation is necessary to explore those connections that have been found to be stimulating for the development of mathematics, or promise to be so in the future.

As appendices to most of the paragraphs, we have written sections with the title “Perspectives”. The aim of those sections is to place the material in a broader context and explain further results and directions without detailed proofs. The material of these Perspectives will not be used in the main body of the text. At the end of each chapter, some exercises for the reader are given.

We assume of the reader sufficient perspicacity to understand our system of numbering and cross-references without further explanation.

The development of the mathematical subject of Geometric Analysis, namely the investigation of analytical questions arising from a geometric context and in turn the application of analytical techniques to geometric problems, is to a large extent due to the work and the influence of Shing-Tung Yau. This book, like its previous editions, is dedicated to him.

Jürgen Jost

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