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On the Proof of the Positive Mass Conjecture in General Relativity

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Abstract. Let M be a space-time whose local mass density is non-negative everywhere. Then we prove that the total mass of M as viewed from spatial infinity (the ADM mass) must be positive unless M is the flat Minkowski space-time. (So far we are making the reasonable assumption of the existence of a maximal spacelike hypersurface. We will treat this topic separately.) We can generalize our result to admit wormholes in the initial-data set. In fact, we show that the total mass associated with each asymptotic regime is non-negative with equality only if the space-time is flat.

0. Introduction

This is the second part of our paper on scalar curvature of a three-dimensional manifold and its relation to general relativity. The problem in general relativity that we address is the following: An isolated gravitating system having non-negative local mass density must have non-negative total mass, measured gravitationally at spatial infinity.

Mathematically, the positive mass conjecture can be described as follows: Let N be a three dimensional Riemannian manifold with metric tensor g_{ij}. Then an initial set consists of N and a symmetric tensor field h_{ij} so that $\mu \geqq \left|\sum_a J^a J_a\right|^{1/2}$ where μ and J are defined by

$$\mu = \frac{1}{2}\left(R - \sum_{a,b} h^{ab}h_{ab} + \left(\sum_a h^a_a\right)^2\right),$$

$$J^a = \nabla_b\left[h^{ab} - \left(\sum_c h^c_c\right) g^{ab}\right]$$

where R is the scalar curvature of our metric.

If N is a spacelike hypersurface in a space time so that g_{ij} is the induced metric and h_{ij} is the second fundamental form, then the above condition says that the apparent energy-momentum of the matter be timelike.

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An initial-data set will be said to be asymptotically flat if for some compact C, $N\backslash C$ consists of a finite number of components $N_1, \ldots, N_k$ such that each N_i is diffeomorphic to the complement of a compact set in R^3. Under such diffeomorphism, the metric tensor will be required to be written in the following form

$$ds^2 = \left(1 + \frac{M}{2r}\right)^4 \left(\sum_i (dx^i)^2\right) + \sum_{i,j} p_{ij} dx^i dx^j$$

where

$$p_{ij} = O\left(\frac{1}{r^2}\right),$$

$$\nabla p_{ij} = O\left(\frac{1}{r^3}\right)$$

and

$$\nabla\nabla p_{ij} = O\left(\frac{1}{r^4}\right).$$

The components of h_{ij} will also be required to be of order $O\left(\frac{1}{r^2}\right)$.

The number M (Arnowitt, Deser and Misner [1], Geroch [9]) is called the mass of the end N_i. This definition is motivated by the observation that the spatial Schwarzschild metric can be written asymptotically in the previous form so that the number M is precisely the Schwarzschild mass. From now on, we shall call N_i an "end" of N and we denote the total mass of N_i by M_i.

In this formulation, the (generalized) positive mass conjecture (Arnowitt et al. [1], Brill and Deser [3], Geroch [2]) states that for an asymptotically flat initial data set, each end has non-negative total mass. If one of the ends has zero total mass, then the initial data set is flat in the sense that the curvature tensor vanishes and the second fundamental form h_{ij} is trivial.

In this paper, we will settle the major case of the conjecture assuming $\sum_a h_a^a = 0$. The most general case will be discussed in a forthcoming paper.

There have been several contributions on this problem prior to our work. (We learned most of these from the excellent survey articles of Geroch [2] and Choquet-Bruhat, Fisher and Marsden [4].) In 1959, Brill settled the problem in case $\sum_a h_a^a = 0$ and the data respect an axial symmetry. In 1968, Brill and Deser [3] showed the conjecture is true up to second order perturbations from flat data. This last result was greatly improved by Choquet-Bruhat and Marsden [5] to the effect that the conjecture is true if the data is close enough to the flat data in a certain smooth norm. In the Stanford conference in differential geometry, Geroch divided the conjecture into several special cases. One case had a direct appeal to the geometers. This case says that if a metric has non-negative scalar curvature in R^3 and if the metric is euclidean outside a compact set, then the metric is flat. In her thesis in 1977, Leite was able to settle this case under the assumption that the manifold can be isometrically embedded into R^4. In 1976, Jang [6] was also able to settle the conjecture if the metric g_{ij} is flat. Finally the conjecture was also known if the data is sperically symmetric (Leibovitz and Israel [7], Misner [8], Jang [6]). However, none of these methods had been carried out to cover the case that we deal with. (It should also be mentioned that Deser had a proof for the supergravity setting and Geroch had an argument to settle the conjecture assuming some statement that remains to be proved.)

The basic idea of our proof is quite simple. It is basically geometric in nature which enables us to deal with the case where the manifold is not diffeomorphic to R^3. While there are more details to be carried out in this paper, the basic ideas are already in our previous paper.

For simplicity, let us assume the manifold is diffeomorphic to R^3. Then assuming the mass is negative we construct a complete surface embedded in R^3 whose area is minimal among all compactly supported deformations of the surface. By using the second variation formula, we prove that the surface is topologically the plane. As in the previous paper, we plan to use the Gauss-Bonnet theorem to arrive at a contradiction. However, as the surface is non-compact, there are technical troubles involved which we are able to overcome. These arguments provide a proof that the total mass is non-negative. If the total mass is zero, then we have a way to reduce it to the previous case unless the Ricci tensor is identically zero. Since M is three-dimensional, Ricci flat implies flat and the reduction finishes the proof of the theorem.

1. Statement of Results

The theorems in this paper deal with asymptotically flat metrics on 3-manifolds. Let N be an oriented three-dimensional manifold (with or without boundary) which has the property that there exists a compact subset K of N so that $N\backslash K$ consists of a finite number of components $N_1, N_2, \ldots, N_r$ with each N_k being diffeomorphic to $\mathbb{R}^3$ minus a ball. We call the N_k *ends* of N. We suppose that N is a manifold of smoothness class C^6. Let ds^2 be a C^5 positive definite metric on N. We say that ds^2 is *asymptotically flat* if each boundary component of N has positive mean curvature with respect to the outward unit normal, and on each N_k there is a coordinate system x^1, x^2, x^3 in which ds^2 has the expansion $ds^2 = \sum_{i,j=1}^{3} g_{ij} dx^i dx^j$ with the g_{ij} satisfying the following inequalities for some k_1, k_2, k_3 positive constants.

$$g_{ij} = \left(1+\frac{M}{2r}\right)^4 \delta_{ij} + h_{ij}, \qquad |h_{ij}| \leqq \frac{k_1}{1+r^2},$$
$$|\partial h_{ij}| \leqq \frac{k_2}{1+r^3}, \qquad |\partial\partial h_{ij}| \leqq \frac{k_3}{1+r^4} \tag{1.1}$$

where $r = \left(\sum_{i=1}^{3} (x^i)^2\right)^{1/2}$ and ∂ is the Euclidean gradient. The number $M = M_k$ is the *total mass* of N_k. We note that (1.1) implies that the Christoffel symbols

$\Gamma^i_{jk}=O(1/r^2)$ and the curvature tensor is $O(1/r^3)$ as $r\to\infty$. Let R be the scalar curvature function for ds^2. We now state our first theorem.

Theorem 1. *Let ds^2 be an asymptotically flat metric on an oriented 3-manifold N. If $R\geqq 0$ on N, then the total mass of each end is nonnegative.*

Our next result concerns the case when total mass on one end is zero. In this case we wish to show that N is flat. In order to prove this we need to add the following assumption to (1.1)

$$|\partial\partial h_{ij}|+|\partial\partial\partial h_{ij}|+|\partial\partial\partial\partial h_{ij}|\leqq \frac{k_4}{1+r^5} \tag{1.2}$$

for a positive constant k_4.

Theorem 2. *Let N be an oriented 3-manifold having an asymptotically flat metric ds^2. Suppose for some end N_k, (1.2) is satisfied and the total mass of N_k is zero. If $R\geqq 0$ on N, then ds^2 is flat. In fact, N is isometric to $\mathbb{R}^3$ with the standard metric.*

2. Proof of Theorem 1

Throughout this section we work on a fixed end N_k, and suppose that x^1, x^2, x^3 are asymptotically flat coordinates on N_k. Suppose these coordinates describe N_k on $\mathbb{R}^3\backslash B_{\sigma_0}(0)$, where $B_{\sigma_0}(0)=\{|x|<\sigma_0\}$ and $r=|x|$ denotes the Euclidean length of $x=(x^1,x^2,x^3)$. We denote the total mass of N_k by M, omitting reference to k. We will suppose that $M<0$ and $R\geqq 0$ in contradiction to Theorem 1. The proof then involves three steps, the first allowing us to assume $R>0$ outside a compact subset of $\mathbb{R}^3\backslash B_{\sigma_0}(0)$, the second is to use the assumption $M<0$ to prove the existence of a complete area minimizing surface, and third to use second variation arguments to show that this is impossible if $R\geqq 0$.

Step 1. If ds^2 is asymptotically flat on N with $R\geqq 0$, and with the total mass of N_k negative, then there is an asymptotically flat metric $d\tilde{s}^2$ conformally equivalent to ds^2 having $\tilde{R}\geqq 0$ on N, $\tilde{R}>0$ outside a compact subset of N_k, and having negative total mass for N_k.

Proof. Let $\mathbb{R}^3\backslash B_{\sigma_0}(0)$ represent N_k as described above. Let Δ be the Laplace operator on functions, so that for a function φ on $\mathbb{R}^3\backslash B_{\sigma_0}(0)$

$$\Delta\varphi=\frac{1}{\sqrt{g}}\sum_{i,j=1}^{3}\frac{\partial}{\partial x^i}\left(\sqrt{g}\,g^{ij}\frac{\partial\varphi}{\partial x^j}\right)$$

where as usual, $g=\det(g_{ij})$ and $(g^{ij})=(g_{ij})^{-1}$. We calculate the asymptotic expansion of $\Delta\frac{1}{r}$ on $\mathbb{R}^3\backslash B_{\sigma_0}(0)$ using (1.1). We see that

$$\begin{aligned}\Delta\frac{1}{r}&=\sum_{i=1}^{3}\frac{\partial}{\partial x^i}\left(\left(1+\frac{M}{2r}\right)^2\frac{\partial}{\partial x^i}\left(\frac{1}{r}\right)\right)+O({}^1/r^5)\\&=\left(1+\frac{M}{2r}\right)\frac{M}{r^4}+O({}^1/r^5)\\&=\frac{M}{r^4}+O({}^1/r^5)\,.\end{aligned}\tag{2.1}$$

It follows that there is a number $\sigma > \sigma_0$ so that

$$\Delta \frac{1}{r} < 0 \quad \text{for} \quad r \geqq \sigma .$$

Choose $t_0 = -\dfrac{M}{8\sigma_0}$, and let $\zeta(t)$ be a C^5 function which satisfies

$$\zeta(t) = \begin{cases} t & \text{for} \quad t < t_0 \\ \dfrac{3t_0}{2} & \text{for} \quad t > 2t_0 , \end{cases} \tag{2.2}$$

$$\zeta'(t) \geqq 0 , \qquad \zeta''(t) \leqq 0 \quad \text{for} \quad t \in (0, \infty) .$$

Define a C^5 function $\varphi : N \to \mathbb{R}$ by

$$\varphi = 1 + \frac{3t_0}{2} \quad \text{on} \quad N \setminus N_k ,$$

$$\varphi(x) = 1 + \zeta\left(-\frac{M}{4r}\right) \quad \text{on} \quad \mathbb{R}^3 \setminus B_{\sigma_0}(0) = N_k .$$

From (2.1) and (2.2) we see that

$$\Delta\varphi \leqq 0 \quad \text{on} \quad N , \quad \text{and} \quad \Delta\varphi < 0 \quad \text{for} \quad r > 2\sigma . \tag{2.3}$$

We now define a new metric

$$d\tilde{s}^2 = \varphi^4 ds^2 .$$

The metric $d\tilde{s}^2$ is asymptotically flat since on all ends other than N_k it is a constant multiple of ds^2, and on N_k we have

$$\begin{aligned} \tilde{g}_{ij} &= \left(1 - \frac{M}{4r}\right)^4 \left(1 + \frac{M}{2r}\right)^4 \delta_{ij} + O(1/r^2) \\ &= \left(1 + \frac{M}{4r}\right)^4 \delta_{ij} + O(1/r^2) . \end{aligned}$$

Thus the new mass of N_k is $\tilde{M} = \dfrac{M}{2} < 0$. The well-known formula for the scalar curvature $\tilde{R}$ is

$$\tilde{R} = \varphi^{-5}[-8\Delta\varphi + R\varphi] .$$

Thus (2.3) implies that $\tilde{R} \geqq 0$ on N and $\tilde{R} > 0$ for $r > 2\sigma$ on N_k. This concludes the proof of Step 1.

We replace our original metric ds^2 by $d\tilde{s}^2$ but maintain the notation ds^2, so that we are assuming $R \geqq 0$ on N, $R > 0$ outside a compact subset of N_k, and $M < 0$.

Step 2. There exists a complete area minimizing (relative to ds^2) surface S properly imbedded in N so that $S \cap (N \setminus N_k)$ is compact, and $S \cap N_k$ lies between two parallel Euclidean 2-planes in the 3-space defined by x^1, x^2, x^3.

Proof. Let $\sigma > 2\sigma_0$, and let C_σ be the circle of Euclidean radius σ centered at 0 in the x^1x^2-plane. Let S_σ be the smooth imbedded oriented surface of least ds^2-area among all competing surfaces regardless of topological type having boundary curve C_σ. A discussion of this known existence result is given in the Appendix to this paper. We wish to extract a sequence $\sigma_i \to \infty$ so that S_{σ_i} converges to the required surface S.

We first show that there is a compact subset $K_0 \subseteqq N$ so that we have

$$S_\sigma \cap (N \backslash N_k) \subseteqq K_0 \quad \text{for every} \quad \sigma > 2\sigma_0 . \tag{2.4}$$

That is, we show that the S_σ cannot run to infinity in an end other than N_k. To see this, let $N_{k'}$ be another end, with asymptotically flat coordinate system y^1, y^2, y^3 associating $N_{k'}$ with $\mathbb{R}^3 \backslash B_{\tau_0}(0)$ where $B_{\tau_0}(0) = \{y : |y| < \tau_0\}$. In this coordinate system, the metric ds^2 has the form $ds^2 = \sum_{i,j=1}^{3} g'_{ij} dy^i dy^j$ with g'_{ij} satisfying (1.1). We calculate the covariant hessian of the function $|y|^2$, that is $D_{ij}|y|^2 = \frac{\partial^2 |y|^2}{\partial y^i \partial y^j} - D_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j}(|y|^2)$. By (1.1) we see

$$D_{ij}|y|^2 = 2\delta_{ij} + O(1/|y|) \quad \text{as} \quad |y| \to \infty$$

where δ_{ij} is the Kronecker delta. In particular, we see that there exists $\tau_1 > \tau_0$ so that the function $|y|^2$ is a convex function for $|y| \geqq \tau_1$. Since $\partial S_\sigma = C_\sigma$ which lies in N_k, we may apply the maximum principle to conclude that

$$S_\sigma \cap N_{k'} \subseteqq B_{\tau_1}(0) .$$

Since $N_{k'}$ was any end of N other than N_k, we have established (2.4).

We now analyze the behavior of $S_\sigma \cap N_k$. In fact, we bound the height of $S_\sigma \cap N_k$ in the x^3 direction. For any $h > 0$, we let

$$E_h = \{x \in \mathbb{R}^3 : |x^3| \leqq h\} .$$

We show that there exists a number $h > \sigma_0$ so that

$$N_k \cap S_\sigma \subseteqq E_h \quad \text{for all} \quad \sigma > 2\sigma_0 . \tag{2.5}$$

To accomplish this, we again use a maximum principle, this time for the function x^3 restricted on $S_\sigma \cap N_k$. We must first compute the asymptotic behavior of the covariant hessian of x^3 on N_k. If D is the Riemannian connection for ds^2, define Γ^l_{ij} by

$$D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{l=1}^{3} \Gamma^l_{ij} \frac{\partial}{\partial x^l} .$$

Then Γ^l_{ij} has the following expression in terms of ds^2

$$\Gamma^l_{ij} = \tfrac{1}{2} g^{lm} \left(\frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m} \right) . \tag{2.6}$$

The hessian of any function φ is given by

$$D_{ij}\varphi = \frac{\partial^2 \varphi}{\partial x^i \partial x^j} - D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}(\varphi) .$$

By direct calculation using (1.1) and (2.6) we have

$$D_{ij}x^3 = -\Gamma_{ij}^3 = \frac{Mx^j}{r^3}\delta_{i3} + \frac{Mx^i}{r^3}\delta_{j3} - \frac{Mx^3}{r^3}\delta_{ij} + O(1/r^3)\,. \tag{2.7}$$

Let $\bar{h}$ be the maximum for x^3 on $S_\sigma \cap N_k$, and suppose this maximum occurs at the point $x_0 \in S_\sigma$. If $\bar{h} \leqq \sigma_0$, we have established (2.5). So suppose $\bar{h} > \sigma_0$. The tangent space to S_σ at x_0 is then spanned by $\frac{\partial}{\partial x^1}(x_0)$, $\frac{\partial}{\partial x^2}(x_0)$. Let $v_1\, v_2$ be tangent vector fields to S_σ defined in a neighborhood of x_0 and satisfying $v_i(x_0) = \frac{\partial}{\partial x^i}(x_0)$ for $i = 1, 2$. Let $(q_{ij})_{1 \leqq i,j \leqq 2}$ be the restriction of ds^2 to S in terms of the base field v_1, v_2. Let ∇ be the induced connection on S_σ, and note

$$\begin{aligned} v_i v_j x^3 - \nabla_{v_i} v_j(x^3) &= v_i v_j x^3 - D_{v_i} v_j(x^3) \\ &\quad + \langle D_{v_i} v_j, \nu \rangle \nu(x^3) \end{aligned}$$

where ν is the unit normal field of S_σ. Evaluating at the point x_0 we have

$$\nabla_{ij} x^3 = D_{ij} x^3 + h_{ij} \nu(x^3)$$

where $h_{ij} = \langle D_{v_i} v_j, \nu \rangle (x_0)$ is the second fundamental form. Contracting with respect to (q_{ij}) we have

$$\sum_{i,j=1}^{2} q^{ij} \nabla_{ij} x^3 = \sum_{i,j=1}^{2} q^{ij} D_{ij} x^3 + \sum_{i,j=1}^{2} q^{ij} h_{ij} \nu(x^3)\,.$$

Since S_σ is minimal we have $\sum_{i,j=1}^{2} q^{ij} h_{ij} = 0$, so applying (2.7) we see

$$\sum_{i,j=1}^{2} q^{ij} \nabla_{ij} x^3 = -\frac{2M\bar{h}}{r^3} + O(1/r^3)\,.$$

Since $M < 0$, we see that $\bar{h}$ sufficiently large implies that $\sum_{i,j=1}^{2} q^{ij} \nabla_{ij} x^3 > 0$ at x_0 contradicting the fact that x^3 attains a maximum there. A similar argument gives a lower bound on $x^3|_{S_\sigma \cap N_k}$, and we have established (2.5)

Now, let $\varrho > 2\sigma_0$ and define the set

$$A_\varrho = (N \backslash N_k) \cup \{x : |x| \geqq \sigma_0, (x^1)^2 + (x^2)^2 \leqq \varrho^2\}\,.$$

For any $\sigma > \varrho$, (2.4) and (2.5) imply

$$S_\sigma \cap A_\varrho \subseteqq (K_0 \cup E_h) \cap A_\varrho \tag{2.8}$$

which is a compact subset of N. We now quote a local interior regularity estimate for area minimizing surfaces which is discussed in the Appendix.

(2.1) *Regularity Estimate.* Let $U_r(x)$ denote the geodesic ball of radius r about $x \in N$. There exists a number $r_0 > 0$ so that for any point $x_0 \in S_\sigma$ with $U_{r_0}(x_0) \cap C_\sigma = \emptyset$, it is true that $S_\sigma \cap U_{r_0}(x_0)$ can be written as the graph of a C^3 function f_σ over the tangent plane to S_σ in a normal coordinate system on $U_{r_0}(x_0)$.

Moreover, there is a constant c_1 depending only on (N, ds^2) which bounds all derivatives of f_σ up to order three in $U_{r_0}(x_0)$. (Note that both r_0 and c_1 are independent of σ.)

It then follows from (2.8) and the Regularity Estimate that we can choose a sequence $\sigma_i^{(\varrho)} \to \infty$ so that $S_{\sigma_i^{(\varrho)}} \cap A_\varrho$ converges in C^2 topology. Since this can be done for any $\varrho > 2\sigma_0$, we can take a sequence $\varrho_j \to \infty$ and by extracting a diagonal sequence we find a sequence $\sigma_i \to \infty$ so that $S_{\sigma_i} \to S$, an imbedded C^2-surface, uniformly in C^2 norm on compact subsets of N. The surface S is properly imbedded by (2.8), and is clearly area minimizing on any compact subset of N. From (2.4) we have $S \cap (N \backslash N_k) \subset K_0$ and hence $S \cap (N \backslash N_k)$ is compact. From (2.5) we have $S \cap N_k \subseteqq E_h$ which is the region between two parallel 2-plane in $\mathbb{R}^3$. This completes the proof of Step 2.

The final step in the proof of Theorem 1 is to use the condition on the scalar curvature to derive a contradiction.

Step 3. The surface S constructed in Step 2 cannot exist.

Proof. For any $\sigma \geqq \sigma_0$, let $S_{(\sigma)}$ be the set

$$S_{(\sigma)} = [S \cap (N \backslash N_k)] \cup [S \cap B_\sigma(0)] .$$

The $S_{(\sigma)}$ form an exhaustion of S, and we can see

$$\text{Area}\,(S_{(\sigma)}) \leqq C_2 \sigma^2 \tag{2.9}$$

for a constant C_2 independent of $\sigma \geqq \sigma_0$. To prove (2.9), we note that if S has transverse intersection with $\partial B_\sigma(0)$ then this intersection is a union of oriented C^2 Jordan curves on $\partial B_\sigma(0)$ which bound $S_{(\sigma)}$. It follows that these curves bound a domain $\Omega \subseteqq \partial B_\sigma(0)$. Thus we have $\partial S_{(\sigma)} = \partial \Omega$, so we can apply the area minimizing property of S to conclude that

$$\text{Area}\,(S_{(\sigma)}) \leqq \text{Area}\,(\Omega) \leqq \text{Area}\,(\partial B_\sigma(0)) .$$

Since (1.1) implies that ds^2 is uniformly equivalent to the Euclidean metric on $\mathbb{R}^3 \backslash B_{\sigma_0}(0)$, (2.9) follows for those $\sigma > \sigma_0$ for which $S \cap \partial B_\sigma(0)$ is transverse. Since this is true except for σ in a set of measure zero, (2.9) follows for any $\sigma \geqq \sigma_0$ by approximation.

We can use (2.9) to bound the integrals of certain functions on S. For $a > 0$ we have

$$\begin{aligned} \int_S \frac{1}{1+r^a} &= \int_{S_{(\sigma_0)}} \frac{1}{1+r^a} + \int_{\sigma_0}^{\infty} \left(\frac{d}{dt} \int_{S_t} \frac{1}{1+r^a} \right) dt \\ &\leqq \text{Area}\,(S_{(\sigma_0)}) + \int_{\sigma_0}^{\infty} \frac{1}{1+t^a} \left(\frac{d}{dt} \text{Area}\,(S_{(t)}) \right) dt . \end{aligned}$$

If $a > 2$, we can integrate by parts and apply (2.9) to get

$$\begin{aligned} \int_S \frac{1}{1+r^a} &\leqq c_2 \sigma_0^2 + a \int_{\sigma_0}^{\infty} \frac{t^{a-1}}{(1+t^a)^2} \text{Area}\,(S_{(t)}) dt \\ &\leqq c_2 \sigma_0^2 + c_2 a \int_{\sigma_0}^{\infty} \frac{t^{a+1}}{(1+t^a)^2} dt . \end{aligned}$$

It follows that we have

$$\int_S \frac{1}{1+r^a} < \infty \quad \text{whenever} \quad a>2\,. \tag{2.10}$$

Applying similar reasoning, we have for $\sigma_0 < \sigma_1 < \sigma_2$,

$$\int_{S(\sigma_2)\setminus S(\sigma_1)} \frac{1}{r^2} \leqq 2C_2 \log \sigma_2/\sigma_1\,. \tag{2.11}$$

We now introduce the second variation inequality for S. This inequality expresses the fact that up to second order S has smallest area in a one-parameter compactly supported deformation of S. Let e_1, e_2, e_3 be orthonormal (with respect to ds^2) vector fields defined locally on N. We use the notation

K_{ij} = sectional curvature of the section $\{e_i, e_j\}$.

The Ricci tensor can then be written

$$\mathrm{Ric}(e_i) = \sum_{j=1}^{3} K_{ij}$$

where we let $K_{ii}=0$. The scalar curvature R is then given by

$$R = K_{12} + K_{13} + K_{23}\,.$$

Let v be the unit normal vector field of S, and choose a frame $e_1, e_2, e_3 = v$ adapted to S. Let A denote the second fundamental form of S, i.e. the matrix in terms of e_1, e_2 is

$$h_{ij} = \langle D_{e_i} v, e_j \rangle\,.$$

It is well-known that A is a symmetric quadratic tensor on S. We let $\|A\|^2$ denote the length of A with respect to ds^2, i.e.

$$\|A\|^2 = \sum_{i,j=1}^{2} h_{ij}^2\,.$$

The condition that S is a minimal surface is

$$\mathrm{Trace}(A) = h_{11} + h_{22} = 0\,. \tag{2.12}$$

The second variation inequality (see [10]) for S is

$$\int_S f[\Delta f + (\mathrm{Ric}(v) + \|A\|^2) f] \leqq 0$$

for any C^2 function f with compact support on S. After integration by parts we see

$$\int_S (\mathrm{Ric}(v) + \|A\|^2) f^2 \leqq \int_S \|\nabla f\|^2 \tag{2.13}$$

for any C^2 function f with compact support on S. By approximation we see easily that (2.13) holds for any Lipschitz function f with compact support on S. The Gauss curvature equation expresses the Gauss curvature K of S as

$$K = K_{12} + h_{11}h_{22} - h_{12}^2\,. \tag{2.14}$$

Applying (2.12) and the symmetry of A gives

$$\tfrac{1}{2}\|A\|^2 = K_{12} - K\,.$$

Putting this into (2.13) gives

$$\int_S (\mathrm{Ric}(v) + K_{12} - K + \tfrac{1}{2}\|A\|^2) f^2 \leqq \int_S \|\nabla f\|^2\,.$$

That is, we have

$$\int_S (R - K + \tfrac{1}{2}\|A\|^2) f^2 \leqq \int_S \|\nabla f\|^2\,. \tag{2.15}$$

We now choose a suitable cutoff function for f in our inequalities. For $\sigma > \sigma_0$ define a function φ by

$$\varphi = \begin{cases} 1 & \text{on} \quad S_{(\sigma)} \\ \dfrac{\log \dfrac{\sigma^2}{r}}{\log \sigma} & \text{on} \quad S_{(\sigma^2)} \backslash S_{(\sigma)} \\ 0 & \text{outside} \quad S_{(\sigma^2)} \end{cases}.$$

Let g be a Lipschitz function on S satisfying $|g| \leqq 1$ and $g = 1$ outside a compact subset of S. Setting $f = \varphi g$ in (2.13) and applying the Schwarz inequality gives

$$\begin{aligned} \int_S (\mathrm{Ric}(v) + \|A\|^2) \varphi^2 g^2 &\leqq 2 \int_S g^2 \|\nabla \varphi\|^2 + 2 \int_S \varphi^2 \|\nabla g\|^2 \\ &\leqq \frac{2}{(\log \sigma)^2} \int_{S_{(\sigma^2)} \backslash S_{(\sigma)}} \frac{\|\nabla r\|^2}{r^2} + 2 \int_S \varphi^2 \|\nabla g\|^2\,. \end{aligned}$$

Because of (1.1), there is a constant C_3 with $\|\nabla r\|^2 \leqq C_3$. Thus our inequality implies by rearranging and using the definition of φ and g

$$\int_{S_{(\sigma)}} \|A\|^2 g^2 \leqq \frac{2C_3}{(\log \sigma)^2} \int_{S_{(\sigma^2)} \backslash S_{(\sigma)}} \frac{1}{r^2} + 2 \int_S \|\nabla g\|^2 + \int_S |\mathrm{Ric}(v)| g^2\,.$$

Applying (2.11) we have

$$\int_{S_{(\sigma)}} \|A\|^2 g^2 \leqq 2C_2 C_3 (\log \sigma)^{-1} + 2 \int_S \|\nabla g\|^2 + \int_S |\mathrm{Ric}(v)| g^2\,.$$

Letting $\sigma \to \infty$ we conclude

$$\int_S \|A\|^2 g^2 \leqq 2 \int_S \|\nabla g\|^2 + \int_S |\mathrm{Ric}(v)| g^2 \tag{2.16}$$

for any Lipschitz g with $|g| \leqq 1$, $g \equiv 1$ outside a compact subset of S. By (1.1), we have $\mathrm{Ric}(v) = O(1/r^3)$, so choosing $g \equiv 1$ on S and applying (2.10) and $a = 3$ we conclude from (2.16) that $\int_S \|A\|^2 < \infty$. (The formula (2.16) with $g \not\equiv 1$ will be used later.)

Formula (2.14) implies that $|K| \leqq |K_{12}| + \|A\|^2$. By (1.1) we have $K_{12} = O(1/r^3)$ so (2.10) implies $\int_S |K_{12}| < \infty$. Thus we have

$$\int_S |K| < \infty . \tag{2.17}$$

We now use the function $f = \varphi$ in inequality (2.15) and let $\sigma \to \infty$ as above to conclude

$$\int_S (R - K + \tfrac{1}{2}\|A\|^2) \leqq 0 .$$

Since $R \geqq 0$, and $R > 0$ outside a compact subset of S, we conclude

$$\int_S K > 0 . \tag{2.18}$$

Remark 2.1. The Cohn-Vossen inequality says that $\int_S K \leqq 2\pi\chi(S)$, where $\chi(S)$ is the Euler characteristic of S. Combining this with (2.18) we see immediately that S is homeomorphic to $\mathbb{R}^2$.

In light of (2.18), the proof of Step 3 will be finished if we can show $\int_S K \leqq 0$. Since this is a very important part of our proof of Theorem 1, we give two proofs of this inequality. The first proof is conceptually very clear, and has the advantage of being more general than the second. The first proof, however, uses a deep theorem of R. Finn [11] and A. Huber [12] concerning the Gauss-Bonnet theorem on open Riemann surfaces, while the second uses no outside results and is special to our situation.

Claim $\int_S K \leqq 0$.

First Proof. We first note that inequality (2.17) and Remark 2.1 imply, by a result of A. Huber [13], that S is conformally equivalent to the complex plane. Thus there is a conformal diffeomorphism $F : \mathbb{C} \to S$. Let D_σ denote the disk of radius σ in $\mathbb{C}$, and let C_σ be the circle of radius σ. For $i = 1, 2, \ldots$ let $L_i = \text{length}(F(C_i))$, and let $A_i = \text{Area}(F(D_i))$. The simply connected case of the theorem of R. Finn [11] and A. Huber [12] says that

$$\int_S K = 2\pi - \lim_{i \to \infty} \frac{L_i^2}{2A_i} . \tag{2.19}$$

Thus to show $\int_S K \leqq 0$, it suffices to show

$$\lim_{i \to \infty} \frac{L_i^2}{4\pi A_i} \geqq 1 . \tag{2.20}$$

Since S is properly imbedded in N with $S \cap (N \backslash N_k)$ compact, we see that $F(C_i)$ lies outside any compact subset of N_k for i sufficiently large. Thus for large i, we let $\tilde{L}_i$

be the Euclidean length of $F(C_i)$. Inequality (1.1) implies

$$\tilde{L}_i^2 \leqq (1+o(1))L_i^2 \quad \text{as} \quad i\to\infty\ . \tag{2.21}$$

For the immersed disk Σ_i of least Euclidean area with boundary curve $F(C_i)$, we have the well-known inequality whose proof can be found in [14]

$$\tilde{A}(\Sigma_i) \leqq \frac{\tilde{L}_i^2}{4\pi}$$

where $\tilde{A}(\cdot)$ is Euclidean area. Let $\tilde{\Sigma}_i$ be an oriented surface of least Euclidean area among all surfaces of boundary $F(C_i)$ regardless of topological type (see Appendix). Since $\tilde{A}(\tilde{\Sigma}_i) \leqq \tilde{A}(\Sigma_i)$, we have

$$\tilde{A}(\tilde{\Sigma}_i) \leqq \frac{\tilde{L}_i^2}{4\pi}\ . \tag{2.22}$$

Because $F(C_i)$ lies outside any compact set for i sufficiently large, we can find a sequence $\sigma_i\to\infty$ with $\tilde{\Sigma}_i \cap B_{\sigma_i}(0) \subseteq \tilde{\Sigma}_i \backslash F(C_i)$. Since from Step 2 we know that $F(C_i) \subseteq E_h = \{x\in \mathbb{R}^3 : |x^3| \leqq h\}$, it follows that $\tilde{\Sigma}_i \subseteq E_h$ by the convex hull property of minimal surfaces. Since $\tilde{\Sigma}_i$ does not retract onto its boundary circle, there is a point $x_0 \in \tilde{\Sigma}_i \cap \{(0,0,x^3) : x^3 \in \mathbb{R}\}$. A well-known inequality (see [15]) implies $\tilde{A}(\tilde{\Sigma}_i \cap B_r(x_0)) \geqq \pi r^2$. Thus we clearly have

$$\tilde{A}(\tilde{\Sigma}_i \cap B_{\sigma_i}(0)) \geqq (1+o(1))\pi\sigma_i^2\ . \tag{2.23}$$

We wish to compare the ds^2-area of $\tilde{\Sigma}_i$ with the Euclidean area, but we cannot do it directly since $\tilde{\Sigma}_i \cap B_{\sigma_0}(0)$ may be nonempty, and ds^2 is not defined on this part of $\tilde{\Sigma}_i$. We get around this problem by modifying $\tilde{\Sigma}_i$ near 0. Let $\bar{\sigma} \in [\sigma_0, \sigma_0+1]$ be such that $\partial B_{\bar{\sigma}}(0)$ and $\tilde{\Sigma}_i$ have transverse (or empty) intersection. We can then find a domain Ω_i on $\partial B_{\bar{\sigma}}(0)$ so that

$$\partial \Omega_i = \tilde{\Sigma}_i \cap \partial B_{\bar{\sigma}}(0)\ .$$

We then define a new surface $\hat{\Sigma}_i$ by

$$\hat{\Sigma}_i = (\tilde{\Sigma}_i \backslash B_{\bar{\sigma}}(0)) \cup \Omega_i\ .$$

Now (2.23) implies $\tilde{A}(\tilde{\Sigma}_i)\to\infty$, so we let $\tilde{A}_i = \tilde{A}(\hat{\Sigma}_i)$, and conclude

$$\tilde{A}_i \leqq (1+o(1))\tilde{A}(\tilde{\Sigma}_i)$$

which combines with (2.22) to give

$$\tilde{A}_i \leqq (1+o(1))\frac{\tilde{L}_i^2}{4\pi}\ . \tag{2.24}$$

By the area minimizing property of $\tilde{\Sigma}_i$, we also have $\tilde{A}(\tilde{\Sigma}_i) \leqq \tilde{A}_i$, so $\tilde{A}_i\to\infty$. By choosing σ_i smaller if necessary, we take

$$A(\hat{\Sigma}_i \cap B_{\sigma_i}(0)) \leqq \sqrt{\tilde{A}_i} \tag{2.24a}$$

and $A(\hat{\Sigma}_i \cap B_{\sigma_i}(0))\to\infty$. This can be done because of (2.23) and (1.1). If the σ_i remain bounded, say $\sigma_i \leqq \varrho$ for all i, then by comparison as in the proof of (2.9) we would have

$$\tilde{A}(\tilde{\Sigma}_i \cap B_\varrho(0)) \leqq 4\pi\varrho^2$$

which would imply that $A(\hat{\Sigma}_i \cap B_{\sigma_i}(0))$ is a bounded sequence. Thus we must have $\sigma_i \to \infty$. It then follows by asymptotic flatness

$$A(\hat{\Sigma}_i \backslash B_{\sigma_i}(0)) \leqq (1+o(1))\tilde{A}_i \ .$$

Combining this with (2.24a) we have

$$A(\hat{\Sigma}_i) \leqq (1+o(1))\tilde{A}_i \quad \text{as} \quad i \to \infty \ . \tag{2.25}$$

Using the area minimizing property of S and inequalities (2.21), (2.24), and (2.25) we have

$$\begin{aligned} A_i &\leqq A(\hat{\Sigma}_i) \leqq (1+o(1))\tilde{A}_i \leqq (1+o(1))\frac{\tilde{L}_i^2}{4\pi} \\ &\leqq (1+o(1))\frac{L_i^2}{4\pi} \quad \text{as} \quad i \to \infty \ . \end{aligned}$$

We thus conclude $\lim\limits_{i\to\infty} \dfrac{L_i^2}{4\pi A_i} \geqq 1$ establishing (2.20). This completes the first proof of our claim.

Second Proof. We now give a proof of the claim in which we directly apply the Gauss-Bonnet theorem with boundary, and estimate the boundary terms. For any $x \in \mathbb{R}^3$, let $x' = (x^1, x^2, 0)$, and let $r' = |x'| = ((x^1)^2 + (x^2)^2)^{1/2}$. Consider the cylinder

$$P_\sigma = \{x \in \mathbb{R}^3 : r' \leqq \sigma\} \ .$$

For any $\sigma > \sigma_0$ for which $\partial P_\sigma \cap S$ is transverse, we have by Remark 2.1, at least one circle in this intersection which is not homologous to zero in $\mathbb{R}^3 \backslash P_{\sigma_0}$. Choose one of these circles, and let D_σ be the connected component of this circle in $S \cap [(N \backslash N_k) \cup P_\sigma]$. We claim that for σ sufficiently large, D_σ is a disk. To see this, recall from the first proof that S is conformally equivalent to $\mathbb{C}$, so we have a conformal diffeomorphism $F : \mathbb{C} \to S$. Now $F^{-1}(D_\sigma)$ is a bounded, connected region in $\mathbb{C}$. By transversality, the function r' changes sign across each boundary component of $F^{-1}(D_\sigma)$. If $F^{-1}(D_\sigma)$ is not simply connected, then there is a bounded domain $\mathcal{O}$ contained in $\mathbb{C} \backslash F^{-1}(D_\sigma)$. Thus on $\partial F(\mathcal{O})$ we have $r' = \sigma$, and inside $F(\mathcal{O})$ at some points we have $r' > \sigma$. Thus r' takes a maximum at some point of $F(\mathcal{O})$. We claim that $(r')^2$ is a subharmonic function on S for r' sufficiently large, which will give a contradiction. We calculate

$$\Delta x^i = \sum_{j=1}^{2} e_j e_j x^i - \nabla_{e_j} e_j x^i \ .$$

Now $e_j x^i = \left\langle e_j, \dfrac{\partial}{\partial x^i} \right\rangle + O(1/r)$, so

$$\begin{aligned} \Delta x^i &= \sum_{j=1}^{2} \left[\left\langle \nabla_{e_j} e_j, \frac{\partial}{\partial x^i} \right\rangle + h_{jj} \left\langle V, \frac{\partial}{\partial x^i} \right\rangle - \left\langle \nabla_{e_j} e_j, \frac{\partial}{\partial x^i} \right\rangle \right] + O(1/r^2) \\ &= O(1/r^2) \end{aligned}$$

since $\sum_{j=1}^{2} h_{jj}=0$. Thus we have

$$\Delta(r')^2 = 2\sum_{i,j=1}^{2}\left\langle e_i, \frac{\partial}{\partial x^j}\right\rangle^2 + O(1/r)\,.$$

Since both $\{e_1, e_2\}$ and $\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right\}$ span 2-dimensional subspace of $\mathbb{R}^3$, they must intersect in at least a line, so the norm of their projection is asymptotically bounded below, i.e. $\sum_{i,j=1}^{2}\left\langle e_i, \frac{\partial}{\partial x^j}\right\rangle^2 \geqq 1-O(1/r)$. So we have

$$\Delta(r')^2 \geqq 2-O(1/r) \quad \text{as} \quad r\to\infty\,.$$

Thus for r sufficiently large, in particular for r' sufficiently large we have $\Delta(r')^2>0$. Thus it follows that D_σ is a disk for σ large.

We may choose the D_σ to be increasing, so that $D_{\bar\sigma}\supset D_\sigma$ when $\bar\sigma>\sigma$. Since S is connected, the D_σ form an exhaustion of S, and we apply the Gauss-Bonnet theorem on D_σ, so that

$$\int_{D_\sigma} K = 2\pi - \int_{\partial D_\sigma} k$$

where k is the geodesic curvature of ∂D_σ relative to the inner normal. Thus, our proof will be complete if we can find a sequence $\sigma_i\to\infty$ so that

$$\int_{\partial D_{\sigma_i}} k \geqq 2\pi - o(1) \quad \text{as} \quad i\to\infty\,. \tag{2.26}$$

In a neighborhood of ∂D_σ, we choose a frame e_1, e_2, e_3 where e_1 the positively oriented unit tangent vector of ∂D_σ, e_2 is the inner normal to D_σ, and $e_3=v$ is the unit normal of S in $\mathbb{R}^3$ relative to ds^2. The geodesic curvature k is given by

$$k = \langle D_{e_1}e_1, e_2\rangle\,.$$

Since $r'=\sigma$ on ∂D_σ, we have $\langle e_1, Dr'\rangle = 0$ on ∂D_σ. Differentiating this with respect to e_1 gives

$$\langle D_{e_1}e_1, Dr'\rangle + \langle e_1, D_{e_1}Dr'\rangle = 0\,.$$

Now (1.1) implies that $Dr' = \frac{x'}{\sigma} + O(1/\sigma)$ and

$$D_{e_1}Dr' = \frac{e_1}{\sigma} - \frac{1}{\sigma}\left\langle e_1, \frac{\partial}{\partial x^3}\right\rangle \frac{\partial}{\partial x^3} + O(1/\sigma^2)\,,$$

so we have

$$\left\langle D_{e_1}e_1, \frac{x'}{\sigma}\right\rangle + 1/\sigma - 1/\sigma\left\langle e_1, \frac{\partial}{\partial x^3}^2\right\rangle^2 = O(1/\sigma)\|D_{e_1}e_1\| + O(1/\sigma^2)\,.$$

Since $D_{e_1}e_1 = ke_2 - h_{11}v$, this gives

$$k\left\langle e_e, \frac{x'}{\sigma}\right\rangle + 1/\sigma - h_{11}\left\langle v, \frac{x'}{\sigma}\right\rangle - \frac{1}{\sigma}\left\langle e_1, \frac{\partial}{\partial x^3}\right\rangle^2 = O(1/\sigma)\|D_{e_1}e_1\| + O(1/\sigma^2)\,. \tag{2.27}$$

Suppose $\sigma\in[\bar{\sigma},2\bar{\sigma}]$. We now apply the divergence theorem for the vector field $\frac{\partial}{\partial x^3}$ on the volume enclosed by $D_{2\bar{\sigma}}\backslash D_{\bar{\sigma}}$, $\partial P_{2\bar{\sigma}}$, $\partial P_{\bar{\sigma}}$, and the plane annulus $\Omega_{\bar{\sigma}}=\{x:x^3=-h,\bar{\sigma}\leqq r'\leqq 2\bar{\sigma}\}$ where h is a bound on $|x^3|$ for $x\in S\cap N_k$ (see Step 2). By (1.1), $\operatorname{div}\frac{\partial}{\partial x^3}=O(1/\bar{\sigma}^2)$, so we have

$$\int_{D_{2\bar{\sigma}}\backslash D_{\bar{\sigma}}}\left\langle v,\frac{\partial}{\partial x^3}\right\rangle-A(\Omega_\sigma)=O(1)$$

where we have used the fact that $\left\langle\frac{\partial}{\partial x^3},\mathbf{n}\right\rangle=O(1/\sigma)$ where $\mathbf{n}$ is the unit normal of $\partial P_{2\bar{\sigma}}$ and $\partial P_{\bar{\sigma}}$. Applying the area minimizing property of S on $D_{2\bar{\sigma}}\backslash D_{\bar{\sigma}}$ as compared with the union of $\Omega_{\bar{\sigma}}$ and the part of $\partial P_{2\bar{\sigma}}\cup\partial P_{\bar{\sigma}}$ between S and $\Omega_{\bar{\sigma}}$, we have

$$A(D_{2\bar{\sigma}}\backslash D_{\bar{\sigma}})\leqq A(\Omega_{\bar{\sigma}})+O(\bar{\sigma})\,. \tag{2.28}$$

Combined with the above inequality this gives

$$\int_{D_{2\bar{\sigma}}\backslash D_{\bar{\sigma}}}1-\left\langle v,\frac{\partial}{\partial x^3}\right\rangle\leqq O(\bar{\sigma})\,.$$

The coarea formula (see [16, p. 258]) gives

$$\int_{\bar{\sigma}}^{2\bar{\sigma}}\int_{D_{2\bar{\sigma}}\cap\partial P_t}\left(1-\left\langle v,\frac{\partial}{\partial x^3}\right\rangle\right)ds\,dt=\int_{D_{2\bar{\sigma}}\cap(P_{2\bar{\sigma}}\backslash P_{\bar{\sigma}})}\|\nabla r'\|\left(1-\left\langle v,\frac{\partial}{\partial x^3}\right\rangle\right)$$

where ds is arclength on $D_{2\bar{\sigma}}\cap\partial P_t$. Since

$$D_{2\bar{\sigma}}\cap(P_{2\bar{\sigma}}\backslash P_{\bar{\sigma}})\subseteqq D_{2\bar{\sigma}}\backslash D_{\bar{\sigma}}\,,$$

we can combine these inequalities

$$\int_{\bar{\sigma}}^{2\bar{\sigma}}\int_{D_{2\bar{\sigma}}\cap\partial P_t}\left(1-\left\langle v,\frac{\partial}{\partial x^3}\right\rangle\right)ds\,dt\leqq O(\bar{\sigma})\,. \tag{2.29}$$

Again using the coarea formula we have

$$\int_{\bar{\sigma}}^{2\bar{\sigma}}L(D_{2\bar{\sigma}}\cap\partial P_t)dt=\int_{D_{2\bar{\sigma}}\cap(P_{2\bar{\sigma}}\backslash P_{\bar{\sigma}})}\|\nabla r'\|\,.$$

Combined with (2.28) this gives

$$\int_{\bar{\sigma}}^{2\bar{\sigma}}L(D_{2\bar{\sigma}}\cap\partial P_t)dt=O(\bar{\sigma}^2)\,. \tag{2.30}$$

We must now bound the second fundamental form of S on ∂D_σ. To do this, we apply inequality (2.16) with the following choice of g

$$g(x)=\begin{cases}0 & \text{for}\quad x\in S\cap[(N\backslash N_k)\cup P_{\sqrt{\bar{\sigma}}}]\\[2ex] \dfrac{\log\dfrac{r'}{\sqrt{\bar{\sigma}}}}{\log\sqrt{\bar{\sigma}}} & \text{for}\quad x\in S\cap(P_{\bar{\sigma}}\backslash P_{\sqrt{\bar{\sigma}}})\\[2ex] 1 & \text{for}\quad x\notin S\cap[(N\backslash N_k)\cup P_{\bar{\sigma}}]\,.\end{cases}$$

This implies

$$\int_{S\setminus[(N\setminus N_k)\cup P_{\bar\sigma}]} \|A\|^2 \leqq \frac{2}{(\log\sqrt{\bar\sigma})^2}\int_{S\cap(P_{\bar\sigma}\setminus P_{\sqrt{\bar\sigma}})} \frac{\|\nabla r'\|^2}{(r')^2} + \int_{S\setminus[N\setminus N_k)\cup P_{\sqrt{\bar\sigma}}]} |\mathrm{Ric}(v)| .$$

This implies by (1.1)

$$\int_{D_{2\bar\sigma}\cap(P_{2\bar\sigma}\setminus P_{\bar\sigma})} \|A\|^2 \leqq \frac{2C}{(\log\sqrt{\bar\sigma})^2}\int_{S\cap(P_{\bar\sigma}\setminus P_{\sqrt{\bar\sigma}})} \frac{1}{(r')^2} + \int_{S\setminus[(N\setminus N_k)\cup P_{\sqrt{\bar\sigma}}]} O(1/r^3) .$$

Similar reasoning as that used in deriving (2.10) and (2.11) using r' in place of r and the fact that x^3 is bounded on $S\cap N_k$ implies

$$\int_{S\cap(P_{\bar\sigma}\setminus P_{\sqrt{\bar\sigma}})} \frac{1}{(r')^2} = O(\log\sqrt{\bar\sigma}) ,$$

$$\int_{S\setminus[(N\setminus N_k)\cup P_{\sqrt{\bar\sigma}}]} O(1/r^3) = O(1/\bar\sigma^{1/2}) .$$

It therefore follows that

$$\int_{D_{2\bar\sigma}\cap(P_{2\bar\sigma}\setminus P_{\bar\sigma})} \|A\|^2 = o(1) .$$

The coarea formula gives

$$\int_{\bar\sigma}^{2\bar\sigma}\int_{D_{2\bar\sigma}\cap\partial P_t} \|A\|^2 dt = \int_{D_{2\bar\sigma}\cap(P_{2\bar\sigma}\setminus P_{\bar\sigma})} \|A\|^2\,\|\nabla r'\| .$$

This then implies

$$\int_{\bar\sigma}^{2\bar\sigma}\int_{D_{2\bar\sigma}\cap\partial P_t} \|A\|^2 dt = o(1) . \tag{2.31}$$

Now (2.29), (2.30), and (2.31) imply that there exists $\sigma\in[\bar\sigma, 2\bar\sigma]$ satisfying

$$\int_{D_{2\bar\sigma}\cap\partial P_\sigma} \left(1-\left\langle v, \frac{\partial}{\partial x^3}\right\rangle\right) \leqq O(1), \qquad L(D_{2\bar\sigma}\cap\partial P_\sigma) = O(\sigma) ,$$

$$\int_{D_{2\bar\sigma}\cap\partial P_\sigma} \|A\|^2 = o(1/\sigma) \quad \text{as} \quad \sigma\to\infty .$$

Applying the Schwarz inequality and the condition on boundary length we have

$$\left(\int_{D_{2\bar\sigma}\cap\partial P_\sigma} \|A\|\right)^2 \leqq L(D_{2\bar\sigma}\cap\partial P_\sigma)\cdot o(1/\sigma) = o(1) .$$

Since ∂D_σ is one component of $D_{2\bar\sigma}\cap\partial P_\sigma$, we have shown

$$\begin{aligned} &\int_{\partial D_\sigma} 1-\left\langle v, \frac{\partial}{\partial x^3}\right\rangle \leqq O(1) \\ &L(\partial D_\sigma) = O(\sigma) \\ &\int_{\partial D_\sigma} \|A\| = o(1)) . \end{aligned} \tag{2.32}$$

We use (2.32) to estimate the terms in (2.27). Note that (2.27) and (2.32) imply

$$\int_{\partial D_\sigma}\left|k\left\langle e_2,\frac{x'}{\sigma}\right\rangle\right| = O(1)+O(1/\sigma)\int_{\partial D_\sigma}\|D_{e_1}e_1\|+\frac{1}{\sigma}\int_{\partial D_\sigma}\left\langle e_1,\frac{\partial}{\partial x^3}\right\rangle^2 . \tag{2.33}$$

To bound the last term, we note that by (1.1) we have

$$\left\langle e_1,\frac{\partial}{\partial x^3}\right\rangle^2+\left\langle e_2,\frac{\partial}{\partial x^3}\right\rangle^2 = 1-\left\langle \nu,\frac{\partial}{\partial x^3}\right\rangle^2+O(1/\sigma)$$

which implies by (2.32) that

$$\int_{\partial D_\sigma}\left\langle e_1,\frac{\partial}{\partial x^3}\right\rangle^2+\left\langle e_2,\frac{\partial}{\partial x^3}\right\rangle^2 = O(1) . \tag{2.34}$$

We now give a pointwise lower bound on $\left|\left\langle e_2,\frac{\partial}{\partial x^3}\right\rangle\right|$. In fact we show that

$$\sup_{\partial D_\sigma}\left(1+\left\langle\frac{x'}{\sigma},e_2\right\rangle\right)=o(1) . \tag{2.35}$$

We first note that by (1.1)

$$\begin{aligned}1-\left\langle\frac{x'}{\sigma},e_2\right\rangle^2 &= \left\langle\frac{x'}{\sigma},\nu\right\rangle^2+O(1/\sigma)=\left\langle\frac{x'}{\sigma},\nu-\frac{\partial}{\partial x^3}\right\rangle^2+O(1/\sigma)\\ &\leqq \left\|\nu-\frac{\partial}{\partial x^3}\right\|^2+O(1/\sigma)\\ &= 2\left(1-\left\langle\nu,\frac{\partial}{\partial x^3}\right\rangle\right)+O(1/\sigma) .\end{aligned}$$

Then applying (2.32) gives

$$\int_{\partial D_\sigma}1-\left\langle\frac{x'}{\sigma},e_2\right\rangle^2=O(1) . \tag{2.36}$$

Now since e_2 is the inner normal to D_σ, and $Dr'=\frac{x'}{\sigma}+O(1/\sigma)$ is the outer normal, we have $\left\langle\frac{x'}{\sigma},e_2\right\rangle\leqq O(1/\sigma)$, so

$$\begin{aligned}1-\left\langle\frac{x'}{\sigma},e_2\right\rangle^2 &= \left(1-\left\langle\frac{x'}{\sigma},e_2\right\rangle\right)\left(1+\left\langle\frac{x'}{\sigma},e_2\right\rangle\right)\\ &\geqq (1-O(1/\sigma))\left(1+\left\langle\frac{x'}{\sigma},e_2\right\rangle\right)+O(1/\sigma) .\end{aligned}$$

Combining with (2.36) then gives

$$\int_{\partial D_\sigma}1+\left\langle\frac{x'}{\sigma},e_2\right\rangle=O(1) . \tag{2.37}$$

Since ∂D_σ is not homologous to zero in $\mathbb{R}^2 \backslash P_{\sigma_0}$, its projection onto the x^1x^2-plane must be the circle of radius σ centered at zero. We therefore have

$$L(\partial D_\sigma) \geqq 2\pi\sigma - O(1) . \tag{2.38}$$

Now (2.37) and (2.38) together imply that there is a point $x_0 \in \partial D_\sigma$ with

$$1 + \left\langle \frac{x'}{\sigma}, e_2 \right\rangle (x_0) = O(1/\sigma) . \tag{2.39}$$

Differentiating $1 + \left\langle \frac{x'}{\sigma}, e_2 \right\rangle$ along ∂D_σ gives

$$\begin{aligned} e_1 \left[1 + \left\langle \frac{x'}{\sigma}, e_2 \right\rangle \right] &= \frac{1}{\sigma} \langle e_1, e_2 \rangle - \frac{1}{\sigma} \left\langle \frac{\partial}{\partial x^3}, e_1 \right\rangle \left\langle \frac{\partial}{\partial x^3}, e_2 \right\rangle \\ &\quad - h_{12} \left\langle \frac{x'}{\sigma}, v \right\rangle + O(1/\sigma)^2 \\ &\leqq \frac{1}{2\sigma} \left(\left\langle \frac{\partial}{\partial x^3}, e_1 \right\rangle^2 + \left\langle \frac{\partial}{\partial x^3}, e_2 \right\rangle^2 \right) + \|A\| + O(1/\sigma^2) . \end{aligned}$$

Applying (2.32) and (2.34) we see

$$\int_{\partial D_\sigma} \left| e_1 \left[1 + \left\langle \frac{x'}{\sigma}, e_2 \right\rangle \right] \right| = o(1) . \tag{2.40}$$

We now write for any $x \in \partial D_\sigma$,

$$1 + \left\langle \frac{x'}{\sigma}, e_e \right\rangle (x) = 1 + \left\langle \frac{x'}{\sigma}, e_2 \right\rangle (x_0) + \int_{x_0}^{x} e_1 \left[1 + \left\langle \frac{x'}{\sigma}, e_2 \right\rangle \right] .$$

Thus combining this with (2.39) and (2.40) we have established (2.35).

Now (2.33), (2.34) and (2.35) together imply

$$\int_{\partial D_\sigma} |k| = O(1) + O(1/\sigma) \int_{\partial D_\sigma} \|D_{e_1} e_1\| . \tag{2.41}$$

Since $D_{e_1} e_1 = k e_2 - h_{11} v$ we have

$$\|D_{e_1} e_1\| \leqq |k| + \|A\| ,$$

so integrating and applying (2.32) we have

$$\int_{\partial D_\sigma} \|D_{e_1} e_1\| \leqq \int_{\partial D_\sigma} |k| + o(1) .$$

Combining this with (2.41) gives

$$\begin{aligned} &\int_{\partial D_\sigma} \|D_{e_1} e_1\| = O(1) \\ &\int_{\partial D_\sigma} |k| = O(1) . \end{aligned} \tag{2.42}$$

We may now rewrite (2.27) using (2.32), (2.34), and (2.42)

$$\int_{\partial D_\sigma} k \geqq \int_{\partial D_\sigma} k\left(1+\left\langle\frac{x'}{\sigma}, e_2\right\rangle\right) + 1/\sigma L(\partial D_\sigma) - o(1)\,.$$

Applying (2.35), (2.38), and (2.42) then gives

$$\int_{\partial D_\sigma} k \geqq 2\pi - o(1)\,.$$

Since this holds for σ arbitrarily large (in any interval $[\bar{\sigma}, 2\bar{\sigma}]$, $\bar{\sigma}$ sufficiently large), we can choose a sequence $\sigma_i \to \infty$ for which (2.26) holds. This completes the second proof of our claim.

This finishes the proof of Theorem 1.

3. Proof of Theorem 2

In this section we prove Theorem 2 which says that if for some end N_k, (1.2) holds, the total mass of M_k is zero, and $R \geqq 0$, then ds^2 is flat. We first note that by throwing away the other ends outside a convex ball, we may assume that N has only one end, so that $N \backslash N_k$ is compact.

We will need the following Sobolev inequality for functions with compact support on N.

Lemma 3.1. *There is a constant $c_1 > 0$ depending on N and the constants k_1, k_2, k_3 of* (1.1) *so that for any function ζ with compact support on N, we have the inequality*

$$\left(\int_N \zeta^6\right)^{1/3} \leqq c_1 \int_N \|D\zeta\|^2\,.$$

Note that we do not require $\zeta = 0$ on ∂N.

Proof. We prove the inequality by contradiction. If it were not true, we could find a sequence of functions f_i with compact support and with

$$\int_N f_i^6 = 1\,, \qquad \int_N \|Df_i\|^2 \leqq 1/i\,. \tag{3.1}$$

Since N_k is identified with $\mathbb{R}^3 \backslash B_{\sigma_0}(0)$ and ds^2 is uniformly equivalent to the Euclidean metric, we have the inequality which follows from the Euclidean inequality (see proof in [17, p. 80–81]).

$$\left(\int_{N_k} (f_i)^6\right)^{1/3} \leqq (\text{const}) \int_{N_k} \|Df_i\|^2\,.$$

Thus by (3.1) we have $\int_{N_k} f_i^6 \to 0$, so we have $f_i \to 0$ in L^6-norm on N_k. If we choose a precompact coordinate neighborhood $\mathcal{O} \subseteqq N$, for any C^1 function g defined on $\mathcal{O}$ we have the following inequality which comes directly from the Euclidean inequality of the same form

$$\inf_{\beta \in \mathbb{R}} \left(\int_{\mathcal{O}} (g-\beta)^6\right)^{1/3} \leqq (\text{const}) \int_{\mathcal{O}} \|Dg\|^2\,.$$

Applying this inequality and (3.1) to the functions $f_i|_{\mathcal{O}}$, we find a sequence β_i so that

$$\int_{\mathcal{O}} (f_i-\beta_i)^6 \to 0 \,.$$

Since $\int_N f_i^6=1$, the sequence β_i is a bounded sequence, and by extracting a subsequence we may assume $\beta_i \to \beta$. Thus we have $f_i \to \beta$ in L^6-norm on $\mathcal{O}$. Since $f_i \to 0$ in L^6-norm on N_k, we must have $\beta=0$ on each coordinate neighborhood $\mathcal{O}$, so we have $f_i \to 0$ in L^6-norm on N in contradiction to (3.1). This proves Lemma 3.1.

We will have need to study equations of the form

$$\Delta v - fv = h \quad \text{on} \quad N \tag{3.2}$$

where f, h are functions which satisfy

$$\begin{aligned} &|f| \leqq k_7(1+r^{+5})^{-1}\,, \quad |h| \leqq k_7(1+r^5)^{-1}\,, \\ &|\partial f| \leqq k_8(1+r^5)^{-1}\,, \quad |\partial h| \leqq k_8(1+r^5)^{-1} \end{aligned} \tag{3.3}$$

on N_k. Let f_+, f_- be the positive and negative parts of f, so that $f=f_+ - f_-$ and $|f|=f_+ + f_-$.

Lemma 3.2. *Suppose* (1.1) *holds and* N_k *has zero total mass. There is a number* $\varepsilon_0>0$ *depending only on* N *and* k_1, k_2, k_3 *of* (1.1) *so that if*

$$\left(\int_N (f_-)^{3/2}\right)^{2/3} \leqq \varepsilon_0 \,,$$

then (3.2) *has a unique solution* v *defined on* N *satisfying* $v=O(1/r)$ *as* $r\to\infty$ *and* $\frac{\partial v}{\partial \mathbf{n}}=0$ *on* ∂N, *where* $\mathbf{n}$ *is the outward unit normal vector to* ∂N. *Moreover, the solution* v *has the properties*

$$v=\frac{A}{r}+\omega\,, \quad |\omega| \leqq k_9(1+r^2)^{-1}\,,$$

$$|\partial\omega| \leqq k_{10}(1+r^3)^{-1}\,, \quad |\partial\partial\omega| \leqq k_{11}(1+r^4)^{-1}$$

on N_k, *where* $A=-\frac{1}{4\pi}\int_N fv+h$, *and the constants* k_9, k_{10}, k_{11} *depend only on* k_1, k_2, k_3.

Proof. Throughout the proof we use $c_1, c_2, c_3 \ldots$ to denote constants depending only on k_1, k_2, k_3.

To prove the existence of v we solve the problem for $\sigma>\sigma_0$

$$\begin{cases} \Delta v_\sigma - f v_\sigma = h \quad \text{on} \quad N^\sigma = (N \setminus N_k) \cup (B_\sigma(0)\cap N_k) \\ v_\sigma = 0 \quad \text{on} \quad \partial B_\sigma(0) \\ \dfrac{\partial v_\sigma}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial N \,. \end{cases} \tag{3.4}$$

If v_σ satisfies (3.4), we can multiply by v_σ and integrate by parts to obtain an integral bound on v_σ as follows

$$\int_{N^\sigma} \|Dv_\sigma\|^2 = -\int_{N^\sigma} f v_\sigma^2 - \int_{N^\sigma} h v_\sigma$$

$$\leqq \int_{N^\sigma} (f_-) v_\sigma^2 + \int_{N^\sigma} |h||v_\sigma|$$

$$\leqq \left(\int_{N^\sigma} f_-^{3/2}\right)^{2/3} \left(\int_{N^\sigma} v_\sigma^6\right)^{1/3} + \left(\int_{N^\sigma} |h|^{6/5}\right)^{5/6} \left(\int_{N^\sigma} v_\sigma^6\right)^{1/6}.$$

We note that if $h=0$, we can apply Lemma (3.1) to obtain

$$\int_{N^\sigma} \|Dv\|^2 \leqq \varepsilon_0 c_1 \int_{N^\sigma} \|Dv_\sigma\|^2 .$$

Thus if we choose $\varepsilon_0 < 1/c_1$, we see that $\Delta v - fv$ has trivial kernel for problem (3.4), and hence standard linear elliptic theory (see [18, p. 262]) implies the existence of a unique smooth solution v_σ of (3.4). By (3.3) we have $\left(\int_{N^\sigma} |h|^{6/5}\right)^{5/6} \leqq c_2$, a constant independent of σ. Applying this, Lemma 3.1, and the hypotheses we have

$$\left(\int_{N^\sigma} v_\sigma^6\right)^{1/3} \leqq \varepsilon_0 c_1 \left(\int_{N^\sigma} v_\sigma^6\right)^{1/3} + c_1 c_2 \left(\int_{N^\sigma} v_\sigma^6\right)^{1/6}.$$

Choosing $\varepsilon_0 = \dfrac{1}{3c_1}$, and using the inequality $|ab| \leqq \frac{1}{3}a^2 + \frac{3}{4}b^2$ we have

$$\left(\int_{N^\sigma} v_\sigma^6\right)^{1/3} \leqq \tfrac{1}{3}\left(\int_{N^\sigma} v_\sigma^6\right)^{1/3} + \tfrac{1}{3}\left(\int_{N^\sigma} v_\sigma^6\right)^{1/3} + \tfrac{3}{4}(c_1 c_2)^2$$

which gives

$$\int_{N^\sigma} v_\sigma^6 \leqq c_3$$

where $c_3 = (\frac{9}{4}(c_1 c_2)^2)^3$. Standard linear elliptic estimates (see [17, p. 161] for the interior estimate and [17, p. 242] for the estimate on ∂N) now imply that $\{v_\sigma : \sigma > \sigma_0\}$ is equicontinuous in C^2 topology on compact subsets of N. Thus we may choose a sequence $\sigma_i \to \infty$ so that $v_{\sigma_i} \to v$ uniformly in C^2-norm on compact subsets of N. Thus v is a solution of (3.2) defined on N satisfying

$$\frac{\partial v}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial N, \quad \text{and}$$

$$\int_N v^6 \leqq c_3, \qquad \sup_N |v| \leqq c_4 . \tag{3.5}$$

(The supremum estimate follows from the L^6 estimate and standard linear theory [17, p. 161].)

To analyze the asymptotic behavior of v, we derive a potential theoretic expression for v. For $x, y \in N_k$ let

$$\varrho_x(y) = \left[\sum_{i,j=1}^{3} g_{ij}(x)(y^i - x^i)(y^j - x^j)\right]^{1/2}.$$

By direct calculation we have

$$\Delta_y[\varrho_x(y)]^{-1} = -4\pi\delta_x(y) + \psi_x(y)$$

where $\delta_x(y)$ is a point mass at x, and by (1.1) $\psi_x(y)$ satisfies

$$|\psi_x(y)| \leqq c_5 |x-y|^{-2}|x|^{-3} \quad \text{for} \quad y \in B_1(x). \tag{3.6}$$

Again by (1.1) we see that for $y \notin B_1(x)$ we have

$$|\psi_x(y)| \leqq c_6 \left[\frac{1}{(1+|y|)^2|x-y|^3} + \frac{1}{|x|^2|x-y|^3} + \frac{1}{(1+|y|)^3|x-y|^2}\right]. \tag{3.7}$$

Also we have

$$\begin{gathered} c_7^{-1}|x-y| \leqq \varrho_x(y) \leqq c_7|x-y|, \quad c_8^{-1} \leqq |\partial_y \varrho_x(y)| \leqq c_8, \\ \lim_{|x|\to\infty} |x|[\varrho_x(y)]^{-1} = 1. \end{gathered} \tag{3.8}$$

Multiplying (3.2) by $[\varrho_x(y)]^{-1}$, integrating by parts twice on the set $D_{\bar\sigma}(x) = \{y \in N_k : \varrho_x(y) \leqq \bar\sigma\}$ where $\bar\sigma \in (\sigma/2, \sigma)$ and $\sigma/2 \gg |x|$, and applying (3.6)

$$\begin{aligned} 4\pi v(x) = &\int_{D_{\bar\sigma}(x)} \psi_x(y)v(y)\sqrt{g(y)}\,dy - \int_{D_{\bar\sigma}(x)} (fv+h)(y)[\varrho_x(y)]^{-1}\sqrt{g(y)}\,dy \\ &+ \frac{1}{\bar\sigma}\int_{\{\varrho_x(y)=\bar\sigma\}} \frac{\partial v}{\partial \mathbf{n}}\,dA(y) - \int_{\{\varrho_x(y)=\bar\sigma\}} v(y)\frac{\partial}{\partial \mathbf{n}}[\varrho_x(y)]^{-1}dA(y) \\ &- \int_{\partial B_{\sigma_0}(0)} \frac{\partial v}{\partial \mathbf{n}}(y)[\varrho_x(y)]^{-1}dA(y) + \int_{\partial B_{\sigma_0}(0)} v(y)\frac{\partial}{\partial \mathbf{n}}[\varrho_x(y)]^{-1}dA(y) \end{aligned} \tag{3.9}$$

where $\sqrt{g}\,dy$ is the volume element of ds^2 and dA is surface area with respect to ds^2. Applying Stokes' theorem we see from (3.2)

$$\begin{aligned} &\int_{\{\varrho_x(y)=\bar\sigma\}} \frac{\partial v}{\partial \mathbf{n}}(y)\,dA(y) \\ &= \int_{\partial B_{\sigma_0}(0)} \frac{\partial v}{\partial \mathbf{n}} + \int_{D_{\bar\sigma}(x)} \Delta v = \int_{\partial B_{\sigma_0}(0)} \frac{\partial v}{\partial \mathbf{n}} + \int_{D_{\bar\sigma}(x)} (fv+h). \end{aligned}$$

From (3.3) and (3.5) we have

$$\left|\int_{D_{\bar\sigma}(x)} fv+h\right| \leqq c_9.$$

Therefore,

$$\left|\int_{\{\varrho_x(y)=\bar{\sigma}\}} \frac{\partial v}{\partial \mathbf{n}}\right| \leqq \left|\int_{\partial B_{\sigma_0}(0)} \frac{\partial v}{\partial \mathbf{n}}\right| + c_{10} \leqq c_{11} . \tag{3.10}$$

From (3.8), Area $\{\varrho_x(y)=\bar{\sigma}\} \leqq c_{12}\bar{\sigma}^2$, so we may apply (3.8)

$$\begin{aligned}\left|\int_{\{\varrho_x(y)=\bar{\sigma}\}} v\frac{\partial}{\partial \mathbf{n}}[\varrho_x(y)]^{-1}\right| &\leqq c_{13}\bar{\sigma}^{-2} \int_{\{\varrho_x(y)=\bar{\sigma}\}} |v| \\ &\leqq c_{13}\bar{\sigma}^{-2}(c_{12}\bar{\sigma}^2)^{5/6}\left(\int_{\{d_x(y)=\bar{\sigma}\}} v^6\right)^{1/6} \\ &\leqq c_{14}\sigma^{-1/3}\left(\int_{\{\varrho_x(y)=\bar{\sigma}\}} v^6\right)^{1/6} .\end{aligned}$$

By (3.5) and the coarea formula we may choose $\bar{\sigma}\in(\sigma/2,\sigma)$ so that

$$\int_{\{\varrho_x(y)=\bar{\sigma}\}} v^6 \leqq c_{15}\sigma^{-1} .$$

Thus we have

$$\left|\int_{\{\varrho_x(y)=\bar{\sigma}\}} v\frac{\partial}{\partial \mathbf{n}}[\varrho_x(y)]^{-1}\right| \leqq c_{16}\sigma^{-1/2} . \tag{3.11}$$

We now let $\sigma\to\infty$ and apply (3.10) and (3.11) in (3.9) to get

$$\begin{aligned}4\pi v(x) = &\int_{N_k} \varphi_x(y)v(y)\sqrt{g(y)}\,dy \\ &- \int_{N_k} (fv+h)(y)[\varrho_x(y)]^{-1}\sqrt{g(y)}\,dy \\ &- \int_{\partial B_{\sigma_0}(0)} \frac{\partial v}{\partial \mathbf{n}}[\varrho_x(y)]^{-1}dA(y) \\ &+ \int_{\partial B_{\sigma_0}(0)} v\frac{\partial}{\partial \mathbf{n}}[\varrho_x(y)]^{-1}dA(y).\end{aligned} \tag{3.12}$$

From (3.5), (3.6), and (3.7) we have

$$\begin{aligned}&\left|\int_{N_k} \varphi_x(y)v(y)\sqrt{g(y)})dy\right| \\ &\leqq c_{17}\left[\int_{B_1(x)} |\varphi_x(y)|dy + \int_{N_k\setminus B_1(x)} |\varphi_x(y)|\,|v(y)|dy\right] \\ &\leqq c_{18}|x|^{-3} + c_{19}\left(\int_{\mathbb{R}^3\setminus B_1(x)}\left(\frac{1}{(1+|y|)^2|x-y|^3} + \frac{1}{(1+|y|)^3|x-y|^2}\right.\right. \\ &\qquad \left.\left. + \frac{1}{|x|^2|x-y|^3}\right)^{6/5} dy\right)^{5/6} .\end{aligned} \tag{3.13}$$

The following inequalities are easily checked

$$\begin{aligned}&\left(\int_{\mathbb{R}^3\setminus B_1(x)} \frac{dy}{(1+|y|)^{12/5}|x-y|^{18/5}}\right)^{5/6}\\ &\leqq \left(\int_{B_{\frac{|x|}{2}}(x)\setminus B_1(x)} \frac{dy}{(1+|y|)^{12/5}|x-y|^{18/5}}\right)^{5/6}\\ &\quad+\left(\int_{B_{\frac{|x|}{2}}(0)} \frac{dy}{(1+|y|)^{12/5}|x-y|^{18/5}}\right)^{5/6}\\ &\quad+\left(\int_{\mathbb{R}^3\setminus(B_{\frac{|x|}{2}}(0)\cup B_{\frac{|x|}{2}}(x))} \frac{dy}{(1+|y|)^{12/5}|x-y|^{18/5}}\right)^{5/6}\\ &\leqq c_{20}(|x|^{-2}+|x|^{-5/2}+|x|^{-5/2})\leqq c|x|^{-2}\,.\end{aligned}$$

Similarly

$$\left(\int_{\mathbb{R}^3\setminus B_1(x)} \frac{dy}{(1+|y|)^{18/5}|x-y|^{12/5}}\right)^{5/6} \leqq c_{21}|x|^{-2}$$

$$\int_{\mathbb{R}^3\setminus B_1(x)} \frac{dy}{|x-y|^{18/5}} \leqq c_{22}\,.$$

Thus it follows from (3.13) that

$$\left|\int_{N_k} \psi_x(y)v(y)\sqrt{g(y)}\,dy\right| \leqq c_{23}|x|^{-2}\,. \tag{3.14}$$

It follows from (3.3), (3.5), and (3.8) that

$$\left|\int_{B_{\frac{|x|}{2}}(x)} (fv+h)(y)\,[\varrho_x(y)]^{-1}\sqrt{g(y)}\,dy\right| \leqq c_{24}|x|^{-3}$$

$$|x|\,[\varrho_x(y)]^{-1} \leqq c_{25} \quad \text{for} \quad y\notin B_{\frac{|x|}{2}}(x)\,.$$

We may thus apply (3.8), (3.3) and the dominated convergence theorem to the functions $(fv+h)(y)|x|\,[\varrho_x(y)]^{-1}\sqrt{g(y)}\,\chi_{N_k\setminus B_{\frac{|x|}{2}}(x)}$ where χ_A denotes the characteristic function of A to conclude that

$$\begin{aligned}&\lim_{x\to\infty} |x| \int_{N_k} (fv+h)(y)\,[\varrho_x(y)]^{-1}\sqrt{g(y)}\,dy\\ &= \int_{N_k} (fv+h)(y)\sqrt{g(y)}\,dy\,.\end{aligned} \tag{3.15}$$

Then by (3.12), (3.14), (3.8), and (3.15) we have

$$\begin{aligned}A = \lim_{|x|\to\infty} 4\pi|x|v(x) = &-\int_{N_k} (fv+h)(y)\sqrt{g}\,dy\\ &-\int_{\partial B_{\sigma_0}(0)} \frac{\partial v}{\partial \mathbf{n}}\end{aligned} \tag{3.16}$$

so we may write

$$v=\frac{A}{r}+\omega$$

where

$$4\pi\omega(x)=\int_{N_k}\psi_x(y)v(y)\sqrt{g(y)}\,dy-\int_{N_k}(fv+h)(y)[\varrho_x(y)^{-1}-|x|^{-1}]\sqrt{g}\,dy$$
$$-\int_{\partial B_{\sigma_0}(0)}\frac{\partial v}{\partial \mathbf{n}}[\varrho_x(y)^{-1}-|x|^{-1}]dA(y)+\int_{\partial B_{\sigma_0}(0)}v\frac{\partial}{\partial \mathbf{n}}[\varrho_x(y)]^{-1}dA(y)\,. \tag{3.17}$$

We see directly that

$$|\varrho_x(y)^{-1}-|x|^{-1}|\leqq c_{26}\frac{|y|}{|x|\,|x-y|}$$

which combined with (3.17), (3.3), and (3.8) shows that

$$|\omega(x)|\leqq c_{27}(1+r^2)^{-1}\quad \text{for}\quad x\in N_k\,. \tag{3.18}$$

To estimate the derivatives of ω we record the following Schauder estimate whose proof can be found in [17, p. 161]. Let L be an elliptic operator on the unit ball of $\mathbb{R}^3$ of the form

$$Lu=\sum_{i,j=1}^{3}a_{ij}(\xi)\frac{\partial^2 u}{\partial\xi^i\partial\xi^j}+\sum_{j=1}^{3}b_j(\xi)\frac{\partial u}{\partial\xi^j}+c(\xi)u(\xi)$$

where $\xi=(\xi^1,\xi^2,\xi^3)$ is the Cartesian coordinate in the ball. For any function $\varphi(\xi)$ defined on an open set Ω and real number λ with $0<\lambda<1$, define the following norms

$$|\varphi|_{0,\Omega}=\sup_{\xi\in\Omega}|\varphi(\xi)|$$

$$|\varphi|_{0,\lambda,\Omega}=\sup_{\xi,\bar\xi\in\Omega}\frac{|\varphi(\xi)-\varphi(\bar\xi)|}{|\xi-\bar\xi|^{\lambda}}$$

$$|\varphi|_{1,\lambda,\Omega}=\sup_{\xi\in\Omega}|\partial\varphi(\xi)|+|\partial\varphi|_{0,\lambda,\Omega}$$

$$|\varphi|_{2,\lambda,\Omega}=\sup_{\xi\in\Omega}|\partial\varphi(\xi)|+\sup_{\xi\in\Omega}|\partial\partial\varphi|(\xi)|+|\partial\partial\varphi|_{0,\lambda,\Omega}\,.$$

Let $B_r=\{\xi:|\xi|<r\}$. Suppose there is a positive number Λ so that

$$\sum_{i,j=1}|a_{ij}|_{0,\lambda,B_1}+\sum_{i=1}^{3}|b_i|_{0,\lambda,B_1}+|c|_{0,\lambda,B_1}\leqq\Lambda\,,$$
$$\Lambda^{-1}|t|^2\leqq\sum_{i,j=1}^{3}a_{ij}(\xi)t^it^j\,\forall t\in\mathbb{R}^3\backslash\{0\}\,,\qquad \forall\xi\in B_1\,. \tag{3.19}$$

It then follows that for any $C^{2,\lambda}$ function u on B_1 we have

$$|u|_{2,\lambda,B_{1/2}} \leqq \bar{C}(|Lu|_{0,\lambda,B_1} + |u|_{0,B_1}) \tag{3.20}$$

where $\bar{C}$ depends only on λ, Λ.

We now fix a point $x \in N_k$, and assume that $\sigma = \frac{1}{2}|x_0| > \sigma_0$. We then let

$$\xi = \frac{1}{\sigma}(y - x)$$

where y is our asymptotic coordinate on N_k. If we let $u(\xi) = \omega(y)$, $a_{ij}(\xi) = g^{ij}(y)$, $b_k(\xi) = \sigma g^{ij}(y)\Gamma^k_{ij}(y)$, and $c(\xi) = -\sigma^2 f(y)$ we have

$$\begin{aligned} Lu(\xi) &= \sum_{i,j} a_{ij}(\xi)\frac{\partial^2 u}{\partial \xi^i \partial \xi^j} + \sum_k b_k(\xi)\frac{\partial u}{\partial \xi^k} + c(\xi)u \\ &= \sigma^2(\Delta\omega(y) - f\omega(y)) . \end{aligned}$$

Now $\Delta\omega - f\omega = Af|y|^{-1} + h - A\Delta|y|^{-1}$ by (3.2) and the definition of ω. From (1.1) we see that

$$|\Delta|y|^{-1}| \leqq c_{28}|y|^{-5} , \qquad |\partial(\Delta|y|^{-1})| \leqq c_{29}|y|^{-6} .$$

These together with (3.3) imply

$$|Lu|_{0,\lambda,B_1} \leqq c_{30}\sigma^{-3+\lambda} .$$

It is clear from (1.1) that (3.19) is satisfied for our operator Lu with a constant Λ independent of σ. Thus (3.20) gives

$$|u|_{2,\lambda,B_{1/2}} \leqq c_{31}(\sigma^{-3+\lambda} + |u|_{0,B_1}) .$$

By (3.18) this gives for any λ, $0 < \lambda < 1$

$$|u|_{2,\lambda,B_{1/2}} \leqq c_{32}\sigma^{-2} .$$

In terms of ω, this implies

$$|\partial\omega(x_0)| \leqq c_{32}|x_0|^{-3} , \qquad |\partial\partial\omega(x_0)| \leqq c_{32}|x_0|^{-4} .$$

This establishes the required growth properties of ω. The expression for A follows by integrating (3.2) over $(N \backslash N_k) \cup (N_k \cap B_\sigma(0))$ using the boundary condition $\frac{\partial v}{\partial \mathbf{n}} = 0$ on ∂N, and letting $\sigma \to \infty$.

To prove uniqueness, suppose $\bar{v}$ is another solution of (3.2) satisfying $\bar{v} = O(1/r)$ and $\frac{\partial \bar{v}}{\partial \mathbf{n}} = 0$ on ∂N. Then $u = v - \bar{v}$ satisfies

$$\Delta u - fu = 0 , \quad u = O(1/r) , \quad \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial N . \tag{3.21}$$

We show $u \equiv 0$. Let $\delta > 0$ be any number, and let $E_\delta = \{x \in N : u(x) \geqq \delta\}$. Because u tends to zero at infinity, we see that E_δ is compact. We multiply (3.21) by u and integrate over E_δ

$$\int_{E_\delta} u\Delta u = \int_{E_\delta} fu^2 .$$

Integrating by parts and applying (3.21) we have

$$\begin{aligned}\int_{E_\delta} |Du|^2 &= -\int_{\partial E_\delta} |Du| - \int_{E_\delta} fu^2 \\ &\leqq \int_{E_\delta} (f_-)u^2 \leqq \left(\int_{E_\delta} f_-^{3/2}\right)^{2/3} \left(\int_{E_\delta} u^6\right)^{1/3} \\ &\leqq \varepsilon_0 \left(\int_{E_\delta} u^6\right)^{1/3} .\end{aligned}$$

Applying Lemma 3.1 with $\zeta = u-\delta$ on E_δ, $\zeta \equiv 0$ on $N \backslash E_\delta$ we have

$$\left(\int_{E_\delta} (u-\delta)^6\right)^{1/3} \leqq c_1 \int_{E_\delta} |Du|^2 .$$

Combining these inequalities and recalling the choice of ε_0

$$\left(\int_{E_\delta} (u-\delta)^6\right)^{1/3} \leqq c_1\varepsilon_0 \left(\int_{E_\delta} u^6\right)^{1/3} \leqq \tfrac{1}{3}\left(\int_{E_\delta} u^6\right)^{1/3} .$$

Since $u = O(1/r)$, we see that $\int_N u^6 < \infty$, so we may let $\delta \to 0$ and deduce a contradiction unless $u \leqq 0$. Since $-u$ also satisfies (3.21) we must have $u \geqq 0$, so that $u \equiv 0$ on N. This concludes the proof of Lemma 3.2.

The next lemma deals with conformal change of metric on N.

Lemma 3.3. *Suppose ds^2 is an asymptotically flat metric on N satisfying* (1.2). *Let R be the scalar curvature function of ds^2, and suppose R satisfies*

$$\tfrac{1}{8}\left(\int_N R_-^{3/2}\right)^{2/3} \leqq \varepsilon_0$$

where ε_0 is defined in Lemma 3.2. *Then there is a unique positive function φ with $\frac{\partial\varphi}{\partial \mathbf{n}} = 0$ on ∂N so that the metric $\overline{ds}^2 = \varphi^4 ds^2$ is asymptotically flat, scalar flat, and has total mass*

$$\bar{M} = -\frac{1}{32\pi}\int_N R\varphi .$$

Proof. In order for the metric $\varphi^4 ds^2$ to be scalar flat, the function φ must satisfy

$$\Delta\varphi - \tfrac{1}{8}R\varphi = 0 . \tag{3.22}$$

The function $v = \varphi - 1$ then satisfies

$$\Delta v - \tfrac{1}{8}Rv = \tfrac{1}{8}R . \tag{3.23}$$

In order for $\varphi^4\, ds^2$ to be asymptotically flat, v must satisfy the asymptotic conditions of Lemma 3.2. Now Lemma 3.2 applies directly to give a v satisfying (3.23) with $\frac{\partial v}{\partial \mathbf{n}} = 0$ on ∂N. Thus $\varphi = v+1$ satisfies (3.22) with $\frac{\partial\varphi}{\partial \mathbf{n}} = 0$ on ∂N. In order to prove that φ is everywhere positive on N, we let $E=\{x\in N:\varphi(x)<0\}$. Since φ is asymptotic to one, we see that $\bar{E}$ is compact, and if E is nonempty, we multiply (3.22) by φ and integrate by parts on E using $\frac{\partial\varphi}{\partial\mathbf{n}}=0$ on ∂N to obtain

$$\begin{aligned}\int_E \|D\varphi\|^2 &= -\tfrac{1}{8}\int_E R\varphi^2\\ &\leqq \int_E (R_-)\varphi^2\\ &\leqq \tfrac{1}{8}\left(\int_E R_-^{3/2}\right)^{2/3}\left(\int_E \varphi^6\right)^{1/3}.\end{aligned}$$

Applying Lemma 3.1 to this inequality we have

$$\left(\int_E \varphi^6\right)^{1/3} \leqq c_1\varepsilon_0\left(\int_E \varphi^6\right)^{1/3}$$

which is a contradiction since $\varepsilon_0 \leqq \frac{1}{3c_1}$. We conclude that $\varphi\geqq 0$ on N. That $\varphi>0$ on N now follows from the Hopf maximum principle. The usual proof works directly on the interior of N, and the boundary condition $\frac{\partial\varphi}{\partial\mathbf{n}}=0$ allows an easy modification to show $\varphi>0$ on ∂N (see [17, p. 61]).

To show that ∂N has positive mean curvature relative to $\overline{ds}^2=\varphi^4 ds^2$ we note that if H is the mean curvature function relative to ds^2 and $\bar{H}$ the mean curvature relative to $\overline{ds}^2$, a direct calculation gives

$$\bar{H}=\frac{1}{\varphi^2}\left(H+\frac{4}{\varphi}\frac{\partial\varphi}{\partial\mathbf{n}}\right)=\frac{1}{\varphi^2}H,$$

so $\bar{H}>0$. This finishes the proof that $\overline{ds}^2$ is an asymptotically flat metric on N. The formula for $\bar{M}$ follows from Lemma 3.2.

A special case of Lemma 3.3 is the following corollary which was proved by O'Murchadka and York [19] in case N is diffeomorphic to $\mathbb{R}^3$.

Corollary 3.1. *If $M=0$, $R\geqq 0$, and R is not identically zero, then there is a metric conformally equivalent to ds^2 which is asymptotically flat, scalar flat, and so that N_k has negative total mass.*

Theorem 1 and Corollary 3.1 imply that an asymptotically flat metric satisfying the hypotheses $M=0$, $R\geqq 0$ must have $R\equiv 0$ on N. We assume now that ds^2 is such a metric and that (1.2) is also satisfied. We define a one-parameter family of metrics ds_t^2 on N by

$$ds_t^2=\sum_{i,j=1}^{3}(g_{ij}+tS_{ij})dx^i dx^j$$

where S_{ij} is the Ricci tensor of ds^2. These metrics are defined in a neighborhood of $t=0$ by (1.1) and (1.2), and $ds_0^2=ds^2$. For t sufficiently small, ds_t^2 is asymptotically flat by (1.2) and because ∂N has positive mean curvature relative to ds^2, so by continuity ∂N also has positive mean curvature relative to ds_t^2 for t small. Let R_t be the scalar curvature function of ds_t^2, so that we have $R_0 = R \equiv 0$. A known formula (see [20]) gives

$$R_0' = \frac{d}{dt} R_t|_{t=0} = -\Delta R + \delta\delta \operatorname{Ric} - \|\operatorname{Ric}\|^2 \tag{3.24}$$

where $\operatorname{Ric} = (S_{ij})$ is the Ricci tensor, and

$$\partial\partial \operatorname{Ric} = S^{ij}_{|i|j|}, \quad \|\operatorname{Ric}\|^2 = g^{ik} g^{jl} S_{ij} S_{kl}.$$

Since $R \equiv 0$, we have $\Delta R \equiv 0$, and a direct application of the second Bianchi identity shows

$$\delta\delta \operatorname{Ric} = 2\Delta R \equiv 0.$$

Thus (3.24) becomes

$$R_0' = -\|\operatorname{Ric}\|^2. \tag{3.25}$$

Since $R_0 \equiv 0$, it follows from (1.2) that for t sufficiently small we have

$$\tfrac{1}{8}\left(\int_N R_{t-}^{3/2}\right)^{2/3} \leqq \varepsilon_0,$$

where ε_0 can be taken independent of t for small t. Applying Lemma 3.3, we find a function φ_t so that the metric $\varphi_t^4 ds_t^2$ is asymptotically flat and scalar flat. The mass $M(t)$ of this metric is

$$M(t) = -\frac{1}{32\pi} \int_N R_t \varphi_t \sqrt{g_t}\, dx \tag{3.26}$$

where $\sqrt{g_t}$ is the volume factor for ds_t^2.

We will prove that $\frac{dM}{dt}$ exists at $t=0$, and can be computed by differentiating (3.26) under the integral sign. For small h, let $\varphi^{(h)}$ be defined by

$$\varphi^{(h)} = \frac{\varphi_h - \varphi_0}{h}.$$

Let Δ_t be the Laplacian for the metric ds_t^2, and let $\Delta^{(h)}$ be the differential operator defined by

$$\Delta^{(h)} v = \frac{1}{h}(\Delta_h v - \Delta_0 v).$$

Let $R^{(h)} = \frac{1}{h}(R_h - R_0)$. The function $\varphi^{(h)}$ satisfies the equation

$$\Delta_0 \varphi^{(h)} - \tfrac{1}{8} R_0 \varphi^{(h)} = -\Delta^{(h)} \varphi_h + \tfrac{1}{8} R^{(h)} \varphi_h. \tag{3.27}$$

By (1.1) and (1.2), we see that this equation satisfies the hypotheses of Lemma 3.2. Since $\varphi^{(h)}$ is $O(1/r)$, that lemma implies

$$|\varphi^{(h)}| \leqq \gamma_1 (1+r)^{-1} \quad \text{on} \quad N_k \tag{3.28}$$

where γ_1 is independent of h. Standard linear theory applied to (3.27) shows that $\varphi^{(h)}$ has a local $C^{2,\alpha}$ bound depending on C^1 bounds on R_0 and $-\Delta^{(h)}\varphi_h+\frac{1}{8}R^{(h)}\varphi_h$. Since these bounds are independent of h, we can find a sequence $\{h_i\}$ tending to zero so that φ_{h_i} converges in $C^{2,\beta}$ norm for any $\beta<\alpha$ uniformly on compact subsets of N to a $C^{2,\alpha}$ function φ'_0 which satisfies

$$\Delta_0\varphi'_0-\tfrac{1}{8}R_0\varphi'_0=-\Delta'_0\varphi_0+\tfrac{1}{8}R'_0\varphi_0$$

where

$$\Delta'_0=\frac{d}{dt}\Delta_t|_{t=0}, \quad \text{and} \quad R'_0=\frac{d}{dt}R_t|_{t=0}.$$

By (3.28) we have $\varphi'_0=O(1/r)$ so the uniqueness part of Lemma 3.2 implies that the limit φ'_0 is independent of the sequence $\{h_i\}$ we have chosen. Thus it follows that $\frac{d}{dt}\varphi_t$ exists at $t=0$ and is equal to φ'_0. From (1.1) and (1.2) we have constants γ_2, γ_3 independent of h so that

$$|R^{(h)}|\leqq\gamma_2(1+r^{3+\alpha})^{-1}g^{(h)}=\gamma_3(1+r^{1+\alpha})^{-1} \quad \text{on} \quad N_k \tag{3.29}$$

where $g^{(h)}=\frac{1}{h}(g_h-g_0)$. We now apply (3.28), (3.29), and the dominated convergence theorem to conclude that $M'(0)=\frac{d}{dt}M(t)|_{t=0}$ exists and

$$M'(0)=-\frac{1}{32\pi}\int_N R_0(\varphi_t\sqrt{g_t})'dx-\frac{1}{32\pi}\int_N R'_0\varphi_0\sqrt{g_0}\,dx\,.$$

Since $R_0\equiv 0$ and $\varphi_0\equiv 1$, we may apply (3.25) to conclude

$$M'(0)=\frac{1}{32\pi}\int_N \|\mathrm{Ric}\|^2\,. \tag{3.30}$$

If Ric is not identically zero, (3.30) implies that $M'(0)>0$ and hence by choosing a suitable $t_0<0$ we would have $M(t_0)<0$. The metric $\varphi_{t_0}^4 ds_{t_0}^2$ would then be asymptotically flat, scalar flat, and N_k would have negative total mass in contradiction to Theorem 1. Hence we conclude that $\mathrm{Ric}\equiv 0$, and because we are working in dimension three, ds^2 is flat. This completes the proof of Theorem 2.

Appendix

In this appendix we give a brief discussion of the Regularity Estimate (2.1), and the existence of smooth solutions of the two-dimensional problem of least area for surfaces (regardless of topological type) having a given boundary curve in a Riemannian 3-manifold with boundary of positive mean curvature. These results are well-known so we mainly give references and briefly indicate a few of the simpler arguments involved. The (interior) Regularity Estimate (2.1) and the existence theorem are part of the powerful approach to minimal surfaces which has developed through the use of geometric measure theory. A thorough account of this field is given by Federer [16] where Chapter 5 discusses the applications to variational problems of area type. We refer the reader to the introduction to that chapter for an account of the people involved in the developments of this theory. The Regularity Estimate (2.1) can be extracted as a very special case of the material in Section 5.3 and Theorem 5.4.15 of [16]. A more differential geometric approach to this estimate can be found in [21].

The existence of an area minimizing current S_σ (surface with singularities) having boundary curve C_σ follows from 5.1.6 of [16]. The above mentioned regularity theory implies that S_σ has no singularities in the interior of $N \cap S_\sigma$. We will show that the positive mean curvature of ∂N implies that S_σ lies entirely in the interior of N, and hence S_σ/C_σ is completely regular. To see this, consider a boundary component B of N. For $\varepsilon > 0$ sufficiently small, the open set $\mathcal{O}_t = \{x \in N: \operatorname{dist}(x, B) < t\}$ for $0 < t < \varepsilon$ retracts smoothly onto B, and the parallel surfaces $B_t = \{x \in N: \operatorname{dist}(x, B) = t\}$ for $0 \leqq t < \varepsilon$ are smooth surfaces diffeomorphic to $B = B_0$. Let v be the outward unit normal vector field to B, and for $0 < t < \varepsilon$, extend v as a vector field on $\mathcal{O}_t$ by parallel translation along geodesics normal to B. Thus on B_t, v is a unit normal vector field. The fact that B has positive mean curvature with respect to v says that $\operatorname{div}(v) > 0$ on B where $\operatorname{div}(\cdot)$ is divergence of a vector field taken on N. By continuity we then have

$$\operatorname{div}(v) > 0 \quad \text{on} \quad \mathcal{O}_t$$

for some $t \in (0, \varepsilon)$. The theory of Chapter 5 of [16] also gives us an open set $V_\sigma \subsetneqq N$ so that

$$S_\sigma = \partial V_\sigma$$

in the sense of geometric measure theory. Let η be the outward (to V_σ) pointing unit normal vector field of S_σ (which exists almost everywhere with respect to Hausdorff 2-dimensional measure), and apply the divergence theorem, Theorem 4.5.6 of [16] to the open set $\mathcal{O}_t \cap V_\sigma$. If this set is not empty we have

$$\int_{S_\sigma \cap \mathcal{O}_t} \langle \eta, v \rangle d\mathcal{H}^2 - \int_{B_t \cap V_\sigma} 1 d\mathcal{H}^2 > 0$$

where $\mathcal{H}^2$ is Hausdorff 2-dimensional measure on N. This implies $\operatorname{Area}(B_t \cap V_\sigma) < \operatorname{Area}(S_\sigma \cap \mathcal{O}_t)$ contradicting the area minimizing property of S_σ. This shows that S_σ lies strictly away from ∂N as claimed.

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