hep-th/9609239
EFI-97-01

Calabi-Yau fourfolds for M- and F-Theory compactifications

A. Klemm[1], B. Lian[2], S-S. Roan[3] and S-T. Yau[4]

[1] *Enrico Fermi Institute, University of Chicago*
5640 S. Ellis Ave., Chicago, IL 60637, USA

[2] *Department of Mathematics, Brandeis University*
Waltham, MA 02154 , USA

[3] *Institute of Mathematics, Academia Sinica Taipei, Taiwan*

[4] *Department of Mathematics, Harvard University*
Cambridge, MA 02138, USA

Abstract We investigate topological properties of Calabi-Yau fourfolds and consider a wide class of explicit constructions in weighted projective spaces and, more generally, toric varieties. Divisors which lead to a non-perturbative superpotential in the effective theory have a very simple description in the toric construction. Relevant properties of them follow just by counting lattice points and can be also used to construct examples with negative Euler number. We study nets of transitions between cases with generically smooth elliptic fibres and cases with ADE gauge symmetries in the N=1 theory due to degenerations of the fibre over codimension one loci in the base. Finally we investigate the quantum cohomology ring of this fourfolds using Frobenius algebras.

aklemm@maxwell.uchicago.edu, maroan@ccvax.sinica.edu.tw,
lian@max.math.brandeis.edu, yau@math.harvard.edu

December 1996

1. Introduction

Geometrizing the expected symmetries in the moduli space of supersymmetric theories has proven to be a simple and successful tool in the investigation of their non perturbative behaviour. Especially the geometric interpretation of the non perturbative $SL(2, \mathbb{Z})$ of type II_B string as coming *really* from a two torus of an (elliptically fibred) compactification of F-theory has helped to uncover many non perturbative properties of string compactifications to dimensions greater then five [1][2][3] [4][5].

Elliptically fibred complex four dimensional Kähler manifolds X with $SU(4)$ holomony, Calabi-Yau fourfolds for short, are the geometry relevant for $N = 1$ compactifications of F-theory to four dimensions [1]. Orbifold constructions [6] [7] [8] [9] of M and F theory are particular useful to get a fast view on the spectrum and the symmetries. Using results of [10] they were considered to compactify F (M, type II) theory to four (three, two) dimensions in [11]. However in order to study the moduli space and in particular transitions, one wishes to have a deformation family *and* the knowledge about the enhanced symmetry points[1]. To get some overview of the possible deformation families of Calabi-Yau fourfolds, especially the elliptically fibred ones, is the first objective of this paper. We will therefore consider a rich class of hypersurfaces and complete intersections in weighted projective spaces and toric varieties. For the hypersurfaces we obtain a large scan (104 021 configurations) over possible Hodge numbers by classifying *all* Fermat type configurations and generic hypersurfaces up to degree 400. The Euler number ranges between $-240 \leq \chi \leq 1\,820\,448$. Examples with negative Euler number could eventually lead to supersymmetry breaking in three dimensions by anti-branes, which have to be included to cancel the tadpoles if $\chi < 0$ and the fourform background flux vanishes [13][14]. A hyperkähler fourfold with $\chi < 0$ was constructed in [13] and and Kähler examples appear in[10], both orbifold constructions. Here we find the first Kähler manifolds with $\chi < 0$ realized as deformation families.

The properties of elliptically fibred fourfolds can be understood from properties of the bases and the degeneration of the fibres. The most basic properties, like the triviality of the canonical bundle, are decided from the degeneration over codimension one. For instance if the singularity here is as mild as possible (I_1 fibres only, compare A.1) one can express the Euler number for the fourfold X by formulas, which refer only to properties of

[1] There are elegant ways of finding such symmetric configurations in the deformation families, see e.g. [12].

the bases and the generic type of fibre. Likewise physically the most basic properties like the unbroken gauge group[2] are decided from the degeneration on codimension one. We therefore aim for examples in which we can control the degeneration at codimension one in a simple way.

We will first study examples, which are simplest in two respects, namely the fibres degenerates homogeneously on a subspace B' of codimension one in the base to an ADE singularity and it does so for generic values of the moduli. In this situation we find formulas for the Euler number, which depend on the cohomology of B' and the invariants of the gauge group. The manifolds provide a realizations of $N = 1$ gauge theories, discussed recently in [15] [16].

F-theory on the fourfolds has beside the complex and the Kähler moduli of the manifold, also the moduli associated with three branes, which live in space-time and intersect the base in points, as well as a choice of discrete back ground fluxes which take (half)integer values in the unimodular selfdual lattice $H^4(X, \mathbb{Z})$, which is even if $\chi = 0$ mod 24. About the global moduli space of the first two types of moduli, we can learn by Kodaira & Spencer deformation theory and mirror symmetry (see e.g. [17] for recent results on dimension> 3). On the moduli space of the three branes one can learn locally in non generic situations with orbifold symmetries [7][18] or more generically in situations as above [16] and at the transitions points which connect the F-theory vacua. Using Batyrev and Watanabes classification of toric Fano threefolds we can construct systematically a rather dense net of such fourfold transitions (fig 1), which are again very simple in that they keep the elliptic fibre structure *and* the generic degeneration type[3]. These extremal transitions correspond to shrinking E_8, E_7, E_6, D_5 Del Pezzo surfaces along one dimensional (T-stable) subsets in the base or generalized elliptic threefolds to (T-fixed) points in the base.

Of course physically one would like to understand perturbative or non-perturbative enhancements of the gauge symmetries, which correspond to codimension one degenerations, which occur only for specific values of the moduli and the meaning of the codimension two (and three) degenerations. A good guidance to these more complicated situations can be obtained by considering those three dimensional elliptic fibrations over Hirzebruch surfaces F_n, for which the degeneration on codimension one and two has been studied in

[2] More exotic theories could also arise from degeneration on codimension one.

[3] The geometrically interesting fourfold transitions considered in [19] are not particularly useful for F-theories, because they behave randomly w.r.t. to the fibre structure (if any).

[2][3][4][5]and [20] and replacing the base $\mathbb{P}^1$ of F_n by a rational surface. In easy cases this can be done so that part of the singularity structure at codimension one and two essentially carries over to fourfolds. Here we can also obtain systematically chains of now more complicated extremal fourfold transitions, which keep the elliptic fibre structure, but frequently violates the evenness of $H^4(X,\mathbb{Z})$.

1.1. Divisors which lead to a non-perturbative superpotential in three dimensions.

Some aspects of the four dimensional theory can be investigated more easily by compactifying first M-theory or type IIB on X to three dimensions or two dimensions and considering decompactification limits to learn about four dimensions. Eleven dimensional M-theory compactifications, on not necessarily elliptically fibred, Calabi-Yau fourfolds X, leads to $N = 2$ supersymmetric theories in three dimensions [21] [22] . There is a general mechanism to generate a non-perturbative superpotential in the three-dimensional theory from supersymmetric instantons, which arise from wrapping the 5-branes of the M-theory around complex divisors D of X.

i.) Under the assumption that D is smooth, the following necessary condition on the arithmetic genus of D for the occurrence of instanton induced terms in the superpotential was derived from the anomaly vanishing requirement in [21]:

$$\chi(D,\mathcal{O}_D)=\sum_{n=0}^{3} h^n(\mathcal{O}_D)=1 \qquad (1.1)$$

ii.) If $h^0(\mathcal{O}_D)=1$ and $h^1(\mathcal{O}_D)=h^2(\mathcal{O}_D)=h^3(\mathcal{O}_D)=0$ a non-perturbative contribution of the form

$$\int d\theta e^{-(V_D+i\phi_D)}T(m_i) \qquad (1.2)$$

must be generated in the superpotential, as no cancellation from extra fermionic zero modes can occur. Here V_D is the volume of D measured in units of the 5-brane tension, $(V_D+i\phi_D)$ are real and complex moduli components of a chiral superfield and $T(m_i)$ is a non-vanishing section of a holomorphic line bundle over the moduli space of the theory on X.

Using the fact that $h^n(\mathcal{O}_D)$ describes the dimension of the deformation space of D it was shown in [21] that divisors given by a polynomial constraints in a Calabi-Yau fourfolds defined as hypersurfaces or complete intersections in (products) of ordinary projective spaces have $\chi(D,\mathcal{O}_D)<1$. The reason is basically that such polynomials have too many possible deformations. These divisors will therefore not lead to nontrivial contributions to the superpotentials. Using the Hirzebruch-Riemann-Roch index formula [23]

$$\chi(D, \mathcal{O}_D) = \int (1 - e^{-[D]}) td(X), \tag{1.3}$$

the explicit expansion of the Todd polynomials $T_0 = 1$, $T_1 = \frac{1}{2}c_1(X)$ $T_2 = \frac{1}{12}(c_2(X) + c_1(X)^2)$ and the fact that $c_1(X) = 0$ for manifolds of $SU(4)$ holonomy we can rewrite (1.1) in the more useful form

$$[D]^4 + c_2(X)[D]^2 = -24. \tag{1.4}$$

With this topological formula the above statement follows from the fact that all intersection numbers on the left of (1.4) come from semi ample divisors in projective spaces and are hence positive. On the other hand the fact that the left hand side of (1.4) has to be negative suggests that D's with the desired properties occurs preferably as exceptional divisors or in situations where the deformation space is for some reasons small. For instance because we make an orbifoldisation and thereby killing most of the deformation space or we work with weighted projective spaces, where the possible deformations are restricted by the weights. This hints that weighted projective space and more generally toric varieties will lead to interesting configurations of such divisors. In fact we will see that the intersection of the T-invariant orbits of the toric ambient space with the Calabi-Yau fourfold will lead under very simple combinatorial conditions, which are explained in section 4, to such divisors. A special situation where one can construct infinitly many divisors, which contribute to the superpotential, was described in [24].

1.2. Preferred physical situations, additional geometrical data and dualities

If the Calabi-Yau manifold[4] X admits an elliptic fibration

$$\mathcal{E} \longrightarrow X \stackrel{\pi}{\longrightarrow} B \tag{1.5}$$

then a compactification of M theory on X is equivalent to F-theory [1] on $X \times S^1$, which in turn is equivalent to Type IIB on $B \times S^1$. If ε is the area of $\mathcal{E}$ one can use for $\varepsilon \to 0$ the

[4] Other interesting compactifications are on manifolds with $Spin(7)$ holonomy, the so-called Joyce manifolds. They lead to $N = 1$ supersymmetry for M-theory compactifications to three dimensions (see [1][25]).

fiberwise equivalence of M theory compactification on $R^9 \times T^2$ with Type IIB on $R^8 \times S^1$. This means that M theory compactification to three dimensions on X has the same moduli as Type IIB compactified to three dimensions on $B \times S^1$. Denoting the radius of the S^1 by R one has $\varepsilon \propto 1/R$ such that the $\varepsilon \to 0$ limit is the decompactification limit for the type IIB theory.

W.r.t. this limit $\varepsilon \to 0$ one has two principally different situations for the location of the divisor D on X to distinguish

a.) $\pi(D_a) = B$, i.e. D_a is a section or multisection. D_a is called *horizontal*.

b.) $D_b = \pi^{-1}(B')$ with B' a divisor in B. D_b is called *vertical*.

As was explained in [21] for *generic fixed* geometry of the base non perturbative superpotentials in the four dimensional Type IIB theory will only occur in case b.). The reason is that the action of the non perturbative configuration in F-theory units is proportional to the volume of the divisors, which for the two types of divisors goes like $D_a \sim 1/\epsilon D_b$ in the limit[5] $\epsilon \to 0$.

For phenomenology it might be more useful to think about the situation in terms of the heterotic $N = 1$ string. This is possible if B admits a holomorphic $\mathbb{P}^1$ fibration

$$\mathbb{P}^1 \longrightarrow B \stackrel{\pi'}{\longrightarrow} B' \tag{1.6}$$

then one can consider an elliptic fibration

$$\mathcal{E}' \longrightarrow Z \stackrel{\pi''}{\longrightarrow} B' \tag{1.7}$$

over B' and get, by fiberwise application of type IIB/heterotic string duality, a description of the heterotic string on the Calabi-Yau threefold Z. The effect of a divisor of type b.) can be interpreted in the heterotic string theory description [21] as worldsheet or as spacetime instanton effect depending of whether D_b maps in Z to a *vertical* or *horizontal* divisor w.r.t. π''. Both types can occur as T-invariant toric divisors as discussed in section 5 and 6.

The organization of the material is as follows. In section two we will summarize the basic topological properties of Calabi-Yau fourfolds. Then we give in section three some overview of the class of complete intersections in weighted projective spaces. In section

[5] Of course one can enhance the contribution of the D_a divisors by going to a singular point in B.

four we explain the toric construction of elliptically fibred toric fourfolds. We extend the formulas of Batyrev and give a characterisation of the divisors on X, which come from the divisors of the ambient space, which are invariant under the torus action. This gives a very easy criterium, when such a divisor contributes to the superpotential. Section five contains a complete list of elliptically fibred Calabi-Yau manifolds over toric Fano bases and the transitions among them. In section six we also discuss degenerations of the fibre, which lead to gauge symmetry in four dimensions. Sections seven and eight contains proofs for the formulas of the Euler number of the fourfolds in terms of the topological properties of the base and the the type of the fibre. Some cases have been already discussed in [13]. In section nine we discuss the quantum cohomology of fourfolds using Frobenius algebras. Especially we give the generalization of the formulas for quantum cohomology ring obtained for threefolds in [26] [27] to the n-fold case. In section (9.6) we discuss in some details examples which are connected by transitions.

Acknowledgements. We would like to thank P. Candelas, X. de la Ossa, S. Katz and C. Vafa for very helpful discussions. We also like to thank S. Hosono for his help and frequent correspondences.

2. General topological properties of Calabi-Yau fourfolds

We will first employ Hirzebruch-Riemann-Roch index theorems to derive some relations and general divisiblity conditions among the topological invariants of Calabi-Yau fourfolds. If W is a vector bundle over X, $\chi(X,W) = \sum_{i=0}^{n}(-1)^i \mathrm{dim} H^i(X,W)$ and $c_0[X],\ldots,c_n[X]$, Chern classes of X and $d_0[W],\ldots,d_r[W]$ Chern classes of W one has [23]

$$\chi(X,W) = \kappa_n \left[\sum_{i=1}^{q} e^{\delta_i} \prod_{i=1}^{n} \frac{\gamma_i}{1-e^{-\gamma_i}} \right], \tag{2.1}$$

where $\kappa_n[]$ means taking the coefficient of the n'th homogeneous form degree, the γ_i and δ_i are the formal roots of the total Chern classes: $\sum_{i=0}^{n} c_i[X] = \prod_{i=1}^{n}(1-\gamma_i)$ and $\sum_{i=0}^{q} d_i[X] = \prod_{i=1}^{q}(1-\delta_i)$. We want to use the index formula to compute the arithmetic genera $\chi_q = \sum_p (-1)^p \mathrm{dim} H^p(X,\Omega^q)$. First we will evaluate (2.1) for $W=T_X$, the tangent bundle of X. One way of to do so is to express the formal roots, via symmetric polynomial, in terms of the Chern classes c_i. This yields for the two,three and four dimensional cases the following formulas for $\chi_q = \sum_{p=1}^{\mathrm{dim}(X)}(-1)^p h^{p,q}$:

$$\mathrm{dim}(X) = 2: \qquad \chi_0 = \frac{1}{12}\int_X (c_1^2 + c_2), \tag{2.2}$$

$$\dim(X) = 3: \qquad \begin{aligned} &a.) \quad \chi_0 = \frac{1}{24}\int_X (c_1 c_2) \\ &b.) \quad \chi_1 = \frac{1}{24}\int_X (c_1 c_2 - 12 c_3), \end{aligned} \tag{2.3}$$

$$\dim(X) = 4: \qquad \begin{aligned} &a.) \quad \chi_0 = \frac{1}{720}\int_X (-c_4 + c_1 c_3 + 3c_2^2 + 4c_1^2 c_2 - c_1^4) \\ &b.) \quad \chi_1 = \frac{1}{180}\int_X (-31 c_4 - 14 c_1 c_3 + 3c_2^2 + 4c_1^2 c_2 - c_1^4) \\ &c.) \quad \chi_2 = \frac{1}{120}\int_X (79 c_4 - 19 c_1 c_3 + 3c_2^2 + 4c_1^2 c_2 - c_1^4) \end{aligned} \tag{2.4}$$

We are mainly interested in Kähler fourfolds with $c_1[X] = 0$. This is equivalent to the statement that a Ricci flat Kähler metric exists and the manifold has holonomy inside $SU(4)$. In the following, by a Calabi-Yau manifold, we mean a manifold for which the holonomy is strictly[6] $SU(4)$. In this case there is a unique holomorphic four-form and no continuous isomorphisms, i.e. $h_{0,0} = 1, h_{1,0} = h_{2,0} = h_{3,0} = 0, h_{4,0} = 1$. Hodge $*$-duality and complex conjugation reduces the independent Hodge numbers in the Hodge square

$$\begin{array}{ccccc} 1 & 0 & 0 & 0 & 1 \\ 0 & h^{3,1} & h^{3,2} & h^{3,3} & 0 \\ 0 & h^{2,1} & h^{2,2} & h^{2,3} & 0 \\ 0 & h^{1,1} & h^{1,2} & h^{1,3} & 0 \\ 1 & 0 & 0 & 0 & 1 \end{array}$$

to four, say $h^{1,1} = h^{3,3}$, $h^{3,1} = h^{1,3}$, $h^{2,1} = h^{3,2}$ and $h^{2,2}$. For Calabi-Yau manifolds in the sense above we have $c_1 = 0$, $\chi_0 = 2$. Using this in (2.4) implies[7] a further relation among the Hodge numbers say

$$h^{2,2} = 2(22 + 2h^{1,1} + 2h^{3,1} - h^{2,1}). \tag{2.5}$$

The Euler number can thus be written as

$$\chi(X) = 6(8 + h^{1,1} + h^{3,1} - h^{2,1}). \tag{2.6}$$

The middle cohomology splits into a selfdual ($*\omega = \omega$) $B_+(X)$ subspace and an anti-selfdual ($*\omega = -\omega$) subspace $B_-(X)$

$$H^4(X) = B_+(X) \oplus B_-(X),$$

[6] Which excludes 4-tori T^8, $K3 \times T^4$, $T^2 \times CY$-3-folds etc. Note however that there are Hyperkähler fourfolds with $h^{2,0} \neq 0$, which are not of of this simple product type.

[7] Beside this it implies $\int_X c_2^2$ is even. It also seems that $c_2^2 \geq 0$, indicating that $\chi \geq -1440$.

whose dimensions are determined by the Hirzebruch signature as

$$\begin{aligned}\tau(X) &= \mathrm{dim}B_+(X) - \mathrm{dim}B_-(X) = \int_X L_2 = \frac{1}{45}\int_X (7p_2 - p_1^2)\\ &= \frac{\chi}{3} + 32.\end{aligned} \tag{2.7}$$

The symmetric inner product $(\omega_1, \omega_2) = \int_X \omega_1 \wedge *\omega_2$ is positive definite on $H^4(X)$ and $H^4(X, \mathbb{Z})$ is by Poincare duality unimodular. The symmetric quadratic form $Q(\omega_1, \omega_2) = \int_X \omega_1 \wedge \omega_2$ is positive definite on $B_+(X)$ and negative on $B_-(X)$. Beside this we expect a split of $H^4(X, \mathbb{Z})$ from mirror symmetry, see section 4 .

We note furthermore that from the definition of the Pontryagin classes $p_i \in H^{4i}(X, \mathbb{Z})$ in terms of the Chern classes

$$\begin{aligned}p &= \sum_{i=0}^{[\dim(X)/2]} (-1)^i p_i = \sum_{i,j} (-1)^i c_i \wedge c_j, \qquad \text{hence}\\ p_1 &= c_1^2 - 2c_2, \qquad p_2 = c_2^2 - 2c_1c_3 + 2c_4, \ldots\end{aligned} \tag{2.8}$$

one has, using the Gauss-Bonnet formula, for Calabi-Yau fourfolds[8] always $\chi = \frac{1}{8}\int_X (4p_2 - p_1^2)$.

It was shown in [13] that $I(R) = -\int_X X_8(R) = \int_X (4p_2 - p_1^2)/192 = \chi/24 \neq 0$ gives rise to a non vanishing contribution one-point function for the two, three or four form in II_A, M- or F-theory compactification on X. Assuming that there are no further non integral contributions to the one-point functions it was argued in [13] that these compactification are unstable if the one-point functions cannot be canceled by introducing integer quanta of string, twobrane or threebrane charge in these theories, that is [9] $\chi = 0 \bmod 24$.

In [14] it was argued that there is a flux quantization $[G] - \frac{p_1}{4} \in H^4(X, \mathbb{Z})$, where G is the four form field strength to which the twobrane of M theory couples [22]. If G is zero that means that $p_1/4$ *has* to be an integral class ($c_2 = 2y$ with $y \in H^4(X, \mathbb{Z})$) and as explained in [14] this implies by the formula of Wu $x^2 = 0 \bmod 2$ for any $x \in H^4(X, \mathbb{Z})$. That means especially by (2.7) that $H^4(X, \mathbb{Z})$ is an even selfdual lattice with signature[10] τ and implies by (2.4) a.) again that $\chi = 0 \bmod 24$. On the other hand if $p_1/4$ is half integral

[8] The later condition was found in [28] requiring the existence of a nowhere vanishing eight dimensional Majorana-Weyl spinor in the 8_c representation of $SO(8)$.

[9] We have collected in table B.2 a couple of non-trivial examples, in which χ *is* actually zero.

[10] Note that $\tau = 0 \bmod 8$ as it must be for even selfdual lattices if $\chi = 0 \bmod 24$.

then $[G]$ has to be half integral and potentially non-integral contributions to the one point function $I(R,G) = -\int_X X_8 - \frac{1}{8\pi^2}\int_X G^2 = \frac{1}{8}\int_X c_2^2 - \frac{1}{2}\int_X G^2 - 60$ can be canceled also for Calabi-Yau's for which $\chi \neq 0 \mathrm{mod}\ 24$.

We will find in chapter three and six various chains of geometrically possible transition between elliptically fibred fourfolds in which the Euler number *is* divisible by 24 in an element of the chain, while it is *not* divisible after the transition (see especially table 6.5). This is somewhat disturbing as the flux G would have to jump by one half unit if one tries to follow this transition in M- or F-theory, suggesting that these transitions are impossible in these theories.

Beside the three brane source terms there are contributions from the fivebranes [29] which can cancel $I(R,G)$. In [13] it has been also suggested to calculate the Euler number of an elliptic fibration by counting locally the three-brane charge which is induced from the seven branes whose world volume W is the discriminant locus $\tilde{\Delta}$ of the projection map $\pi : X \to B$ times the uncompactified space-time. This three brane charge is $Q = \frac{1}{48}\int_W p_1(W) = \frac{1}{48}\int_{\tilde{\Delta}} p_1(\tilde{\Delta})$. It might be that such induced three brane charges can explain the occurrence of three brane charge quanta in $\mathbb{Z}/4$ if one tries to follow the transition.

3. Constructions of Calabi-Yau fourfolds

The classification of Calabi-Yau manifolds with dimension $d \geq 3$ is an open problem[11]. The purpose of this section is to get a preliminary overview over Calabi-Yau fourfolds by investigating simple classes: namely hypersurfaces in weighted projective spaces, Landau-Ginzburg models and some complete intersections in toric varieties. Some examples of Calabi-Yau fourfolds appear in [10](orbifolds) [21][19][31] (hypersurfaces and complete intersections) [29] (toric hypersurfaces).

3.1. Hypersurfaces in weighted projective spaces

There is well studied connection between $N = 2$ (gauged) Landau-Ginzburg theories and conformal σ-models on Calabi-Yau complete intersections in weighted projective spaces [32], [33]. For example consider a Landau-Ginzburg models which flows in the infrared to

[11] It was shown in [30] that there are, up to birational equivalence, only a finite number of families of *elliptically fibred* Calabi-Yau threefolds.

a conformal theory with $c = 3 \cdot d$. If such a model has a transversal quasi-homogeneous superpotential of degree m, and $r = d + 2$ chiral super-fields with positive charges (w.r.t. the $U(1)$ of the $N = 2$ algebra) $q_i = w_i/m$ subject to the constraint

$$\sum_{i=1}^{r}(1-2q_i) = d \tag{3.1}$$

then it corresponds to a σ-models on the Calabi-Yau hypersurfaces $X_m(w_1, \ldots, w_r)$ of degree m in a weighted projective space $\mathbb{P}^{r-1}(w_1, \ldots, w_r)$. Due to fixed sets of the $\mathbb{C}^*$-action of the weighted projective space, the Calabi-Yau hypersurface $X_m(w_1, \ldots, w_r)$ is in general singular. The Hodge numbers of the resolved Calabi-Yau hypersurface can be obtained from the Landau-Ginzburg model formula for the Poincarè polynomial of the canonical twisted LG model [32], i.e.

$$\mathbf{tr}\ t^{mJ_0}\bar{t}^{m\bar{J}_0} = \sum_{l=0}^{m-1} \prod_{l\frac{w_i}{m} \bmod \mathbb{Z}=0} \frac{1-(t\bar{t})^{m-w_i}}{1-(t\bar{t})^{w_i}} \prod_{l\frac{w_i}{d} \bmod \mathbb{Z}\neq 0} (t\bar{t})^{m/2-w_i} \left(\frac{t}{\bar{t}}\right)^{m(l\frac{w_i}{d} \bmod \mathbb{Z}-\frac{1}{2})} . \tag{3.2}$$

Here the Hodge numbers $h^{p,q}$ are given simply by the degeneracy of states with $(J_0, \bar{J}_0)$-charges $(d-p, q)$. For $d \leq 3$ there is always a geometrical desingularization of theses singularities [34]. For $d \geq 4$ there need not be such a geometrical resolution. However we note that for all Landau-Ginzburg models described in the following the relation derived from the index theorem (2.5)(2.6) holds, *independent* of whether a geometrical resolution exist or not. This is a hint that the index theorem (and many other apparently geometrical aspects relevant to M and F-theory compactifications) could be stated in terms of an internal $N = 2$ topological field theory.

To get some overview of this class of Calabi-Yau fourfolds we classify first the Fermat type constraints. In these cases, all weights divide the degree. It is easy to see that the maximal allowed degree of these configurations growth with $m_d = m_{d-1}(m_{d-1}+1)$ ($m = 6$ for tori, $m = 42$ for K_3 etc.) much faster then factorial in the dimension. In fact the maximal configuration in dimension d is a fibration with maximal number of branch points over $\mathbb{P}^1$ as base, whose fibre is in turn the maximal configuration in dimension $d-1$. The extreme Calabi-Yau fourfold[12] with degree $m = 326548$ is hence the top of the following

[12] Let us use the notation $X_m(w_1, \ldots, w_r)^{h^{1,1}, h^{d-1,1}}_{h^{2,1}}$ to summarize the three independent Hodge numbers of a fourfold

vertical chain of self mirrors[13] ($h^{d-1,1}=h^{1,1}$) in $d=1,\ldots,4$

$$\begin{array}{ccc} X_{3265248}(1,1806,75894,466206,108714,1631721)^{151700,151700}_{0} & & X_{3612}(1,1,84,516,1204,1806)^{252,303148}_{0} \\ \downarrow & \swarrow & \\ X_{1806}(1,42,258,602,903)^{251,251} & X_{84}(1,1,12,28,42)^{11,491} & \\ \downarrow & \swarrow & \\ X_{42}(1,6,14,21) & X_{12}(1,1,4,6) & \\ \downarrow & \swarrow & \\ X_{6}(1,2,3)^{1,1}, & & \end{array} \tag{3.3}$$

and has $\chi=1\,820\,448=24\cdot 75852$. It is the Calabi-Yau fourfold with the highest Euler number in this class. There are in total 3462 Fermat type fourfolds[14] (to be compared with $147,14,3$ in $d=3,2,1$). The bounds on the topological numbers for the Fermat Calabi-Yau fourfold hypersurfaces are

$$\begin{aligned} 288 \leq &\chi \leq 1\,820\,448, \qquad 1\leq h^{1,1}\leq 151\,700, \qquad 0\leq h^{2,1}\leq 1008\\ 284 \leq &h^{2,2}\leq 1\,213\,644 \qquad 60\leq h^{3,1}\leq 303\,148. \end{aligned}$$

Note that all upper bounds up to the last one are saturated by the $X_{3265248}$ case, while the configuration with maximal $h^{n-1,1}$

$$X_{3612}(1,1,84,516,1204,1806)^{252,303148}_{0},$$

is constructed by taking the minimal number of branch points over $\mathbb{P}^1$ for the top fibration in (3.3). Configurations for which $\delta=h^{n-1,1}-h^{1,1}$ is maximal[15] fit as branches in the chain of d-fold fibrations over $\mathbb{P}^1$ as indicated in (3.3). Among the 3462 Fermat cases there are 59(7) for which the Euler number is not divisible by 24(12).

Using the transversality conditions [36][37] one can show similarly as in [37] that the number of all quasi homegenous hypersurface fourfolds is finite. It is straightforward but

---

[13] For the $K3$ case in the chain the statement is that the Picard lattice of X_{42} can be identified with the Picard lattice of the mirror. Especially half of the Picard-lattice has to be invariant under automorphismus by which the mirror K_3 is constructed see e.g. [35].

[14] $N=2$ Landau-Ginzburg models with $c=3\cdot d$ can have maximally $3\cdot d$ nontrivial ($q_i<\frac{1}{2}$) fields. For $d=4$ one has $157,43,14,10,2,1$ Fermat examples for $r=7,\ldots,12$.

[15] For K^3 the statement is that the invariant part of the Picard Lattice under the mirror automorphism is maximal [35].

very time consuming to enumerate all of them. To get an estimate on the number of these configurations we note that there are 100 559 configurations with[16] $m \leq 400$, which exhibits topological numbers in the range

$$-240 \leq \chi \leq 239\,232, \qquad 1 \leq h^{1,1} \leq 173 \qquad 0 \leq h^{2,1} \leq 716,$$
$$82 \leq h^{2,2} \leq 159\,506, \qquad 6 \leq h^{3,1} \leq 39\,840.$$

Among them there are 21641 (9654) cases for which the Euler number is not divisible by 24 (12).

Furthermore a small fraction of it, 138 cases, are examples of Calabi-Yau fourfolds with negative Euler number, which give the possibility to break supersymmetry at least for the M-theory compactification to three dimensions. E.g. the hypersurface

$$X_{180}(10,17,36,36,36,45)^{30,36}_{78}$$

has Euler number $\chi = -24$. Because of (2.6) for χ to be negative $h^{2,1}$ has to be large. Elements in $H^{2,1}$, or by the Hodge $*$ and Poincare duality we may actually count elements of $H_{3,2}$, are generally generated if we have a singular curve C of genus g in the unresolved space X_{sing}. In this example we have a genus 6 singular curve $X_5(1,1,1)$ living in the x_3, x_4, x_5 stratum of the weighted projective space with a $\mathbb{Z}_{36}$ action on its transversal space in X_{sing}. Putting the curve in the origin the singularity in the transverse direction is a $\mathbb{C}^3/\mathbb{Z}_{36}$, where the $\mathbb{Z}_{36}$ acts by phase multiplication by $\exp(2\pi i\frac{10}{36}), \exp(2\pi i\frac{17}{36}), \exp(\frac{2\pi i 45}{36})$ on the $\mathbb{C}^3$ coordinates. The resolution of this singularity can be described easily torically (see section below). It gives rise to a 2 dimensional toric variety E whose fan Σ_E is spanned by $\nu^*_1 = (-1;-1,-1)$ $\nu^*_2 = (-1;1,-1)$, $\nu^*_3(-1;11,19)$ from the orign. The triangle in the $(-1;0,0)$ plane contains 13 points in the interior which correspond to rational surfaces with an intersection form which will depend on the triangulation of Σ_E. Thus X contains a divisor which has the fibre structure of a fibre bundle $E \to Y \to C$ which contributes $13 \cdot 6$ independent $H_{2,3}$-cycles, all of them made up from a $(1,0)$-cycles of the base and the $(2,2)$-cycles of the rational surfaces in the fibre. This reasoning will be generalized in the toric description to yield formula (4.9). Further examples with $\chi \leq 0$ appear in table B.2. For many cases constructed in the literature as orbifold examples we obtain candidates of deformation families[17]. For example the Hodge numbers of the model discussed in [11][38] coincide with the deformation family $X_{47}(3,5,7,8,11,13)^{100,4}_{0}$.

[16] Nine examples appear in [31]. Some Other examples of complete intersections in products of projective spaces are considered in [19].

[17] A list containing admissible weights and the dimensions of $H^{*,*}$ is available on request.

3.2. Elliptic fibrations with sections and multisections as complete intersection CY

We will describe here a method for constructing elliptic fibred Calabi-Yau spaces as hypersurfaces in weighted projective spaces. The starting point are the elliptic curves

$$\begin{aligned}
E_6 &: \quad X_3(1,1,1) = \{x^3 + y^3 + z^3 - sxyz = 0 \mid (x,y,z) \subset \mathbb{P}^2(1,1,1)\} \\
E_7 &: \quad X_4(1,1,2) = \{x^4 + y^4 + z^2 - sxyz = 0 \mid (x,y,z) \subset \mathbb{P}^2(1,1,2)\} \\
E_8 &: \quad X_6(1,2,3) = \{x^6 + y^3 + z^2 - sxyz = 0 \mid (x,y,z) \subset \mathbb{P}^2(1,2,3)\} \\
D_5 &: \quad X_{2,2}(1,1,1,1) = \left\{ \begin{array}{c} x^2 + y^2 - szw = 0 \\ z^2 + w^2 - sxy = 0 \end{array} \middle| \, (x,y,z,w) \subset \mathbb{P}^3(1,1,1,1) \right\},
\end{aligned} \tag{3.4}$$

which will appear as the generic fibers. Here we included the complete intersection case D_5. We will focus in the following mainly on the first three cases.

In the third case there are birational equivalent representations, which give rise to additional possibilities to construct the fibration space. To find them, consider the $\mathbb{C}^*$-action $\sigma : (x \to \rho x, y \to \rho^2 y, x \to \rho^3 x)$, with $\rho^6 = 1$ and construct the possible fractional transformations, which are well defined under this action. There are two series of fractional transformations,

$$(1): \quad \begin{array}{ll} x & = \xi^{\frac{2}{3}+k} \\ y & = \xi^{\frac{1}{3}} \eta \\ z & = \zeta \end{array} , \qquad (2): \quad \begin{array}{ll} x & = \xi^{\frac{1}{2}+k} \\ y & = \eta \\ z & = \xi^{\frac{1}{2}} \zeta \end{array} \tag{3.5}$$

which identify $X_6(1,2,3)$ with the following representations

$$\begin{aligned}
E_8' : \quad & X_{4+6k}(1, 1+2k, 2+3k) = \\
& \qquad \{\xi^{4+6k} + \xi\eta^3 + \zeta^2 - s\xi^k \eta\zeta = 0 | (\xi,\eta,\zeta) \subset \mathbb{P}^2(1, 1+2k, 2+3k)\} \\
E_8'' : \quad & X_{3+6k}(1, 1+2k, 1+3k) = \\
& \qquad \{\xi^{3+6k} + \eta^3 + \xi\zeta^2 - s\xi^k \eta\zeta = 0 | (\xi,\eta,\zeta) \subset \mathbb{P}^2(1, 1+2k, 1+3k)\}.
\end{aligned} \tag{3.6}$$

Our construction of elliptic fibred Calabi-Yau hypersurfaces (complete intersections) will proceed by the following general process

$$X^{(0)}_{d_1,\ldots,d_k}(w^{(0)}_1, w^{(0)}_2, \ldots w^{(0)}_{r^{(0)}}) \to X^{(1)}_{pd_1,\ldots,pd_k}(w^{(1)}_1, w^{(1)}_2, \ldots, w^{(1)}_{r^{(1)}}, pw^{(0)}_2, \ldots, pw^{(0)}_{r^{(0)}}) \tag{3.7}$$

with $\sum_{i=1}^{r^{(1)}} w_i^{(1)} + p \sum_{i=2} w_i^{(0)} = p \sum_{i=1}^k d_i$. In this cases the base is given by

This construction is a simple generalisation of the one used in [39] to get threefolds with K_3 fibre. It was used in [40] to produce more such examples and in [31] to get some fourfold configurations. Iteration of this process, with say $r^{(i)} = 2\,i > 0$, lead to sequences, e.g. for the X_3 case,

$$
\begin{array}{lclclcll}
 & & & & & & X_{24}(1,1,2,4,8,8) & \rightarrow \ldots \\
 & & & & X_{12}(1,1,2,4,4) & \rightarrow & X_{24}(1,2,3,6,12,12) & \\
 & & & & X_{18}(1,2,3,6,6) & & \vdots & \\
 & & X_{6}(1,1,2,2) & \rightarrow & X_{24}(1,3,4,8,8) & & & \\
 & & X_{9}(1,2,3,3) & & X_{30}(1,4,6,6,6) & & & \\
X_3(1,1,1) & \rightarrow & X_{12}(1,3,4,4) & & \vdots & & & \\
 & & X_{15}(2,3,5,5) & & & & &
\end{array}
$$

in which fiber of the threefold is itself an elliptic fibered K_3 and so on. The birational equivalent cases (3.6) can be treated similarly. The table B.3 contains a complete list of all K_3 hypersurfaces which are obtained in the first step from this process.

Let us investigate some general properties of these types of fibrations. The condition for triviality of the canonical bundle of X follows from the analysis in [41]. As summarized in [2] one can choose a birational model to get a Calabi-Yau with $K_X = 0$ if

$$K_B = -\sum a_i [B_i'], \tag{3.8}$$

where B_i' is a divisor in the base B and a_i follows from the type of singularity of the fibre over B_i' according to Kodaira's list of singular fibres for Weierstrass models in table A.1.

Our first aim is to relate the Euler number of the total space to topological data of the base. In the following we first concentrate on cases which have a section (or multisection) and for which the fibre degenerates no worse than with the I_1 fibre over codimension one in the base. That means that the discriminant Δ of the normal form of the elliptic fibre vanishes with ord $\Delta = 1$, while the coefficient functions e, f, g are generic (see section 5.1). Proofs of the formulas for the Euler numbers can be found in section 7,8. The $d = 4$ X_6 case was first treated in [13].

If the dimension of the total space X is $d = 3$ we have the following formula

$$\chi(X) = -2 \cdot C_{(G)} \cdot \int_B c_1^2(B), \tag{3.9}$$

where $\int_B c_1^2(B)$ is the integral of the square of the first Chern class over the base and $C_{(G)}$ is the dual Coxeter number of the group associated with the elliptic fibre (3.4),

$$C_{(E_8)} = 30, \quad C_{(E_7)} = 18, \quad C_{(E_6)} = 12, \quad C_{(D_5)} = 8.$$

Using (3.7) , with $r^{(1)} = 3$, we can provide examples with $B = \mathbb{P}^2$ for these cases

$$X_{18}(1,1,1,6,9)^{2(0),272}, \quad X_{12}(1,1,1,3,6)^{3(1),165}, \quad X_9(1,1,1,3,3)^{4(2),112},$$
$$X_{6,6}(1,1,1,3,3,3)^{5(3),77.}$$

From the index theorem (2.2) and $\chi(\mathbb{P}^2) = c_2^B = 3$ we conclude $\int_B c_1^2(B) = 9$ and application of (2.3) gives $\chi = 2(h^{1,1} - h^{2,1})$. We can represent these manifolds torically as described in the next section. $\mathbb{P}^2$ is then encoded in the fan spanned by $(1,0),(0,1),(-1,-1)$ and the blow up can be represented torically by adding the successively the vectors $(-1,0),(0,-1)$ and $(1,1)$ to the $\mathbb{P}^2$ fan. This enhances $h^{1,1}(B)$ and hence the Euler number of the bases by 1, but does not introduce singularities of the fibre in higher codimension therefore $h^{1,1}(X) \to h^{1,1}(X) + 1$ and by (3.9) we get chains of models with $(h^{1,1}_{(i+1)}(X), h^{2,1}_{(i+1)}(X))= (h^{1,1}_{(i)}(X) + 1, h^{2,1}_{(i)}(X) - C_{(G)} + 1)$. Transitions of this type involve the vanishing of real 2 (d-1)-cycles and for $d = 3$ they have been analysed in [3][42][43] and we generalise this situation to $d = 4$ in section 5.

For the general dimension d of X we show that

$$\chi(X) = a \sum_{r=1}^{d-1} (-1)^{r-1} b^r \int_B c_1^r(B) c_{d-r-1}(B) \tag{3.10}$$

with $a = 2,3,4$, $b = 6,4,3$ for the E_8, E_7, E_6 fibre respectively. For D_5 the Euler number likewise only depends on the Chern classes of the base. Let us summarize the formulas for $d = 4$

$$\begin{aligned} E_8: \ \chi(X) &= 12 \int_B c_1 c_2 + 360 \int_B c_1^3, \qquad & E_7: \ \chi(X) &= 12 \int_B c_1 c_2 + 144 \int_B c_1^3, \\ E_6: \ \chi(X) &= 12 \int_B c_1 c_2 + 72 \int_B c_1^3, \qquad & D_6: \ \chi(X) &= 12 \int_B c_1 c_2 + 36 \int_B c_1^3. \end{aligned} \tag{3.11}$$

The study of examples with low Picard numbers has helped a lot to establish the $N = 2$ Type II/hetetoric duality in four dimensions. Fourfold cases with low Picard numbers are expected to play a role in the investigation of the dynamics of M theory compactifications to three dimensions and $N = 1$ F-theory/heterotic duality in four dimensions. For the general LG-models we found respectively $31, 108, 255, 411, 508, 800$ configurations with $h^{1,1} = 1,2,3,4,5,6$. The ones which have an elliptic fibration of type $E_6, E_7, E_8, E_8', E_8''$, which is apparent in the patches of the weighted projective space are collected in table B.4.

It is clear from table B.4 and (3.11) that the cases in which the fibre degenerates only to I_1 are very rare. Such cases are for instance (5,9,27), where the base is $\mathbb{P}^3$ with $\int_{\mathbb{P}^3} c_1^3 = 64$. Let us check for these manifolds (3.8) and the fact that the fibre degenerates with I_1 over a generic point of the codimension one locus. $c_1(\mathbb{P}^n) = n[H]$, where $[H]$ is the hyperplane class. So $K_B = -n[H]$ and from section (5.1) we see that the discriminant $\tilde{\Delta} = 0$ is a singular degree $12n$ polynomial in $\mathbb{P}^n$, i.e. $[\tilde{\Delta}] = -12K_B$. However f, g, h are are generic such that over codimension one the fibre degenerates to I_1. As $d\tilde{\Delta} = 3f df + 2g dg$ (e.t.c) $\tilde{\Delta}$ will degenerate in codimension two at $f = g = 0$ to a cusp, but this does not contribute to (3.8). So $a = 12$ and hence $[\tilde{\Delta}] = -K_B$. Similar cases are (23,41,79) where the base is a $\mathbb{P}(\mathcal{O}_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(3)) \to B \to \mathbb{P}^2$ bundle with $\int_B c_1^3 = 72$ and case (109) where the base has a $\mathbb{P}(\mathcal{O}_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(4)) \to B \to \mathbb{P}^2$ structure with $\int_B c_1^3 = 86$ etc.

The E_8', E_8'' cases are very interesting because the Weierstrass form degenerates for them over codimension one in the base. For example for the E_8' case (60) in table B.4 the Weierstrass form degenerates to a conic bundle for $x_4 = 0$, which splits over codimension two in the base into pairs of lines. In this respect it is very similar to the case $X_{20}(1,1,2,6,10)$ described in [20]. Similar as in [20] it is part of a chain of transitions (110) $\to$ (60) $\to$ (23), which is analogous to the $X_{18}(1,1,2,6,8) \to X_{20}(1,1,2,6,10) \to X_{24}(1,1,2,8,12)$ transitions. Note that the Euler number of (110) and (23) is divisible by 24 while the one of (60) only by six. We will discuss such chains further in the toric setup in section 6.

Most of the time the models of table B.4 have a much more intricate singularity structure over the base. As these give rise to gauge groups, matter spectrum and more exotic physics in the low energy field theory, it is very important to investigate these cases. It turns out however that the realisation of simple generalisations e.g. to gauge groups without matter are easier to engineer in the toric framework, which we will do in the next section.

4. Toric construction and mirror symmetry for Calabi-Yau Fourfolds

Next we consider a generalization of the previous construction namely a d-dimensional hypersurfaces X in a compact toric variety [18] $\mathbb{P}_{\Delta^*}$. This hypersurface is defined by the zero set of the Laurent polynomial [45]

$$P = \sum_{\nu^{(i)}} a_i U_i = 0, \text{ where } U_i = \prod_{k=1}^{d+1} X^{\nu_k^{(i)}} \tag{4.1}$$

[18] See e.g. [44] and section 2.6 for the construction of $\mathbb{P}_{\Delta^*}$.

and $\nu^{(i)}$ are the integral points in $M \sim \mathbb{Z}^{d+1}$, whose convex hull defines the polyhedron Δ. The hypersurface (4.1) defines a Calabi-Yau space if Δ contains the origin as the only interior point [45]. The polar polyhedron $\Delta^* = \{y \in M^* | \langle x, y \rangle = -1,\ \forall x \in \Delta\}$ is likewise the convex hull of integral points $\nu^{*(i)} \in M^*$ with this property. Such a pair of polyhedra (Δ, Δ^*) is called reflexive pair. Note that $(\Delta^*)^* = \Delta$.

In [45] Batyrev has given the following combinatorial formulas for $h^{1,1}$ and $h^{d-1,1}$ in terms of the numbers of points in (Δ, Δ^*):

$$
\begin{aligned}
h^{1,1}(X_\Delta) &= h^{d-1,1}(X_{\Delta^*}) \\
&= l(\Delta^*) - (d+2) - \sum_{\mathrm{dim}\Theta^* = d} l'(\Theta^*) + \sum_{\mathrm{codim}\Theta_i^* = 2} l'(\Theta_i^*) l'(\Theta_i), \qquad a.) \\
h^{d-1,1}(X_\Delta) &= h^{1,1}(X_{\Delta^*}) \\
&= l(\Delta) - (d+2) - \sum_{\mathrm{dim}\Theta = d} l'(\Theta) + \sum_{\mathrm{codim}\Theta_i = 2} l'(\Theta_i) l'(\Theta_i^*), \qquad b.)
\end{aligned}
\tag{4.2}
$$

where Θ (Θ^*) denotes faces of Δ (Δ^*), $l(\Theta)$ is the number of all points of a face Θ and $l'(\Theta)$ is the number of points inside that face. In the last term the sum is over dual pairs (Θ_i, Θ_i^*) of faces. The fan $\Sigma(\Delta^*)$ *over* Δ^* defines in the standard way [44] a toric variety $\mathbb{P}_{\Delta^*}(\Sigma(\Delta^*)) = \mathbb{P}_{\Delta^*}$ in which X is embedded.

The following facts are relevant for the discussion of the divisors

i.) *Divisors and sub-manifolds in* $\mathbb{P}_{\Delta^*}$: Every ray τ_k through a point P_k in Δ^* (or more generally a cone in $\Sigma(\Delta^*)$) defines a $\mathbb{Q}$-Cartier divisor (or more generally a sub-manifold) in $\mathbb{P}_{\Delta^*}$, denoted $D'_k := V(\tau_k)$, which by itself has a very simple toric description. Take all cones $\mathcal{S}_k = \{\sigma_{k_i}\}$ for which τ_k is a face and consider the image of $\mathcal{S}_k$ in $M^*(\tau) = M^*/M^*_{\tau_k}$, where $M^*_{\tau_k}$ is the sub-lattice of M^* generated by vectors in τ_k. This image is called star(τ_k) and can be visualized as the projection of the $\mathcal{S}_k$ along τ_k on the hyperplane perpendicular to τ_k. Now $V(\tau_k)$ is the toric variety constructed from the fan over star(τ_k). Especially all these divisors in $\mathbb{P}_{\Delta^*}$ have $h^{0,0} = h^{d,d} = 1$ and $h^{i,j} = 0$ for $i \neq j$ and one can construct $l(\Delta^*) - (d+2)$ independent divisors classes which are a basis for $H^d(\mathbb{P}_{\Delta^*})$.

ii.) *Divisors and sub-manifolds in* X: The intersections $D_K = D'_K \cap X$ leads to divisors in X. In fact the divisors classes $[D_K]$ obtained this way generate $H^{d-1}(X)$. The manifold X can be thought as being constructed from a singular variety X_{sing} with quotient singularities along subsets R_K of *codim* > 1, which are induced from quotient singularities of the ambient space $\mathbb{P}_{\Delta^*}$. The divisors D_K will therefore be bundles of exceptional components

E_k coming from the desingularisation of the ambient space over the regular component R_k. The dimension of the regular and singular components depend simply on the dimension of the face of Δ^* on which the point P_i lies. The real dimension $d_{\Theta_k^*}$ of the face Θ_k^* is the complex dimension d_{E_k} of the exceptional component of D_k, while the complex dimension of R_k is $d_{R_k} = d - 1 - d_{E_k}$. In fact E_k and R_k have a very simple toric description. If Θ_k^* is a face of the $d+1$ dimensional polyhedron Δ_i^* then the dual face Θ_k, of dimension $\dim(\Theta) = d - \dim(\Theta_k^*)$, is defined as

$$\Theta_k = \{u \in \Delta | \langle u, v \rangle = -1, \forall v \in \Theta_k^*\}. \tag{4.3}$$

The sets R_i can be viewed as $D_i' \cap X_{sing}$ and are constructed as follows. Remember that the coordinate ring of the singular ambient space is generated by the corners E_i of Δ^*, especially X_{sing} is given in this coordinates by the vanishing of

$$p = \sum_{i=1}^{l(\Delta)} a_i \prod_{j=1}^{\#E} x_j^{\langle \nu^i, E_j \rangle}. \tag{4.4}$$

Now from the construction of D_i' as above it is clear that $D_k \cap X_{sing}$ is given by the vanishing of

$$p_k = \sum_{i=1}^{l(\Theta_k)} a_{k_i} \prod_{j=1}^{\#E(\Theta_k^*)} x_j^{\langle \nu^i, E_j(\Theta_k^*) \rangle}, \tag{4.5}$$

where $E_j(\Theta_k^*)$ are the corners of the face Θ_k^*. The structure of the exceptional component of D_k is given by the toric variety constructed from $\text{star}'(\tau_k)$; the projection of $\mathcal{S}_k$ on Θ_k^* along τ_k. This implies especially that $h^{0,0} = h^{d_{\theta_k^*}, d_{\theta_k^*}} = 1$ and $h_{i,j} = 0$ if $j \neq j$ [44]. Particularly useful is the fact that number of parameters by which we can move R_k in X namely $l(\Theta_K)$ is also the dimension of $H^{d_{R_k},0}(R_k)$, i.e.

$$h^{d_{R_k},0}(R_k) = l(\Theta_k). \tag{4.6}$$

This structure gives a useful classification of the divisors in X in types (a-d) below just according to the dimension of the face on which τ_k lies.

o.) $d_{\Theta_k^*} = d$, then Θ_k is a point and $R_k = \{p_k = 0\} = \emptyset$. Therefore divisors associated with these points have no intersection with X and the corresponding points are therefore subtracted in the third term in (4.2) a).

a.) $d_{\Theta_k}^* = d - 1$, then Θ^k is one dimensional and $R_k = \{Q_i | i = 1, \ldots, deg(p_k)\}$ are points in X whose number is given by the the degree of p_k or equivalently by $l(\Theta_k) + 1$. The fact that one has $l(\theta_k)+1$ divisor components D^i_k of the type $p_i \times E_k$ explains addition of the fourth term in (4.2) a). That E_k is toric variety implies $h^{i,j}(D^i_k)=0$ if $i\neq j$ and in particular $\chi(D^i_k,\mathcal{O}_{D^i_k})=1$. So this case leads to divisors for which a non-perturbative superpotential due to five fivebrane wrappings is generated.

b.) $d_{\Theta^*_k}=2$, E_k are rational surfaces, while R_k are Riemann surfaces whose genus g is by (4.6) the number of points inside Θ_k i.e. $l(\Theta_k)$. In this case we get $l(\Theta^*_k)\cdot l(\Theta_k)$ $(3,2)$-forms from the pairing of the $(1,0)$-forms on R_k with the $(2,2)$-forms of the E_k, which leads to the generalization of (4.2) given below. Especially we have for the irreducible component of the divisor $h^{0,0}(D_k)=1$, $h^{1,0}(D_k)=l(\Theta_k)$, $h^{2,0}(D_k)=0$, $h^{3,0}(D_k)=0$.

c.) $d_{\Theta^*_k}=1$ E_k is a $\mathbb{P}^1$ (in general in a Hirzebruch Sphere three) and R_k is a hypersurface in a three dimensional toric variety with $h^{2,0}(R_k)=l(\theta_k)$, moreover we can use the Lefschetz theorem to conclude $h^{1,0}(R_k)=0$. In this case we get additional $(3,1)$ forms form the pairing of $(2,0)$-forms of R_k with the $(1,1)$-forms of E_k, which gives rise to the fourth term in (4.2) b). A superpotential is generated if $l(\Theta_k)=0$.

d.) $d_{\Theta^*_k}=0$ in this case $D_k=R_k$. Similar as in [21] one can argue with the Lefschetz theorem that $h^1(D)=h^2(D)$ is zero, so that $\chi(D_k,\mathcal{O}_{D_k})=1-l(\Theta_k)$. Usually $h^3(D_k)$ is expected to be very positive so that D is movable and $\chi(D_k,\mathcal{O}_{D_k})\leq 1$. However in toric varieties due to conditions imposed by the weights this deformation space can be actually very restricted so that one can easily construct cases in which $h^3(D)=0$ for divisors of type d.), i.e. this divisors can lead to a non-perturbative superpotential. To summarize we have

$$\chi(D_k,\mathcal{O}_{D_k})=1-(-1)^{dim(\Theta_k)}l(\Theta_k). \qquad (4.7)$$

It should be clear by the above that $\chi(D,\mathcal{O}_D)=1$ divisors classes can be made abundant in the toric constructions of CY-manifolds. To illustrate this point take the mirror of any fourfold with small Picard number, e.g. the mirror of the sixtic in $\mathbb{P}^5$. Δ^* is now the Newton polyhedron of the sixtic which has $6,75,200,150,30,1$ points on dimension $0,1,2,3,4,5$ faces, which lead, as the Δ has only 6 corners and the inner point such that $l(\Theta_k)=0$, all to $\chi(D,\mathcal{O}_D)=1$ divisors. Some examples of this type of divisors have been considered in [21][19][29]. Very frequently one encounters the situation were $l(\Theta_k)=1$, which means Θ_K is a reflexive polyhedron of lower dimension and $c_1(R_k)=0$. The compactification of the fivebrane on such a divisor could lead to a sub-sector in the $N=1$ theory with enhanced supersymmetry.

Mirror symmetry implies for the Hodge diamonds of a mirror pair X, X^* that

$$h^{p,q}(X) = h^{d-p,q}(X^*). \tag{4.8}$$

For threefolds this property follows from (4.2) as $h^{2,1}(X)$ and $h^{1,1}(X)$ are the only independent Hodge numbers, if we construct $X^* = X_{\Delta^*}$ from Δ^* in the same way as $X = X_\Delta$ is constructed from Δ.

For fourfolds we have from (4.2) $h^{3,1}(X) = h^{1,1}(X^*)$ but since we have one more independent Hodge number we also have to establish $h^{2,1}(X) = h^{2,1}(X^*)$. This follows from the discussion of c.) above, which gives the formula

$$h^{d-r,1}(X) = h^{r,1}(X^*) = \sum_{\text{codim } \Theta_i = r+1} l'(\Theta_i) \cdot l'(\Theta_i^*), \text{ for } d-1 > r > 1. \tag{4.9}$$

Together with (2.5) it shows for four-folds that X, X^* as constructed from Δ, Δ^* have indeed the mirror Hodge diamond.

It is somewhat more complicated to obtain $h^{2,2}(X) = h^{2,2}(X^*)$ directly from the polyhedron. If mirror symmetry is true however, then one expects to have very good control over $H^{2,2}(X)$ as

$$H^{2,2}(X) = H^{2,2}_{prim}(X) \oplus H^{2,2}_{prim}(X^*), \tag{4.10}$$

were $H^{2,2}_{prim}(.)$ denotes the primitive part of the cohomology. This gives of course also a way of counting $h^{2,2}$ directly[19].

To every quasi homogeneous polynomial p in $d+2$ variables, like the one discussed in the last section, we can associate a Newton polyhedron Δ_p by considering the $(d+2)$-tuples of the exponents of the monomials of p as coordinates of points in $\mathbb{R}^{d+2}$ and building the convex hull of them. Quasi homogeneity of p implies that Δ_p lies in a hyperplane in $\mathbb{R}^{d+2}$, while (3.1) implies that $(1, \ldots, 1)$ is always an interior point of Δ_p, which we shift in the origin of $\mathbb{R}^{d+1}$. For $d \leq 3$ transversality of p implies reflexivity of Δ_p. That was actually shown by construction [46] (see also [47]). For $d \geq 4$ this property does not hold. A simple counter example is the manifold $X_7(1, 1, 1, 1, 1, 2)$.

---

19 E.g. for the sixtic in $\mathbb{P}^5$ ($h^{1,1} = 1, h^{2,1} = 0, h^{3,1} = 426$) it is easy to see that $h^{2,2}(X) = h^{2,2}_{prim}(X) + h^{2,2}_{prim}(X^*) = 1 + \dim(\mathbb{C}[x_1, \ldots, x_6]/\partial P|_{\deg=12}) = 1752$, as it also follows from the index theorem. Here P is a degree 6 polynomial in $x_1, \ldots, x_6$.

5. Toric four-folds over Fano Bases.

Fano varieties of dimension two, so called del Pezzo surfaces, are $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ blown up in up to eight points. There are five toric del Pezzo surfaces classified in [48]. $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, the Hirzebruch surface F_1, the equivariant blow up of $\mathbb{P}^2$ at two points B_2, and the equivariant blow up of $\mathbb{P}^2$ at three points B_3.

There are 84 Fano varieties of dimension three which were classified by Iskovskih and Mori-Mukai [49]. From these we will consider the 18 which can be represented in toric varieties (see [50] for a review). From [48][51] we have

(1) $\mathbb{P}^3$
(2) $\mathbb{P}^1 \times \mathbb{P}^2$
(3) The $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_{B'} \oplus \mathcal{O}(1)_{B'})$ over $B' = \mathbb{P}^2$
(4) The $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_{B'} \oplus \mathcal{O}(2)_{B'})$ over $B' = \mathbb{P}^2$
(5) The $\mathbb{P}^2$-bundle $\mathbb{P}(\mathcal{O}_{B'} \oplus \mathcal{O}_{B'} \oplus \mathcal{O}(1)_{B'})$ over $B' = \mathbb{P}^1$
(6) $(\mathbb{P}^1)^3$
(7) The $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_{B'} \otimes \mathcal{O}_{B'}(f_1 + f_2))$ over $B' = (\mathbb{P}^1)^2$, where f_1 and f_2 are fibres of the two projections from B' to $\mathbb{P}^1$
(8) The $\mathbb{P}(\mathcal{O}_{B'} \otimes \mathcal{O}_{B'}(f_1 - f_2))$ bundle over $\mathbb{P}^1 \times \mathbb{P}^1$.
(9) $\mathbb{P}^1 \times F_1$ where F_1 is the Hirzebruch surface.
(10) The $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_{B'} \otimes \mathcal{O}(s + f))$ over F_1, where f is the fibre from F_1 to $\mathbb{P}^1$, while s is the minimal cross section for the projection with -1 as self-intersection number.
(13) $\mathbb{P}^1 \times B_2$ with B_2 as above
(17) $\mathbb{P}^1 \times B_3$ with B_3 as above

The other cases are equivariant blow ups of the ones mentioned. This can be seen from the concrete fans below and is depicted in figure 1. Let us denote by $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$ unit vectors which span a rectangular lattice in $\mathbb{R}^3$. Then we can represent the toric varieties by the complete fans spanned by the following vectors

(1) $(e_1, e_2, e_3, -e_1-e_2-e_3)$, (2) $(e_1, e_2, e_3, -e_1-e_2, -e_3)$,
(3) $(e_1, e_2, e_3, -e_2, -e_1-e_2-e_3)$, (4) $(e_1, e_2, e_3, -e_2, -e_1-2e_2-e_3)$,
(5) $(e_1, e_2, e_3, -e_1-e_2-e_3, -e_1-e_3)$ (6) $(e_1, e_2, e_3, -e_1, -e_2, -e_3)$,
(7) $(e_1, e_2, e_3, -e_1-e_3, -e_2-e_3, -e_3)$, (8) $(e_1, e_2, e_3, -e_1-e_3, e_3-e_2, -e_3)$,
(9) $(e_1, e_2, e_3, -e_2, e_2-e_1, -e_3)$, (10) $(e_1, e_2, e_3, -e_1-e_3, e_1-e_2, -e_3)$,
(11) $(e_1, e_2, e_3, e_3-e_2, -e_2, -e_1-e_2-e_3)$, (12) $(e_1, e_2, e_3, e_3-e_2, -e_1-e_3, -e_2)$,
(13) $(e_1, e_2, e_3, e_2-e_1, -e_2, e_1-e_2, -e_3)$, (14) $(e_1, e_2, e_3, e_2-e_1, -e_2, e_1-e_2, e_1-e_2-e_3)$
(15) $(e_1, e_2, e_3, e_2-e_1, -e_2, e_1-e_2, -e_2-e_3)$, (16) $(e_1, e_2, e_3, e_2-e_1, -e_2, e_1-e_2, e_1-e_3)$,
(17) $(e_1, e_2, e_3, -e_1, -e_2, -e_3, e_1-e_2, e_2-e_1)$ (18) $(e_1, e_2, e_3, e_2-e_1, -e_1, -e_2, e_1-e_2, -e_1-e_3)$

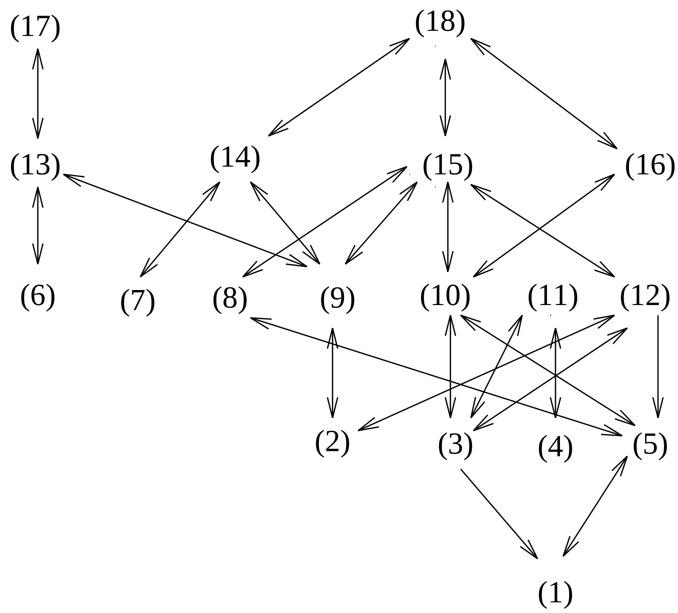

Fig.1:

The net of equivariant blow ups (downs) among the fano bases as in [51]. The blow ups are either at points ↓ or along one dimensional closed irreducible subvarities ↕, which are stable under the torus action. By the construction below, they will be promoted to transitions between elliptically fibred CY fourfolds.

To construct d-dimensional elliptic fibration Calabi-Yau manifolds X over this base spaces B we consider polyhedra which are obtained from the toric description of the base spaces as follows. We define the vectors in the rectangular $\mathbb{Z}^{d+1}$ latticed in $\mathbb{R}^{d+1}$

$$v_A = (\underbrace{0,\ldots,0}_{d-1},2,3), \quad v_B = (\underbrace{0,\ldots,0}_{d-1},1,2), \quad v_C = (\underbrace{0,\ldots,0}_{d+1},1,1)$$

$$\text{as well as } e_d = (\underbrace{0,\ldots,0}_{d-1},1,0) \text{ and } e_{d+1} = (\underbrace{0,\ldots,0}_{d+1},0,1).$$

Let $\nu^{(i)}$ $i = 1,\ldots,r$ be the vectors of the complete fan of the base space embedded in the $1,\ldots,d-1$-plane in $\mathbb{R}^{d+1}$, and $\nu^{(r+1)} = (0,\ldots,0)$ the origin. Then we can define for any given base space B (1)-(18) three reflexive polyhedra Δ^*_I, $I = A, B, C$ with vertices

$$\{\nu^{(i)*} = \nu^{(i)} + v_I \cdot (\sum_j \nu^{(i)}_j - 1), i = 1,\ldots,r+1; e_d, e_{d+1}\}. \tag{5.1}$$

The hypersurfaces as defined by (4.1) in $\mathbb{P}^{d+1}_{\Delta^*}$ correspond to elliptic fibrations over the base space Σ with generic fibre of the type $X_6(1,2,3)$, $X_4(1,1,2)$ and $X_3(1,1,1)$. The topological data of these manifolds are summarized in table 6.1:

	Bases				X_3-fibrations				X_4-fibrations				X_6-fibrations			
B	χ^B	h^B_{11}	$(c_1^3)_B$	$(c_1c_2)_B$	χ^X	$(c_2^X)^2$	h^X_{11}	h^X_{31}	χ^X	$(c_2^X)^2$	h^X_{11}	h^X_{31}	χ^X	$(c_2^X)^2$	h^X_{11}	h^X_{31}
$\mathbb{P}^3$	4	1	64	24	4896	2112	4(2)	804	9504	3648	3(1)	1573	23328	8256	2	3878
$F_0^{(2)}$	6	2	54	24	4176	1872	5(2)	683	8064	3178	4(1)	1332	19728	7056	3	3277
$F_1^{(2)}$	6	2	56	24	4320	1920	5(2)	701	8352	3264	4(1)	1380	20448	7296	3	3397
$F_2^{(2)}$	6	2	62	24	4752	2064	5(2)	779	9216	3552	4(1)	1524	22608	8016	3	3757
(5)	6	2	54	24	4176	1728	5(2)	683	8064	3168	4(1)	1332	19728	7056	3	3277
$(\mathbb{P}^1)^3$	8	3	48	24	3744	1728	6(2)	610	7200	2880	5(1)	1187	17568	6336	4	2916
(7)	8	3	52	24	4032	1824	6(2)	658	7776	3072	5(1)	1283	19008	6816	4	3156
(8)	8	3	44	24	3456	1632	6(2)	562	6624	2688	5(1)	1091	16128	5856	4	2676
$\mathbb{P}^1 \times F_1^{(1)}$	8	3	48	24	3744	1728	6(2)	610	7200	2880	5(1)	1187	17568	6336	4	2916
(10)	8	3	50	24	3888	1776	6(2)	634	7488	2976	5(1)	1235	18288	6576	4	3036
(11)	8	3	50	24	3888	1776	6(2)	634	7488	2976	5(1)	1235	18288	6576	4	3036
(12)	8	3	46	24	3600	1680	6(2)	586	6912	2784	5(1)	1139	16848	6048	4	2796
$\mathbb{P}^1 \times B_2$	10	4	42	24	3312	1584	7(2)	537	6336	2592	6(1)	1042	15408	5616	5	2555
(14)	10	4	44	24	3456	1632	7(2)	561	6624	2688	6(1)	1090	16128	5856	5	2675
(15)	10	4	40	24	3168	1536	7(2)	513	6048	2496	6(1)	994	14688	5376	5	2435
(16)	10	4	46	24	3600	1680	7(2)	585	6912	2784	6(1)	1138	16848	6096	5	2795
$\mathbb{P}^1 \times B_3$	12	5	36	24	2880	1440	8(2)	464	5472	2302	7(1)	897	13248	4896	6	2194
(18)	12	5	36	24	2880	1440	8(2)	464	5472	2302	7(1)	897	13248	4896	6	2194

Tab. 6.1: *Elliptic fibred fourfolds over toric Fano bases with fibre X_3, X_4 and X_6. All topological numbers*[20] *are calculated independently. From (2.2) and $\chi_0 = 1$ for Fano d-folds, follows $\int_B c_1 c_2 = 24$. As further checks serve (2.4) a.) and (2.6). $h^X_{21} = 0$ for all fibre types and all bases.*

By construction Δ^* has a prominent reflexive face Θ^*_B, which is the convex hull of $\nu^{*(i)}$, with $\tau = \nu^*_{r+1} = (0,0,0,-2,-3)$ as the only interior point. $B = V(\tau) \cap X$ gives divisors of type a.), which describes sections of the fibration in X. The two endpoints of Θ_B are $\nu^{\pm} = (-1,-1,-1,a^{\pm},b^{\pm})$, with $(a^+,b^+) = (2,-1),(3,-1),(2,-1)$; $(a^-,b^-) = (-1,2),(-1,1),(-1,-2)$ for the X_3, X_4, X_6 fibres, i.e. $l(\Theta_B) + 1 = 3,2,1$ reflecting the fact that the fibrations have $3,2,1$ sections. The discussion of the other divisors is equally simple. For instance for the model $\mathbb{P}^1 \times B_3$ (13) we see that all divisors $D_i = V(\nu^*_i) \cap X$, (up to $D_{r+1} = B$) with ν^*_i from (5.1) are of type d.), with $\chi(D_i, \mathcal{O}_{D_i}) = 1,1,1,0,0,0,0$ for $i = 1,\dots,r$ and $\chi(D_{e_4}, \mathcal{O}_{D_{e_4}}) = -109$, $\chi(D_{e_5}, \mathcal{O}_{D_{E_5}}) = -324$. Especially $D_1,\dots,D_3$

are divisors which lead to superpotentials while $D_4, \ldots, D_7$ correspond to embeddings of Calabi-Yau threefolds in X.

Let us finally comment on the transitions. Model (1)-(3) and (5)-(12) above are connected by the blow up of a fixed point under the torus action. We can blow up $\mathbb{P}^3$ in generic points by adding successively the vertices $-e_1$, $-e_2$, $-e_3$ and $e_1+e_2+e_3$ to the $\mathbb{P}^3$ polyhedron. If $\hat{B}$ is obtained from B by blowing up such fixed points then for the canonical bundles one has ($n = d-1 = \dim(B)$)

$$\hat{K} = \pi^* K + (n-1)[E] \tag{5.2}$$

and since $[E]$ ($[E]^2 = -1$) does not intersect with classes of B one has $c_1^n(\hat{K}) = c_1^n(K) + (n-1)^n [E]^n$ so that $\int_{\hat{B}} c_1^n(\hat{B}) = \int_B c_1^n(B) - (n-1)^n$. In our case the effect of the transition is $\chi(\hat{B}) = \chi(B) + 2$, $h^{1,1}(\hat{B}) = h^{1,1}(B) + 1$, $\int_{\hat{B}} c_1^3(\hat{B}) = \int_B c_1^3(B) - 8$ and since $\int_B c_1 c_2$ is invariant one has $\chi(\hat{X}) = \chi(X) - 8 \cdot 360$ for the X_6 fibre (360 has to be replaced by $144, 72, 36$ for the other fibres). As $h^{1,1}(\hat{X}) = h^{1,1}(X) + 1$ and $h^{2,1}(\hat{X}) = h^{2,1}(X)$ this means by the index theorem (2.6) especially that $h^{3,1}(\hat{B}) = h^{3,1}(B) - 471$ for the X_6 fibre (471 has to be replaced by $183, 87, 39$ for the other fibres). The seven branes at the discrimante induce a three brane charge (comp. section 2 and [13]) . The contribution from the generic member in the class $-12[\tilde{\Delta}]$ is for $\mathbb{P}^3$ $Q(X) = -2300$ and each blow up changes this number by $Q(\hat{X}) = Q(X) + 286$.

For generic moduli values in the above examples we have no codimension three enhancements of the elliptic fibre singularities over the base. However if we restrict the complex 471 moduli as we must do in order to follow a transition, then enhanced singularities at codimension three emerge, which should roughly localize the induced negative threebrane charges to points in the base were they annihilate with the positive threebranes. Figure 1 also shows that the F-theory vacua under consideration are multiple connected by paths in the moduli space. The associated fourfold polyhedra are embedded into each other, which implies that there are (extremal) transitions among them[21][52]. We will discuss the geometrical aspects of the (5) $\leftrightarrow$ (1) $\leftarrow$ (3) transitions in more detail in (9.6).

One might wonder what are in general the allowed modifications of the three dimensional fan Σ_B of the base for which the property of the elliptic fibration $K_B = -\frac{1}{12}[\tilde{\Delta}]$ is kept. From the Weierstrass form and $K_B = -\sum_i D_i$ for reflexive polyhedra we expect this to be the case when Σ_B comes from a reflexive polyhedron, which would mean that any $K3$ polyhedron can be used in this construction.

[21] Such embeddings are expected to connect all fourfolds constructed by reflexive polyhedra .

5.1. The Weierstrass form of X

To study the elliptic fibration and it's possible degeneration let us first describe the Weierstrass forms of X. Recall that the toric variety $\mathbb{P}^{d+1}_{\Delta^*}$ is defined as follows. We associate to every integral point $\nu_i \neq (0,\ldots,0)$ in Δ^* a coordinate x_i $i=1,\ldots,q=l(\Delta^*)$ in $\mathbb{C}^q$. Next we choose a complete triangulation $\mathcal{T}$ of Δ^* in $d+1$-dimensional simplices, whose vertices are the ν_i points. The Stanley-Reisner ideal is defined as the common zero of all those coordinate sets $\{x_{i_1},\ldots,x_{i_p}\}$, for which every subset S of points $S\subset\{\nu_{i_1},\ldots,\nu_{i_p}\}$ does not lie on a common k dimensional simplex, we denote these zero sets as $\mathcal{S}^{(k)}_j$. Linear relations between points in Δ^*, like $\sum l^{(k)}_i \nu_i=(0,\ldots,0)$ define $(q-d-2)$ independent $\mathbb{C}^*$-actions on the coordinates x_i: $(x_1,\ldots,x_q)\sim(\lambda_{(k)}^{l_1^{(k)}}x_1,\ldots,\lambda_{(k)}^{l_q^{(k)}}x_q)$, with $\lambda_{(k)}\in\mathbb{C}^*$. The toric variety is then $\mathbb{P}^{d+1}_{\Delta^*}=(\mathbb{C}^q-\cup_{k,j}\mathcal{S}^{(k)}_j)/(\mathbb{C}^*)^{q-d-2}$. For every regular triangulation $\mathcal{T}$ there is a canonical choice of $l^{(k)}$ such that all $l^{(k)}_i$ are semi-positive. Given such a choice we can write the hypersurface $p=0$ as the polynomial in the x_i which scales homogeneously and with the minimal integers $\sum_i l^{(k)}_i$ with respect to all the $k=1,\ldots,q-d-1$ $\mathbb{C}^*$-actions. Suppose such $\hat{l}^{(k)}$ $k=1,\ldots,s$ have been constructed for a triangulation of the fan of B, then there exist always a triangulation $\mathcal{T}$ of Δ^* such that the following scaling vectors $l^{(k)}$ appear among the $l^{(k)}$ for $(\Delta^*,\mathcal{T})$:

$$\begin{array}{l} l^{(1)}=(0,\ldots,0;1,n_1,n_2)\\ l^{(k+1)}=(\quad \hat{l}^{(k)}\quad ;0,n_1\sum_I l^{(k)}_i,n_2\sum_I l^{(k)}_i)\end{array},\quad (n_1,n_2)=\left\{\begin{array}{l}(2,3)\text{ for the }E_8-\text{fibre}\\(1,2)\text{ for the }E_7-\text{fibre}\\(1,1)\text{ for the }E_6-\text{fibre,}\end{array}\right. \tag{5.3}$$

where k runs from 1 to s.

This implies that p can be written, at least in a certain patch, in the following Weierstrass [22] form ($y:=x_q$, $x:=x_{q-1}$ and $z:=x_{q-2}$)

$$y^2=x^3+xz^4f(x_1,\ldots,x_{q-3})+z^6g(x_1,\ldots,x_{q-3}), \tag{5.4}$$

with discriminant

$$\Delta=27g^2+4f^3. \tag{5.5}$$

As one can see from the table 6.1 the Euler number of X fulfills always (3.11), so one expects that the elliptic fibre does not degenerate over codimension one or two in the base. This can in fact easily be checked in the Weierstrass models.

[22] Here one omits the first sub-leading terms in x and y to avoid redundant deformations of the equation as it is familiar in singularity theory. Writing down normal forms compatible with (5.3) for the other cases is straightforward: $E_7:\ y^2=x^4+x^2z^2e(x_i)+xz^3f(x_i)+z^4g(x_i)$ with $\Delta=2^8g^3-2^7e^2g^2+2^43^2egf^2+2^4e^3f^2-3^3f^4$ and $E_6:\ y^3+x^3=yz^2e(x_i)+xz^2f(x_i)+zg^3(x_i)$ with $\Delta=2^4(f^6+e^6)-2^33^3g^2(e^3+f^3)-2^5e^3f^3+3^6g^4$.

6. Gauge groups in four dimensions and more general degeneration of the elliptic fibres

The degenerations of the fibre are described by Kodaira (table A.1) and a practical way to identify or construct such degenerations from the functions f and g of the Weierstrass form is Tate's alogarithm [53]. This was used in [4], to analyze the physics associated to the degenerations of the elliptic fibre for F-theory compactifications to six dimensions. Here we will be interested in the simplest situation were the fibre degenerates homogeneously over a codimension one locus B' in the base. In this situation the enhancement of the gauge group in four dimensions can be, at least for A_n singular fibres, explained with parallel 7-branes whose world-volume fills $B' \times \mathbb{R}^4$. We will study situations in which B admits a itself a fibration $\mathbb{P}^1 \to B \to B'$, such that we get a $N=1$ heterotic theory on $\mathcal{E}' \to Z \to B'$.

Let us consider for this purpose a generalization of the models (2)-(3), i.e. we consider as base B the fibration $\mathbb{P}(\mathcal{O}_{\mathbb{P}^r} \otimes \mathcal{O}_{\mathbb{P}^r}(n)) \to B \to \mathbb{P}^r$, which we denote as $F_n^{(r)}$, such that $F_n^{(1)}$ are the ordinary Hirzebruch surfaces F_n. The fan Σ_B for $F_n^{(r)}$ is spanned by $(e_i, i=1,\ldots,r+1; -e_{r+1}, -e_1 - \ldots - e_r - ne_{k+1})$. For the relevant case $r=2$ we have the following topological properties of the base

$$\chi(F_n^{(2)}) = 6, \quad \int_{F_n^{(2)}} c_1 c_2 = 24, \quad \int_{F_n^{(2)}} c_1^3 = 54 + 2n^2. \tag{6.1}$$

From (5.3) with

$$\hat{l}^{(1)} = (1,\ldots,1,n,0)$$
$$\hat{l}^{(2)} = (1,\ldots,0,1,1)$$

follows ($y := x_q$, $x := x_{q-1}$, $z := x_{q-2}$)

$$y^2 = x^3 + xz^4 \sum_{l=0}^{\left[\frac{4(n+r+1)}{n}\right]} v^l u^{8-l} f_{4(n+r+1)-nl} + z^6 \sum_{l=0}^{\left[\frac{6(n+r+1)}{n}\right]} v^l u^{12-l} g_{6(n+r+1)-nl},$$

where $u = x_{r+3}$, $v = x_{r+2}$ are the coordinates of the $\mathbb{P}^1(\mathcal{O}_{\mathbb{P}^r} \otimes \mathcal{O}(n)_{\mathbb{P}^r})$ fibre, $[a/b]$ denotes the integer part of a/b and f_k and g_k are polynomials homogeneous of degree k in the coordinates of the $\mathbb{P}^r$ $(x_1,\ldots,x_{r+1})$. To discuss the simplest degenerations of the the fibres, which lead to generic gauge groups in space time, we have now just to look at the leading behavior of the Weierstrass form near $(z,u)=(0,0)$. The basic behavior is determined by the divisibility properties of $4(n+r+1)$, $6(n+r+1)$ by n; the leading singularity is

$$x f_{(4(n+r+1)\ mod\ n)} u^{8-\left[\frac{4(n+r+1)}{n}\right]} + g_{(6(n+r+1)\ mod\ n)} u^{12-\left[\frac{6(n+r+1)}{n}\right]}. \tag{6.2}$$

The general discussion is exactly as in [53][3] for $r=1$ apart from the fact that one gets for the four-folds much richer singularity structure if the functions f_k, g_k are not forced to be constant over the $\mathbb{P}^2$ for the leading term in u. Let us focus on the simple cases with pure gauge group and no additional matter. As it is obvious from (6.2) the pure $SO(8)$, E_6, E_7 and E_8 singularities which occur for $r=1$ over the base $\mathbb{P}^1$ in $F^{(1)}_{n^{(1)}}$ for $n^{(1)}=4,6,8,12$, will occur in general over the base $\mathbb{P}^r$ of $F^{(r)}_{(r+1)n^{(1)}/2}$. Especially in four dimensions $r=2$ this gives the following examples

B		B'			X_6-fibrations			
B	$\int_B c_1(B)^3$	B'	$\int_{B'} c_1^2(B')$	G	$\chi(X)$	h_{11}^X	h_{21}^X	h_{31}^X
$F_6^{(2)}$	126	$\mathbb{P}^2$	9	D_4	44136	7(2)	0	7341(0)
$F_9^{(2)}$	216	$\mathbb{P}^2$	9	E_6	69624	9(2)	0	11587(0)
$F_{12}^{(2)}$	342	$\mathbb{P}^2$	9	E_7	101862	10(0)	0	16959(0)
$F_{18}^{(2)}$	702	$\mathbb{P}^2$	9	E_8	186048	11(0)	0	30989(0)

Tab. 6.3: *Elliptic fibrations over $F_n^{(2)}$, with pure gauge groups. Note that $\chi = 24\cdot 4244 + \frac{1}{4}$ for the E_7 case.*

The enhancement of the gauge group can easily studied in detail if we recognize that this cases are closely related to the hypersurfaces $X_{6(n+3)}(1,1,1,n,2(n+3),3(n+3))$ which in turn are K_3 fibrations with generic fibre $X_{2(n+3)}(1,n/3,2(n+3)/3,(n+3))$ over $\mathbb{P}^2$ of the type discussed in section (2.3). That is the intersection form, which lead to the gauge symmetry enhancement comes from vanishing of the corresponding cycles in the K_3. We can also see this from the embedding of the polyhedra. Notice that the points in the $3,4,5$ plane cutting Δ^*

$$\nu_1^*=(0,0,-n/3,-2(n+3)/3,-(n+3)), \nu_2^*=(0,0,1,0,0), \nu_3^*(0,0,0,1,0), \nu_4^*(0,0,0,0,1),$$

span the K_3 polyhedron. E.g. in the case of the E_8 $K3$ ($n=7$) one has six points on the edge between ν_1^* and ν_2^*, two between ν_1^* and ν_3^* and one between ν_1^* and ν_4^*. Together with the hyperplane class the $V(\tau)\cap K_3$ the divisors associated to these points make up

Pic of the K_3 in question and have the intersection form $E_8 \times U$ [54] (for the other cases see Kondos's list [55][54]). The nine points on the edges of the K_3 gives rise divisors $D = V(\tau) \cap X$ of the four fold of type b.) in addition the point ν_1^* gives rise to a divisor of type c.). All of them have $\chi(D, \mathcal{O}_D) = 1$ from (4.7). One is horizontal w.r.t. π of (1.5) the other are are $\mathbb{P}^1$ bundles over $\mathbb{P}^2$ and vertical w.r.t. π and but horizontal w.r.t. π'' of (1.7), i.e. they will lead to a non-perturbative superpotential of the heterotic string. In fact we have here a realization of the situation described in [15][16] for the E_8 group.

As a further simple generalization[23] of (7), we chose B such that is it is a $\mathbb{P}^1$ bundle $\mathbb{P}(\mathcal{O}_{B'} \otimes \mathcal{O}(bf_1 \otimes cf_2)_{B'})$ over B' with $\mathbb{P}(\mathcal{O} \otimes \mathcal{O}(a)_{\mathbb{P}^1}) \to B' \to \mathbb{P}^1$. This base B , say $F^{(2)}_{k,m,n}$ has as fan $(-e_1 - ke_2 - me_3, -e_2 - ne_3, e_1, e_2, e_3, -e_3)$ with coordinates (p, s, q, t, v, u) and the topological properties

$$\chi(F^{(2)}_{k,m,n}) = 8, \quad \int_{F^{(2)}_{a,b,c}} c_1 c_2 = 24, \quad \int_{F^{(2)}_n} c_1^3 = 48 + 4mn - 2m^2 k. \tag{6.3}$$

In particular if $k = 0$ ($B' = \mathbb{P}^1 \times \mathbb{P}^1$) and $m = n$ the elliptic fibre degenerates homogeneously over $\mathbb{P}^1 \times \mathbb{P}^1$ as can be seen from the Weierstrass form

$$y^2 = x^3 + xz^4 \sum_{l=0}^{\left[\frac{4(n+2)}{n}\right]} v^l u^{8-l} f^{(s,t;p,q)}_{4(n+2)-nl;4(n+2)-nl} + z^6 \sum_{l=0}^{\left[\frac{6(n+2)}{n}\right]} v^l u^{12-l} g^{(s,t;p,q)}_{6(n+2)-nl;6(n+2)-nl}, \tag{6.4}$$

such that we get as before the matter free degenerations, but this time at $n = 3, 4, 6, 8, 12$. The case $n = 8$ leads however not to reflexive polyhedra hence not to a model with a geometrical resolution.

B		B'			X_6-fibrations			
B	$\int_B c_1(B)^3$	B'	$\int_{B'} c_1^2(B')$	G	$\chi(X)$	h^X_{11}	h^X_{21}	h^X_{31}
$F^{(2)}_{0,3,3}$	84	$\mathbb{P}^1 \times \mathbb{P}^1$	8	A_2	30336	6(1)	0	5042(0)
$F^{(2)}_{0,4,4}$	112	$\mathbb{P}^1 \times \mathbb{P}^1$	8	D_4	39264	8(2)	0	6528(0)
$F^{(2)}_{0,6,6}$	192	$\mathbb{P}^1 \times \mathbb{P}^1$	8	E_6	61920	10(2)	0	10302(0)
$F^{(2)}_{0,12,12}$	624	$\mathbb{P}^1 \times \mathbb{P}^1$	8	E_8	165498	12(0)	0	27548(0)

Tab. 6.3: *Elliptic fibrations over $\mathbb{P}^1 \times \mathbb{P}^1$, with pure gauge groups.*

[23] The generation of a superpotential in ten examples of this kind with generically I_1 degeneration are discussed in great detail in [29].

Again the $X_{3(n+2)}(1, n/2, (n+2), (n/2+1)3)$ K_3 is embedded in the $(2,3,4)$ plane and the divisors of X leading to the enhanced gauge symmetry have very similar properties to the ones discussed before.

The general formula for the Euler number for the elliptic fibred four fold X for which the X_6-fibration degenerates to a singularity of type G over a codimension one subspace B' in the base B is

$$\chi(X) = 12 \int_B c_1(B)c_2(B) + 360 \int_B c_1^3(B) - \delta^{d=4}(B', G), \qquad \text{with}$$
$$\delta^{d=4}(B', G) = r_{(G)}c_{(G)} \left(c_{(G)} \int_{B'} c_1(B')^2 + (6 - \int_{B'} c_2(B')) \int_{B'} c_2(B') \right). \tag{6.5}$$

For $d = 3$ the correction term is

$$\delta^{d=3} = r_{(G)}c_{(G)} \int_{B'} c_1(B'), \tag{6.6}$$

while for $d = 5$ we observe for $B' = \mathbb{P}^3$

$$\delta^{d=5} = r_{(G)} \left(c_{(G)}^3 \int_{B'} c_1^3(B') + 3c_{(G)}^2 \int_{B'} c_1(B')c_2(B') + 2(3c_{(G)} - c_{(G)}^2) \int_{B'} c_3(B') \right), \tag{6.7}$$

e.g. the elliptic fivefold fibration over the four dimensional base $F_{18}^{(3)}$ for which the generic fibre X_6 degenerates over a $\mathbb{P}^3$ has by (3.10),(6.7) the Euler number $\chi = -55556832$.

If the degeneration of the fibre is not of the same type over a subspace of codimension one in the base, but there are codimension two loci where the degeneration increases, a positive correction to the Euler number (3.10) is expected. As example we consider $F_{0,0,n}^{(2)}$. Now the functions $f_8'(p,q), g_{12}'(p,q)$ do not become constants, when we consider the leading singularity around $(x,u) = (0,0)$ and we get extra singularities when these functions vanish. Application of (3.10) gives $\chi_s = 17568$, while the actual data are

B		B'			X_6-fibrations			
B	$\int_B c_1(B)^3$	B'	$\int_{B'} c_1^2(B')$	G	$\chi(X)$	h_{11}^X	h_{21}^X	h_{31}^X
$F_{0,0,3}^{(2)}$	48	$\mathbb{P}^1 \times \mathbb{P}^1$	8	A_2	18240	5(0)	5	3032(0)
$F_{0,0,4}^{(2)}$	48	$\mathbb{P}^1 \times \mathbb{P}^1$	8	D_4	19680	6(0)	10	3276(0)
$F_{0,0,6}^{(2)}$	48	$\mathbb{P}^1 \times \mathbb{P}^1$	8	E_6	23328	8(0)	10	3882(0)
$F_{0,0,12}^{(2)}$	48	$\mathbb{P}^1 \times \mathbb{P}^1$	8	E_8	35808	24(11)	0	5936(0)

Tab. 6.4: *Elliptic fibrations over* $F_{0,0,n}^{(2)}$.

In $F_n^{(2)}$ and the $F_{0,n,n}^{(2)}$ cases we considered a specific point $p = (u = 0, v = 1)$ in the rational fibre over B' and configurations such that the degeneration of the elliptic fibre was homogeneous over B'. B' is of course just one component of the discriminant locus and away from p the fibre will degenerate over codimension one to I_1, but more complicated in higher codimensions. If we allow for special values of the moduli, there will be also more complicated degenerations over codimension one surfaces in the base, which will lead to non generic gauge group enhancement. In particular one can design examples with ADE sphere tree's over a $\mathbb{P}^2$ in the base in which non generic gauge groups arise in M theory compactifications to three dimensions similarly as in [56].

Let us discuss in extension of the last examples in table 3.3 situations where we have a generic ADE fibre over B', but additional enhancements over lines and points in B'. These cases can be designed, by "upgrading" the corresponding F_n fibrations in three dimensions, which were studied in great detail in [4][57] to four dimensions.

These three dimensional Calabi-Yau spaces Y are elliptic fibration over F_n: $\mathcal{E} \to Y \to F_n$ *and* K_3 fibrations $K_3 \to Y \to \mathbb{P}^1$. Furthermore the the K_3 is itself a elliptic fibration over the fibre $\mathbb{P}^1$ of F_n, i.e. $\mathcal{E} \to K_3 \to \mathbb{P}^1$. These fibration structure[24] are reflected in the geometry of the four dimensional polyhedron Δ^* (cf. [57]). It has the polar polyhedron of the Newton polyhedron of $X_6(1,2,3)$ in the (say) $(4,5)$ plane, which is augmented to a K_3 polyhedron in the $(3,4,5)$ plane. Now in the threefold polyhedron there are two points $p_1 = (0,-1,0,2,3)$ and $p_2 = (0,1,2n,2,3)$ outside the $(3,4,5)$ plane such that a corner of the K_3 polyhedron $c = (0,0,n,2,3)$ is in the middle of the line $\overline{p_1 p_2}$. The coordinates associated to p_1 and p_2 are the homogeneous coordinates of the base $\mathbb{P}^1$. It is now very easy to replace the base $\mathbb{P}^1$ by a rational surface S. E.g. we can replace it by $\mathbb{P}^2$ by adding instead of p_1, p_2 the points $p_0 = (-1,0,0,2,3)$, $p_1 = (0,-1,0,2,3)$ and $p_2 = (1,1,3n,2,3)$ so that e represents the canonical class of $\mathbb{P}^2$ (or S). It is important that the only modification in the scaling relations (5.3) from the three to the four dimensional case is that the Mori generator with the two 1's on the $\mathbb{P}^1$ coordinates $l = (1,1,n,0,\ldots,0)$ is replaced by $l = (1,1,1,n,0,\ldots,0)$ with three 1's on the $\mathbb{P}^2$ coordinates, all other linear relations between the points $K3$-plane are obviously the same. This implies that the Weierstrass form is essentially the same but f and g depend now homogeneously on three coordinates. That is the generic codimension one singularity at $(u = 1, v = 0)$ is as

[24] The complete process is the generalization of (3.7) with $X_6^{(0)}(1,2,3)$, $r^{(1)} = 2$, $r^{(2)} = 2$, and $r^{(3)} = 3$ to the polyheder description.

analysed in [4][20]and indicated in table (6.5), while the additional singularities which give matter in the six dimensional compactification are now at codimension one in the $\mathbb{P}^2$. Let us "upgrade" a couple of examples from table 3.2 of [57] to four dimensions in order to demonstrate the effect of "unhiggsing" of the $(u=1, v=0)$ locus in the fibre of $F_n^{(2)}$.

B^0	$SU(1)$	$SU(2)$	$SU(3)$	$SU(4)$	$SU(5)$
$F_3^{(2)}$	$({}^0 26208;3,1)$	$({}^3 17082;4,1)$	$({}^0 13032;5,1)$	$({}^2 10116;6,1)$	$({}^3 7578;7,1)$
$F_6^{(2)}$	$({}^0 44136;7,0)$	$({}^3 24642;8,0)$	$({}^0 16704;9,0)$	$({}^0 11520;10,0)$	$({}^0 7416;11,0)$
$F_9^{(2)}$	$({}^0 69624;9,0)$	$({}^3 35874;10,0)$	$({}^0 22752;11,0)$	$({}^2 14652;12,0)$	$({}^2 8604;13,0)$

Tab. 6.5: *Topological invariants* $(\chi; h^{1,1}, h^{1,2})$ *of the chains of elliptic fibrations over* $F_n^{(2)}$. *We indicate by the prefix* ^{n} *on the Euler number it's divisibility* $6n = \chi$ *mod* 24.

We will discuss in section (9.6) in detail how the aspects of the discussion of the transitions [20] carries over.

7. Euler number of Elliptic CY manifolds

For a complex manifold M, we denote the tangent bundle, canonical bundle, the total Chern class of M by T_M, K_M and $c(M)$ respectively.

Lemma 1. Let M be a m-dimensional compact complex manifold, and D be an irreducible smooth divisor of M such that $\mathcal{O}_M(D)$ is the d-th power of the canonical sheaf of M for some rational number d, $\mathcal{O}_M(D) = \omega_M^d$. Then

$$\chi(D) = -\sum_{k=1}^{m} d^k c_1^k c_{m-k},$$

where c_j is the j-th Chern class of M.

Let N be the normal bundle of D in M. It is known that the Chern class of D, $1 + c_1(D) + \cdots + c_{m-1}(D)$, is related to $c_1(N)$ and c_j's bt he following relations:

$$c_j(D) + c_1(N) c_{j-1}(D) = c_{j|D} \ ,$$

hence

$$c_j(D) = \sum_{k=0}^{j} (-1)^k c_1(N)^k c_{j-k|D}$$

for $1 \leq j \leq m-1$. By $c_1(N) = -dc_{1|D}$, the result follows from the above relation for $j = m-1$. $\square$

Lemma 2. Let X be a n-dimensional CY manifold, which is a l-fold cyclic cover of a manifold Y for $l \geq 2$. Then

$$\frac{1}{l-1}\chi(X) = \frac{l}{l-1}\chi(Y) + \sum_{k=1}^{n}(\frac{-l}{l-1})^k c_1(Y)^k c_{n-k}(Y)$$

Proof. Let D be the branched locus for the double cover of X over Y. D is a smooth divisor with $\mathcal{O}_Y(D) = \omega_Y^{\frac{-l}{l-1}}$. The result follows immediately from Lemma 1. $\square$

Let E be a vector bundle over a complex manifold M of rank r, and $\mathbb{P}$ be the associated projective bundle,

$$\pi : \mathbb{P} = \mathbb{P}(E) \longrightarrow M \ .$$

Note that $\mathbb{P} = \mathbb{P}(E \otimes L)$ for any line bundle L over M. We have the exact sequence of vector bundles over $\mathbb{P}(E)$:

$$0 \longrightarrow \mathbb{P} \times \mathbb{C} \longrightarrow \pi^* E(1) \longrightarrow T_{\mathbb{P}} \longrightarrow \pi^*(T_M) \longrightarrow 0 \ .$$

where $(\pi^* E)(1)$ is the tensor bundle $\pi^* E \otimes \mathcal{O}(1)$ with $\mathcal{O}(1)$ the inverse of the tautological bundle over $\mathbb{P}$ for the bundle E. Hence

$$K_{\mathbb{P}} = \pi^*(K_M \otimes \det(E^*)) \otimes \mathcal{O}(-r) \tag{7.1}$$

We have the relation

$$c(\mathbb{P}) = c(M)c(\pi^* E(1)) \ .$$

Consider the cohomology ring $\mathrm{H}^*(M)$ as a subring of $\mathrm{H}^*(\mathbb{P})$. $\mathrm{H}^*(\mathbb{P})$ is a $\mathrm{H}^*(M)$-algebra generated by the Chern class

$$\eta = c_1(\mathcal{O}(1))$$

with the relation

$$c_d(\pi^* E(1)) = \sum_{k=0}^{r} c_k(E)\eta^{r-k} = 0 \ . \tag{7.2}$$

As $c(\pi^* E(1))$ is a projective invariant in $\mathrm{H}^*(\mathbb{P})$,(i.e. an invariant under changing E to $E \otimes L$), one can in principal derive all the projective invariants of E in $\mathrm{H}^*(M)$. For later purpose, let us work out the cases for $r = 2, 3$. For $r = 2$, we have

$$c_1(\pi^* E(1)) = c_1(E) + 2\eta \ .$$

Using (7.2) to eliminate η, we have the well-known projective invariant E in $\mathrm{H}^*(M)$:

$$i(E) := c_1(E)^2 - c_2(E) = c_1(\pi^* E(1))^2 \in \mathrm{H}^*(M) \ . \tag{7.3}$$

For $r = 3$, by

$$c_1(\pi^* E(1)) = c_1(E) + 3\eta \ , \quad c_2(\pi^* E(1)) = c_2(E) + 2c_1(E)\eta + 3\eta^2 \ , \tag{7.4}$$

we obtain the the projective invariant of E:

$$\begin{aligned} i_2(E) :=& c_1(E)^2 - 3c_2(E) = c_1(\pi^* E(1))^2 - 3c_2(\pi^* E(1)) \ , \\ i_3(E) :=& 2c_1(E)^3 - 9c_1(E)c_2(E) + 27c_3(E) = 2c_1(\pi^* E(1))^3 - 9c_1(\pi^* E(1))c_2(\pi^* E(1)). \end{aligned} \tag{7.5}$$

One can always express the Chern numbers of $\mathbb{P}$ in terms of those of M and projective invariants of E. We are going to derive the relations for $r = 2, 3$. For $r = 2$, we have

$$c_i(\mathbb{P}) = c_i(M) + c_{i-1}(M)(c_1(E) + 2\eta) \ ,$$

which implies $\chi(\mathbb{P}) = 2\chi(M)$ for $i = m + 1$. Using (7.3), we have

$$c_k(\mathbb{P})c_{m+1-k}(\mathbb{P}) = 2c_k(M)c_{m-k}(M) + 2c_{k-1}(M)c_{m+1-k}(M) \quad \text{for } 1 \leq k \leq m \ .$$

All the relations of Chern numbers for $r = 2, m = 2, 3$ are given by

$$\begin{aligned} m = 2 : & \begin{cases} c_2(\mathbb{P})c_1(\mathbb{P}) = & 2c_2(M) + 2c_1(M)^2 \\ c_1^3(\mathbb{P}) = & 6c_1(M)^2 + 2i(E) \ ; \end{cases} \\ m = 3 : & \begin{cases} c_3(\mathbb{P})c_1(\mathbb{P}) = & 2c_3(M) + 2c_2(M)c_1(M) \\ c_2(\mathbb{P})^2 = & 4c_2(M)c_1(M) \\ c_2(\mathbb{P})c_1(\mathbb{P})^2 = & 4c_2(M)c_1(M) + 2c_1(M)^3 + 2c_1(M)i(E) \\ c_1(\mathbb{P})^4 = & 8c_1(M)^3 + 8c_1(M)i(E) \end{cases} \end{aligned} \tag{7.6}$$

For $r = 3$, we have

$$c_i(\mathbb{P}) = c_i(M) + c_{i-1}(M)c_1(\pi^* E(1)) + c_{i-2}(M)c_2((\pi^* E)(1)) \ ,$$

which implies $\chi(\mathbb{P}) = 3\chi(M)$ for $i = m + 1$. By (7.5) we have

$$\begin{aligned} c_k(\mathbb{P})c_{m+2-k}(\mathbb{P}) =& 3c_k(M)c_{m-k}(M) + 3c_{k-2}(M)c_{m+2-k}(M) + 9c_{k-1}(M)c_{m+1-k}(M) \\ &+ c_{k-2}(M)c_{m-k}(M)i_2(E) \end{aligned}$$

The relations of Chern numbers for $r=3, m=2,3$ are given as follows:

$$m=2:\begin{cases} c_3(\mathbb{P})c_1(\mathbb{P}) = & 9c_2(M)+3c_1(M)^2 \ , \\ c_2(\mathbb{P})^2 = & 6c_2(M)+9c_1(M)^2+i_2(E) \ , \\ c_2(\mathbb{P})c_1(\mathbb{P})^2 = & 9c_2(M)+21c_1(M)^2+6i_2(E) \ , \\ c_1(\mathbb{P})^4 = & 54c_1(M)^2+27i_2(E) \ ; \end{cases}$$

$$m=3:\begin{cases} c_4(\mathbb{P})c_1(\mathbb{P}) = & 9c_3(M)+3c_2(M)c_1(M) \ , \\ c_3(\mathbb{P})c_2(\mathbb{P}) = & 9c_3(M)+12c_2(M)c_1(M)+c_1(M)i_2(E) \ , \\ c_3(\mathbb{P})c_1(\mathbb{P})^2 = & 9c_3(M)+18c_2(M)c_1(M)+3c_1(M)^3+6c_1(M)i_2(E) \ , \\ c_2(\mathbb{P})^2c_1(\mathbb{P}) = & 9c_1(M)^3+24c_2(M)c_1(M)+13c_1(M)i_2(E)-i_3(E) \ , \\ c_2(\mathbb{P})c_1(\mathbb{P})^3 = & 27c_2(M)c_1(M)+30c_1(M)^3+45c_1(M)i_2(E)-3i_3(E) \ , \\ c_1(\mathbb{P})^5 = & 90c_1(M)^3+135c_1(M)i_2(E)-9i_3(E) \ . \end{cases} \tag{7.7}$$

We now discuss the n-dimensional CY manifolds X which is either a hypersurface or a cyclic branched cover of a projective bundle $\mathbb{P}(E)$ over a complex manifold M. Such X is always an elliptic fibration over M. By Lemma 1 and 2, the Euler number $\chi(X)$ can be expressed by the Chern numbers of M and the projective invariants of E. By (7.6) and (7.7), we have the following results for $n=3,4$:

Proposition 1. Let X be a n-dimensional CY manifold.

(I) If X is a double cover of a projective bundle $\mathbb{P}$ associated to a rank 2 bundle E over a $(n-1)$-dimensional complex manifold M for $n=3,4$, then

$$\chi(X)=\begin{cases} -28c_1(M)^2-8i(E) & \text{for } n=3 \ , \\ 12c_2(M)c_1(M)+72c_1(M)^3+72c_1(M)i(E) & \text{for } n=4 \ . \end{cases}$$

(II) If X is a hypersurface of a projective bundle $\mathbb{P}$ associated to a rank 3 bundle E over a $(n-1)$-dimensional complex manifold M for $n=3,4$, then

$$\chi(X)=\begin{cases} -18c_1(M)^2-6i_2(E) & \text{for } n=3 \ , \\ 12c_2(M)c_1(M)+27c_1(M)^3+39c_1(M)i_2(E)-3i_3(E) & \text{for } n=4 \ \square \end{cases}$$

Remarks. (1) For $n=4$ in (I), by $12|c_2(M)c_1(M)$, we have

$$72|\chi(X) \ .$$

When $E=K_M^{-2}\oplus 1$, one obtains the formula (2.12) in [13].

(2) For $n=4$ and $E=$ the trivial bundle in (II), we have the following criterion for the integral property of $\frac{\chi(X)}{24}$:

$$24|\chi(X) \iff 8|c_1(M)^3 \ .$$

Note that above condition do not hold for $M=\mathbb{P}^1\times\mathbb{P}^2$, in which case, $c_1(M)^3=54$ and X is an elliptic CY 4-fold in $\mathbb{P}^1\times\mathbb{P}^2\times\mathbb{P}^2$ with $\chi(X)=1746$ $\square$

8. Elliptic CY manifolds with sections

In this section we consider the structure of elliptic CY n-fold $\pi : X \longrightarrow M$ with an involution σ, and a (holomorphic) section $s : M \longrightarrow X$. Here the involution σ means an order 2 automorphism of X commuting with π having the non-empty fixed points on the general fiber of π, and the section s will always assume its image $s(M)$ lying outside the critical points of π.

For a line bundle L over M, we shall denote $\overline{L}$ the $\mathbb{P}^1$-bundle $\mathbb{P}(L \oplus 1)$ over M,

$$\pi_0 : \overline{L} \longrightarrow M \ ,$$

M_0 the zero-divisor $\mathbb{P}(0 \oplus 1)$ and M_∞ the infinity-divisor $\mathbb{P}(L \oplus 0)$ in $\overline{L}$. As $\mathcal{O}(1)$ and $\pi_0^* L^{-1}$ are the line bundles associated to the divisor M_∞ and $-M_0 + M_\infty$ respectively, by (7.1) we have

$$K_{\overline{L}} = \pi_0^* K_M \otimes \mathcal{O}(-M_0 - M_\infty) \ .$$

Now set $L = K_M^{-2}$, and consider a smooth divisor D contained in L such that the restriction of π_0 defines a 3-fold branched covering over M. Then D is defined by the equation:

$$\xi^3 + a_1\xi^2 + a_2\xi + a_3 = 0 \ , \quad \xi \in L \ , \quad a_i \in \Gamma(M, L^i) \ \text{ for } i = 1, 2, 3. \tag{8.1}$$

Hence D is linearly equivalent to $3M_0$ in $\overline{L}$ and we have

$$K_{\overline{L}}^{-2} = \mathcal{O}(D + M_\infty) \ .$$

The double cover of $\overline{L}$ branched at $D + M_\infty$ becomes an elliptic CY n-fold over M, denoted by $Z(2)$, with the involution σ and the projection given by the following diagram:

$$\begin{array}{ccc} Z(2) & \longrightarrow & \overline{L} = Z(2)/ < \sigma > \\ \pi \downarrow & & \downarrow \pi_0 \\ M & = & M \ . \end{array}$$

The infinity-section of $\overline{L}$ over M induces a section of the fibration $Z(2)$ over M fixed by σ.

Proposition 2. The Euler number of $Z(2)$ is given by

$$\chi(Z(2)) = 2 \sum_{k=1}^{n-1} (-1)^{k-1} 6^k c_1(M)^k c_{n-1-k}(M)$$

Proof. We have

$$\chi(Z(2)) = 2\chi(\overline{K_M^{-2}}) - \chi(M_\infty) - \chi(D) = 3\chi(M) - \chi(D) \ .$$

We may assume $a_1 = a_2 = 0, a_3 \neq 0$ in the equation (8.1) of D. Hence D is a 3-fold cyclic cover of M branched at the zeros of a_3, which is a divisor in M for K_M^{-6}. By Lemma 1,

$$\chi(D) = 3\chi(M) + 2\sum_{k=1}^{n-1}(-6)^k c_1(M)^k c_{n-1-k}(M) \ ,$$

hence the result follows immediately □

For $L = K_M^{-1}$, and a smooth divisor D contained in L with the restriction of π_0 defining a 4-fold branched covering over M. Then D is linearly equivalent to $4M_0$ in $\overline{L}$, and

$$K_{\overline{L}}^{-2} = \mathcal{O}(D) \ .$$

Denote $Z(1)$ the double cover of $\overline{L}$ branched at D, and σ the involution. $Z(1)$ is an elliptic CY n-fold over M with the projection

$$\pi : Z(1) \longrightarrow M$$

induced by π_0. Since the infinity-section M_∞ of $\overline{L}$ does not intersect D, it gives rise to two disjoint sections of $Z(1)$ over M permuted by σ. With the similar argument in Propostion 2, we have the following result:

Proposition 3. The Euler number of $Z(1)$ is given by

$$\chi(Z(1)) = 3\sum_{k=1}^{n-1}(-1)^{k-1}4^k c_1(M)^k c_{n-1-k}(M) \ \square$$

Remarks

(1) The formulas of $\chi(Z(i))$ for small N are as follows:

$$\chi(Z(2)) = \begin{cases} -60c_1^2(M) & \text{for } \ n = 3 \ , \\ 12c_1(M)c_2(M) + 360c_1^3(M) & \text{for } \ n = 4 \ , \\ 12c_1(M)c_3(M) - 72c_1(M)^2c_2(M) - 2160c_1^4(M) & \text{for } \ n = 5 \ . \end{cases}$$

and

$$\chi(Z(1)) = \begin{cases} -36c_1^2(M) & \text{for } \ n = 3 \ , \\ 12c_1(M)c_2(M) + 144c_1^3(M) & \text{for } \ n = 4 \ , \\ 12c_1(M)c_3(M) - 48c_1(M)^2c_2(M) - 576c_1^4(M) & \text{for } \ n = 5 \ . \end{cases}$$

The above $\chi(Z(2)$ for $n=4$ is the formula in [13].
(2) When $M=\mathbb{P}^{n-1}$, $Z(2), Z(1)$ are the CY manifolds for the hypersurface $X_{6n}(\underbrace{1,\cdots,1}_{n},2n,3n)$ and $X_{4n}(\underbrace{1,\cdots,1}_{n},n,2n)$ respectively. In general, a CY n-fold $X_{6k}(w_1,\cdots,w_{n+2})$ with $\sum_{j=1}^{n} w_j = k$ and $w_{n+1} = 2k, w_{n+2} = 3k$ has the above $Z(2)$ structure with M as a non-singular toric variety dominating $\mathbb{P}(w_1,\cdots,w_n)$. Similarly the CY n-fold $X_{4k}(w_1,\cdots,w_{n+2})$ with $\sum_{j=1}^{n} w_j = k$ and $w_{n+1} = k, w_{n+2} = 2k$ for $Z(1)$. However the construction of $Z(2)$ can also be applied to a non-toric variety M, e.g. a del Pezzo surface.

The above elliptic CY fibration $Z(i)$ has the following characterization:

Proposition 4. Let X be an elliptic CY fibration over a complex manifold M with an involution σ such that all the fibers are irreducible.
(I) If there is a section of X over M fixed by σ, and $H^1(M, K_M^{-2})=0$, then X isomorphic to $Z(2)$ over M.
(II) If there exist two disjoint sections of X over M permuted by σ, and $H^1(M, K_M^{-1})=0$, then X isomorphic to $Z(1)$ over M.

Proof. Let π be the projection of X onto M, and s a section fixed by σ. The image $s(M)$ is a smooth divisor of X isomorphic to M. By the irreducibility of fibers of π, $\pi_*\mathcal{O}(2s(M))$ is a rank 2 vector bundle M, and denote its dual bundle by E. The section of $\pi_*\mathcal{O}(2s(M))$ determined by $2s(M)$ gives rise the trivial line sub-bundle of $\pi_*\mathcal{O}(2s(M))$, hence one has the extension

$$0 \longrightarrow L \longrightarrow E \longrightarrow 1 \longrightarrow 0$$

where L is a line bundle over M. The ratio of values of local sections of $\pi_*\mathcal{O}(2s(M))$ induces a double cover of X over $\mathbb{P}(E)$, in which $\mathbb{P}(L)$ lies as a component of the branched locus corresponding to $s(M)$. As the normal bundle of $s(M)$ in X is equal to s^*K_M, one obtains $L=K_M^{-2}$. By $H^1(M, K_M^{-2})=0$, the above extension of E splits and we have $E=K_M^{-2}\oplus 1$, hence (I) follows immediately. By the same argument one obtains (II) □

9. Topological correlation functions and mirror symmetry

The A and the B models are topological $N=2$ supersymmetric σ-models with a Calabi-Yau d-fold X as their target space. They correspond to two possibilities to twist the $N=2$, $c=3{\cdot}\mathrm{d}$ superconformal σ-model on the world-sheet [58]. The algebra of observable

(BRST invariants) of the A model is identified with the quantum deformation of the classical intersection algebra on $\mathcal{A} = \oplus_{p=0}^{n} H^p(X, \wedge^p T^*)$. More precisely the corresponding cubic forms has the form

$$Q(a,b,c) = \int_X a \wedge b \wedge c + \sum N_d(a,b,c) \frac{q^d}{1-q^d} \tag{9.1}$$

where $N_d(a,b,c)$ can be defined as certain intersection numbers on a moduli space of mappings. Here $q^d = q_1^{d_1} \cdots q_m^{d_m}$, $m = h^{1,1}(X)$, where $q_1, .., q_m$ are some local coordinates on the complexified Kähler cone of X. The series above is expected to converge for small $|q_i|$. The algebra of observables of the B model is identified with an algebra on[25] $\mathcal{B} = \oplus_{p=0}^{n} H^p(X, \wedge^p T) \sim \oplus_{p=0}^{n} H^p(X, \wedge^{d-p} T^*)$, whose structure constants can be analyzed using Griffith's transversality of the Gauss-Manin connection on the middle dimensional cohomology of X. Especially the marginal operators of the A and B model are identified with elements of $H^{1,1}(X)$ and $H^{d-1,1}(X)$ respectively. All correlations functions of the topological theories can be obtained from these structure constants or equivalently from the 2- and 3-point correlators. The 2-point correlator in a topological field theory is purely topological: in the present cases it is simply the Poincaré pairing on $\mathcal{A}$ or $\mathcal{B}$ respectively. Relative to any given base, we denote the matrix value of this inner product $\langle , \rangle : H^p(X) \otimes H^{d-p}(X) \to \mathbf{C}$ by $\eta^{\alpha\beta}_{(p)}$. It's inverse is denoted $\eta^{(p)}_{\alpha\beta}$. By the identification of the marginal operators, the 3-point correlators will depend on $h^{1,1}(X)$ complexified Kähler moduli in the A model and $h^{d-1,1}(X)$ complex structure moduli in the B model. Our goal here is to show *all 3-point correlators of the B model can be written explicitly in terms of the periods for the middle dimensional cohomology of the Calabi-Yau d-fold X.* We give explicit expression for the periods and 3-point correlators containing two marginal operators, which are a direct generalisation of the formulas in [26]. Using further properties of the Frobenius algebra one can derive explicit expressions for all correlators of the B-model on X from them. By mirror symmetry the formalism can therefore be used to obtain the A-model correlation functions on X, after suitable identification, from the B-model correlation functions on the mirror manifold X^*. This is in fact the main application we have in mind. For one moduli Calabi-Yau of arbitrary dimensions this was discussed in [59]. Some aspects of the generalization to multimoduli cases can be found in [60] [61][62],[29]. Here we generalize the $d = 4$ case treated in [29] to d-folds.

[25] The equivalence is due to the unique holomorphic $(d, 0)$-form Ω present on every Calabi-Yau d-fold.

9.1. The B-model algebra

Let $\pi : \mathcal{X} \to S$ be a family whose generic fiber is a Calabi-Yau n-fold X_z. One writes now the 3-point correlators as a cubic form on the groups $H^p(X_z, \wedge^p T)$. Put $\mathcal{B}_z = \oplus H^p(X_z, \wedge^p T)$. The cubic forms are defined by

$$C(a,b,c) = \int \Omega(a \wedge b \wedge c) \wedge \Omega \tag{9.2}$$

where $\Omega(a \wedge b \wedge c)$ is the contraction along the tangent direction producing an n-form on X_z.

Mirror symmetry provides a vector space isomorphism $\phi_z : \mathcal{B}_z \to \mathcal{A}$, a mapping $z \mapsto z(q)$ and a normalization $\frac{1}{f}$ such that near the large radius limit $q = 0$, we have

$$\frac{1}{f} C(\phi_z a, \phi_z b, \phi_z c) = Q(a,b,c)(q(z)), \tag{9.3}$$

where Q is the quantum corrected cubic form on $\mathcal{A}$. It's clear that Q should be independent of the choice of Ω. But C depends on Ω quadratically. Thus we expect that $\frac{1}{f}$ must be a holomorphic function near $q = 0$ which cancels this dependence. Near the large radius limit, there is a unique holomorphic period $\omega_0(z) = \int_\gamma \Omega(z)$. The choice $\frac{1}{f} = \frac{1}{\omega_0^2}$ therefore provides a natural resolution to this cancellation problem. Equivalently we can replace Ω by $\frac{1}{\omega_0}\Omega$ and set $f = 1$. This is what we shall do. We shall first fix a base point $0 \in S$, a topological base of homology cycles and the dual base $\gamma_a^{(p)}$ on $H^n(X_0)$ with the property that $\langle \gamma_a^{(p)}, \gamma_b^{(q)} \rangle = 0$ for $p+q \leq n$. For fixed p, the label a in $\gamma_a^{(p)}$ takes $h^{n-p,p}(X_0)$ different values. Due to mirror symmetry such a base will be the image of a base on $\mathcal{A}$ under ϕ_0. In fact in practice, there is usually a canonical choice of such a base on the A-model side.

There is a filtration of holomorphic vector bundles over S: $F_{(0)} \subset F_{(1)} \subset \cdots \subset F_{(n)}$, where the fiber over $z \in S$ of $F_{(k)}$ is the vector space $\oplus_{p=0}^k H^p(X_z, \wedge^p T)$. We now provide a set of frames for the these bundles. We shall express these frames as linear combinations in the base $\gamma_a^{(p)}$ with holomorphically varying coefficients. We shall see that these coefficients completely determine the cubic form C. For each k, let $\{\alpha^{(0)} := \Omega, \alpha_a^{(1)}, .., \alpha_b^{(k)}\}$ be a frame of $F_{(k)}$ having the following upper-triangular property with respect to the $\gamma_a^{(p)}$:

$$\alpha_a^{(k)} = \gamma_a^{(k)} + \sum_{p>k} g_a^{(p)c} \gamma_c^{(p)}. \tag{9.4}$$

(The $g^{(p)}$ actually depends on k, which we have suppressed in the notation above.) These frames can be obtained by row reduction on a given arbitrary base of sections. (See

[59].) Note that for $k = 0$ the coefficients $g^{(p)}$ are exactly the periods of the above given homology cycles. These periods are solutions to the Picard-Fuchs equations (in an appropriate gauge). We will give explicit formulas later for these periods for Calabi-Yau complete intersections in a toric variety. Note that in $\alpha^{(0)}$ the coefficients $t_a := g_a^{(1)}$ are regarded as local coordinates on S. These are the so-called flat coordinates. In these coordinates the Gauss Manin connection ∇_a becomes ∂_{t_a}, and the cubic form of type $(1, k, d-k-1)$ is given by

$$C_{a,b,c}^{(1,k,d-k-1)} = \int_X \alpha_a^{(d-k-1)} \wedge \partial_{t_a} \alpha_b^{(k)} =: \langle \partial_{t_a} \alpha_b^{(k)}, \alpha_c^{(n-k-1)} \rangle. \tag{9.5}$$

Using the upper-triangular property of the $\alpha_a^{(k)}$ and the topological basis $\gamma^{(k)}$, it is easy to show that

$$\eta_{ab}^{(k)} := \langle \alpha_a^{(k)}, \alpha_b^{(d-k)} \rangle = \langle \gamma_a^{(k)}, \gamma_b^{(n-k)} \rangle. \tag{9.6}$$

In particular these matrix coefficients are independent of t. Furthermore we claim that

$$\partial_{t_a} \alpha_b^{(k)} = C_{a,b,c}^{(1,k,d-k-1)} \eta^{cd}_{(d-k-1)} \ \alpha_d^{(k+1)}. \tag{9.7}$$

By Griffith's transversality, we have $\partial_{t_a} \alpha_b^{(k)} \in F_{(k+1)} = Span\{\alpha^{(0)}, .., \alpha_a^{(k+1)}\}$. But because of the upper triangular form of $\alpha_b^{(k)}$, $\partial_{t_a} \alpha_b^{(k)}$ has zero component along $\gamma^{(0)}, .., \gamma_a^{(k)}$. Thus it can be expressed as a linear combination (with holomorphically varying coefficients) of the $\alpha_b^{(k+1)}$. To determine the coefficients, we take its inner product with $\alpha_c^{(n-k-1)}$ and apply eqns (9.5), (9.6). The claim above then follows.

To summarize, our strategy for computing the A-model cubic form Q on X by mirror symmetry is as follows. Actually we will only do it for a Frobenius subalgebra $\mathcal{A}$ (see below) of the A-model algebra. First we fix a topological basis on $\mathcal{A}$ (In the case of toric hypersurfaces, this basis will come from toric geometry). We define our isomorphism ϕ_z so that it sends this basis to the holomorphically varying basis $\alpha_a^{(k)}$ of the B-model with $1 \mapsto \alpha^{(0)}$. Then we shall use eqns (9.5),(9.6) and (9.7) as our crucial ingredients for computing the B-model cubic forms C explicitly. For this we shall need some elementary theory of Frobenius algebras which we now discuss.

9.2. Frobenius algebras

In this section, all vector spaces are finite dimensional. A Frobenius algebra is a commutative graded algebra $A = \oplus_{i=0}^n A_{(i)}$, generated by $A_{(1)}$, has $A_{(0)} = \mathbf{C}\cdot 1$, and a nondegenerate degree n bilinear symmetric invariant pairing $\langle , \rangle : A \times A \to \mathbf{C}$. Note that because we require generation by $A_{(1)}$, this notion is slightly stronger than the usual notion of a Frobenius algebra. We give some well-known examples from geometry. Let $\mathbf{P}$ be a complete toric variety, and $A^*(\mathbf{P})$ be its Chow ring. Then $A^*(\mathbf{P})\otimes\mathbf{C}$ is a Frobenius algebra. The pairing here is the Poincaré pairing. If X is a hypersurface in $\mathbf{P}$, then it can be shown that the ring

$$\tilde{A}^*(X) := Im(A^*(\mathbf{P}) \to A^*(X)) = A^*(\mathbf{P})/Ann([X]) \tag{9.8}$$

tensored with $\mathbf{C}$ is a Frobenius algebra. More generally, if A is a Frobenius algebra, and $x \in A_{(1)}$ is a nonzero element, then $\tilde{A} := A/Ann(x)$ is a Frobenius algebra with the induced pairing $\langle a + Ann(x), b + Ann(x)\rangle := \langle a, b\cdot x\rangle$ having degree $n-1$.

Let V_1, V_2, V_3 be vector spaces, and $C : V_1 \otimes V_2 \otimes V_3 \to \mathbf{C}$ be a cubic form. It is call V_1-nondegenerate if that $C_{(a,b,c)} = 0$ for all b, c implies that $a = 0$. Similar notion of V_i-nondegeneracy applies. We call the form nondegenerate if it is V_i-nondegenerate for all i. Now suppose C is V_3-nondegenerate. Then we have the following invertibility property. Let $D : V_3^* \otimes V_4 \to \mathbf{C}$ be any bilinear form. Then the knowledge of the 3-form $E_{(a,b,d)} := C_{(a,b,c_i)} D_{(\gamma^i,d)}$ ($\{c_i\}, \{\gamma^i\}$ being dual bases), allows us to determine D completely. In fact, there exists (in general not unique) a 3-form F such that $D_{(\gamma,d)} = F_{(\gamma,\alpha^i,\beta^j)} E_{(a_i,b_j,d)}$. This is just the statement that the V_3-nondegenerate cubic form C defines an onto map $V_1 \otimes V_2 \to V_3^*$, hence choosing a section gives us a left inverse F to this map.

We now return to a Frobenius algebra A. it determines a collection of cubic forms $C^{(ijk)} : A_{(i)} \otimes A_{(j)} \otimes A_{(k)} \to \mathbf{C}$ with $i, j, k \geq 0, i + j + k = n$. These cubic forms are $A_{(i)}$-nondegenerate whenever either $j = 1$ or $k = 1$ because $A_{(1)} \cdot A_{(i)} = A_{(i+1)}$.

9.3. Reconstruction

Let $A = \oplus_{i=0}^n A_{(i)}$ be a graded space with $A_{(0)} = \mathbf{C}$ and equipped with a degree n nondegenerate symmetric bilinear form η. Suppose we are given cubic forms: $C^{(ijk)} : A_{(i)} \otimes A_{(j)} \otimes A_{(k)} \to \mathbf{C}$, $i, j, k \geq 0$ with the following properties:

(a) (Degree) $C^{(ijk)} = 0$ unless $i + j + k = d$.

(b) (Unit) $C^{(0ij)}_{(1,b,c)} = \eta^{(i)}_{b,c}$.

(c) (Nondegeneracy) $C^{(1ij)}$ is nondegenerate in the second slot.
(d) (Symmetry) For any permutation σ of 3 letters, $C^{(ijk)}_{(a,b,c)} = C^{\sigma(ijk)}_{\sigma(a,b,c)}$.
(e) (Associativity)

$$C^{(i,j,n-i-j)}_{(a,b,c_p)}\eta^{pq}_{(n-i-j)}C^{(i+j,k,n-i-j-k)}_{(d_q,e,f)} = C^{(i,k,n-i-k)}_{(a,e,c'_p)}\eta^{pq}_{(n-i-k)}C^{(i+k,j,n-i-j-k)}_{(d'_j,b,f)}$$

where the c and the d are bases of the appropriate spaces.

Then A is a Frobenius algebra with the product

$$a \cdot b = C_{(a,b,c_p)}\eta^{pq}d_q. \tag{9.9}$$

The rules above are known as fusion rules. One can also build a k-form by fusing together 2- and 3-forms. The associativity law says that there will often be many ways to build a given k-form. Similarly the 3-forms are not independent. We claim that the forms of type $(i, j, n-i-j)$ for $i, j > 1$ are determined by the those of type $(1, r, n-r-1)$. To see this without loss of generality, we can assume $1 < n-i-j \leq i, j$. Now by the associativity law above with $k = n-i-j-1$ and the invertibility property of $C^{(i+j,k,n-i-j-k)} = C^{(i+j,k,1)}$, it follows that $C^{(i,j,n-i-j)}$ are determined in terms of forms of type $(i, n-i-j-1, j+1)$ and $(i+k, j, 1)$. By the symmetry property, $(i, n-i-j-1, j+1)$ is equivalent to $(i, j+1, n-i-j-1)$. Thus we have reduced the value of $n-i-j$ by 1. By induction, we see that all $(i, j, n-i-j)$ can be expressed in terms of those of type $(1, r, n-r-1)$. In terms of the algebra A itself, an alternative way to state the result is that all the products $A_{(i)} \otimes A_{(j)} \to A_{(i+j)}$ is determined by those of the form $A_{(1)} \otimes A_{(r)} \to A_{(r+1)}$ because A is generated by $A_{(1)}$ and that

$$(a_1 \cdots a_i)(a_{i+1} \cdots a_{i+j}) = a_1(a_2 \cdots a_{i+j}). \tag{9.10}$$

9.4. Application

Let X be a Calabi-Yau n-fold, and let $\mathcal{A}$ be a Frobenius subalgebra of $\oplus_{p=0}^n H^p(X, \wedge^p T^*)$. Suppose mirror symmetry holds: there is a mirror family X^* whose B-model algebra coincides with the A-model algebra of X. We shall now compute the Frobenius subalgebra $\mathcal{B}$ of the B-model algebra corresponding to $\mathcal{A}$. From our general discussion of Frobenius algebras, it is enough to compute the cubic forms C of types $(1, r, n-r-1)$ which come with $\mathcal{B}$. Once we have a period expansion in the topological base (9.4) these can be easily obtained using eqns (9.5), (9.6)and (9.7). To obtain the coefficients in (9.4) we will use the fact[63][27]that the universal structure of the solution of the Picard-Fuchs equation on X^* at the large radius point mirrors the primitive part of the vertical cohomology of X and the leading structure of logarithm enables us to associate this solutions with the expansion of the periods in a topological base. This leads to a direct generalisation of the formulas of [63] to some correlation functions on d-folds.

More precisely there are $h^{r,r}_{prim}(X)$ solutions $0 < r < d$ with leading degree r in the $\log(z_i)$, which have the form

$$\tilde{\Pi}^{(r)}_k = \sum_{\Pi} {}^0C^{d-r,1\ldots 1}_{k,i_1,\ldots,i_r}\big(\frac{1}{r!}l_{i_1}\ldots l_{i_r}S_0 + \frac{1}{(r-1)!}l_{i_1}\ldots l_{i_{r-1}}S_{i_r} + \ldots + S_{i_1,\ldots,i_r}\big), \qquad (9.11)$$

here we defined $l_i := \log(z_i)$ and the $S_{i_1,\ldots i_r}$ are holomorphic series in the z_i, whose explicit form are given below. The map to an specific element of the cohomology $H^{d-r,d-r}$ of X can be made precise by noting that the ${}^0C^{d-r,1\ldots 1}_{k,i_1,\ldots i_r}$ are given by the classical intersection of that specific element with the intersection of divisors $J_{i_1}\cdot\ldots\cdot J_{i_r}$. We discuss the primitive part of the (co)homology generated by $J_1\ldots J_{h^{1,1}}$ only and by Poincare duality, this data fix the element in $H^{d-r,d-r}$ completely.

As mentioned above the covariant derivative ∇_a in [59] becomes the ordinary derivative in the flat complexified Kähler structure coordinates t_k. The coordinate change from the natural complex structure coordinates z_a to the t_k variables is given by the mirror map $t_k = \frac{\tilde{\Pi}^{(1)}_k(z_i)}{\tilde{\Pi}^{(0)}(z_i)} = \log(z_k) + \frac{S_k}{S_0}$. If we substitute this coordinate transformation in the normalized periods $\Pi^{(r)}_i = \frac{\tilde{\Pi}^{(r)}_i}{\tilde{\Pi}^{(0)}}$ some simplifications occur as the first subleading terms in the t_i cancel out:

$$\Pi^{(r)}_k = \sum_{\Pi} {}^0C^{d-r,1\ldots 1}_{k,i_1,\ldots,i_r}\big(\frac{1}{r!}t_{i_1}\ldots t_{i_r} + \frac{1}{(r-2)!}t_{i_1}\ldots t_{i_{r-2}}\hat{S}_{i_{r-1}}\hat{S}_{i_r} + \ldots + \hat{S}_{i_1,\ldots,i_r}\big). \quad (9.12)$$

Now we notice from the monodromy around $z_i = 0$ ($t_i \to t_i + 1$) that the periods $\Pi^{(r)}_k$ correspond to a expansion of $\alpha^{(0)} = \Omega$ in terms of the topological basis[26] $\gamma^k_{(r)}$ of (9.4) $\alpha^{(0)} = \sum_{k,r}\Pi^{(r)}_k\gamma^k_{(r)}$.

The coupling $C^{(1,1,d-2)}_{a,b,c} : H^{1,1}\times H^{1,1}\times H^{d-2,d-2} \to \mathbb{C}$ is especially simple to obtain. Applying (9.7) in the case $k = 0$ we have $\partial_{t_a}\alpha^{(0)} = \alpha^{(1)}_a$. This determines $\alpha^{(1)}_a$, hence

[26] This is actually only true up to the addition of solutions with subleading logarithms, which however does not affect the holomorphic couplings discussed below. It will affect however the non-holomorphic Weil-Peterson metric.

all its coefficients. Now using (9.5) for $k=1$, (9.4) for $k=1,d-2$, and the fact that $\langle\gamma_a^{(k)},\gamma_b^{(l)}\rangle=0$ for $k+l>d$, we see that

$$C^{(1,1,d-2)}_{a,b,c}=\partial_{t_a}g_b^{(2)d}\eta^{(2)}_{dc}=\partial_{t_b}\partial_{t_b}\Pi^{(2)}_c, \tag{9.13}$$

where the $g^{(2)}$ are the coefficients of the $\gamma^{(2)}$ in the $\alpha^{(1)}$. Note that the last equation follows from the fact that $\Pi_a^{(r)}$ is an expansion in the dual base $\gamma^a_{(r)}$ and that the associativity of the classical parts in (9.13) is manifest. Eqs. (9.12)(9.13) are direct generalizations of eqs. (4.9) and (4.18) to the d-fold case. For $d=4$ an equivalent description has been given in [29]. For $H^{1,1}$ we have always a canonical choice of the basis say $J_1\dots J_{h^{1,1}}$, as there is a canonical basis for the tangent space of the moduli space corresponding to elements $H^{d-1,1}(X^*)$, which is mapped by the monomial divisor mirror map to $H^{1,1}(X)$ and (9.13) reduces for $d=3$ to the expressions given in [63]. For $d>3$ there is a priori no canonical choice for the basis of $H^{d-2,d-2}$. However toric geometry can be used as in [27] to show that the graded ring

$$\mathcal{R}=\mathbb{C}[\theta_1,\dots,\theta_{h^{1,1}}]/\mathcal{J},$$

where $\mathcal{J}$ is the ideal generated by the leading θ-terms of Picard-Fuchs equations, gives, by the identification $\theta_i\to J_i$, a presentation of the primitive part of $H^{*,*}$. Because of Poincare duality it is of course sufficient to pick a basis of half of $H^{*,*}$ and as mentioned above the choice of the basis in $H^{1,1}$ is canonical. It was shown in [63][27] that any element of $\mathcal{R}$ can be mapped to a solution (9.11), i.e. the ${}^0C^{d-r,1\dots1}_{i_1,\dots,i_r}$ are determined by the principal part of the Picard-Fuchs equation. This can be viewed as a proof of mirror symmetry at the level of the classical intersections, which readily generalizes to d-folds.

Now proceed by induction. Suppose we know (the coefficients of) the $\alpha_{(i)}$ and the cubic forms of types $(1,i,n-i-1)$ for $i=0,1,..,k$. Then by the invertibility property of a cubic form of type $(1,k,n-k-1)$ in a Frobenius algebra, we can solve for the $\alpha_{(k+1)}$ using (9.7). Thus the $\alpha^{(k+1)}$ are determined. By (9.4), we can write $\partial_{t_a}\alpha_b^{(k+1)}=\partial_{t_a}g_b^{(k+2)d}\gamma_d^{(k+2)}+\cdots$ (which is now known), arguing as before using (9.5) with k replaced by $k+1$, and using the inner product property of the γ, we find that $C^{(1,k+1,n-k-2)}_{abc}=\partial_{t_a}g_b^{(k+2)d}\eta^{(k+2,n-k-2)}_{dc}$. Thus the cubic form of type $(1,k+1,n-k-2)$ is also determined. This shows that all cubic forms of type $(1,k,n-k-1)$ for $k=1,2,..,n-1$ can be expressed in terms of the coefficients of $\alpha_{(0)}$ alone.

9.5. Explicit expressions for periods and instanton sums for complete intersections in toric varieties

Following [63] we can determine the holomorphic series $S_{i_1,\ldots,i_r}$ from the generators of the Mori cone. Consider a Calabi-Yau d-fold defined as complete intersection with p polynomial constraints in a toric variety of dimension $d+p$. The generators of the Mori cone will be of the form

$$l^{(i)} = (\hat{l}_0^{(i)}, \ldots, \hat{l}_{p-1}^{(i)}; l_1^{(i)}, \ldots, l_q^{(i)}),$$

where $q = d + p + h^{d-1,1}$. The series $S_{i_1,\ldots,i_r}$ are obtained by the Frobenius method from the coefficients of the holomorphic function $\omega(\vec{z}, \vec{\rho})$

$$\begin{aligned} \omega(z, \vec{\rho}) &= \sum c(\vec{n}, \vec{\rho}) \prod_{j=1}^{h^{1,D-1}} z_j^{n_j+\rho_j} \\ c(\vec{n}, \vec{\rho}) &= \frac{\prod_{k=1}^{p} \Gamma(1 - \sum_{i=1}^{h^{1,D-1}} \hat{l}_k^{(i)}(n_i + \rho_i))}{\prod_{k=1}^{q} \Gamma(1 - \sum_{i=1}^{h^{1,D-1}} l_k^{(i)}(n_i + \rho_i))} \\ S_{i_1,\ldots,i_r} &= \partial_{\rho_{i_1}} \ldots \partial_{\rho_{i_r}} \omega(\vec{z}, \vec{\rho})|_{\vec{\rho}=\vec{0}} \end{aligned}$$

Notably with leading behavior $S_0 = 1 + \ldots$, $S_i = z_i + \ldots$.

This gives the explicit expansion of $C_{A,b,c}^{(d-2,1,1)} =^0 C_{A,b,c}^{(d-2,1,1)} + \mathcal{O}(q_i)$, with $q_i = e^{t_i}$. The latter has a conjectural interpretation as being the counting function for invariants of maps from the two sphere into X. These maps are defined such that two fixed points P_b, P_c are mapped to the divisors $\mathcal{D}_b$, $\mathcal{D}_c$, while one point P_A is mapped to the codimension r subvariety A in a class of $H^{r,r}(X)$. And the invariant is the Euler class of the moduli space of that curve, weighted by $(-1)^{\dim \mathcal{M}}$. From the definition of the degree a generic rational curves of degree d_l will pass through the divisor $\mathcal{D}_l$ in d_l points, but a generic curve does not pass through the submanifold A of higher codimension then one. If we require the latter this imposes a restriction and the invariants of that specific curves will be labeled by the class of A. Moreover in the path integral definition of $C_{A,b,c}^{(d-2,1,1)}$ one integrates over the points P_i and has accordingly to divide by a combinatorial factor of $d_b d_c$ in order to extract the invariant for the elementary rational curves $n_{\vec{d}}^{(A)}$ from the three-point function. By a similar reasoning as in [64] is was described in [59] how to subtract the multiple wrapping contributions from the lower degree curves in order to get the invariants of the elementary curves at given multidegree $\vec{d}$. Taking both effects into account the expansion of the three-point function in terms of invariants $n_{\vec{d}}$ is as follows[27]

$$C^{(d-2,1,1)}_{A,b,c} =^0 C^{(d-2,1,1)}_{A,b,c} + \sum_{\vec{d}} \frac{d_a d_b n^{(A)}_{\vec{d}}}{1-\prod_{i=1}^{h^{1,1}} q_i^{d_i}} \prod_{i=1}^{h^{1,1}} q_i^{d_i}. \qquad (9.14)$$

9.6. Examples of the quantum cohomology rings and transitions

Let us discuss as the simplest example case (1) of chapter 5, the elliptic fibration with $X_6(1,2,3)$ fibre over $\mathbb{P}^3$ and its transition by the blow up at an equivariant fix point in $\mathbb{P}^3$ to model (3) and along the irreducible subvariety to model (5). Evaluation of the explicit quantum cohomology in other cases can be found in [29].

The toric representation of the mirror of (1) is defined by (4.1) were Δ^*, is given by (5.1) as the convex hull of the following points

$$\begin{aligned} \nu_0^* &= (\ 0,\ 0,\ 0,\ 0,\ 0) \\ \nu_1^* &= (\ 1,\ 0,\ 0,\ 0,\ 0) \\ \nu_2^* &= (\ 0,\ 1,\ 0,\ 0,\ 0) \\ \nu_3^* &= (\ 0,\ 0,\ 1,\ 0,\ 0) \\ \nu_4^* &= (-1,-1,-1,-8,-12) \\ \nu_6^* &= (\ 0,\ 0,\ 0,\ 1,\ 0) \\ \nu_7^* &= (\ 0,\ 0,\ 0,\ 0,\ 1) \\ \nu_8^* &= (\ 0,\ 0,\ 0,-2,-3). \end{aligned} \qquad (9.15)$$

The manifold itself can be described by considering the vanishing of the Newtonpolynom of the polar polyhedron Δ in P^*_Δ. It turns out to be a degree 24 Fermat hypersurface in a weighted projective space $X_{24}(1,1,1,1,8,12)$.

There is a unique triangulation of the polyhedron Δ^* from its origin $\nu_0^* = (0,0,0,0,0)$. Note that the points $\nu_1^*, \nu_2^*, \nu_3^*, \nu_4^*, \nu_7^*$ all lie on a codim 2 face of Δ^*, with ν_7 the interior point of that face, while the points $\nu_5^*, \nu_6^*, \nu_7^*$ and ν_0^* lie on a codim 3 plane, which cuts the polyhedron. The two linear relation implied by this lead to the two generators of the Mori cone.

[27] For all toric varieties these invariants can be calculated with a updated version of the program INSTANTON (which is available on request) from the Mori generators and the classical intersections.

$$l^{(1)} = (\ \ 0;1,1,1,1,0,0,-4)$$
$$l^{(2)} = (-6;0,0,0,0,2,3,\ \ 1)$$

The two Kähler classes J_1, J_2 dual to this Mori generators measure classically the volume of the base $\mathbb{P}^3$ and the size of the fiber respectively. While the the divisor D_1 associated to the first Mori cone represents the section and is horizontally, D_2 is a vertical divisor, which intersects the base $\mathbb{P}^3$ in codim 2. Since three planes do not intersect generically in $\mathbb{P}^3$ the classical 4-point coupling $\mathcal{D}_1 \cdot \mathcal{D}_1 \cdot \mathcal{D}_1 \cdot \mathcal{D}_2 = \int J_1^3 J_2$ is zero. The other classical 4-point couplings $\int J_i J_k J_l J_m$ and the evaluation $\int c_2 J_i J_k$, and $\int c_3 J_i$ are summarized by the coefficients in the following formal polynomials

$$\begin{aligned}
\mathcal{C}_0 &= J_2 J_1^3 + 4J_2^2 J_1^2 + 16 J_2^3 J_1 + 64 J_2^4 \\
\mathcal{C}_2 &= 48 J_1^2 + 182 J_1 J_2 + 728 J_2^2 \\
\mathcal{C}_3 &= -960 J_1 - 3860 J_2
\end{aligned}$$

The Picard-Fuchs equations for the mirror manifold are

$$\begin{aligned}
\mathcal{L}_1 &= \theta_1^4 - (4\theta_1 - \theta_2 - 4)(4\theta_1 - \theta_2 - 3)(4\theta_1 - \theta_2 - 2)(4\theta_1 - \theta_2 - 1) z_1 \\
\mathcal{L}_2 &= \theta_2(\theta_2 - 4\theta_1) - 12(6\theta_2 - 5)(6\theta_2 - 1) z_2
\end{aligned},$$

have the following discriminant

$$\begin{aligned}
\Delta_1 &= (1 - 256 z_1) \\
\Delta_2 &= (1 - 432 z_2)^4 - z_1 z_2^4.
\end{aligned}$$

The mirror map $z_2(q_1 = 0, q_2) = P(J(t_2))$ is defined by the ratio of two periods of holomorphic 1-form on the elliptic curve $X_6(1,2,3)$, while mirror map $z_1(q_1, q_2 = 0)$ is described by the ratio of periods over a meromorphic differential on the K_3 surface $X_4(1,1,1,1)$.

The basis of $H^{1,1}$ are denoted by $J_1, \ldots, J_r$. We choose then a basis of $H^{2,2}$

$$\begin{aligned}
b_1^{(2)} &= J_1^2 \\
b_2^{(2)} &= J_1 J_2 + 4 J_2^2.
\end{aligned}$$

The intersection matrix between elements of $H^{2,2}$ in this basis is

$$\eta_{(2,2)} = \begin{pmatrix} 0 & 17 \\ 17 & 1156 \end{pmatrix}.$$

If we determine the basis of $H^{(3,3)}$ by the requirement that Poincarè bilinear pairing takes the simplest form $\eta^{i,j}_{(1,3)} = \delta^{i,h^{1,1}-i+1}$ with the canonical basis of $H^{1,1}$, then we get

$$\begin{aligned} b_1^{(3)} &= J_1^3 \\ b_2^{(3)} &= \frac{1}{273}(J_1^2 J_2 + 4J_1 J_2^2 + 16J_2^3) - 4J_1^3. \end{aligned}$$

The basis for $H^{4,4}$ is fixed up to a volume normalization of the d-fold, which we choose so that $\eta^{1,1}_{0,d} = 1$. In our case above $b^{(4)} = \frac{1}{75}\mathcal{C}_0$.

The leading order logarithms in the periods are according to (9.11)

$$\begin{aligned} \Pi_1^{(2)} &= S_0(l_1 l_2 + 2l_1^2) + \mathcal{O}(l) \\ \Pi_2^{(2)} &= S_0(\frac{17}{2}l_1^2 + 68l_1 l_2 + 136l_2^2) + \mathcal{O}(l). \end{aligned}$$

The invariants for the genus zero curves from the normalized three-point functions listed in the two tables below

$b_1^{(2)} = J_1^2$, $\frac{1}{20}C^{(2,1,1)}_{1,i,j}$:

m	$n^{(1)}_{0,m}$	$n^{(1)}_{1,m}$	$n^{(1)}_{2,m}$	$n^{(1)}1_{3,m}$	$n^{(1)}_{4,m}$
0	0	0	0	0	0
1	−1	384	−90000	13919744	31152804996
2	−41	24576	−7990080	1785169920	−301991420880
3	−3403	2812800	−1230118560	369021660288	−84154079407488
4	−374322	397171200	−219729224832	83117668597760	−23932769831261760
5	−48251945	62575303680	−41951914533360	19174105171468800	−6670224866876828160

$b_2^{(2)} = J_1 J_2 + 4J_2^2$, $\frac{1}{16320}C^{(2,1,1)}_{2,i,j}$:

m	$n^2_{0,m}$	$n^2_{1,m}$	$n^2_{2,m}$	$n^2_{3,m}$	$n^2_{4,m}$
0	0	1	2	3	4
1	0	6	−1893	439256	2661669198
2	0	189	−102750	31221300	−6618229812
3	0	14366	−11162250	4632513522	−1326773710832
4	0	1518750	−1537867338	816075268892	−297124091742240
5	0	191238192	−238866784083	154724059936392	−68479975849390752

Adding of the point $\nu_5^* = (0, -1, -1, -6, -9)$ correspond to an blow up of $\mathbb{P}^3$ along an $\mathbb{P}^1$ and leads to model (5). This transition has a close similarity to the transition by shrinking (blowing) a Del Pezzo surface studied in [3][42] as in the fourfold a *six-cycle* shrinks along the E_8 Del Pezzo[28] surface to T invariant orbit in the base. In fact we will see the E_8 partition function

$$\hat{\Lambda}_{E_8} = \frac{1}{2}\sum_{\alpha=even}\frac{\theta^8_\alpha(\tau)}{\eta(\tau)^{12}} = 1 + 252q + 5130q^2 + \ldots$$

[28] Similarly one can observe the shrinking of E_7, E_6, (D_5) Del Pezzo surface in the corresponding fibrations types.

appearing as counting functional of the instantons in the appropriate normalized threepoint functions, marked by the $*$ in the table below (as well as the higher degree invariants of the shrinking Del Pezzo, marked with the $\diamond$). This model has two phases and in the first the Stanley Reisner ideal is given by $\mathcal{S} = \{x_2x_5, x_1x_3, x_1x_3x_4, x_6x_7x_8\}$. The Mori generators below correspond to the classes of the curve in the $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus_{\mathbb{P}^1} (1))$ bundle (2), a section of the $\mathbb{P}^1$ base in this bundle (1) and the class of the elliptic fibre over B (3):

$$\begin{aligned} l^{(1)} &= (0; 0, 1, 0, -1, 1, 0, 0, -1), \\ l^{(2)} &= (0; 1, 0, 1, 1, 0, 0, 0, -3), \\ l^{(3)} &= (-6; 0, 0, 0, 0, 0, 2, 3, 1), \end{aligned} \tag{9.16}$$

The classical couplings are

$$\begin{aligned} \mathcal{C}_0 &= J_1J_2^2J_3 + J_2^3J_3 + 3J_1J_2J_3^2 + 4J_2^2J_3^2 + \\ &\quad 9J_1J_3^3 + 15J_2J_3^3 + 54J_3^4 \\ \mathcal{C}_2 &= 36J_1J_2 + 102J_1J_3 + 48J_2^2 + 172J_2J_3 + 618J_3^2 \\ \mathcal{C}_3 &= -540J_1 - 900J_2 - 3258J_3. \end{aligned} \tag{9.17}$$

Analogous as in [3] one has to flop the $\mathbb{P}^1$ in B first. As such flops were not discussed in the fourfolds context let us give the data of this transition to the second phase whose Mori generators are $l'^{(1)} = -l(1)$, $l'^{(2)} = l^{(1)}+l^{(2)}$ and $l'^{(3)} = l^{(1)}+l^{(3)}$. The Stanley Reisner Ideal changes to $\mathcal{S} = \{x_4x_8, x_1x_3x_4, x_6x_7x_8, x_1x_2x_3x_5, x_2x_5x_6x_7\}$ while the classical couplings become

$$\begin{aligned} \mathcal{C}_0 &= J'_3J'^3_2 + 4J'^2_2J'^2_3 + 15J'^3_3J'_2 + 54J'^4_3 + J'_1J'^3_2 + 4J'_1J'^2_2J'_3 + 16J'_1J'^2_3J'_2 \\ &\quad + 60J'_1J'^3_3 + 4J'^2_1J'^2_2 + 16J'_2J'_3J'^2_1 + 64J'^2_1J'^2_3 + 16J'^3_1J'_2 + 64J'^3_1J'_3 + 64J'^4_1 \\ \mathcal{C}_2 &= 48J'^2_2 + 172J'_2J'_3 + 182J'_2J'_1 + 618J'^2_3 + 688J'_3J'_1 + 728J'^2_1 \\ \mathcal{C}_3 &= -900J'_2 - 3258J'_3 - 3620J'_1. \end{aligned} \tag{9.18}$$

The positive scaling relations on the variables $x_1, \ldots, x_8$ are

$$\begin{aligned} &(-18; 1, 0, 1, 1, 0, 6, 9, 0), \\ &(-24; 1, 1, 1, 0, 1, 8, 12, 0), \\ &(-6; 0, 0, 0, 0, 0, 2, 3, 1), \end{aligned} \tag{9.19}$$

and the Weierstrass form

$$x_7^2 = x_6^3 + x_6 x_8^4 \sum_{\mu,\nu,\rho} x_1^\mu x_3^\rho x_2^\nu x_5^{16-\mu-\nu-\rho} x_4^{12-\mu-\rho} + x_8^6 \sum_{\mu,\nu,\rho} x_1^\mu x_3^\rho x_2^\nu x_5^{24-\mu-\nu-\rho} x_4^{18-\mu-\rho}.$$

The singularity at D_4, near $x_2 = x_5 = 0$ and along (x_1, x_3) is recognized as the *canonical singularity with crepant blowup* which signals the collapse of the E_8 Del Pezzo surface [3] and is smoothed to a generic member of the family $X_{24}(1,1,1,1,8,12)$ by perturbing with those terms, which were forbidden by the first scaling relation. This completes the transition to the fibration over $\mathbb{P}^3$.

With the choice of basis

$$b_1^{(2)} = J_1 J_2, \quad b_2^{(2)} = J_1 J_3 + J_3^2, \quad b_3^{(2)} = J_2^2, \quad b_4^{(2)} = J_2 J_3 + 3J_3^2, \tag{9.20}$$

we have the following data for the quantum cohomology ring

$$\eta_{2,2} = \begin{pmatrix} 0 & 3 & 0 & 10 \\ 3 & 72 & 5 & 207 \\ 0 & 5 & 0 & 13 \\ 10 & 207 & 13 & 580 \end{pmatrix}$$

$b_1^{(2)} = J_1 J_2,\ C^{(2,1,1)}_{1,i,j}$:

m	$n^{(1)}_{m,0,0}$	$n^{(1)}_{m,0,1}$	$n^{(1)}_{m,0,2}$	$n^{(1)}_{m,0,3}$	$n^{(1)}_{m,0,4}$	$n^{(1)}_{m,1,0}$	$n^{(1)}_{m,1,1}$	$n^{(1)}_{m,1,2}$	$n^{(1)}1_{m,2,0}$	$n^{(1)}_{m,2,1}$	$n^{(1)}_{m,2,2}$
0	0	0	0	0	0	3	-1080	143370	-12	5400	-1149120
1	1^*	252^*	5130^*	54760^*	419895^*	-19	6840	-1578960	344	-182520	5206830
2	0	0	$-2\cdot 9252^\diamond$	$-2\cdot 673760^\diamond$	$-2\cdot 20534040^\diamond$	1	-360	156060	-798	447480	-140472720

$b_2^{(2)} = J_1 J_3 + J_3^2,\ \frac{1}{12} C^{(2,1,1)}_{(2,i,j)}$:

m	$n^{(2)}_{m,0,0}$	$n^{(2)}_{m,0,1}$	$n^{(2)}_{m,0,2}$	$n^{(2)}_{m,1,0}$	$n^{(2)}_{m,1,1}$	$n^{(2)}_{m,1,2}$	$n^{(2)}_{m,2,0}$	$n^{(2)}_{m,2,1}$	$n^{(2)}_{m,2,2}$
0	315	630	945	0	-630	167265	0	1575	-670320
1	0	249	9495	0	1890	-577485	0	34020	16320375
2	0	0	-17268	0	0	56970	0	59535	-31350510

$b_3^{(2)} = J_2^2,\ \frac{1}{2} C^{(2,1,1)}_{3,i,j}$:

m	$n^{(3)}_{m,0,0}$	$n^{(3)}_{m,0,1}$	$n^{(3)}_{m,0,2}$	$n^{(3)}_{m,1,0}$	$n^{(3)}_{m,1,1}$	$n^{(3)}_{m,1,2}$	$n^{(3)}_{m,2,0}$	$n^{3}_{m,2,1}$	$n^{(3)}_{m,2,2}$
0	0	0	0	4	-1260	236520	-19	7920	-1624950
1	0	0	0	-10	3600	-831600	256	-133560	38111040
2	0	0	0	0	0	20520	-410	230400	-72511020

$b_4^{(2)} = J_2 J_3 + 3J_3^2,\ \frac{1}{12} C^{(2,1,1)}_{4,i,j}$:

m	$n^{(4)}_{m,0,0}$	$n^{(4)}_{m,0,1}$	$n^{(4)}_{m,0,2}$	$n^{(4)}_{m,1,0}$	$n^{(4)}_{m,1,1}$	$n^{(4)}_{m,1,2}$	$n^{(4)}_{m,2,0}$	$n^{(4)}_{m,2,1}$	$n^{(4)}_{m,2,2}$
0	0	885	1770	0	-1770	469935	0	4425	-1883280
1	0	489	18945	0	-5310	-1606995	0	-95580	45813825
2	0	0	-34383	0	0	113670	0	167265	-87245010

The blow up to (3), which is the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O})_{\mathbb{P}^2} \otimes \mathcal{O}(1)_{\mathbb{P}^2}$ over $\mathbb{P}^2$ is described torically by adding the point $\nu_5^* = (0,-1,0,-4,-6)$ to (9.15). For this case we have the Mori generators:

$$\begin{aligned} l^{(1)} &= (0;1,0,1,-1,1,0,0,-2), \\ l^{(2)} &= (0;0,1,0,1,0,0,0,-2), \\ l^{(3)} &= (-6;0,0,0,0,0,2,3,1), \end{aligned} \tag{9.21}$$

The associated Kähler classes control the volume of the $\mathbb{P}^2$, the volume of the $\mathbb{P}^1$ fibre and the volume of the elliptic fibre. The Picard-Fuchs equations are :

$$\begin{aligned} \mathcal{L}_1 &= -\theta_1^3 - (-1+\theta_1-\theta_2)(-2+2\theta_1+2\theta_2-\theta_3)(-1+2\theta_1+2\theta_2-\theta_3)z_1 \\ \mathcal{L}_2 &= \theta_2(-\theta_1+\theta_2) - (-2+2\theta_1+2\theta_2-\theta_3)(-1+2\theta_1+2\theta_2-\theta_3)z_2 \\ \mathcal{L}_3 &= \theta_3(-2\theta_1-2\theta_2+\theta_3) - 12(-5+6\theta_3)(-1+6\theta_3)z_3 \end{aligned} \tag{9.22}$$

The classical couplings

$$\begin{aligned} \mathcal{C}_0 &= J_3J_1^2J_2 + J_3J_2^2J_1 + J_3J_2^3 + 2J_3^2J_1^2 + 4J_2J_1J_3^2 \\ &\quad + 4J_3^2J_2^2 + 12J_3^3J_1 + 16J_3^3J_2 + 56J_3^4 \\ \mathcal{C}_2 &= 24J_1^2 + 48J_1J_2 + 138J_1J_3 + 48J_2^2 + 182J_2J_3 + 640J_3^2 \\ \mathcal{C}_3 &= -720J_1 - 960J_2 - 3378J_3. \end{aligned} \tag{9.23}$$

show that there is also a K_3 fibration over the $\mathbb{P}^2$. Basis of $H^{2,2}$:

$$b_1^{(2)} = J_1^2, \quad b_2^{(2)} = J_1J_2 + J_2^2, \quad b_3^{(2)} = J_1J_3 + 2J_3^2, \quad b_4^{(2)} = J_2J_3 + 2J_3^2, \tag{9.24}$$

with

$$\eta_{2,2} = \begin{pmatrix} 0 & 0 & 4 & 5 \\ 0 & 0 & 18 & 18 \\ 4 & 18 & 274 & 284 \\ 5 & 18 & 284 & 292 \end{pmatrix}$$

The following invariants are read off from the normalized threepoint functions $b_1^{(2)} = J_1^2$, $C_{1,i,j}^{(2,1,1)}$:

m	$n^{(1)}_{m,0,0}$	$n^{(1)}_{m,0,1}$	$n^{(1)}_{m,0,2}$	$n^{(1)}_{m,1,0}$	$n^{(1)}_{m,1,1}$	$n^{(1)}_{m,1,2}$	$n^{(1)}1_{m,2,0}$	$n^{(1)}_{m,2,1}$	$n^{(1)}_{m,2,2}$
0	0	0	0	0	0	0	0	0	0
1	−1	240	141444	−14	5040	−1096200	51	22800	−5263920
2	1	−240	28200	−6	2640	−703800	−616	356160	−110457000

$b_2^{(2)} = J_1 J_2 + J_2^2,\ \frac{1}{2} C^{(2,1,1)}_{(2,i,j)}$:

m	$n^{(2)}_{m,0,0}$	$n^{(2)}_{m,0,1}$	$n^{(2)}_{m,0,2}$	$n^{(2)}_{m,1,0}$	$n^{(2)}_{m,1,1}$	$n^{(2)}_{m,1,2}$	$n^{(2)}_{m,2,0}$	$n^{(2)}_{m,2,1}$	$n^{(2)}_{m,2,2}$
0	0	0	0	−1	720	424332	0	0	1440
1	0	0	0	−20	7680	−1716840	−138	−62400	−15292440
2	0	0	0	0	0	0	−820	491520	−155976240

$b_3^{(2)} = J_1 J_3 + 2J_3^2,\ \frac{1}{24} C^{(2,1,1)}_{3,i,j}$:

m	$n^{(3)}_{m,0,0}$	$n^{(3)}_{m,0,1}$	$n^{(3)}_{m,0,2}$	$n^{(3)}_{m,1,0}$	$n^{(3)}_{m,1,1}$	$n^{(3)}_{m,1,2}$	$n^{(3)}_{m,2,0}$	$n^{3}_{m,2,1}$	$n^{(3)}_{m,2,2}$
0	0	310	620	0	310	501273	0	0	620
1	0	0	64710	0	1860	−586830	0	9300	−3818580
2	0	0	0	0	0	0	0	58590	−31852500

$b_4^{(2)} = J_2 J_3 + 2J_3^2,\ \frac{1}{24} C^{(2,1,1)}_{4,i,j}$:

m	$n^{(4)}_{m,0,0}$	$n^{(4)}_{m,0,1}$	$n^{(4)}_{m,0,2}$	$n^{(4)}_{m,1,0}$	$n^{(4)}_{m,1,1}$	$n^{(4)}_{m,1,2}$	$n^{(4)}_{m,2,0}$	$n^{(4)}_{m,2,1}$	$n^{(4)}_{m,2,2}$
0	0	130	260	0	130	235266	0	0	260
1	0	−260	69030	0	−2080	761670	0	−7020	3118050
2	0	320	640	0	320	547029	0	0	640

Let us finally discuss the transition between the first two models in table (6.5). The four parameter model has as polyhedron the convex hull of

$$\begin{aligned}
&\nu_1^* = (-1,0,0,2,3),\ \nu_2^* = (0,-1,0,2,3),\ \nu_3^* = (0,0,0,0,-1),\ \nu_4^* = (0,0,0,-1,0)\\
&\nu_5^* = (0,0,0,2,3),\ \nu_6^* = (0,0,1,2,3),\ \nu_7^* = (1,1,3,2,3),\ \nu_8^* = (0,0,-1,2,3),\\
&\nu_9^* = (0,0,-1,1,2)
\end{aligned} \tag{9.25}$$

$$\begin{aligned}
l^{(1)} &= (-2;0,0,1,0,\quad 1,\quad 0,0,-2,\quad 2),\\
l^{(2)} &= (\ \ 0;1,1,0,0,\quad 0,-3,1,\quad 0,\quad 0),\\
l^{(3)} &= (\ \ 0;0,0,0,0,-2,1,\quad 0,\quad 1,\quad 0),\\
l^{(4)} &= (-2;0,0,1,1,\quad 0,\quad 0,0,\quad 1,-1).
\end{aligned} \tag{9.26}$$

$$\begin{aligned}
\mathcal{C}_0 &= 12J_2J_1^3 + 6J_2^2J_4^2 + 18J_1^2J_3^2 + 324J_1J_4^3 + 9J_1J_3^3 + 18J_4J_3^3 +\\
&\quad 54J_4^2J_3^2 + 162J_3J_4^3 + 72J_1^4 + 54J_2J_4^3 + 216J_1^2J_4^2 + 36J_3J_1^3 +\\
&\quad 144J_4J_1^3 + 2J_2^2J_1^2 + 6J_2J_4J_3^2 + 3J_2J_1J_3^2 + 36J_2J_1J_4^2 + 6J_2J_3J_1^2 +\\
&\quad 24J_2J_4J_1^2 + 18J_2J_3J_4^2 + 108J_1J_3J_4^2 + 2J_2^2J_3J_4 + J_2^2J_3J_1 + 4J_2^2J_1J_4 +\\
&\quad 36J_1J_4J_3^2 + 72J_4J_3J_1^2 + 486J_4^4 + 12J_2J_4J_3J_1\\
\mathcal{C}_2 &= 216J_3^2 + 582J_3J_4 + 408J_3J_1 + 72J_3J_2 + 1746J_4^2\\
&\quad + 1164J_4J_1 + 198J_4J_2 + 816J_1^2 + 138J_1J_2 + 24J_2^2\\
\mathcal{C}_3 &= -1674J_3 - 5076J_4 - 3366J_1 - 558J_2.
\end{aligned} \tag{9.27}$$

The transition to the three parameter model is described by the omission of the point ν_9^* from the polyhedron (9.25). The Mori generators of the three parameter model are $l^{(1')} = 2l^{(4)} + l^{(1)}$, $l^{(2')} = l^{(2)}$ and $l^{(3')} = l^{(3)}$. We have adapted our notation to [20], so that the indices of x_i are shifted by one to make place for the additional coordinate of the $\mathbb{P}^2$ (instead of $\mathbb{P}^1$) at x_1. The elliptic fibre has again type $(1, 0, 0, 2)$. The conic bundle at $D_9 = 0$ is $x_3^2 f_8 + x_4^2 + x_8^2 f_{20} + x_3 x_4 f_4 + x_3 x_8 f_{14} + x_4 x_8 f_1 0$ over $\mathbb{P}^2$ with x_1, x_2, x_6 coordinates degenerates over a curve of genus 351. The contraction of the conic bundle to a singular form of the parameter model is given by the map $(x_1, \ldots, x_9) \mapsto (x_1, x_2, x_3 x_9, x_4 x_9, x_5, x_6 x_7, x_8 x_9)$.

The classical couplings of the three parameter model are essentially obtained by restricting (9.27) to $J_4 = 0$, only $\mathcal{C}_3$ changes to $\mathcal{C}_3 = -4338 J_1 - 720 J_2 - 2160 J_3$.

Appendix A: *Kodaira's classification of elliptic fibre singularities.*

ord(f)	ord(g)	ord(Δ)	fibre	singularity	a_i
≥ 0	≥ 0	0	*smooth*	*none*	$-$
0	0	n	I_n	A_{n-1}	$\frac{n}{12}$
≥ 1	1	2	II	*none*	$\frac{1}{6}$
≥ 1	≥ 2	3	III	A_1	$\frac{1}{4}$
≥ 2	2	4	IV	A_2	$\frac{1}{3}$
2	≥ 3	$n+6$	I_n^*	D_{n+4}	$\frac{1}{2}+\frac{n}{12}$
≥ 2	3	$n+6$	I_n^*	D_{n+4}	$\frac{1}{2}+\frac{n}{12}$
≥ 3	4	8	IV^*	E_6	$\frac{5}{6}$
3	≥ 5	9	III^*	E_7	$\frac{3}{4}$
≥ 4	5	10	II^*	E_8	$\frac{2}{3}$

Tab. 1 *Classification of the singular fibres occurring in an non-singular elliptic surface with section [65][2]. The last entry is the Euler number of the singular fibre divided by* 12. *For* ord($\Delta > 10$) *there exist no resolution with trivial canonical bundle.*

Appendix B: *Tables of Calabi-Yau manifolds*

Table B.1 CY – Fourfolds with negative Euler number

* indicates that no reflexive polyhedron exists.

N^o	χ	h_{11}	h_{21}	h_{22}	h_{31}	w_1	w_2	w_3	w_4	w_5	w_6	m
1	−240	54	228	308	126	9	9	70	72	80	120	360
2	−198	22	272	424	209	5	5	5	18	24	33	90
3	−198	30	101	82	30	21	24	25	25	25	30	150
4	−192	24	147	178	83	9	9	18	28	32	48	144
5	−192	47	195	274	108	7	13	27	27	27	88	189
6	−192	47	195	274	108	13	14	54	54	54	189	378
7	−168	81	144	188	27	21	24	52	97	97	97	388
8	−144	22	165	246	111	7	7	7	12	18	33	84
9	−144	23	242	400	187	5	5	10	24	32	44	120
10	−144	24	141	198	85	9	11	11	11	15	42	99
11	−144	24	141	198	85	15	18	22	22	22	99	198
12	−144	28	91	98	31	15	18	19	19	19	24	114
13	−144	29	147	210	86	8	14	15	15	15	53	120
14	−144	29	102	120	41	12	15	22	22	22	39	132
15	−144	31	91	98	28	25	25	28	32	40	50	200
16	−144	33	273	462	208	5	5	20	36	48	66	180
17	−144	41	102	120	29	25	25	42	48	60	100	300
18	−144	44	322	560	246	5	5	30	48	64	88	240
19	−138	22	144	208	91	9	9	9	10	16	37	90
20	−138	76	462	844	355	5	5	55	78	104	143	390
21	−120	26	135	202	81	9	9	9	19	23	30	99
22	−120	26	135	202	81	12	12	12	23	33	40	132
23	−120	58	198	328	112	9	9	61	63	71	102	315
24	−120	81	135	202	26	30	33	37	100	100	100	400
25	−96	23	90	128	43	11	13	13	13	16	25	91
26	−96	23	90	128	43	11	15	15	15	21	28	105
27	−96	23	147	242	100	7	7	14	16	24	44	112
28	−96	24	240	428	192	5	5	5	16	23	31	85
29	−96	24	240	428	192	6	6	6	17	32	35	102
30	−96	24	240	428	192	10	10	10	23	32	85	170
31	−96	25	126	200	77	11	11	12	20	22	56	132
32*	−96	27	90	128	39	20	20	20	21	24	35	140
33	−96	29	82	112	29	19	19	20	24	32	38	152
34	−96	30	132	212	78	11	11	15	25	33	70	165
35	−96	30	92	132	38	20	24	33	33	66	88	264
36	−96	30	132	212	78	22	22	25	30	66	165	330
37	−96	32	166	280	110	7	7	24	28	36	66	168
38	−96	32	146	240	90	9	9	20	32	36	74	180
39	−96	36	144	236	84	10	18	19	19	38	86	190
40	−96	38	92	132	30	19	19	30	36	48	76	228
41	−96	39	90	128	27	24	25	35	42	42	42	210
42	−96	42	150	248	84	15	15	16	28	60	106	240
43	−96	43	90	128	23	31	35	36	36	42	72	252
44	−96	45	180	308	111	6	13	25	25	25	81	175
45	−96	45	180	308	111	7	12	26	26	26	85	182
46	−96	45	168	284	99	11	11	24	40	66	112	264
47	−96	45	180	308	111	12	13	50	50	50	175	350
48	−96	50	165	278	91	12	22	23	23	69	127	276
49	−96	52	228	404	152	7	7	40	56	60	110	280
50*	−96	59	150	248	67	8	15	54	54	54	85	270
51	−96	61	228	404	143	9	9	40	64	90	148	360
52	−96	68	204	356	112	9	17	35	35	70	149	315
53	−96	78	450	848	348	5	5	55	76	103	141	385
54	−90	26	81	114	32	12	17	17	17	18	21	102
55	−90	27	92	136	42	9	15	19	19	19	33	114
56	−84	30	126	208	74	9	9	18	26	31	42	135
57	−84	30	126	208	74	12	12	24	31	45	56	180
58*	−72	27	84	132	37	16	16	18	21	32	41	144
59*	−72	34	90	144	36	16	16	31	37	44	48	192
60*	−72	34	90	144	36	20	20	37	48	55	60	240
61*	−72	36	90	144	34	12	17	33	33	33	37	165
62*	−72	36	90	144	34	15	16	36	36	36	41	180
63*	−72	36	90	144	34	16	17	44	44	44	55	220
64*	−72	37	84	132	27	22	25	25	29	49	75	225
65*	−72	37	84	132	27	27	29	32	32	72	96	288
66	−72	56	144	252	68	7	15	51	51	51	80	255
67	−72	68	144	252	56	16	16	65	75	100	128	400
68	−72	78	135	234	37	17	18	61	96	96	96	384
69	−66	27	220	408	174	5	5	10	22	31	42	115
70	−66	27	220	408	174	6	6	12	23	44	47	138
71	−66	27	220	408	174	10	10	20	31	44	115	230
72	−60	50	96	164	28	20	24	49	49	54	98	294
73	−48	27	73	126	30	16	17	17	24	28	34	136
74	−48	28	83	146	39	12	19	19	20	38	44	152
75	−48	28	83	146	39	16	20	27	27	54	72	216
76	−48	35	82	144	31	17	17	24	36	42	68	204
77	−48	37	94	168	41	18	19	19	30	66	76	228
78	−48	39	83	146	28	19	19	25	30	40	57	190
79	−48	41	94	168	37	20	22	22	25	65	66	220
80	−48	41	94	168	37	25	30	33	33	99	110	330
81*	−48	45	114	208	53	13	13	33	48	75	91	273
82	−48	46	153	286	91	11	11	21	35	55	98	231
83	−48	46	280	540	218	6	6	30	41	80	83	246
84*	−48	53	210	400	141	7	7	36	49	54	99	252
85*	−48	53	114	208	45	14	19	40	40	80	87	280
86	−48	61	120	220	43	14	22	61	61	86	122	366
87	−48	63	108	196	29	12	28	33	73	73	73	292
88	−36	24	80	148	42	11	11	16	20	22	30	110
89	−36	25	82	152	43	9	12	17	17	17	30	102
90	−36	25	182	352	143	5	10	15	36	48	66	180
91*	−36	26	120	228	80	10	10	13	20	24	53	130
92	−36	33	71	130	24	25	42	48	50	60	75	300
93	−36	62	108	204	32	12	21	40	73	73	73	292
94*	−30	31	120	232	76	10	13	13	19	26	62	143
95*	−30	31	120	232	76	11	14	14	20	28	67	154
96*	−30	31	120	232	76	19	20	26	26	52	143	286
97	−24	30	78	152	36	10	17	36	36	36	45	180
98	−24	30	72	140	30	16	32	39	45	48	60	240
99	−24	39	117	230	66	9	18	45	56	64	96	288
100	−24	66	192	380	114	8	17	33	33	66	140	297
101	−24	66	192	380	114	9	16	34	34	68	145	306
102	−24	66	117	230	39	12	17	52	81	81	81	324
103	−24	69	108	212	27	24	28	39	91	91	91	364
104	−12	33	108	220	65	9	18	18	35	40	60	180
105	−12	65	108	220	33	18	29	34	81	81	81	324
106	−6	36	74	156	29	17	17	20	30	35	51	170
107	−6	38	85	178	38	15	19	19	25	55	57	190
108	−6	38	85	178	38	20	25	27	27	81	90	270

Table B.2 CY – Fourfolds with vanishing Euler number

* indicates that no reflexive polyhedron exists.

N^o	χ	h_{11}	h_{21}	h_{22}	h_{31}	w_1	w_2	w_3	w_4	w_5	w_6	d
109	0	21	75	162	46	9	11	11	11	14	21	77
110	0	23	84	180	53	12	12	12	14	21	25	96
111	0	24	126	264	94	7	7	13	14	23	41	105
112*	0	24	80	172	48	10	10	18	20	23	29	110
113	0	24	126	264	94	8	8	13	16	30	45	120
114	0	24	126	264	94	8	8	15	16	26	47	120
115*	0	24	80	172	48	12	12	22	24	27	35	132
116*	0	24	80	172	48	15	15	23	27	30	55	165
117	0	24	126	264	94	14	14	23	26	28	105	210
118*	0	25	72	156	39	12	13	13	21	26	32	117
119	0	25	111	234	78	7	14	21	24	36	66	168
120	0	26	74	160	40	12	16	17	17	34	40	136
121	0	27	84	180	49	15	15	15	16	24	35	120
122	0	27	96	204	61	11	18	22	30	33	84	198
123*	0	30	72	156	34	16	16	23	28	29	32	144
124*	0	30	72	156	34	20	20	27	28	40	45	180
125*	0	30	72	156	34	20	20	29	35	36	40	180
126	0	31	64	140	25	19	30	36	38	48	57	228
127	0	33	108	228	67	12	12	24	29	39	52	168
128	0	34	84	180	42	17	17	18	24	60	68	204
129	0	37	108	228	63	9	18	36	49	56	84	252
130	0	40	120	252	72	12	12	36	37	51	68	216
131	0	41	108	228	59	12	12	38	57	60	61	240
132	0	49	84	180	27	28	32	38	49	49	98	294
133	0	53	84	180	23	35	37	40	56	56	112	336
134	0	56	108	228	44	14	18	55	55	78	110	330
135	0	62	168	348	98	12	12	61	72	87	116	360
136	0	63	108	228	37	12	25	41	78	78	78	312
137	0	63	108	228	37	14	25	52	91	91	91	364
138	0	64	114	240	42	9	20	48	77	77	77	308

Table B.3 Elliptic fibred K3

d	w_1	w_2	w_3	w_4	P	$\mathcal{E}$
6	1	1	2	2	$x_1^6 + x_2^6 + x_3^3 + x_4^3$	E_6
9	1	2	3	3	$x_1^9 + x_2^4 x_1 + x_3^3 + x_4^3$	E_6
12	1	3	4	4	$x_1^{12} + x_2^4 + x_3^3 + x_4^3$	E_6
15	2	3	5	5	$x_1^6 x_2 + x_2^5 + x_3^3 + x_4^3$	E_6
8	1	1	2	4	$x_1^8 + x_2^8 + x_3^4 + x_4^2$	E_7
12	1	2	3	6	$x_1^{12} + x_2^6 + x_3^4 + x_4^2$	$E_8\, E_7$
16	1	3	4	8	$x_1^{16} + x_2^5 x_1 + x_3^4 + x_4^2$	E_7
20	2	3	5	10	$x_1^{10} + x_2^6 x_1 + x_3^4 + x_4^2$	E_7
20	1	4	5	10	$x_1^{20} + x_2^5 + x_3^4 + x_4^2$	E_7
28	3	4	7	14	$x_1^8 x_2 + x_2^7 + x_3^4 + x_4^2$	E_7
9	1	1	3	4	$x_1^9 + x_2^9 + x_3^3 + x_4^2 x_1$	E_8'
15	1	2	5	7	$x_1^{15} + x_2^4 x_4 + x_3^3 + x_4^2 x_1$	E_8'
21	1	3	7	10	$x_1^{21} + x_2^7 + x_3^3 + x_4^2 x_1$	E_8'
10	1	1	3	5	$x_1^{10} + x_2^{10} + x_3^3 x_1 + x_4^2$	E_8''
16	1	2	5	8	$x_1^{16} + x_2^8 + x_3^3 x_1 + x_4^2$	E_8''
18	1	3	5	9	$x_1^{18} + x_2^6 + x_3^3 x_2 + x_4^2$	E_8''
22	1	3	7	11	$x_1^{22} + x_2^5 x_3 + x_3^3 x_1 + x_4^2$	E_8''
28	1	4	9	14	$x_1^{28} + x_2^7 + x_3^3 x_1 + x_4^2$	E_8''

d	w_1	w_2	w_3	w_4	P	$\mathcal{E}$
12	1	1	4	6	$x_1^{12} + x_2^{12} + x_3^3 + x_4^2$	E_8
18	2	3	4	9	$x_1^9 + x_2^6 + x_3^4 x_1 + x_4^2$	E_8
18	1	2	6	9	$x_1^{18} + x_2^9 + x_3^3 + x_4^2$	E_8
24	1	3	8	12	$x_1^{24} + x_2^8 + x_3^3 + x_4^2$	E_8
30	4	5	6	15	$x_1^6 x_3 + x_2^6 + x_3^5 + x_4^2$	E_8
30	1	4	10	15	$x_1^{30} + x_2^5 x_3 + x_3^3 + x_4^2$	E_8
36	1	5	12	18	$x_1^{36} + x_2^7 x_1 + x_3^3 + x_4^2$	E_8
42	3	4	14	21	$x_1^{14} + x_2^7 x_3 + x_3^3 + x_4^2$	E_8
42	2	5	14	21	$x_1^{21} + x_2^8 x_1 + x_3^3 + x_4^2$	E_8
42	1	6	14	21	$x_1^{42} + x_2^7 + x_3^3 + x_4^2$	E_8
48	3	5	16	24	$x_1^{16} + x_2^9 x_1 + x_3^3 + x_4^2$	E_8
54	4	5	18	27	$x_1^9 x_3 + x_2^{10} x_1 + x_3^3 + x_4^2$	E_8
66	5	6	22	33	$x_1^{12} x_2 + x_2^{11} + x_3^3 + x_4^2$	E_8

Table B.4 Elliptic fibred CY – Fourfolds with small Picard number

* indicates that χ not divisible by 24.

N^o	F	χ	h_{11}	h_{21}	h_{22}	h_{31}	w_1	w_2	w_3	w_4	w_5	w_6	d
1	$E8$	*13362	2	0	8920	2217	1	3	4	5	26	39	78
2	$E8$	16176	2	0	10796	2686	1	2	3	5	22	33	66
2	$E8$	20832	2	0	13900	3462	1	1	2	3	14	21	42
4	$E8$	22776	2	0	15196	3786	1	1	1	2	10	15	30
5	$E8$	23328	2	0	15564	3878	1	1	1	1	8	12	24
6	$E8$	8424	3	0	5628	1393	2	3	4	5	28	42	84
7	$E7$	*8484	3	0	5668	1403	1	1	2	3	7	14	28
8	$E7$	*9276	3	0	6196	1535	1	1	1	2	5	10	20
9	$E7$	9504	3	0	6348	1573	1	1	1	1	4	8	16
10	$E8'$	10992	3	0	7340	1821	1	2	3	5	17	28	56
11	$E8$	*12810	3	0	8552	2124	1	4	5	7	34	51	102
12	$E8$	14232	3	0	9500	2361	1	3	5	7	32	48	96
13	$E8'$	*14484	3	0	9668	2403	1	1	2	3	11	18	36
14	$E8$	15240	3	0	10172	2529	1	2	3	4	20	30	60
15	$E8$	15624	3	0	10428	2593	1	2	2	3	16	24	48
16	$E8'$	*16242	3	0	10840	2696	1	1	1	2	8	13	26
17	$E8'$	18528	3	0	12364	3077	1	1	2	3	13	20	40
18	$E8'$	*18954	3	0	12648	3148	1	1	1	1	7	11	22
19	$E8$	19056	3	0	12716	3165	1	2	3	7	26	39	78
20	$E8'$	*19308	3	0	12884	3207	1	1	1	2	9	14	28
21	$E8$	19728	3	0	13164	3277	1	1	2	2	12	18	36
22	$E8$	*22122	3	1	14762	3677	1	1	3	4	18	27	54
23	$E8$	26208	3	1	17486	4358	1	1	1	3	12	18	36
24	$E6$	4368	4	0	2924	716	1	1	2	3	7	7	21
25	$E8$	4704	4	0	3148	772	4	5	6	7	44	66	132
26	$E6$	4776	4	0	3196	784	1	1	1	2	5	5	15
27	$E6$	4896	4	0	3276	804	1	1	1	1	4	4	12
28	$E8'$	*6228	4	0	4164	1026	2	3	4	5	23	37	74
29	$E7$	6240	4	0	4172	1028	1	2	3	4	10	20	40
30	$E7$	6408	4	0	4284	1056	1	2	2	3	8	16	32
31	$E8'$	*6708	4	0	4484	1106	2	3	5	7	29	46	92
32	$E7$	8064	4	0	5388	1332	1	1	2	2	6	12	24
33	$E8''$	8640	4	0	5772	1428	1	2	3	5	17	23	51
34	$E8$	8640	4	0	5772	1428	2	4	5	6	34	51	102
35	$E8$	8856	4	12	5940	1476	2	3	3	5	26	39	78
36	$E8$	*8916	4	1	5958	1475	2	3	5	8	36	54	108
37	$E7$	9000	4	1	6014	1489	1	1	3	4	9	18	36
38	$E8$	10248	4	1	6846	1697	2	3	5	11	42	63	126
39	$E8'$	10344	4	0	6908	1712	1	3	4	5	22	35	70
40	$E8$	10608	4	1	7086	1757	2	2	3	5	24	36	72
41	$E7$	10656	4	1	7118	1765	1	1	1	3	6	12	24
42	$E8'$	*10884	4	0	7268	1802	1	2	3	4	16	26	52
43	$E8'$	11376	4	0	7596	1884	1	2	2	3	13	21	42
44	$E8''$	11568	4	0	7724	1916	1	1	2	3	11	15	33
45	$E8$	11568	4	0	7724	1916	2	2	3	4	22	33	66
46	$E8'$	*11850	4	0	7912	1963	1	2	3	4	17	27	54
47	$E8'$	12000	4	0	8012	1988	1	2	3	7	19	32	64
48	$E8$	12480	4	1	8334	2069	1	4	5	6	32	48	96
49	$E8'$	*12708	4	0	8484	2106	1	2	2	3	14	22	44
50	$E8'$	*12900	4	0	8612	2138	1	2	3	5	19	30	60
51	$E8''$	13200	4	0	8812	2188	1	1	1	2	8	11	24
52	$E8$	*13434	4	10	8988	2237	1	3	3	4	22	33	66
53	$E8'$	*14028	4	0	9364	2326	1	2	3	4	19	29	58
54	$E8$	14856	4	0	9916	2464	1	2	3	3	18	27	54
55	$E8'$	14928	4	0	9964	2476	1	1	2	2	10	16	32
56	$E8'$	15168	4	0	10124	2516	1	1	3	4	14	23	46
57	$E8$	15792	4	1	10542	2621	1	2	4	5	24	36	72
58	$E8''$	16776	4	0	11196	2784	1	1	1	1	7	10	21
59	$E8$	16776	4	0	11196	2784	1	2	2	2	14	21	42
60	$E8'$	*17082	4	1	11402	2836	1	1	1	3	9	15	30
61	$E8'$	17184	4	0	11468	2852	1	1	2	2	11	17	34
62	$E8''$	17328	4	0	11564	2876	1	1	2	3	13	19	39
63	$E8$	17328	4	0	11564	2876	1	2	4	6	26	39	78
64	$E8''$	17544	4	0	11708	2912	1	1	1	2	9	13	27
65	$E8$	17544	4	0	11708	2912	1	2	2	4	18	27	54
66	$E8'$	20208	4	0	13484	3356	1	1	3	4	17	26	52
67	$E8$	20688	4	7	13818	3443	1	1	3	3	16	24	48
68	$E8'$	*22854	4	0	15248	3797	1	1	1	3	11	17	34
69	$E8$	23328	4	1	15566	3877	1	1	2	4	16	24	48
70	$E8$	*24234	4	0	16168	4027	1	1	4	5	22	33	66
71	$E8$	24264	4	2	16192	4034	1	1	3	5	20	30	60
72	$E8$	*31194	4	0	20808	5187	1	1	1	4	14	21	42
73	$E6$	3240	5	0	2172	527	1	2	3	4	10	10	30
74	$E6$	3336	5	0	2236	543	1	2	2	3	8	8	24
75	$E6$	3408	5	0	2284	555	1	2	3	5	11	11	33
76	$E7$	*3516	5	0	2356	573	2	3	5	7	17	34	68
77	$E6$	4176	5	0	2796	683	1	1	2	2	6	6	18
78	$E8$	*4938	5	0	3304	810	4	5	7	13	58	87	174
79	$E6$	5472	5	1	3662	900	1	1	1	3	6	6	18
80	$E7$	5976	5	1	3998	984	1	3	4	7	15	30	60
81	$E7$	*6108	5	0	4084	1005	1	2	3	3	9	18	36
82	$E8'$	7416	5	0	4956	1223	2	3	5	7	31	48	96
83	$E8'$	7440	5	0	4972	1227	2	2	3	5	19	31	62
84	$E8'$	7512	5	0	5020	1239	2	3	4	5	26	40	80
85	$E7$	*7764	5	0	5188	1281	1	2	3	7	13	26	52
86	$E8$	8328	5	7	5578	1382	2	3	5	6	32	48	96
87	$E8$	8760	5	0	5852	1447	2	3	7	8	40	60	120
88	$E8''$	8856	5	0	5916	1463	1	2	3	4	16	22	48
89	$E8''$	*8874	5	0	5928	1466	1	3	4	5	22	31	66
90	$E8''$	8928	5	0	5964	1475	1	2	3	7	19	25	57
91	$E8$	8928	5	0	5964	1475	2	4	6	7	38	57	114
92	$E7$	9504	5	1	6350	1572	1	1	2	4	8	16	32
93	$E7$	9864	5	2	6592	1633	1	1	3	5	10	20	40
94	$E8'$	10032	5	0	6700	1659	1	4	5	7	29	46	92
95	$E8$	10176	5	9	6814	1692	2	2	3	3	20	30	60
96	$E8$	10464	5	0	6988	1731	2	3	7	13	50	75	150
97	$E8'$	10704	5	9	7166	1780	1	3	3	4	19	30	60
98	$E7$	*10788	5	0	7204	1785	1	1	2	5	9	18	36
99	$E8'$	*11094	5	0	7408	1836	1	2	4	5	19	31	62
100	$E8''$	11256	5	0	7516	1863	1	2	2	3	14	20	42
101	$E8'$	11256	5	0	7516	1863	1	2	3	3	15	24	48
102	$E8''$	11280	5	0	7532	1867	1	2	3	5	19	27	57
103	$E8$	11280	5	0	7532	1867	2	3	4	10	38	57	114
104	$E8'$	*11748	5	0	7844	1945	1	3	4	7	26	41	82
105	$E8''$	*11994	5	0	8008	1986	1	1	3	4	14	19	42
106	$E8'$	12144	5	0	8108	2011	1	2	2	4	14	23	46
107	$E8$	12432	5	19	8338	2078	1	4	4	5	28	42	84
108	$E8''$	12624	5	0	8428	2091	1	1	2	2	10	14	30
109	$E7$	12672	5	0	8460	2099	1	1	1	4	7	14	28
110	$E8''$	13032	5	1	8702	2160	1	1	1	3	9	12	27
111	$E8$	13032	5	1	8702	2160	2	2	2	3	18	27	54
112	$E8$	13248	5	12	8868	2207	1	4	5	8	36	54	108
113	$E8'$	13320	5	0	8892	2207	1	2	2	2	12	19	38
114	$E8$	13896	5	6	9288	2309	1	3	4	6	28	42	84
115	$E8''$	14256	5	0	9516	2363	1	1	2	3	12	17	36
116	$E8'$	*15078	5	6	10076	2506	1	1	3	3	13	21	42
117	$E8'$	15216	5	1	10158	2524	1	1	2	4	12	20	40
118	$E8$	*15354	5	0	10248	2546	1	4	5	11	42	63	126
119	$E8'$	15744	5	0	10508	2611	1	2	3	7	23	36	72
120	$E8$	*16554	5	5	11058	2751	1	3	4	9	34	51	102

Table B.4 (continued) CY – Fourfolds with small Picard number

N^o	F	χ	h_{11}	h_{21}	h_{22}	h_{31}	w_1	w_2	w_3	w_4	w_5	w_6	d
121	$E8'$	16728	5	1	11166	2776	1	1	3	4	15	24	48
122	$E8$	18288	5	2	12208	3037	1	2	2	5	20	30	60
123	$E8$	*18930	5	0	12632	3142	1	3	4	11	38	57	114
124	$E8'$	18960	5	0	12652	3147	1	1	2	4	14	22	44
125	$E8'$	*19044	5	0	12708	3161	1	1	1	4	10	17	34
126	$E8$	*20844	5	0	13908	3461	1	2	3	8	28	42	84
127	$E8'$	*21054	5	0	14048	3496	1	1	2	4	15	23	46
128	$E8''$	21120	5	0	14092	3507	1	1	1	3	11	16	33
129	$E8$	21120	5	0	14092	3507	1	2	2	6	22	33	66
130	$E8'$	*22374	5	0	14928	3716	1	1	3	5	19	29	58
131	$E8$	22704	5	13	15174	3784	1	1	4	4	20	30	60
132	$E8'$	*24228	5	0	16164	4025	1	1	2	5	17	26	52
133	$E8$	26880	5	5	17942	4472	1	1	3	6	22	33	66
134	$E8$	27072	5	0	18060	4499	1	2	3	11	34	51	102
135	$E8$	*28554	5	0	19048	4746	1	1	4	7	26	39	78
136	$E6$	1848	6	0	1244	294	2	3	4	5	14	14	42
137	$E6$	2832	6	0	1900	458	1	3	4	5	13	13	39
138	$E6$	3192	6	0	2140	518	1	2	3	3	9	9	27
139	$E7$	3672	6	1	2462	599	2	3	5	8	18	36	72
140	$E7$	4392	6	1	2942	719	2	2	3	5	12	24	48
141	$E7$	4440	6	0	2972	726	2	2	3	4	11	22	44
142	$E6$	4632	6	1	3102	759	1	1	3	4	9	9	27
143	$E6$	4896	6	1	3278	803	1	1	2	4	8	8	24
144	$E6$	5064	6	2	3392	832	1	1	3	5	10	10	30
145	$E7$	5112	6	1	3422	839	1	4	5	6	16	32	64
146	$E8''$	5808	6	0	3884	954	2	3	5	7	29	41	87
147	$E8$	5808	6	0	3884	954	4	5	6	14	58	87	174
148	$E8$	6048	6	1	4046	995	3	4	5	6	36	54	108
149	$E8$	6048	6	2	4048	996	3	4	7	10	48	72	144
150	$E8'$	6144	6	0	4108	1010	2	3	5	8	28	46	92
151	$E8$	*6210	6	0	4152	1021	3	4	7	11	50	75	150
152	$E7$	*6612	6	0	4420	1088	1	2	2	2	7	14	28
153	$E7$	6840	6	0	4572	1126	1	2	5	6	14	28	56
154	$E7$	6960	6	0	4652	1146	1	2	2	4	9	18	36
155	$E8$	*7002	6	2	4684	1155	3	3	4	5	30	45	90
156	$E8''$	7032	6	0	4700	1158	2	3	4	5	26	38	78
157	$E8'$	7080	6	0	4732	1166	2	3	5	8	31	49	98
158	$E8$	7272	6	3	4866	1201	2	5	6	9	44	66	132
159	$E8'$	7320	6	11	4914	1217	2	3	3	5	23	36	72
160	$E8$	*7380	6	0	4932	1216	2	5	8	11	52	78	156
161	$E7$	7488	6	2	5008	1236	1	2	2	5	10	20	40
162	$E8'$	*7794	6	0	5208	1285	2	3	5	7	32	49	98
163	$E8'$	7896	6	8	5292	1310	2	2	3	3	17	27	54
164	$E8''$	7944	6	0	5308	1310	1	3	4	5	21	29	63
165	$E8$	7944	6	0	5308	1310	2	5	6	8	42	63	126
166	$E8'$	8160	6	0	5452	1346	2	3	5	8	34	52	104
167	$E7$	8496	6	0	5676	1402	1	2	3	8	14	28	56
168	$E8'$	8592	6	1	5742	1419	2	2	3	5	21	33	66
169	$E8'$	8664	6	0	5788	1430	2	2	3	4	19	30	60
170	$E8$	*8754	6	0	5848	1445	3	4	5	17	58	87	174
171	$E8$	8928	6	1	5966	1475	2	3	3	4	24	36	72
172	$E8'$	9120	6	0	6092	1506	1	4	5	6	26	42	84
173	$E8'$	*9174	6	0	6128	1515	1	5	6	7	32	51	102
174	$E8'$	*9276	6	0	6196	1532	2	2	3	5	22	34	68
175	$E8''$	*9366	6	0	6256	1547	1	2	2	3	13	18	39
176	$E8$	*9366	6	0	6256	1547	2	3	4	4	26	39	78
177	$E8''$	9528	6	0	6364	1574	1	2	3	3	15	21	45
178	$E8$	9528	6	0	6364	1574	2	3	4	6	30	45	90
179	$E8'$	9672	6	5	6470	1603	1	3	4	6	22	36	72
180	$E7$	9864	6	0	6588	1630	1	1	4	5	11	22	44
181	$E8''$	9888	6	0	6604	1634	1	3	4	5	23	33	69
182	$E8$	9888	6	0	6604	1634	2	3	8	10	46	69	138
183	$E8'$	*10050	6	0	6712	1661	2	2	3	4	20	31	62
184	$E8$	10128	6	0	6764	1674	2	3	3	3	22	33	66
185	$E8'$	10152	6	1	6782	1679	1	4	5	6	28	44	88
186	$E8'$	10512	6	4	7028	1742	1	3	4	9	25	42	84
187	$E8$	10800	6	5	7222	1791	2	3	5	12	44	66	132
188	$E8'$	10992	6	0	7340	1818	2	2	2	3	16	25	50
189	$E8''$	11616	6	1	7758	1923	1	1	2	4	12	16	36
190	$E8'$	*11628	6	6	7776	1930	1	3	4	6	25	39	78
191	$E8'$	11640	6	0	7772	1926	1	2	5	6	22	36	72
192	$E8''$	11760	6	0	7852	1946	1	2	3	4	18	26	54
193	$E8'$	11784	6	0	7868	1950	1	2	4	6	20	33	66
194	$E8'$	11904	6	2	7952	1972	1	2	2	5	15	25	50
195	$E8$	12024	6	3	8034	1993	2	2	3	7	28	42	84
196	$E8$	12192	6	2	8144	2020	1	5	6	8	40	60	120
197	$E8''$	12432	6	6	8312	2064	1	1	3	3	13	18	39
198	$E8$	12432	6	6	8312	2064	2	2	3	6	26	39	78
199	$E8'$	12432	6	1	8302	2059	1	3	4	7	27	42	84
200	$E8'$	12744	6	0	8508	2110	1	2	3	8	20	34	68
201	$E8''$	12792	6	0	8540	2118	1	2	3	5	20	29	60
202	$E8$	13248	6	1	8846	2195	1	3	4	4	24	36	72
203	$E8'$	13344	6	0	8908	2210	1	2	2	6	16	27	54
204	$E8$	13344	6	6	8920	2216	1	5	8	12	52	78	156
205	$E8$	*13410	6	3	8958	2224	1	4	7	9	42	63	126
206	$E8'$	13464	6	0	8988	2230	1	2	4	6	22	35	70
207	$E8$	13752	6	2	9184	2280	1	3	5	6	30	45	90
208	$E8''$	*13842	6	0	9240	2293	1	1	1	4	10	13	30
209	$E8'$	13896	6	0	9276	2302	1	3	4	7	29	44	88
210	$E8''$	14136	6	1	9438	2343	1	1	3	4	15	21	45
211	$E8$	14136	6	1	9438	2343	2	2	3	8	30	45	90
212	$E8'$	14424	6	0	9628	2390	1	2	2	2	13	20	40
213	$E8$	14424	6	0	9628	2390	1	3	3	3	20	30	60
214	$E8''$	14424	6	0	9628	2390	1	2	3	5	21	31	63
215	$E8$	14424	6	0	9628	2390	1	4	6	10	42	63	126
216	$E8'$	14664	6	0	9788	2430	1	2	2	4	16	25	50
217	$E8'$	*14754	6	0	9848	2445	1	2	4	5	23	35	70
218	$E8'$	15408	6	0	10284	2554	1	2	3	11	23	40	80
219	$E8$	15408	6	6	10296	2560	1	3	7	9	40	60	120
220	$E8'$	15528	6	0	10364	2574	1	2	2	5	18	28	56
221	$E8'$	15600	6	0	10412	2586	1	2	2	4	17	26	52
222	$E8$	15600	6	0	10412	2586	1	3	3	6	26	39	78
223	$E8''$	15864	6	0	10588	2630	1	1	2	2	11	16	33
224	$E8$	15864	6	0	10588	2630	1	2	4	4	22	33	66
225	$E8$	16128	6	10	10784	2684	1	4	5	12	44	66	132
226	$E8'$	16176	6	12	10820	2694	1	1	4	4	16	26	52
227	$E8'$	16464	6	0	10988	2730	1	1	4	5	17	28	56
228	$E8'$	*16806	6	4	11224	2791	1	1	3	6	16	27	54
229	$E8$	17568	6	1	11726	2915	1	2	3	6	24	36	72
230	$E8'$	*17844	6	0	11908	2960	1	1	4	5	18	29	58
231	$E8$	17904	6	3	11954	2973	1	2	4	7	28	42	84
232	$E8'$	*17994	6	0	12008	2985	1	1	5	6	20	33	66
233	$E8$	18672	6	3	12466	3101	1	2	5	8	32	48	96
234	$E8''$	19200	6	0	12812	3186	1	1	3	4	17	25	51
235	$E8$	19200	6	0	12812	3186	1	2	6	8	34	51	102
236	$E8'$	*22074	6	0	14728	3665	1	1	2	5	16	25	50
237	$E8$	22560	6	0	15052	3746	1	2	2	7	24	36	72
238	$E8$	26208	6	3	17490	4357	1	1	4	6	24	36	72
239	$E8'$	27744	6	0	18508	4610	1	1	1	4	13	20	40
240	$E8$	30336	6	0	20236	5042	1	1	2	6	20	30	60
241	$E8'$	*33594	6	0	22408	5585	1	1	1	5	15	23	46

References

[1] C. Vafa, *Evidence for F-Theory*, hep-th/9602022

[2] D. Morrison, C. Vafa, hep-th/9602114

[3] D. Morrison and C. Vafa, hep-th/9603161

[4] M. Bershadsky, K. Intriligator, S. Kachru, D. R. Morrison, V. Sadov and C. Vafa, hep-th/9605200

[5] M. Bershadsky and A. Johansen, hep-th/961011

[6] E. Witten, Nucl. Phys. B463 (1996) 506, P. Horava and E. Witten, Nucl. Phys. B460 (1996) 506, K. Dasgupta and S. Mukhi, Nucl. Phys. B465 (1996) 399

[7] A. Sen, Mod. Phys. Lett. A11 (1996) 1339, A. Sen, *F-theory and Orientifolds*, hep-th/9605150

[8] A. Kumar and K. Ray, hep-th/9602144, hep-th/9604133

[9] J. Blum and A. Zaffaroni, hep-th/9607019, J. Blum, hep-th/9608053

[10] C. Borcea, *K3 Surfaces with and Mirror Pairs of Calabi-Yau Manifolds*, in Mirror Symmetry II, Ed. B. Greene and S.T. Yau, International Press (1991)

[11] R. Gopakumar and S. Mukhi, *Orbifold and Orientifold Compactifications of F-Theory and M-theory to six and four Dimensions*, hep-th/9607057

[12] F. Hirzebruch and J. Werner, *Some examples of Threefolds with trivial canonical bundle*, Preprint Max-Planck-Institut Bonn MPI/85-58

[13] S. Sethi, C. Vafa and E. Witten, Constraints on low-dimensional string compactifications, hep-th 9606122

[14] E. Witten, *On Flux Quantization in M-theory and the effective Action*, hep-th/9609122

[15] S. Katz and C. Vafa, *Geometrical Engineering of* $N = 1$ *Quantum Field Theories*, hep-th/96011091

[16] M. Berschadsky, A. Johanson, T. Pantev, V. Sadov and C. Vafa,*F-theory, Geometrical Engineering and* $N = 1$ *dualities* , hep-th/9612052

[17] Y. Hayakawa, *Degeneration of Calabi-Yau Manifold with Weil-Petersen Metric*, alg-geom/9507016

[18] T. Banks, M. Douglas, N. Seiberg, *Probing F-theory with Branes*, hep-th/9605199

[19] I. Brunner, M. Lynker and R. Schimmrigk, hep-th/9610195.

[20] P. Berglund, S. Katz, A. Klemm and P. Mayr, *New Higgs Transitions between Dual N=2 Models*, hep-th/9605154

[21] E. Witten, *Non-Perturbative Superpotentials in String Theory*, hep-th9604030

[22] K. Becker, M. Becker, M-Theory on Eight-Manifolds, hep-th/9605053

[23] F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Springer-Verlag (1966), Berlin, Heidelberg, New York

[24] R. Donagi, A. Grassi and E. Witten, a non-perturbative superpotential with E_8-symmetry, hep-th 9607091

[25] B. S. Acharya, *N=1 M-theory Heterotic duality in Three-Dimensions and Joyce Manifolds*, hep-th/9604133; *M-theory Compactification and Two Brane/Five Brane Duality*, hep-th/9605047

[26] S. Hosono, A. Klemm, S. Theisen, S. Yau, Nucl. Phys. B433 (1985) 501

[27] S. Hosono, B. Lian and S. T. Yau, alg-geom-9511001, to appear in CMP.

[28] C. J. Isham, C. N. Pope, Class. Quant. Grav. **5** (1988) 257; C. J. Isham, C. N. Pope and N. P. Warner Class. Quant. Grav. **5** (1988) 257

[29] P. Mayr, hep-th/9610162

[30] M. Gross, Duke Math. Jour. **74** (1994) 271

[31] I. Brunner, R. Schimmrigk, hep-th/9606148

[32] C. Vafa, Mod. Phys. Let. $\underline{A4}$ (1989) 1169 , Mod. Phys. Let. $\underline{A4}$ (1989) 1615

[33] E. Witten, Nucl. Phys. B403 (1993) 159

[34] S.-S. Roan, Int. J. Math. **2** (1991) 439

[35] S. Kondo, J. Math. Soc. Japan **44** (1992) 75

[36] A. R. Fletcher, *Working with Weighted complete Intersections*, Preprint MPI/89-34, Max-Planck-Institut f. Math. Bonn (1989)

[37] A. Klemm, R. Schimmrigk, Nucl. Phys. B (411 (1994) 559, M. Kreuzer, H. Skarke Nucl. Phys. B388, (1993) 113

[38] E. Gimeon and C. Johnson, *Multiple Realizations of $N = 1$ Vacua in Six-Dimensions* hep-th/9606176

[39] A. Klemm, W. Lerche and P. Mayr, Phys. Lett. 357 B (1995) 313

[40] S. Hosono, B. Lian S.-T. Yau,alg-geom/9603020

[41] Y. Kawamata, J. Fac. Sci. Univ. Tokyo Sec. IA **30** (1983) 1; T. Fujita, J. Math. Soc. Japan **38** 20; N. Nakayama, in Algebraic Geometry and Commutative Algebra vol. II, Kinokuniya, Tokyo (1988) 405, A. Grassi, Math. Ann. **290** (1991) 287

[42] A. Klemm, P. Mayr and C. Vafa, *BPZ States of Exceptional Non-Critical Strings*, hep-th/9607139

[43] J. Louis, J. Sonnenschein, S. Theisen and S. Yankielowicz, *Non-perturbative properties of heterotic String Vacua conpactified on $K3 \times T^2$*, hep-th/9606049

[44] W. Fulton, *Introduction to Toric Varieties* Princeton University Press, Princeton 1993

[45] V. Batyrev, *Dual Polyhedra and Mirror Symmetry for Calabi-Yau Hypersurfaces in Toric Varieties*, J. Alg. Geom. 3 (1994) 493

[46] P. Candelas, X. d. Ossa, S. Katz, Nucl. Phys. B372 (1995) 127

[47] S. Hosono, A. Klemm, S.Theisen, S. T. Yau, Comm. Math. Phys. 167 (1995) 301

[48] V. V. Batyrev, *Toroidal Fano 3-folds*, Math. USSR-Izv **19** (1982), 13-25, Izv. Akad. Nauk SSSR, Ser. Mat. **45** (1981), 704-717

[49] S. Mori and S. Mukai, *On Fano 3-folds with* $B_2 \geq 2$, in Algebraic Varieties and Analytic Varieties, Ed. S. Iitaka, Adv. Studies in Pure Math. 1 (1983) 101

[50] T. Oda, *Convex Bodies and Algebraic Geometry*, Springer Verlag, Berlin Heidelberg (1988)

[51] K. Watanabe, M. Watanabe, *The classification of Fano 3-folds with torus embeddings*, Tokyo J. Math. **5** (1982) 37-48

[52] P. Berglund, S. Katz, A. Klemm, Nucl. Phys. B456 (1995) 153

[53] J. Tate, in Modular Functions in one variable IV, Lect. Notes in Math., **476**, Springer Verlag, Berlin, Heidelberg (1975) 33

[54] B. Hunt and R. Schimmrigk, hep-th/9512138

[55] S. Kondo, J. Math. Soc. Japan **44** (1992) 75

[56] S. Katz, A. Klemm and C. Vafa, hep-th/9609239

[57] P. Candelas, A. Font, *Duality Between the Webs Heterotic and Type II Vacua*, hep-th/9603170

[58] E. Witten, *Mirror symmetry and topological field theory*, Essays on Mirror Manifolds (ed. S.-T. Yau), Int. Press (1992) Hong Kong

[59] B. Greene, D. R. Morrison, R. Plesser, Comm. Math. Phys. 173 (1995) 559

[60] M. Jinzenij and M. Nagura Int. J. Mod. Phys. A11 (1996) 455

[61] M. Nagura, Mod. Phys. Let. $\underline{A10}$ (1995) 1667

[62] K. Sugiyama, hep-th 9504114; hep-th/9504115

[63] S. Hosono, A. Klemm, S. Theisen and S.T. Yau, Nucl. Phys. B433 (1995) 501-554

[64] P. Aspinwall, D. Morrison, Comm. Math. Phys. 151 (1993) 245

[65] K. Kodaira, Annals of Math. **77** (1963) 563; **78** (1963) 1