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On Univalent Harmonic Maps Between Surfaces

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Let f be a Lipschitz map between two compact Riemannian manifolds (M, dS_M^2) and $(M', dS_{M'}^2)$. Then the energy of f is defined to be

$$E(f) = \int_M \mathrm{tr}_{dS_M^2}(f^* dS_{M'}^2).$$

Hence the energy defines a functional on the space of Lipshitz maps between M and M'. Critical points of this functional are called harmonic maps. These maps were studied by Bochner, Morrey, Rauch, Eells and Sampson, Hartman, Uhlenbeck, Hamilton, Hildebrandt and others. The first fundamental result was due to Eells and Sampson [3] who proved that, in case M' has non-positive sectional curvature, each map from M to M' is homotopic to a harmonic map. (This result was then extended by Hamilton [8] to the case where both M and M' are allowed to have boundary.) Later Hartman [7] was able to prove the harmonic map is unique in each homotopy class if M' has strictly negative curvature.

This last result of Hartman leads one to believe that harmonic maps between compact manifolds with negative curvature must enjoy a lot of nice properties. In fact, a few years ago, *B.* Lawson and the second author conjectured the following statement: If f is a harmonic map between two compact Riemannian manifolds of negative curvature and if f is a homotopy equivalence, then f is a diffeomorphism.

In this paper, we demonstrate that the above statement is true at least when $\dim M = \dim M' = 2$. In other words, we prove that when M' has non-positive curvature and genus $M = g \geq 1$, then every degree one harmonic map from M into M' is a diffeomorphism. We also generalize this theorem to the case where both M and M' have boundary, $\partial M'$ has non-negative geodesic curvature and the harmonic map restricted to ∂M is a homeomorphism from ∂M to $\partial M'$.

It should be noted that in case both M and M' are bounded simply connected domins in the plane, the last theorem was an old theorem and was

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due to Rado [10] and Choquet [11]. In this case, harmonic maps are simply pairs of harmonic functions and the linear structure is available. For the properties of univalent harmonic maps, one should mention the result of H. Lewy [1] and E. Heinz [4]. Lewy proved the Jacobian of such a map does not change sign while Heinz gave an estimate of the Jacobian from below for a certain class of harmonic maps.

At least, we should mention that in 1963, K. Shibata [5] claimed to prove that every diffeomorphism between two compact Riemann surfaces (without boundary) is homotopic to a harmonic diffeomorphism. However there are serious gaps in several crucial steps of his argument. For example, in Lemma 3 of page 180, he ignored the fact that M' is not a plane domain and the standard argument cannot be applied. His paper, which is hard to comprehend, is therefore subject to various criticism.

§ 1. Notation and Basic Formulae

We let M and M' be Riemann surfaces. Suppose $f\colon M \to M'$ is a C^∞ map. Let $z = x^1 + ix^2$ be a local complex coordinate on M and $u = u^1 + iu^2$ a complex coordinate on M'. Let $\sigma(z)|dz|^2$ and $\rho(u)|du|^2$ be conformal metrics on M and M'. In this section we express the condition that f be a harmonic map, and derive important local formulae.

Define local one-forms $\theta = \sqrt{\sigma(z)}\, dz$ on M and $\omega = \sqrt{p(u)}\, du$ on M'. We write the first structural equations

$$d\theta = \theta_c \wedge \theta \qquad \text{on } M$$

and (1)

$$d\omega = \omega_c \wedge \omega \qquad \text{on } M'.$$

Here θ_c and ω_c are the Riemannian connection one-forms given by

$$\theta_c = \frac{\partial \log \sqrt{\sigma(z)}}{\partial \bar{z}}\, d\bar{z} - \frac{\partial \log \sqrt{\sigma(z)}}{\partial z}\, dz \tag{2}$$

and

$$\omega_c = \frac{\partial \log \sqrt{\rho(u)}}{\partial \bar{u}}\, d\bar{u} - \frac{\partial \log \sqrt{\rho(u)}}{\partial u}\, du.$$

The curvature functions K on M and K' on M' are defined by

$$d\theta_c = -\frac{K}{2}\, \theta \wedge \bar{\theta}$$

and (3)

$$d\omega_c = -\frac{K'}{2}\, \omega \wedge \bar{\omega}.$$

It is straightforward to check that K and K' are independent of choice of coordinates. More explicitly we have

$$K = -\frac{4}{\sigma}\frac{\partial^2 \log\sqrt{\sigma}}{\partial z\,\partial \bar{z}} \quad \text{and} \quad K' = -\frac{4}{\rho}\frac{\partial^2 \log\sqrt{\rho}}{\partial u\,\partial \bar{u}}.$$

We now consider the map $f\colon M \to M'$ and define the first derivatives of f by the equations

$$f^*\omega = u_\theta \theta + u_{\bar\theta}\bar\theta$$
and (5)
$$f^*\bar\omega = \bar u_\theta \theta + \bar u_{\bar\theta}\bar\theta$$

where f^* denotes pullback of differential forms, and $u_\theta, u_{\bar\theta}, \bar u_\theta, \bar u_{\bar\theta}$ are locally defined functions. By conjugating the first equation of (5) we see that $\overline{u_\theta} = \bar u_{\bar\theta}$ and $\bar u_\theta = \overline{u_{\bar\theta}}$. We define the energy density and Jacobian of f by

$$e(f) = |u_\theta|^2 + |u_{\bar\theta}|^2$$
and (6)
$$J(f) = |u_\theta|^2 - |u_{\bar\theta}|^2$$

where $|\ |$ denotes the absolute value of complex numbers. It follows that $e(f)$ and $J(f)$ are globally defined functions on M independent of coordinate choice on M and on M'. We next define the second covariant derivatives of f by the equations

$$du_\theta + u_\theta \theta_c - u_\theta f^*\omega_c = u_{\theta\theta}\,\theta + u_{\theta\bar\theta}\,\bar\theta$$
and (7)
$$du_{\bar\theta} + u_{\bar\theta}\bar\theta_c - u_{\bar\theta} f^*\omega_c = u_{\bar\theta\theta}\theta + u_{\bar\theta\bar\theta}\bar\theta.$$

By conjugating (7) we also have $\bar u_{\theta\theta}$, $\bar u_{\theta\bar\theta}$, $\bar u_{\bar\theta\theta}$, $\bar u_{\bar\theta\bar\theta}$ defined locally on M. Exterior differentiating (5), rearranging, and applying (7) we obtain $u_{\theta\bar\theta}\bar\theta \wedge \theta + u_{\bar\theta\theta}\theta \wedge \bar\theta = 0$, i.e.

$$u_{\theta\bar\theta} = u_{\bar\theta\theta}. \tag{8}$$

We say that f is a harmonic map if the following equation is satisfied

$$u_{\theta\bar\theta} = 0. \tag{9}$$

Equation (9) is independent of coordinate change on M and M'. Moreover, it is shown in [3] that f is a harmonic map if and only if f is a critical point of the energy functional $E(f) = \int_M e(f)\,dV_M$ where dV_M denotes the volume element of M. A direct computation shows that (9) can be written explicitly as

$$u_{z\bar z} + \frac{\partial \log \rho(u(z))}{\partial u} u_z u_{\bar z} = 0. \tag{10}$$

We remark that (10) implies that f being harmonic does not depend on the metric $\sigma|dz|^2$ on M, but only on the conformal structure of M. It follows from (10) that if f is harmonic, then the quadratic differential $\psi(z)\,dz^2$ defined by

$$\psi(z)\,dz^2 = \rho(u(z))\,u_z \bar{u}_z\,dz^2 \tag{11}$$

is a holomorphic quadratic differential on M.

We now compute the Laplacian of the (well-defined) function $|u_\theta|^2$ and $|\bar{u}_\theta|^2$ on M. Before doing so, however, we need to define the third covariant derivatives of f

$$du_{\theta\theta} + 2u_{\theta\theta}\theta_c - u_{\theta\theta} f^*\omega_c = u_{\theta\theta\theta}\theta + u_{\theta\theta\bar{\theta}}\bar{\theta}$$

and (12)

$$du_{\theta\bar{\theta}} - u_{\theta\bar{\theta}} f^*\omega_c = u_{\theta\bar{\theta}\theta}\theta + u_{\theta\bar{\theta}\bar{\theta}}\bar{\theta}.$$

Exterior differentiating the first formula in (7) we have

$$du_\theta \wedge \theta_c + u_\theta\,d\theta_c - du_\theta \wedge f^*\omega_c - u_\theta f^*\,d\omega_c = du_{\theta\theta} \wedge \theta + u_{\theta\theta}\,d\theta + du_{\theta\bar{\theta}} \wedge \bar{\theta} + u_{\theta\bar{\theta}}\,d\bar{\theta}.$$

Applying (1), (3), (5), (7), and rearranging terms we obtain

$$(u_{\theta\theta}\theta + u_{\theta\bar{\theta}}\bar{\theta}) \wedge \theta_c - (u_{\theta\theta}\theta + u_{\theta\bar{\theta}}\bar{\theta}) \wedge f^*\omega_c - u_\theta \frac{K}{2}\theta \wedge \bar{\theta} + u_\theta \frac{K'}{2}(f^*\omega) \wedge (f^*\omega)$$
$$= (du_{\theta\theta} + u_{\theta\theta}\theta_c) \wedge \theta + (du_{\theta\bar{\theta}} + u_{\theta\bar{\theta}}\bar{\theta}_c) \wedge \bar{\theta}.$$

Using (10), (5), and the fact that $\bar{\theta}_c = -\theta_c$ this becomes

$$u_{\theta\theta\bar{\theta}}\bar{\theta} \wedge \theta + u_{\theta\bar{\theta}\theta}\theta \wedge \bar{\theta} = -u_\theta \frac{K}{2}\theta \wedge \bar{\theta} + u_\theta \frac{K'}{2}(f^*\omega) \wedge (f^*\omega). \tag{13}$$

It follows from (5) and (6) that $(f^*\omega) \wedge (f^*\omega) = J(f)\,\theta \wedge \bar{\theta}$, so (13) implies

$$u_{\theta\theta\bar{\theta}} - u_{\theta\bar{\theta}\theta} = -u_\theta \frac{K'}{2} J(f) + u_\theta \frac{K}{2}. \tag{14}$$

We let Δ denote the Laplacian on M and note that for a function χ on M we have $\Delta\chi = 4\chi_{\theta\bar{\theta}}$ where χ_θ, $\chi_{\bar{\theta}}$ are defined by $d\chi = \chi_\theta\theta + \chi_{\bar{\theta}}\bar{\theta}$ and the Hessian of χ by $d\chi_\theta + \chi_\theta\theta_c = \chi_{\theta\theta}\theta + \chi_{\theta\bar{\theta}}\bar{\theta}$. We now assume that f is a harmonic map (i.e. $u_{\theta\bar{\theta}} = 0$) and compute

$$\Delta|u_\theta|^2 = 4(u_\theta \bar{u}_{\bar{\theta}})_{\theta\bar{\theta}} = 4(u_{\theta\theta}\bar{u}_{\bar{\theta}})_{\bar{\theta}} = 4u_{\theta\theta\bar{\theta}}\bar{u}_{\bar{\theta}} + 4u_{\theta\theta}\bar{u}_{\bar{\theta}\bar{\theta}}.$$

Applying (14) in this formula we have

$$\Delta|u_\theta|^2 = 4|u_{\theta\theta}|^2 - 2K' J(f)|u_\theta|^2 + 2K|u_\theta|^2. \tag{15}$$

If $|u_\theta|^2 \neq 0$, then (15) can be written more concisely

$$\Delta \log|u_\theta|^2 = -2K' J(f) + 2K. \tag{16}$$

It is a similar computation to show that if $|\bar{u}_\theta|^2 \neq 0$ we have

$$\Delta \log |\bar{u}_\theta|^2 = 2K' J(f) + 2K. \tag{17}$$

That is, whether or not $|\bar{u}_\theta|^2 = 0$ we have

$$\Delta |\bar{u}_\theta|^2 = 4|\bar{u}_{\theta\theta}|^2 + 2K' J(f)\,|\bar{u}_\theta|^2 + 2K\,|\bar{u}_\theta|^2. \tag{18}$$

Adding (15) and (18), noting (6), we obtain the formula for $\Delta e(f)$ derived in [3]

$$\Delta e(f) = 4(|u_{\theta\theta}|^2 + |\bar{u}_{\theta\theta}|^2) - 2K'(J(f))^2 + 2K\,e(f). \tag{19}$$

§ 2. Singularities and Some Local Results

In this section we derive a few basic facts concerning singular points of harmonic maps, and analyze the zeroes of $|u_\theta|^2$ and $|\bar{u}_\theta|^2$.

Proposition 2.1. Let M and M' be Riemann surfaces, and let $\rho(u)\,|du|^2$ be a conformal metric on M'. Let Ω be an open connected subset of M, and $f: \Omega \to M'$ be a harmonic map. The function $|u_\theta|^2$ (resp. $|\bar{u}_\theta|^2$) vanishes identically or has at most isolated zeroes on Ω. Moreover, if it does not vanish identically, there exist integers $n_p \geqq 0$ (resp. $m_p \geqq 0$) with $n_p = 0$ (resp. $m_p = 0$) except for isolated points $p \in \Omega$ such that if z is a coordinate centered at p, then $|z|^{-n_p}|u_\theta|^2$ (resp. $|z|^{-m_p}|\bar{u}_\theta|^2$) is a nonzero C^∞ function in a neighborhood of p.

Proof. We see from (1.10) that in a neighborhood of any $p \in \Omega$ we have $|(u_z)_{\bar{z}}| \leqq c_1 |u_z|$ for some constant c_1. It follows from the similarity principle (see [2]) that in a neighborhood, say $D = \{z: |z| \leqq a\}$, of p we have $u_z(z) = \zeta(z)\,h(z)$ where $h(z)$ is an analytic function of z, and $\zeta(z)$ is a nonvanishing Hölder continuous function. The analyticity of h enables us to shrink D so for $z \in D$ with $z \neq 0$ we have $\zeta(z) = u_z(z)/h(z)$ is a C^∞ function of z. Therefore, it follows from (1.16) that on $D \setminus \{0\}$ we have

$$\Delta \log |\zeta|^2 = -2K' J(f) + 2K. \tag{1}$$

To show that $|\zeta|^2$ is C^∞ on D, let η be the solution of the Dirichlet problem, $\Delta\eta = -2K' J(f) + 2K$ on D

$$\eta = \log|\zeta|^2 \quad \text{on } \partial D.$$

It follows from (1) that η-$\log|\zeta|^2$ is harmonic on $D \setminus \{0\}$ and Hölder continuous on all of D. Therefore, by a standard theorem on removable singularities η-$\log|\zeta|^2$ is harmonic and C^∞ on D. So we have $\log|\zeta|^2 = \eta$ is smooth which implies $|\zeta|^2$ is C^∞ on D. Thus we have

$$|u_\theta|^2 = \frac{\rho}{\sigma}\,|u_z|^2 = \frac{\rho}{\sigma}\,|\zeta|^2\,|h|^2 \quad \text{in } D.$$

Since h is analytic, this proves Proposition 2 for $|u_\theta|^2$. The proof for $|\bar{u}_\theta|^2$ is analogous.

The next proposition appears also in J. Wood's thesis, but we prefer to give a different proof which has more potential to generalize to higher dimensions.

Proposition 2.2. *If $f: \Omega \to M'$ is a harmonic map defined on an open connected subset $\Omega \subset M$ satisfying $J(f) \geqq 0$ on Ω, then either J is identically zero or the zeroes of $J(f)$ are isolated. Moreover, if there is a number l so that $\#(f^{-1}(q)) \leqq l$ for each regular value $q \in M'$ of f, then each isolated zero of $J(f)$ is a nontrivial branch point of f.*

Proof. Suppose J is not identically zero on Ω. We claim that for any point $p \in \Omega$, $J(f)(p) = 0$ if and only if $u_\theta(p) = \bar{u}_\theta(p) = 0$. The first statement will then follow from Proposition 2.1.

By (1.16) and (1.17), we have

$$\Delta \log \frac{|u_\theta|^2}{|\bar{u}_\theta|^2} = -4K' J(f) \tag{2}$$

when both $|u_\theta|^2$ and $|\bar{u}_\theta|^2$ are not zero.

Suppose now $J(f)(p) = 0$. Then $|u_\theta(p)|^2 = |\bar{u}_\theta(p)|^2$. If our assertion were false, $|u_\theta(p)|^2 = |\bar{u}_\theta(p)|^2 > 0$ and we can choose a small neighborhood V around p so that the inequalities continue to hold.

Since $J \geqq 0$ on Ω, and both $\bar{u}_\theta$ and K' are smooth, we can find positive constants $\tilde{c}$ and c so that

$$\begin{aligned} -4JK' &\leqq \tilde{c}\left[\frac{|u_\theta|^2}{|\bar{u}_\theta|^2} - 1\right] \\ &\leqq c \log \frac{|u_\theta|^2}{|\bar{u}_\theta|^2} \end{aligned} \tag{3}$$

holds in V.

Putting (2) and (3) together, we see that the non-negative function $h = \log \frac{|u_\theta|^2}{|\bar{u}_\theta|^2}$ verifies the inequality

$$\Delta h \leqq c h \tag{4}$$

in V.

By Lemma 6′ of [4], we find

$$\int_{|z| \leqq R} h(z) \leqq \tilde{\tilde{c}} h(0) \tag{5}$$

for some constants $\tilde{\tilde{c}} > 0$ and $R > 0$. Here z is a coordinate system around p.

Since h is non-negative and $h(0) = 0$, h is identically zero in a neighborhood of p. This fact easily demonstrates the claim.

To prove the final statement of Proposition 2.2, we suppose $p \in \Omega$ is an isolated zero of $J(f)$. Consider the set $f^{-1}(f(p))$. It follows from the fact that $J > 0$ near p, and our assumption $\#(f^{-1}(q)) < l$ for each regular value q of f, that

p is an isolated point of $f^{-1}(f(p))$. We choose a neighborhood U of p containing no other points of $f^{-1}(f(p))$. If the local degree of f at p is greater than one, then p is a nontrivial branch point of f. If the local degree of f at p is one, then f is one-one in a neighborhood of p, and by a theorem of Lewy [1] we have $J(f)(p)>0$, a contradiction. This completes the proof of Proposition 2.2.

In case M, M' are compact surfaces of genus g, g' and $f: M \to M'$ is a harmonic map of degree s, one may derive the following formulas by a standard residue argument from formulas (1.16), (1.17), and Proposition 2.1. These formulas have also been derived by Eells and Wood [6].

$$\sum_{p\in M} n_p = -s(4g'-4)+(4g-4) \qquad \text{provided } |u_\theta|^2 \not\equiv 0 \text{ on } M \tag{6}$$

$$\sum_{p\in M} m_p = s(4g'-4)+(4g-4) \qquad \text{provided } |u_\theta|^2 \not\equiv 0 \text{ on } M. \tag{7}$$

(n_p, m_p are defined in Proposition 2.1.)

3. Harmonic Diffeomorphisms of Compact Surfaces

In this section we examine the case M, M' compact, $s=1$, and $K' \leqq 0$. We prove that in this case the harmonic map f is in fact a diffeomorphism. Note that we do not assume a priori that f is homotopic to a diffeomorphism, but only a degree one map.

Theorem 3.1. *Suppose $f: M \to M'$ is harmonic and suppose M, M' are compact of genus g, g' with $g=g' \geqq 1$, $s=1$, and $K' \leqq 0$. Then f is a diffeomorphism with $J(f)>0$ on M.*

Proof. Since $s=1$, it follows that $|u_\theta|^2$ is not identically zero, so formula (2.6) becomes $\sum_{p\in M} n_p = 0$. That is, we have

$$|u_\theta|^2 > 0 \qquad \text{on } M. \tag{1}$$

We will now show that $J(f) \geqq 0$ on M. Suppose to the contrary that $D = \{p \in M: J(f)(p) < 0\}$ is not empty. We recall that $|u_\theta|^2 = \frac{1}{2}(e(f)+J(f))$ and $|\bar{u}_\theta|^2 = \frac{1}{2}(e(f)-J(f))$, so that $|\bar{u}_\theta|^2 > 0$ on D. Moreover, (1) implies that $|\bar{u}_\theta|^2 > 0$ on ∂D since $J(f)=0$ on ∂D. We subtract (1.17) from (1.16) to obtain

$$\Delta \log \frac{|u_\theta|^2}{|\bar{u}_\theta|^2} = -4K' J(f) \qquad \text{on } D \cup \partial D. \tag{2}$$

Thus $\log \frac{|u_\theta|^2}{|\bar{u}_\theta|^2}$ is smooth and superharmonic on D. Also $\log \frac{|u_\theta|^2}{|\bar{u}_\theta|^2} < 0$ in D and $\log \frac{|u_\theta|^2}{|\bar{u}_\theta|^2} = 0$ on ∂D. By the minimum principle it follows that $\log \frac{|u_\theta|^2}{|\bar{u}_\theta|^2} \equiv 0$ on D, i.e. $J(f) \equiv 0$ on D. This contradiction implies $J(f) \geqq 0$ on M.

To finish the proof we apply Proposition 2.2 which shows that $J(f)$ has at most isolated zeroes which are nontrivial branch points. Since $s=1$ these do not exist, so we have $J(f)>0$ on M and f is a diffeomorphism.

Corollary. *Suppose M and M' are Riemann surfaces of genus $g>1$. Let $\rho(u)|du|^2$ be a metric of nonpositive curvature on M'. Any map $\Phi: M \to M'$ of degree one is homotopic to a unique map $f: M \to M'$ which is harmonic with respect to $\rho(u)|du|^2$. Moreover f is a diffeomorphism with positive Jacobian.*

Proof. A restatement of the existence Theorem [3], uniqueness Theorem [7], and Theorem 1.

§ 4. Compact Surfaces with Boundary

In this section we consider the boundary value problem for harmonic mapping. We give conditions under which the solution of this problem is a diffeomorphism (see Theorem 5.1). Let M be a compact Riemann surface with boundary, and let M' be a compact Riemann surface (with or without boundary). Suppose $\rho(u)|du|^2$ is a metric of nonpositive curvature on M', and suppose $\varphi: M \to M'$ is a given map such that φ is a diffeomorphism of M onto its image. Moreover, suppose $\varphi(\partial M)$ is a curve (or union of curves) having non-negative geodesic curvature with respect to $\varphi(M^0)$, where $M^0 =$ interior of M. Let $f: M \to M'$ be a solution of the boundary value problem: f is harmonic with respect to ρ, $f=\varphi$ on ∂M, and f is homotopic to φ relative to ∂M. We will show that f is a diffeomorphism in M^0.

It follows from the maximum principle for the heat equation (see Hamilton [8]) that $f(M) \subset \varphi(M)$, so we may replace M' by $\varphi(M) \subset M'$ with the induced conformal structure. Having done this, we replace M' by its double, thus taking M' to be boundaryless. We now consider two cases. If M' has genus zero, our task is relatively simple, and we indicate the proof below in the proof of Theorem 5.1. We now concern ourselves with the case in which the genus of M' is at least one. We take a smooth extension of $\rho(u)|du|^2$ to the double, and note that although the curvature may not be non positive on all of M', it is still so on $f(M)$. Let $\lambda(u)|du|^2$ be the Poincare metric on M' if the genus of M' is greater than one, and the flat metric if M' has genus one.

Lemma 4.1. *With respect to $\lambda(u)|du|^2$, $f(\partial M)$ is a union of closed geodesics.*

Proof. Since M' is the double of $\varphi(M)$, M' has an anticonformal automorphism which fixes $\varphi(\partial M)$. The lemma thus follows from the invariance of the metric $\lambda(u)|du|^2$ and the fact that the fixed point set of an isometry is a union of geodesics.

We now consider the conformal structure on $\varphi(M)$ given by pulling back the conformal structure from M via φ^{-1}. Let M'' denote the double of this Riemann surface. Having done this, we consider a smooth path of conformal structures on $\varphi(M)$ so that, denoting the doubles by M'_t for $t \in [0,1]$, we have

$$M'_0 = M'' \quad \text{and} \quad M'_1 = M'.$$

Let λ_t be the Poincare metric on M'_t. As in Lemma 5.1 we conclude that $\varphi(\partial M)$ is a union of closed geodesics relative to λ_t for each $t\in[0,1]$. Let $f_\lambda: M\to M'$ be a solution of the boundary value problem for $\lambda(u)|du|^2$, and for each $t\in[0,1]$, let $f_t: M\to M'_t$ be a solution for λ_t on M'_t.

Lemma 4.2. *The family* $\mathscr{F}=\{f_t: t\in[0,1]\}$ *is a smooth family of maps satisfying* $f_0=\varphi$ and $f_1=f_\lambda$.

Proof. In order to prove smoothness, we prove that $\mathscr{F}$ is compact in the C^∞ topology, and we prove a uniqueness theorem.

To prove compactness, we note that $E(f_t)\leqq E_t(\varphi)$ where $E_t(\cdot)$ means energy taken with respect to λ_t. Thus it follows that $E(f_t)\leqq c_1$ for $t\in[0,1]$. The work of Hamilton [8] could now be used to prove compactness, but we prefer to give a direct proof.

We first establish a uniform Hölder estimate on f_t. The interior estimate follows from the inequality

$$\Delta e(f_t)\geqq -c_2\, e(f_t) \tag{1}$$

which holds under our curvature condition by (1.19). Standard estimates (see [13, Thm. 5.3.1]) imply that $e(f_t)$ is pointwise bounded on the interior of M, so that

$$\sup_K e(f_t)\leqq c_3 \tag{2}$$

for any compact $K\subset M^0$, where c_3 depends on the distance from K to ∂M.

To give an estimate near ∂M, we consider the function d defined on M as follows: Let $\tilde M'_t$ be the universal cover of M'_t, $\tilde M$ the universal cover of M, and Γ, Γ_t be groups of isometries on $\tilde M$, $\tilde M'_t$ so that $M=\tilde M/\Gamma$ and $M'_t=\tilde M'_t/\Gamma_t$. Consider the function $\tilde d$: $\tilde M'_t\times \tilde M'_t\to\mathbb{R}$ given by $\tilde d(p,q)=\text{distance}(p,q)$. Now Γ_t acts on $\tilde M'_t\times\tilde M'_t$ by acting jointly on each factor, and d is invariant under this action, so we have an induced function $\delta: (\tilde M'_t\times\tilde M'_t)/\Gamma_t\to\mathbb{R}$. By lifting f_t,φ we get maps $\tilde f_t$, $\tilde\varphi$: $\tilde M\to\tilde M'_t$, and we consider the map $\tilde M\to\tilde M_t\times\tilde M_t$ given by

$$p\mapsto(\tilde f_t(p),\,\tilde\varphi(p)).$$

Since f_t and φ are homotopic relative to ∂M, we get an induced map $\alpha: M=\tilde M/\Gamma\to(\tilde M'_t\times\tilde M'_t)/\Gamma_t$. Now define $d: M\to\mathbb{R}$ by $d=\delta\circ\alpha$. Now d has the property that $d\equiv 0$ on ∂M, and a direct calculation which we will give in a later paper implies that

$$\Delta d^2\geqq e(f_t)-c_4(1+d) \tag{3}$$

where c_4 is a positive constant depending on an upper bound for first and second derivatives of φ. (Note that d depends on t. But since the metric varies smoothly for $t\in[0,1]$, the constant c_4 can be chosen independent of t.) Choose a boundary coordinate z on M centered at $p\in\partial M$, and multiply (3) by a non-negative function ζ^2 to give

$$\zeta^2\Delta d^2\geqq\zeta^2 e(f_t)-c_4\zeta^2(1+d).$$

Integrating this over M, and integrating by parts we have

$$\int_M f^2 e(f_t)\,dV_M \leqq -2\int_M d\nabla\zeta^2\cdot\nabla d\,dV_M + c_4\int_M \zeta^2(1+d)\,dV_M \tag{4}$$

where we choose ζ to be a function satisfying

$$\zeta(z)=\begin{cases}1 & \text{for } |z|<\sigma/2, \quad |\nabla\zeta|\leqq\dfrac{c_5}{\sigma}\\ 0 & \text{for } |z|>\sigma.\end{cases}$$

Since $|\nabla d|^2\leqq e(f_t)+\tilde{c}_5$, inequality (4) implies

$$\int_{H_{\sigma/2}} e(f_t)\,dV_M \leqq \frac{c_6}{\sigma^2}\int_{H_\sigma\sim H_{\sigma/2}} d^2\,dV_M + c_6\left(\sigma^2+\sigma\left(\int_{H_\sigma} d^2\right)^{\frac{1}{2}}\right) \tag{5}$$

where we let $H_r=M\cap\{z:|z|<r\}$ for any $r>0$ which is sufficiently small. Since d vanishes on ∂M, the Poincare inequality gives

$$\int_{H_\sigma\sim H_{\sigma/2}} d^2\,dV_M \leqq c_7\sigma^2\int_{H_\sigma\sim H_{\sigma/2}} |\nabla d|^2\,dV_M.$$

Using the fact that $|\nabla d|^2\leqq e(f_t)+\tilde{c}_5$, we have

$$\int_{H_\sigma\sim H_{\sigma/2}} d^2\,dV_M \leqq c_8\sigma^2\int_{H_\sigma\sim H_{\sigma/2}} e(f_t)+c_8\sigma. \tag{6}$$

Combining (5) and (6) we obtain

$$\int_{H_{\sigma/2}} e(f_t)\,dV_M \leqq c_9\int_{H_\sigma\sim H_{\sigma/2}} e(f_t)\,dV_M + c_9\sigma$$

which implies

$$\int_{H_{\sigma/2}} e(f_t)\,dV_M \leqq \frac{c_9}{1+c_9}\int_{H_\sigma} e(f_t)\,dV_M + \frac{c_9}{1+c_9}\sigma. \tag{7}$$

Since $\dfrac{c_9}{1+c_9}<1$ is independent of σ, we can iterate (7) starting with some fixed value $\sigma=\sigma_0$, and we obtain

$$\begin{aligned}\int_{H_r} e(f_t)\,dV_M &\leqq c_{10}r^{2\alpha}\int_{H_{\sigma_0}} e(f_t)\,dV_M + c_{10}r\\ &\leqq c_{11}r^{2\alpha}\end{aligned} \tag{8}$$

for some number $\alpha\in(0,1)$. It follows from a well known lemma of C.B. Morrey that (8) implies a Holder estimate on f_t with exponent α. Combining this with the interior estimate (2) we have

$$\|f_t\|_\alpha \leqq c_{12} \tag{9}$$

where $\|.\|_\alpha$ represents the Holder norm with exponent α. In particular d is bounded and in (3) $c_3(1+d)$ can be replaced by a constant, so that (3) implies

$$\Delta d^2 \geqq e(f_t) - \tilde{c}_4. \tag{3'}$$

We next give a pointwise bound on $e(f_t)$. To estimate the normal derivatives of f_t on ∂M, we use a device Hildebrandt, Kaul, Widman [12] which they attribute to Bochner. Let $p \in \partial M$, and let $\mathbf{n}$ be the outward unit normal on M. The normal derivative $\partial f_t/\partial \mathbf{n}$ at p represents a vector at $f_t(p) \in M'_t$, so we move along the geodesic through $f_t(p)$ tangent to $\frac{\partial f_t}{\partial \mathbf{n}}(p)$ a fixed positive distance τ to a point $p' \in M'$, and we consider the function g defined on a neighborhood of $p \in M$ by $g(q) = \text{distance}\,(f_t(q), p')$. Because of our Hölder estimate on f_t, the function g is defined on a fixed neighborhood V (independent of t) of p. We note that since the distance function to $P' \in M'$ is convex on $f_t(V)$ (provided τ is sufficiently small, we have

$$\Delta g \geqq 0.$$

Following [12] we write $g = h + s$ *where* h is harmonic on V and satisfies $h \equiv g$ on ∂V, while s is subharmonic on V and vanishes on ∂V. We note that

$$\frac{\partial g}{\partial \mathbf{n}}(p) = -\left|\frac{\partial f_t}{\partial \mathbf{n}}(p)\right|$$

and that

$$\frac{\partial s}{\partial \mathbf{n}}(p) \geqq 0$$

by the maximum principle. We therefore have

$$\left|\frac{\partial f_t}{\partial \mathbf{n}}(p)\right| \leqq \frac{\partial h}{\partial \mathbf{n}}(p). \tag{10}$$

Since h is harmonic on V, Schauder boundary estimates imply that

$$\left|\frac{\partial h}{\partial \mathbf{n}}(p)\right| \leqq c_{13}$$

where c_{13} depends on a Hölder estimate for the tangential derivative of h along $\partial V \cap \partial M$. Now $h \equiv g$ on ∂V, and it is easily checked that the tangential derivatives of g are bounded along ∂M by the corresponding derivatives of φ (with the bound depending on τ). We therefore have

$$\left|\frac{\partial h}{\partial \mathbf{n}}(p)\right| \leqq c_{14} \tag{11}$$

where c_{14} is independent of t. Combining (10) and (11) gives the desired bound on the normal derivative of f_t. To complete our bound on $e(f_t)$ we note that (1) and (3′) can be combined to give

$$\Delta(e(f_t)+c_{15}d^2)\geqq -c_{16} \tag{12}$$

so that if γ is the solution of the Dirichlet problem

$$\Delta\gamma+c_{16}=0 \quad \text{on } M$$
$$\gamma=0 \quad \text{on } \partial M$$

we have $e(f_t)+c_{15}d^2-\gamma$ is a subharmonic function on M, so by the maximum principle

$$\sup_M e(f_t)\leqq \sup_M |\gamma|+\sup_{M\partial} e(f_t)$$

which by (11) implies

$$\sup_M e(f_t)\leqq c_{17}. \tag{13}$$

It is now a standard matter to estimate all higher derivatives of f_t. This completes the proof of compactness of $\mathscr{F}$.

To prove uniqueness of f_t, we suppose there are two maps f_t and f_t' both harmonic with respect to λ_t, and satisfying

$$f_t=f_t'=\varphi \quad \text{on } \partial M$$
$$f_t, f_t' \quad \text{homotopic to } \varphi \text{ relative to } \partial M.$$

Constructing the function $\hat{d}$ for f_t, f_t' in the same way that d was constructed for f_t, φ, we find

$$\Delta\hat{d}^2\geqq 0,$$

and $\hat{d}\equiv 0$ on ∂M. Therefore $\hat{d}\equiv 0$ on M, and $f_t\equiv f_t'$.

To prove that $\mathscr{F}$ is a smooth family, we suppose $t_0\in[0,1]$, and $t_i\to t_0$. Then every subsequence of $\{f_{t_i}\}$ has a subsequence which converges in C^∞ topology to f_{t_0} by compactness and uniqueness. This implies that $f_{t_i}\to f_{t_0}$ in C^∞ topology, proving that $\mathscr{F}$ is a smooth family. This completes the proof of Lemma 3.

Lemma 4.3. If f is a harmonic map on M with $f|_{\partial M}$ an orientation preserving homeomorphism onto a curve having non-negative geodesic curvature with respect to $\varphi(M)$, then

$$J(f)\geqq 0 \quad \text{on } \partial M.$$

Proof. It follows from the fact that f can be gotten from the heat equation, and from the maximum principle for the heat equation (see Hamilton [8]) that $f(M)\subset\varphi(M)$ which implies the lemma.

Proposition 4.1. f_λ is a diffeomorphism of M^0.

Proof. We use the family $\mathscr{F}$, and let

$$S=\{t\in[0,1]: f_t \text{ is a diffeomorphism of } M^0\}.$$

Note that $0\in S$, and we show that S is both open and closed. To prove S is closed, let $t_i\in S$ and $t_i\to t_0$, then by Lemma 5.2 we have $J(f_{t_0})\geqq 0$ which implies, by Proposition 2.2, that $J(f_{t_0})>0$ on M^0 and hence $t_0\in S$. (See also the last part of Section 2.) This proves S is closed.

To prove S is open, let $t_0\in S$, and note that (as $e(f_{t_0})\neq 0$ on ∂M) we have $e(f_{t_0})+J(f_{t_0})>0$ on M, so by Lemma 5.2 it follows that $e(f_t)+J(f_t)>0$ for t in a neighborhood of t_0. Thus we have $|(u_t)_\theta|^2>0$ on M, so we let $D=\{p\in M: J(f_t)<0\}$. By Lemma 4.3 we have $D\subset M^0$, so the argument of Proposition 4.1 is applicable, and we conclude that $J(f_t)>0$ on M^0, i.e. $t\in S$ for t in a neighborhood of t_0. This proves that S is open.

Therefore $S=[0,1]$, and f_λ is a diffeomorphism on M^0 finishing the proof of Proposition 4.1.

To prove that f is a diffeomorphism, we consider a one-parameter family of conformal metrics on M' given by

$$\lambda_t(u)\,|du|^2=\rho^t(u)\,\lambda^{1-t}(u)\,|du|^2.$$

Lemma 4.4. $\lambda_t(u)\,|du|^2$ is a conformal metric on M' satisfying $\lambda_0(u)=\lambda(u)$ and $\lambda_1(u)=\rho(u)$. Moreover, for $t\in[0,1]$, the curvature of λ_t is nonpositive, and $\varphi(\partial M)$ has non-negative geodesic curvature relative to λ_t with respect to $\varphi(M)$.

Proof. The first statement is clear, and (1.4) implies that

$$K'_t=tK'-(1-t)\leqq 0 \qquad \text{for } t\in[0,1].$$

Now if $\gamma(t)$ is a parametrization of the curve $\varphi(\partial M)$, we note that the geodesic curvature of $\varphi(\partial M)$ is given by

$$\gamma''+\text{linear term in connection form},$$

and since the connection form ω_c becomes additive for λ_t (see (1.2)), we see that up to a positive function, the geodesic curvature is a convex linear combination of that for λ and ρ. This proves the last statement of the lemma.

We now consider the family $\mathscr{F}_0=\{f_t:\ t\in[0,1]\}$ where $f_t: M\to M'$ is the harmonic map with respect to λ_t satisfying

$$f_t=\varphi \quad \text{on } \partial M$$
$$f_t \quad \text{homotopic to } \varphi.$$

Now the same reasoning as that in Lemma 4.2 implies that $\mathscr{F}_0$ is a smooth family of maps with $f_0=f_\lambda$ and $f_1=f$. We now prove the main result for the boundary value problem.

Theorem 4.1. Let M be a compact Riemann surface with boundary. Let M' be a compact Riemann surface (with or without boundary), and let $\varphi: M\to M'$ be a diffeomorphism of M^0 onto its image, and a homeomorphism of ∂M onto its image. Suppose M' has a metric $\rho(u)|du|^2$ of nonpositive curvature, and that $\varphi(\partial M)$ is a union of curves each having nonnegative geodesic curvature with respect to $\varphi(M)$. There exists a unique map $f\colon M \to M'$ which is harmonic with respect to ρ, which is a diffeomorphism of M^0 onto its image, and which satisfies

$$f=\varphi \quad \text{on } M$$

$$f \quad \text{homotopic to } \varphi \text{ relative to } \partial M.$$

Proof. The existence of f is due to Hamilton [8], and the uniqueness follows from the uniqueness part of the proof of Lemma 4.2. To prove that f is a diffeomorphism of M^0, we first consider the case in which M' has genus at least one. This case follows from the argument of Proposition 4.1 with the family $\mathscr{F}_0$ in place of $\mathscr{F}$.

In case M' has genus zero, we note that $\varphi(M)$ is simply connected, so that the uniformization theorem implies the existence of a conformal map $\psi\colon M \to \varphi(M)$ (anti-conformal if φ reverses orientation). We then choose a smooth family of homeomorphisms $\varphi_t\colon \partial M \to \varphi(\partial M)$ for $t \in [0, 1]$ so that $\varphi_0 = \psi$ and $\varphi_1 = \varphi$ (for example write ψ, φ in terms of arclength along $\varphi(\partial M)$ and take a convex combination). Let $f_t\colon M \to M'$ be the solution of the harmonic mapping problem with

$$f_t = \varphi_t \quad \text{on } \partial M \quad \text{and} \quad f_t \text{ homotopic to } \varphi.$$

Let $\mathscr{F}_1 = \{f_t\colon t \in [0, 1]\}$. Thus we have $f_0 = \psi$ and $f_1 = f$. The proof now follows by the techniques used in the proof of Proposition 4.1.

This completes the proof in case φ is a diffeomorphism of M, and the case where φ is merely a homeomorphism of ∂M follows by approximation.

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