

THE CLASSICAL PLATEAU PROBLEM AND THE TOPOLOGY OF THREE-DIMENSIONAL MANIFOLDS

THE EMBEDDING OF THE SOLUTION GIVEN BY DOUGLAS-MORREY AND AN ANALYTIC PROOF OF DEHN'S LEMMA

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INTRODUCTION

LET γ be a rectifiable Jordan curve in three-dimensional euclidean space. Answering an old question, whether γ can bound a surface with minimal area, Douglas[11] and Radó[45] (independently) found a minimal surface spanning γ which is parametrized by the disk. This minimal surface has minimal area among all Lipschitz maps from the disk into R^3 which span γ . The question whether this solution has branch points or not was finally settled by Osserman[42], who proved that there are no interior “true” branch points, and by Gulliver[15], who proved that there are no interior “false” branch points.

In 1948, Morrey[35] devised a new method to solve the Plateau problem for a map from the disk into a “homogeneously regular” Riemannian manifold. Moreover, he proved the interior regularity of the map in case the ambient manifold is regular, and that the map is real analytic if the ambient manifold is real analytic. The arguments of Osserman and Gulliver in addition show that Morrey’s solution has no interior branch point when the ambient manifold is three-dimensional. In 1951, Lewy[29] showed that if the Jordan curve γ is also real analytic, in a real analytic manifold, then any minimal surface with boundary γ is real analytic up to the boundary. Hence in this case Morrey’s map is real analytic map on the closed disk. (For a proof of Lewy’s Theorem in a general real analytic manifold, see Hildebrandt[23].)

In 1969, Hildebrandt[23] proved that the Douglas solution is smooth up to the boundary if the Jordan curve is smooth and regular. (Further improvements are due to Kinderlehrer[25], Nitsche[40], and Warschawski[55].) In [22], Heinz and Hildebrandt extended Hildebrandt’s result to minimal surfaces in general Riemannian manifolds.

Once we have boundary regularity, it makes sense to ask whether Douglas’ or Morrey’s solution of the Plateau problem has a boundary branch point or not. To date, this problem has not been settled. The first partial result in this direction is due to Nitsche[40], who showed that there are only a finite number of boundary branch points for minimal surfaces with smooth boundary in R^3 . This was then generalized by Heinz and Hildebrandt to smooth manifolds. Gulliver and Lesley[16] have also observed that the Douglas–Morrey solution for a real analytic curve in a real analytic manifold has no boundary branch point, using the previously mentioned result of Lewy.

Despite all these results, an interesting topological question remained unsolved, namely, under what conditions is the Douglas solution an embedded surface? It has been generally conjectured that when the Jordan curve is extremal, i.e. lies on the

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boundary of its convex hull, then the Douglas solution is indeed embedded [17]. The first progress in this direction was due to Radó [46], who proved that if the Jordan curve has a one-to-one convex projection onto a plane, then the Douglas solution is both unique and embedded. In case the Jordan curve is extremal and has absolute total curvature not greater than 4π , the question was also solved in the affirmative by Gulliver and Spruck [17]. Here the uniqueness result of Nitsche [39] (which in turn depends on a result of Barbosa and Do Carmo [5]) was used in an essential way, so that the proof surely cannot be extended to cover the general case. Recently Almgren and Thurston [3] constructed an unknotted Jordan curve which does not bound any embedded minimal disk. Hence some geometric assumption besides the topological obstruction is required if one is to show the minimal boundary surface is embedded.

There is a well-known theorem in topology which deals with similar situations. In 1910 Dehn [10] published a proof showing, for a Jordan curve on the boundary of a compact three-dimensional manifold, that if it is homotopically trivial in the manifold, then it bounds an embedded disk. Later, a mistake was found [26] in Dehn's paper, though the gap was eventually filled by Papakyriakopoulos [43]. In 1957, shortly after Papakyriakopoulos' proof appeared, Whitehead and Shapiro [49] gave another proof of Dehn's Lemma by using partial covering space arguments to resolve the singularities of an immersed disk. This proof of Whitehead and Shapiro will be crucial for us in determining the singularities of the Douglas-Morrey solution of Plateau's problem. In fact, our approach also gives a different proof of Dehn's Lemma.

To be precise, we prove that if a Jordan curve on the boundary of a three-dimensional compact convex manifold is homotopically trivial, then every Morrey solution to the Plateau problem for this Jordan curve is embedded. Our definition of convex manifold will be general enough to include the case of a bounded convex set in R^3 . In particular, every Douglas solution for an extremal Jordan curve is embedded. In a later paper we shall consider the more general situation when the boundary of the manifold has nonnegative mean curvature. In any case, since there is no topological obstruction for a manifold to have convex boundary, we realize the conclusion of Dehn's Lemma by a minimal disk.

Our proof goes as follows. First of all, we reduce the problem to the case of a real analytic Jordan curve on the boundary of a real analytic convex manifold. This reduction depends on a careful approximation procedure together with the estimates of Morrey, Hildebrandt, Heinz and Hildebrandt, and others.

In the real analytic case, we know from the above mentioned theorems of Lewy and Morrey that the map from the disk into the manifold is real analytic and hence simplicial with respect to some triangulations. Then we use the partial covering space argument of Whitehead and Shapiro to construct a tower of two-sheeted partial covering spaces, with the property that when we lift Morrey's map to the manifold at the top of the tower, the boundary γ of the lifted disk is contained in the boundary of this manifold, which itself is a disjoint union of spheres. If the lifted disk on the top of the tower is not embedded, we can push a disk on the boundary sphere which contains γ into the lifted disk, to acquire a folding curve. This will enable us to find a disk with area less than the original disk, which is a contradiction. Once we show that the lifted disk on the top of the tower is embedded, we are sure that the singularities corresponding to the next level of the tower consists of double points only. We can then use a cutting and pasting argument to prove that these double points do not exist. This process will be carried out in §4.

Since Morrey stated his theorem only for homogeneously regular manifolds, we show in detail in §1 how his solution can be used to provide a solution for compact convex manifolds with boundary. In §5, we solve the embedding problem for plane

domains in compact orientable convex manifolds. This embedding theorem gives a new topological result, namely a generalization of Dehn's Lemma from the disk to plane domains, when the three dimensional manifold is orientable. In §6 we prove that solutions of Plateau's problem for an extremal Jordan curve are either equal up to conformal reparametrization or else intersect along the Jordan curve. It should be noted that in our proof of the embedding of the solution of Douglas and Morrey, we only require the closure of the self-intersection set of the solution to be disjoint with the boundary of the disk. If the boundary curve is real analytic and extremal, this is automatic.

In our paper [34], using recent results of Sacks and Uhlenbeck [48] on the existence of minimal spheres, we prove that for a convex manifold there is a generating set consisting of embedded spheres, or of doubly covered embedded projective planes, for the second homotopy group considered as a π_1 -module. This implies the sphere and projective plane theorems in three-manifold theory. We also prove the loop theorem in [34], by considering a free boundary problem for minimal disks. Dehn's Lemma, the sphere theorem and the loop theorem are important in three-manifold theory. Our results give a differential geometric interpretation and a more or less canonical representation for the solutions by means of minimal surfaces. (In [34] we exploit this representation to prove some new theorems for finite group actions on a three-manifold.)

Finally, we mention that Almgren and Simon [2] proved that there exists an embedded minimal disk spanning an extremal Jordan curve in R^3 . Whether their solution is a Douglas solution is, however, not presently known. Tromba and Tomi [54] proved a similar result as Almgren and Simon, though by a different method of independent interest. Our work was finished in the fall of 1977.

§1. THE EXISTENCE OF MINIMAL SURFACES IN A CONVEX RIEMANNIAN MANIFOLD

Throughout this paper, we shall refer to both the Douglas solution and the Morrey solution simply as "a solution of Plateau's problem".

In order to apply Morrey's solution of Plateau's problem and to generalize the situation of an extremal curve in euclidean space, we define the concept of *convex manifold*.

We say that a smooth manifold is strictly convex if the second fundamental form (with respect to the outer normal) of its boundary is positive definite. We say that M is a *convex manifold* if M is a compact Lipschitz domain on some compact strictly convex smooth manifold N with the following properties. (i) There is a continuous function g defined on N which is convex in a neighborhood of the closure of N/M and satisfies $M = \{x \in N | g(x) \leq 0\}$ and $\partial M = \{x \in N | g(x) = 0\}$. (ii) There is a bi-Lipschitz homeomorphism φ from $\partial M \times [-1, 1]$ to a neighborhood of ∂M such that, for $x \in \partial M$ and $1 \geq t_2 \geq t_1 \geq -1$, we have $\varphi(x, 0) = x$ and $-c(t_2 - t_1) \geq g[\varphi(x, t_2)] - g[\varphi(x, t_1)]$ where c is a positive constant independent of t_1, t_2 and x . (iii) There is a smooth function \bar{g} defined on N , which is strictly convex in a neighborhood of the closure of N/M .

Note that according to our definition, any compact convex set M in euclidean space with nonempty interior is a convex manifold. Indeed, we may take N to be a large ball that contains this convex set and g to be the distance function from ∂M , for points outside M , and the negative of the distance function from ∂M for points inside M . Also, by assuming the origin is in the interior of M we may take φ to be the radial deformation. The function \bar{g} can be taken to be the square of the distance from the origin.

Let $\Gamma = \{\Gamma_1, \Gamma_2, \dots, \Gamma_k\}$ be a collection of disjoint oriented Jordan curves in a convex manifold M . Let $d_M(\Gamma)$ be the infimum of the areas of all possible C^1 -maps into M from a fixed plane domain bounded by k disjoint circles whose restriction on each properly oriented circle gives a parametrization of Γ_i . Let $d_M^*(\Gamma) = \infty$ if $k = 1$. Otherwise let $d_M^*(\Gamma) = \min(\sum_{i=1}^p d_M(\Gamma^i))$ where each Γ^i is a subcollection of curves selected from $\{\Gamma_1, \dots, \Gamma_k\}$ and the minimum is obtained by letting Γ^i vary in such a way that $\Gamma^1 \cup \dots \cup \Gamma^p = \{\Gamma_1, \dots, \Gamma_k\}$, $p > 1$ and $\Gamma^i \cap \Gamma^j = \emptyset$ for $i \neq j$.

THEOREM 1. *Suppose $d_M(\Gamma) < d_M^*(\Gamma)$. Then there exists a connected plane domain B bounded by circles so that f maps B into M and f maps each properly oriented boundary circle C_i of B monotonically onto Γ_i . Moreover f is harmonic and conformal on \tilde{B} , the interior of B , with area given by $d(\Gamma)$. Furthermore either f maps B into ∂M or f maps \tilde{B} into the interior of M .*

Proof. Since M is a domain of some smooth strictly convex manifold N , we shall first treat the special case when M is replaced by N . Clearly N is a smooth domain of some smooth Riemannian manifold \tilde{N} so that ∂N is the zero set of some smooth function h which is negative on N and is strictly convex in a neighborhood of the closure of $\tilde{N} - N$. Define a smooth function \tilde{h} on \tilde{N} by requiring \tilde{h} to be 1 on N and $1 + \exp(-1/h)$ on \tilde{N}/N . We may assume that $\tilde{h}|_{\partial \tilde{N}} = a > 1$. Then we define a new smooth metric on \tilde{N} by simply multiplying the original metric by the function $(a - 1)^2 (a - \tilde{h})^{-2}$. We claim that the resulting metric is homogeneously regular in the sense of Morrey [28].

To see this, we notice that there is a constant $\epsilon > 0$ such that for all $x \in \tilde{N}$ with (old) distance $\geq \epsilon$ from $\partial \tilde{N}$, the geodesic ball with center x and radius $\epsilon/2$ is diffeomorphic to the euclidean ball of radius $\epsilon/2$ so that under this diffeomorphism, the metric has the form $\sum_{i,j} g_{ij} dx^i \otimes dx^j$ where (g_{ij}) has eigenvalues bounded from above and below by two positive constants c_1 and c_2 , independent of x . By using a radial deformation, we may map the unit ball onto the $\epsilon/2$ -ball so that the pulled back metric tensor becomes $(\epsilon^2/4) \sum_{i,j} g_{ij} dx^i \otimes dx^j$.

Since we are assuming h is strictly convex, $|\nabla \tilde{h}|$ is not zero in a neighborhood of $\partial \tilde{N}$ and the function $(a - \tilde{h}(x))d(x, \partial \tilde{N})^{-1}$ is bounded from above and below by a positive constant in a neighborhood of $\partial \tilde{N}$. Here $d(x, \partial \tilde{N})$ is the (old) distance of x from $\partial \tilde{N}$. Consequently the new metric in the unit ball is obtained by multiplying $(\epsilon^2/4) \sum_{i,j} g_{ij} dx^i \otimes dx^j$ by a positive function bounded from above and below by a constant compatible to ϵ^{-2} . Hence the new metric on the unit ball is uniformly equivalent to the euclidean metric. This means that \tilde{N} is homogeneously regular in the sense of Morrey.

Let us now prove Theorem 1 for N in case $k = 1$. According to Morrey [35], we can prove the existence of f mentioned in the theorem except that $f(B)$ may not be a subset of N . To prove that f maps \tilde{B} into the interior of N , we consider the real valued function $h \circ f$ defined on B .

It is clear that for any tangent vector X in a neighborhood of $\tilde{N} \setminus N$ which is orthogonal to ∇h , the Hessian of h in the direction of X is positive. Then by direct computation, one can show that $\Delta(h \circ f) \geq -b|\nabla(h \circ f)|^2$ on a neighborhood of $\tilde{N} \setminus N$ where b is positive constant. We can now apply the following lemma.

LEMMA 1. *Let h be a continuous function defined on a bounded open set B such that the Dirichlet integral of h is finite and for some constant b , $\Delta h \geq -b|\nabla h|^2$ in the sense of distribution. Then $\sup_B h \leq \sup_{\partial B} h$.*

Proof of the Lemma. Suppose, on the contrary, $\sup_B h > \sup_{\partial B} h$. Let c be a number so that $\sup_B h > c > \sup_{\partial B} h$ and $b(\sup_B h - c) < 1$. Then over the set $B_c = \{x | h(x) \geq c\}$ we multiply the inequality $\Delta h \geq -b|\nabla h|^2$ by $(h - c)$. After integrating by parts, we find $-\int_{B_c} |\nabla h|^2 + b \int_{B_c} (h - c) |\nabla h|^2 \geq 0$ which implies that $\nabla h = 0$ over B_c . This is a contradiction and we have proved the lemma.

For our function $h \circ f$, we let $\Omega_\epsilon = \{x \in \bar{B} | h \circ f(x) > \epsilon\}$. Then when $\epsilon > 0$, the set $\bar{\Omega}_\epsilon$ is disjoint from ∂B . Since $\Delta(h \circ f) \geq -b|\nabla h \circ f|^2$ on Ω_ϵ , the above lemma shows that Ω_ϵ is empty. As $\epsilon > 0$ is arbitrary, we have therefore proved that $h \circ f \leq 0$ on B and f maps B into N .

To show that f maps \bar{B} to the interior of N or ∂N , we recall that h is strictly convex in a neighborhood of ∂N and the nonpositive function $h \circ f$ is subharmonic in a neighborhood of the set $\{x | h \circ f(x) = 0\}$. An easy application of the standard maximum principle shows that either f maps \bar{B} to the interior of N or $f(B) \subset \partial N$. In the latter case, $h \circ f$ is a constant and the strict convexity of h forces f to be a constant function. This contradiction shows the validity of Theorem 1 for strictly convex manifolds in case $k = 1$.

For $k > 1$, we proceed as follows. If $d_N(\Gamma) < d_N^*(\Gamma)$, then the above argument goes through without any changes. Otherwise $d_N(\Gamma) \geq d_N^*(\Gamma)$ and we may find $p > 1$ so that $\Gamma^1 \cup \dots \cup \Gamma^p = \{\Gamma_1, \dots, \Gamma_k\}$ with $\sum_{i=1}^p d_N(\Gamma^i) \geq d_N(\Gamma)$. Let p be the largest integer ($\leq k$) chosen in such a way. Then for each i , $d_N(\Gamma^i) > d_N^*(\Gamma^i)$ for otherwise we can split Γ^i and increase p .

Hence we can solve Plateau's problem for each Γ^i . According to our previous argument, we know that the solution must stay in N and so $d_N(\Gamma^i) = d_N^*(\Gamma^i)$. This implies that $\sum_{i=1}^p d_N(\Gamma^i) = \sum_{i=1}^p d_N^*(\Gamma^i) \leq d_N(\Gamma) \leq d_N^*(\Gamma)$ and $d_N^*(\Gamma) \leq d_N(\Gamma)$. This contradicts our assumption and we have proved Theorem 1 for strictly convex manifolds.

To prove Theorem 1 for a general convex manifold M , we consider M as a subdomain of N as in the definition of convex manifold. Then according to the above, we can solve the Plateau problem for N . It remains to prove that the solution stays in M . The proof is almost the same as above except that g is only assumed to be continuous convex. However we can still prove that $g \circ f$ is continuous and subharmonic in the sense of distribution in the domain $\Omega_\epsilon = \{x | g \circ f(x) \geq -\epsilon\}$ and this will enable us to apply the previous arguments. To see that $g \circ f$ is subharmonic in the sense of distribution, we recall that Green and Wu[14] have proved that g can be approximated uniformly on a neighborhood of $f(\Omega_{\epsilon/2})$ by smooth functions g_i such that the lowest eigenvalue of the Hessian of g_i is greater than ϵ_i with ϵ_i tending to zero when i tends to infinity. Using this fact, one can show by direct computation that $\lim_{i \rightarrow \infty} \inf_{x \in \Omega_{\epsilon/2}} \Delta(g_i \circ f)(x) \geq 0$. Therefore $g \circ f$ is subharmonic in sense of distribution over $\Omega_{\epsilon/2}$. This finishes the proof of Theorem 1.

§2. LOCAL PROPERTIES OF MINIMAL SURFACES

In this section, we record basic properties of minimal surfaces. Some of the lemmas are well-known. However, as they cannot be found readily in the literature, we report them here for our later references.

LEMMA 2. *Let B be an open plane domain. Let $f: B \rightarrow M$ be a minimal immersion of B into a three-dimensional manifold. Suppose that for some $p \neq q$, $f(p) = f(q)$. Then there are neighborhoods U_1 and U_2 of p and q respectively such that either $f(U_1) = f(U_2)$ or $f(U_1)$ and $f(U_2)$ intersect along a finite number of curves passing through $f(p)$ and the intersection is transversal at points other than $f(p)$.*

Proof. If the self-intersection is transversal at $f(p)$, the assertion is clear. Otherwise, the tangent planes at p and q are identical. We can introduce coordinates (x^1, x^2, x^3) so that the common tangent plane is the (x^1, x^2) plane. By taking small neighborhoods V_1 and V_2 of p and q respectively, we may assume that $f(V_1)$ and $f(V_2)$ are given by graphs of functions φ_1 and φ_2 over a small neighborhood U of the origin in (x^1, x^2) plane. Since both φ_1 and φ_2 satisfy the minimal surface equation in M , it is easy to verify that $\varphi_1 - \varphi_2$ satisfies a second order linear homogeneous elliptic equation with smooth coefficients.

By unique continuation and the theory of asymptotic behavior of the solution of elliptic equation ([2, 13]), we know that if φ_1 is not identical to φ_2 , $\varphi_1(x) - \varphi_2(x) = p_N(x) + O(|x|^{N-1+\epsilon})$ and $\nabla \varphi_1(x) - \nabla \varphi_2(x) = \nabla p_N(x) + O(|x|^{N-2+\epsilon})$ where $0 < \epsilon < 1$. Here $p_N(x)$ is a nonzero homogeneous polynomial of degree $N \geq 2$ satisfying a second order linear homogeneous elliptic equation with constant coefficients. It follows that by taking U smaller, we may assume that the origin is the only critical point of $\varphi_1 - \varphi_2$. A theorem of Kuo [27] (see [8]) shows that, up to a C^1 -change of coordinate system on x^1 and x^2 , $\varphi_1 - \varphi_2$ is given by $p_N(x)$. As $\varphi_1 - \varphi_2$ has no critical point besides the origin, the intersection of the graphs of φ_1 and φ_2 over U is the union of finite number of smooth curves intersecting at $f(p)$. The lemma follows from this.

LEMMA 3. *If $f: D \rightarrow M$ is a conformal harmonic immersion from the disk D into a three-dimensional Riemannian manifold such that f is continuous on D , $f|_{\partial D}$ is one to one, and $f(x) \notin f(\partial D)$ for $x \in \mathring{D}$, then for any two disjoint open sets U_1 and U_2 in D , $f(U_1) \neq f(U_2)$.*

Proof. Suppose this is not true. Then we can find two disjoint open sets U_1 and U_2 so that the restriction of f to either U_1 or U_2 is an embedding and $f(U_1) = f(U_2)$. Clearly we can find a one-to-one conformal map $h: U_1 \rightarrow U_2$ so that $f(x) = f \circ h(x)$ for all $x \in U_1$. We may also assume that U_1 is a disk such that $U_1 \subset D_\epsilon = \{x \mid |x| < 1 - \epsilon\}$ for some $\epsilon > 0$.

Let U be a largest open disk in D_ϵ so that U contains U_1 and there exists a locally one-to-one conformal map $k: U \rightarrow D$ with $f(x) = f(k(x))$ for all $x \in U$ and $k(x) = h(x)$ for $x \in U_1$. We claim that $U = D_\epsilon$. Otherwise there is a point $x \in \partial U \cap \mathring{D}_\epsilon$. As f is an immersion on \mathring{D} and $f(x) \notin f(\partial D)$, $f^{-1}(f(x))$ is a finite set of points $\{x_1, \dots, x_m\}$ in \mathring{D} . Let N be a neighborhood of $f(x)$ so that $f^{-1}(N)$ is a disjoint union of neighborhoods N_1, \dots, N_m of x_1, \dots, x_m respectively and f is an embedding on a neighborhood of each N_i . We may assume that $x = x_1$ and we define D_x to be a disk around x so that $D_x \subset N_1$, $f(D_x)$ is a piece of graph over the tangent plane of $f(D_x)$ at $f(x)$ and $f(\bar{D}_x) \subset N$. As $k(D_x \cap U)$ is connected, $k(D_x \cap U) \subset N_i$ for some i . Since $f(k(y)) = f(y)$ for $y \in D_x \cap U$, it is clear that k is one-to-one, continuous on $\overline{D_x \cap U}$ and k^{-1} is continuous on $k(\overline{D_x \cap U})$.

Since $f(x) = f(k(x))$, Lemma 2 shows that either $f(D_x) \cap f(N_i)$ is a union of finite number of curves or $f(D_x) \subset f(N_i)$ when we shrink the radius of D_x a little more. As $U \cap D_x$ is an open set and $f(U \cap D_x) \subset f(N_i)$, it must be true that $f(D_x) \subset f(N_i)$. We can then extend k to the disk D_x in an obvious manner so that k is conformal, continuous, locally one-to-one on D_x and $f(y) = f(k(y))$ for all $y \in D_x$.

Extending k on each boundary point, we can enlarge the definition of k . By the maximality of U , we must have $U = D_\epsilon$. Since $\epsilon > 0$ can be taken to be arbitrarily small, we can use unique continuation of conformal maps to extend k to D .

Since $f(x) \notin f(\partial D)$ and f is a homeomorphism on ∂D , the equation $f(k(x)) = f(x)$ shows that k is continuous on D and $k(x) = x$ for all $x \in \partial D$. Therefore k is the

identity mapping and the assumption that $k(U_1) \subseteq U_2$ with $U_1 \cap U_2 = \emptyset$ is violated. This finishes the proof of Lemma 3.

Lemma 2 and Lemma 3 together give the following corollary.

COROLLARY. *Under the assumption of Lemma 3, the self-intersection set of g cannot be a point, a curve with an end point in the interior of D or a set with nonempty interior.*

Remark. A rather complete study of the theory of branched minimal surfaces has been carried out by Gulliver *et al.* [18]. Their Theorem 6.3 implies the following: Let $f: \Omega \rightarrow M$ be a continuous conformal harmonic map of a compact Riemann surface into a three dimensional manifold such that $f|_{\partial\Omega}$ is one-to-one. Then if U_1 and U_2 are two disjoint open sets with $f(U_1) = f(U_2)$, then there exist smaller open subsets U'_1 contained in U_i for $i = 1, 2$ and an orientation reversing conformal transformation $g: \tilde{U}'_1 \rightarrow U'_2$ so that $f \circ g|_{U'_1} = f|_{U'_2}$. In particular f has no false branch points and no two disjoint open sets overlap in an orientation preserving way. Thus if two open sets overlap, there is a naturally induced $G: \tilde{\Omega} \rightarrow \tilde{\Omega}$ anti-conformal diffeomorphism with $f(G(x)) = f(x)$ and $G \circ G = id$. As in the proof of Lemma 3, G extends continuously to $\partial\Omega$ which is impossible. Hence Lemma 3 is true in much greater generality.

The following lemma can be considered as a generalization of Lemma 3.

LEMMA 4. *Let $f: D \rightarrow M$ and $\bar{f}: D \rightarrow M$ be two conformal harmonic immersions from the disk into a three-dimensional Riemannian manifold M so that both f and \bar{f} are continuous on D , $f|_{\partial D}$ and $\bar{f}|_{\partial D}$ are one-to-one and $f(\partial D) = \bar{f}(\partial D)$. Suppose $f(x) \notin \bar{f}(\partial D)$ for all $x \in \dot{D}$ and for some nonempty open sets U and V in D , $f(U) = \bar{f}(V)$. Then $f(D) \subset \bar{f}(D)$ and there exists a continuous one-to-one conformal map $k: D \rightarrow D$ such that $f(x) = \bar{f}(k(x))$ for $x \in D$. When \bar{f} is a Douglas-Morrey solution or when $\bar{f}(x) \notin \bar{f}(\partial D)$ for $x \in \dot{D}$, the map k is surjective.*

Proof. First of all, let us prove that $f(D) \subseteq \bar{f}(D)$. In fact, let \mathcal{O} be the interior of the set $f^{-1}[\bar{f}(D)]$ in \dot{D} . Then the hypothesis guarantees that \mathcal{O} is not empty. We claim that $\mathcal{O} = \dot{D}$.

Otherwise let $x \in \partial\mathcal{O} - \partial D$. Then the hypothesis guarantees that $f(x) \notin f(\partial D)$. Let $\bar{x} \in D$ be a point such that $\bar{f}(\bar{x}) = f(x)$. Then we can find two disks D_x and $D_{\bar{x}}$ around x and \bar{x} respectively so that $f|_{D_x}$ and $\bar{f}|_{D_{\bar{x}}}$ are embeddings with $f(D_x)$ and $\bar{f}(D_{\bar{x}})$ being graphs over the tangent planes of $f(x)$ and $\bar{f}(\bar{x})$ respectively. According to Lemma 1, either $f(D_x) \cap \bar{f}(D_{\bar{x}})$ is a union of finite curves or $f(D_x) \subset \bar{f}(D_{\bar{x}})$ when we shrink D_x a little. As $x \in \partial\mathcal{O}$, the first possibility is excluded and so we can enlarge the open set \mathcal{O} unless $\mathcal{O} = D$. This proves our claim $f(D) \subseteq \bar{f}(D)$.

Let W be a maximal open disk in $D_\epsilon = \{x | |x| < 1 - \epsilon\}$ so that we can find a locally one-to-one conformal map k from W into D with $f(x) = \bar{f}(k(x))$. Then for every point $x \in \partial W$, $f(x) \notin f(\partial D) = \bar{f}(\partial D)$. Since $\bar{f}^{-1}(f(x)) \in \dot{D}$, we can apply the previous arguments to conclude that $W = D$. Letting $\epsilon \rightarrow 0$, we obtain a locally one-to-one conformal map k from \dot{D} into \dot{D} so that $f(x) = \bar{f}(k(x))$ for $x \in D$. We assert that k is continuous on D . In fact if $x \in \partial D$ and $f(x) \in f(\partial D) = \bar{f}(\partial D)$, then there is a neighborhood N (in M) of $f(x)$ such that $\bar{f}^{-1}(N)$ is a subset of the disjoint union of open sets U_1, \dots, U_m in D where the diameters of each U_i can be arbitrary small depending on the choice of N . (As \bar{f} is one-to-one on ∂D , $\bar{f}^{-1}(f(x))$ is either a finite set

or a sequence of points converging to a unique point on ∂D . We may take U_1 to be a subset of a neighborhood of this unique point and the other U_i 's to be a subset of neighborhoods of the other points in the sequence. Clearly f maps a connected neighborhood of x into N so that the image of this neighborhood under k must be a subset of one of the above U_i 's. This means that k is continuous. The univalence of f on ∂D easily implies that k is one-to-one on ∂D . Being a conformal map from D into $k(D)$, one can use the argument principle to prove that k is globally one-to-one on D .

If $\bar{f}(x) \notin \bar{f}(\partial D)$ for $x \in \bar{D}$, it is clear that $k(D) = D$. In the other case when \bar{f} is a solution to Plateau's problem for $\bar{f}(\partial D)$, we can also conclude that $k(D) = D$ because the restriction of \bar{f} to $k(D)$ defines another minimal immersion of a disk into M which bounds $\bar{f}(\partial D)$ and which has smaller area than $\bar{f}|_D$ unless $k(D) = D$. The Jordan curve theorem then implies $k(\bar{D}) = \bar{D}$ and $k(\partial D) = \partial D$. This completes the proof of the lemma.

The same proof also shows the following.

LEMMA 4'. *Let f and \bar{f} be two conformal harmonic maps from S^2 into a three-dimensional Riemannian manifold. Suppose f is an immersion and suppose for some nonempty open sets U and V in S^2 , $f(U) = \bar{f}(V)$. Then there is a conformal map k from S^2 into itself, so that $\bar{f}(x) = f(k(x))$ for all $x \in S^2$.*

LEMMA 4''. *Let Ω and Ω' be two circular domains bounded by circles $\{\gamma_1, \dots, \gamma_m\}$ and $\{\gamma'_1, \dots, \gamma'_m\}$ respectively. Let f and \bar{f} be two conformal harmonic immersions from Ω into a three-dimensional Riemannian manifold such that both $f|_{\bigcup_{i=1}^m \gamma_i}$ and $\bar{f}|_{\bigcup_{i=1}^m \gamma'_i}$ are one-to-one maps. Suppose $f(\bar{\Omega}) \cap f(\bigcup_{i=1}^m \gamma_i) = \emptyset$ and \bar{f} is a solution to Plateau's problem for the system of Jordans curves $\bigcap_{i=1}^m f(\gamma'_i)$. If there are nonempty open sets U and V so that $f(U) = \bar{f}(V)$, then there is a conformal map k from Ω onto Ω' which is continuous and one-to-one on $\bar{\Omega}$ with $\bar{f}(x) = f(k(x))$ for $x \in \Omega$.*

Proof. We assume the outer circle of the circular domain Ω is the unit circle. By shrinking the unit circle a little and expanding the other circles a little, we obtain circular domains Ω_ϵ so that Ω_ϵ is in the interior of Ω and Ω_ϵ approaches to Ω as ϵ tends to zero. Let p be a point in the interior of one of those inner circles different from the boundary of Ω so that any ray issued from p will not be tangential to more than two inner circles. Then we consider a maximal simply connected region ω in Ω_ϵ bounded by part of circular arcs of Ω_ϵ and line segments $\sigma_1^\epsilon, \dots, \sigma_n^\epsilon$ of rays issuing from p so that we can define a locally one-to-one conformal map k from this region to Ω' with $\bar{f}(x) = f(k(x))$ for x in the region. (It is easy to use the argument of Lemma 4 to show that ω is nonempty.) We claim that this region is equal to Ω_ϵ minus the segments $\sigma_1^\epsilon, \dots, \sigma_n^\epsilon$. Otherwise, one of these segments, say σ_1^ϵ , will separate an open disk around every point of σ_1^ϵ into two pieces, one piece belongs to ω and one piece does not. If the segment σ_1^ϵ is not tangential to any inner circle or if there is no other σ_i^ϵ which is on the same ray as σ_1^ϵ , then the arguments in Lemma 4 can easily be adapted to prove a contradiction to the maximality of ω . Hence we may assume that there is another line segment, say σ_2^ϵ , such that both σ_1^ϵ and σ_2^ϵ are on the same ray and both of them are tangential to an inner circle at the same point q . Our choice of p makes sure that $\sigma_1^\epsilon \cup \sigma_2^\epsilon$ is the part of the above ray that is in Ω_ϵ . It is clear that in a neighborhood of this ray, ω must be on the same side of σ_1^ϵ and σ_2^ϵ . We can then fix σ_2^ϵ and continue the domain of definition of k starting from σ_1^ϵ to the other side of σ_2^ϵ . As a result, we have formed larger domain of the same type as ω where we replaced $\sigma_1^\epsilon \cup \sigma_2^\epsilon$ by σ_1^ϵ . This contradicts the maximality of ω and our claim is proved.

A moment's reflection shows that n , the number of line segments, is not greater

than m . Hence by passing to a subsequence of ϵ , we may assume that the number n is the same and the line segments converge when ϵ tends to zero. In the above construction, we could have fixed the value of k in a small open set for all $\epsilon > 0$. Hence by unique continuation, k has the same value at those points not on the limit of the above line segments, when ϵ tends to zero. Finally we have constructed a locally one-to-one conformal map k from a domain ω , which is $\dot{\Omega}$ minus some line segments, to Ω' so that $f(x) = \bar{f}(k(x))$.

We claim that k is in fact one-to-one. Otherwise there are two disjoint open sets U and V in Ω so that $f(U) = f(V)$. Then the above argument can be used again to find a non-trivial locally one-to-one conformal map h from Ω minus some line segments into Ω so that $f(x) = f(h(x))$. Since $f(\dot{\Omega}) \cap f(\cup \gamma_i) = \emptyset$, the arguments in Lemma 4 show that we can extend h to $\partial\Omega$ continuously with $h(x) = x$ for $x \in \partial\Omega$. Hence h is the identity map from Ω to Ω and we have arrived at a contradiction.

If there is an open set of Ω' in the complement of the image of k , then the energy of f will be strictly less than the energy of \bar{f} . This contradicts the assumption that \bar{f} is a solution to Plateau's problem and so $\Omega' = \overline{k(\dot{\Omega})}$.

Let x be a point on $\partial\Omega$ which is not an end point of the line segments. Then the arguments of Lemma 4 show that we can extend k continuously to a neighborhood N_x of x such that the extended k is still one-to-one. For convenience, we take N_x to be the intersection of a small disk with Ω . Then we claim that k maps the part of $\partial\Omega$ in N_x to $\partial\Omega'$. In fact, as $f(\dot{\Omega}) \cap f(\cup \gamma_i) = \emptyset$, there is a neighborhood $\tilde{N}_{f(x)}$ of $f(x)$ so that $f^{-1}(\tilde{N}_{f(x)}) \subset N_x$. If $k(x)$ were an interior point of Ω' , we choose a disk $D_{k(x)}$ around $k(x)$ so that \bar{f} is an embedding on $D_{k(x)}$ and $\bar{f}(D_{k(x)}) \subseteq \tilde{N}_{f(x)}$. As the image of k is dense in Ω' , it follows that $D_{k(x)} \subseteq \overline{k(N_x)}$. Since k is one-to-one, $k(N_x)$ is a simply connected domain bounded by the Jordan curve $k(\partial N_x)$. Therefore $D_{k(x)} \subseteq k(N_x)$. As $k(x) \in k(\partial N_x)$, this is a contradiction. Hence we have established that $k(\partial\Omega \cap N_x) \subset \partial\Omega'$. From this fact, it is easy to prove that k can be extended continuously to every point on $\partial\Omega$ and the extended map is still one-to-one.

Now we claim that $\bar{f}(\dot{\Omega}') \cap \bar{f}(\partial\Omega') = \emptyset$. In fact, for any point $y \in \dot{\Omega}'$, we can find a sequence $\{x_i\} \subset \omega$ so that $\lim x_i = x$, $\lim_{i \rightarrow \infty} k(x_i) = y$ and $f(x) = \bar{f}(y)$. From the assertion $k(\partial\Omega) \subset \partial\Omega'$, we conclude that $x \in \dot{\Omega}$. Our assumption of f then implies that $\bar{f}(y) = f(x) \notin f(\partial\Omega) = \bar{f}(\partial\Omega')$. This proves our claim.

Once we have established $\bar{f}(\dot{\Omega}') \cap \bar{f}(\partial\Omega') = \emptyset$, we can repeat the above process to construct a one-to-one conformal map \bar{k} from Ω' minus some line segments to Ω so that the equation $f(\bar{k}(x)) = \bar{f}(x)$ holds. Furthermore we may require $\bar{k} \circ k(x) = x$, $k \circ \bar{k}(x) = x$ for x in some small open sets of Ω and Ω' respectively. We may also assume that the line segments form a subset of a finite union of rays emanating from a fixed point in the interior of an inner circle of Ω' . By the unique continuation of conformal map, $\bar{k} \circ k(x) = x$ and $k \circ \bar{k}(x) = x$ for all points x where $k \circ \bar{k}$ and $\bar{k} \circ k$ are defined.

Along each line segment on the boundary of ω , the map k is smooth when we approach from each side to an interior point. Let p be an interior point where k is not smooth. Then there are two possible values of $k(p)$ obtained by considering them as limiting values of k from each side of the line segment. We claim that both these values are points on one of the line segments that appear in the definition of \bar{k} . Otherwise one of these values, say $k(p-)$, is in $\dot{\Omega}'$ minus those line segments. The mapping \bar{k} is smooth in a neighborhood of $k(p-)$. Since $\bar{k}[k(p-)] = p$, \bar{k} maps a small neighborhood of $k(p-)$ biholomorphically onto a neighborhood of p . The equation $k \circ \bar{k}(x) = x$ shows that k is smooth at p which contradicts our definition of p . This proves our claim and the image of the nonsmooth points under k is always a subset of the line segments for the definition of \bar{k} .

Since we may change the domain of definition of \bar{k} by changing the point for emulating rays, we conclude that the image of nonsmooth points under k is a set of finite number of points. Therefore k can only be non-smooth at finite numbers of points. The Riemann extension theorem shows that k is smooth on $\hat{\Omega}$ and the previous arguments show that k is continuous one-to-one on Ω . This completes the proof of Lemma 4".

We shall need the following unique continuation property of minimal surfaces.

LEMMA 5. *Let Σ_1 and Σ_2 be two minimal surfaces in a three-dimensional manifold M so that $\partial\Sigma_1 = \partial\Sigma_2$ and some part of $\partial\Sigma_1$ is a smooth curve σ . Suppose that at each point of σ where Σ_1 and Σ_2 are immersed, the tangent planes of both Σ_1 and Σ_2 coincide and the inward normal of Σ_1 and Σ_2 agree along σ . Then some nonempty open set of Σ_1 is equal to some nonempty open set of Σ_2 .*

Proof. As in Lemma 2, we can choose a local coordinate system (x^1, x^2, x^3) around a point p of σ so that σ is given by the line $x^1 = x^3 = 0$ and Σ_1 and Σ_2 are given by the graphs of smooth functions φ_1 and φ_2 respectively. (Note that by the boundary regularity of Hildebrandt[23], Σ_1 and Σ_2 are smooth in a neighborhood of σ . On the other hand, Nitsche[40] (see also Heinz–Hildebrandt[22]) proves that there are only finite number of branch points on a smooth arc so that we can assume both Σ_1 and Σ_2 are immersed at p .) The hypothesis that Σ_1 and Σ_2 are tangent to each other along σ means that $\partial\varphi_1/\partial x^2 = \partial\varphi_2/\partial x^2$ along σ . Since $\varphi_1 - \varphi_2 = 0$ on σ and $\varphi_1 - \varphi_2$ satisfies a linear homogeneous elliptic equation (with smooth coefficient in a neighborhood of p), it is clear that, by successively differentiating the equation, all derivatives of $\varphi_1 - \varphi_2$ vanish along σ . One can extend $\varphi_1 - \varphi_2$ to be zero on the other side of the domain in the (x^1, x^2) plane so that $\varphi_1 - \varphi_2$ is a smooth solution of a linear homogeneous elliptic equation. The unique continuation property[2] then implies that $\varphi_1 - \varphi_2 \equiv 0$ in the neighborhood where both φ_1 and φ_2 are defined. This proves Lemma 5.

For later purposes, we define the concept of *folding curve* for a surface in the following manner. Let f be a Lipschitz map from the disk $D(r)$ of radius r into a three-dimensional manifold M such that the restriction of f to either the right hand disk $\{(x, y) | x \geq 0, x^2 + y^2 \leq r\}$ or the left hand disk $\{(x, y) | x \leq 0, x^2 + y^2 \leq r\}$ is C^1 up to the boundary and is an immersion. If for each point $(0, y)$ with $y < r$, either the plane spanned by $f_*(\partial/\partial y)|_{(0, y)}$ and $\lim_{x \rightarrow 0} f_*(\partial/\partial x)|_{(x, y)}$ is transversal to the plane spanned by $f_*(\partial/\partial y)|_{(0, y)}$ and $\lim_{x < 0} f_*(-(\partial/\partial x))|_{(x, y)}$, or $\lim_{x < 0} f_*(\partial/\partial x)|_{(x, y)}$ is a positive multiple of $\lim_{x < 0} f_*(-(\partial/\partial x))|_{(x, y)}$. Then we say that the mapping f has a *folding curve* along the image of the y -axis.

There is a situation where the folding curve arises very frequently later. We describe it as follows. Suppose we have a Lipschitz map f from the unit disk D into a three-dimensional manifold M such that the restriction of f to both the right hand disk and the left hand disk is C^1 up to the boundary and is an immersion. Suppose we can choose a local coordinate (x^1, x^2, x^3) around the point $f(0)$ such that the image of the y -axis under f is the x^3 -axis. Suppose there are distinct planes P_1, P_2, \dots, P_l ($l \geq 2$) in (x^1, x^2, x^3) space which pass through the x^3 -axis and decompose a small ball B with center at the origin into many cells. Suppose there is a sequence of Lipschitz maps $\{f_i\}$ from D into M so that (i) the image of f_i does not intersect $\bigcup_{j=1}^l P_j$ for all i . (ii) Each f_i is C^1 on both the right closed half disk and the left closed half disk. (iii) The sequence $\{f_i\}$ converges in C^1 -norm on both closed half disks to f . In this case, we have the following lemma.

LEMMA 6. If $f(D) \cap B \subseteq \bigcup_{j=1}^l P_j$, then either $\lim_{\substack{x < 0 \\ x \rightarrow 0}} f_*(\partial/\partial x)|_{(x,y)}$ is a positive multiple of $\lim_{\substack{x < 0 \\ x \rightarrow 0}} f_*(\partial/\partial x)|_{(x,y)}$ or there are distinct planes P_i and P_j so that for y small, $\lim_{\substack{x < 0 \\ x \rightarrow 0}} f_*(\partial/\partial x)|_{(x,y)} \in P_i$ and $\lim_{\substack{x < 0 \\ x \rightarrow 0}} f_*(-\partial/\partial x)|_{(x,y)} \in P_j$. In particular, f has a folding curve along the x^3 -axis when we restrict f to a small disk around the origin.

Proof. As $f(D) \cap B \subseteq \bigcup_{j=1}^l P_j$, it is clear that along y -axis, $\lim_{\substack{x < 0 \\ x \rightarrow 0}} f_*(\partial/\partial x)|_{(x,y)} \in P_i$ and $\lim_{\substack{x < 0 \\ x \rightarrow 0}} f_*(-\partial/\partial x)|_{(x,y)} \in P_j$ for some i, j . (As $f_*(\partial/\partial y) = \partial/\partial x^3$, the planes P_i and P_j are determined by the above property.) If $\lim_{\substack{x < 0 \\ x \rightarrow 0}} f_*(\partial/\partial x)|_{(x,y)}$ were not a positive multiple of $\lim_{\substack{x < 0 \\ x \rightarrow 0}} f_*(\partial/\partial x)|_{(x,y)}$ and P_i were equal to P_j , then the image of the x -axis under f is a nontrivial curve contained in P_i whose projection into (x^1, x^2) plane is a line segment which contains the origin in its interior. When n is large, the image of the x -axis under f_n is then a nontrivial curve whose projection into (x^1, x^2) plane is close to the above line segment. Clearly this will mean that the projected curve must intersect some P_k with $P_k \neq P_i$ on the (x^1, x^2) plane. In particular in B , the image of the x -axis under f_n is not disjoint from $\bigcup_{i=1}^l P_i$ which is a contradiction.

LEMMA 7. Let f be a map from a plane domain D into a three-dimensional manifold. If f has a folding curve as defined above, then f cannot have minimal area among all Lipschitz maps which are piecewise C^1 from D into M and have the same boundary value as f .

Proof. Suppose not, then it is clear that at points where f is an immersion, the mean curvature of f is zero there. We shall find a deformation of f in M which decreases the area of f . We construct a (continuous) deformation vector field so that E has compact support in D and so that along the y -axis, $\langle E, f_*(\partial/\partial y) \rangle|_{(0,y)} = 0$, $\lim_{\substack{x > 0 \\ x \rightarrow 0}} \langle E, f_*(\partial/\partial x) \rangle|_{(x,y)} > 0$ and $\lim_{\substack{x < 0 \\ x \rightarrow 0}} \langle E, f_*(-\partial/\partial x) \rangle|_{(x,y)} > 0$. This is possible because of the definition of folding curve. We can also require E to be C^1 on each half disk of the (x, y) plane and C^1 on the y -axis. As the mean curvature of f is zero on each half disk, the first variation formula (see [22]) shows that the first variation on the right half disk is given by the integral of $-\langle E, f_*(\partial/\partial x) \rangle$ along the folding curve and the first variation on the left half disk is given by the integral of $-\langle E, f_*(-\partial/\partial x) \rangle$ along the folding curve. Hence when we deform f by E , the area is strictly decreasing. This gives a contradiction and proves the lemma.

Remark. If the self-intersection of a minimal immersed disk is nontrivial and does not go up to the boundary, then according to the corollary of Lemma 4, we can find a point p in M so that the set $f^{-1}(p) = \{p_1, \dots, p_k\}$ has the following property. There are neighborhoods U_i of each p_i so that the U_i 's are mutually disjoint and $f(U_i)$'s are mutually transversal to each other. If we know that the map f is real analytic, then by the triangulating property of real analytic set, we can assume that near p all the $f(U_i)$'s pass through a real analytic curve which contains p in its interior. If we know that the self-intersection set of f is a compact subset of D , then it is easy to verify that in a small neighborhood of p , the set $f(D)$ is given by the union of the $f(U_i)$'s. Hence the local picture of $f(D)$ near p is the same as the one described in the paragraph before Lemma 6. If we push the boundary of a regular neighborhood into $f(D)$, we shall therefore obtain a folding curve. Lemma 7 can then be applied in this situation.

When we pass from real analytic metric to smooth metric, we need the following.

LEMMA 8. *Let M be a three-dimensional manifold (possibly with boundary). Let D be a bounded plane domain whose boundary is a disjoint union of Jordan curves $\{\gamma_i\}$ and $f: D \rightarrow M$ be a Douglas-Morrey solution to Plateau's problem for the disjoint union of Jordan curves $\{f(\gamma_i)\}$. Let D' be a proper sub-domain of D so that D' is diffeomorphic to D and f restricted to each component of $\partial D'$ is one-to-one. Suppose there is a smooth arc σ contained in both $\partial D'$ and the interior of D . If $f(\partial D') \cap f(\dot{D}') = \emptyset$ and g is any Douglas-Morrey solution with boundary given by $f(\partial D')$, then g is equal to f up to a conformal reparametrization of D' .*

Proof. By nonconformal reparametrization and applying the boundary regularity theorem of Hildebrandt, (Hildebrandt[23], Heinz-Hildebrandt[22]), we may assume $f(x) = g(x)$ for all $x \in \partial D'$ and both f and g are smooth on σ . Since the theorems of Nitsche[40] and Heinz-Hildebrandt[22] show that there are only finite number of branch points of f or g on σ , we may assume both f and g are immersion in a neighborhood of σ . Define a new map \tilde{f} from D into M by requiring $\tilde{f}(x) = f(x)$ for $x \in D \setminus D'$ and $\tilde{f}(x) = g(x)$ for $x \in D'$. Then it is clear that \tilde{f} is of class H_2^1 as defined in Morrey[35]. If \tilde{f} does not have a folding curve along σ , then Lemma 5 is applicable to the minimal surfaces $f(D')$ and $g(D')$ and one concludes that some nonempty open sets of $f(D')$ is equal to some nonempty open sets of $g(D')$. Lemma 4'' then shows up to a conformal reparametrization of D' (before reparametrization) our original mapping f is equal to our original mapping g . It remains to prove that \tilde{f} has no folding curve along σ . Otherwise by applying Lemma 7 to a proper subarc of σ , we can find a H_2^1 continuous map from D into M which has the same boundary values as \tilde{f} and which has area strictly less than the area of \tilde{f} . Since the area of g is equal to the area of $f|_{D'}$, the area of f is equal to the area of \tilde{f} . Hence we have found a contradiction to the fact that f is a Douglas-Morrey solution to $f|\partial D$. This proves Lemma 8.

Finally, we note that if f maps a bounded plane domain D with smooth boundary into a smooth manifold M so that the following conditions are satisfied

- (i) $f \in C(D) \cap C^2(\dot{D})$
- (ii) f restricted to each properly oriented boundary of D described a monotonic representation of an oriented Jordan curve. This means that as a boundary point of ∂D describes the boundary component monotonically, the image point describes the image curve monotonically.
- (iii) f is harmonic, i.e. for a local coordinate system (u, v) in \dot{D} and local coordinate system (x^1, \dots, x^n) in M , we have the elliptic system

$$\begin{aligned} \frac{\partial}{\partial u} \left(\sum_i g_{ij}(x) \frac{\partial x^j}{\partial u} \right) + \frac{\partial}{\partial v} \left[\sum_i g_{ij}(x) \frac{\partial x^j}{\partial v} \right] \\ = \frac{1}{2} \sum_{k,l} \frac{\partial g_{kl}(x)}{\partial x^j} \left[\frac{\partial x^k}{\partial u} \frac{\partial x^l}{\partial u} + \frac{\partial x^k}{\partial v} \frac{\partial x^l}{\partial v} \right] \end{aligned} \quad (2.1)$$

for $j = 1, \dots, n$.

- (iv) f is conformal, i.e.

$$\sum_{k,l} g_{kl}(x) \frac{\partial x^k}{\partial u} \frac{\partial x^l}{\partial u} = \sum_{k,l} g_{kl}(x) \frac{\partial x^k}{\partial u} \frac{\partial x^l}{\partial v} \quad (2.2)$$

and

$$\sum_{k,l} g_{kl}(x) \frac{\partial x^k}{\partial u} \frac{\partial x^l}{\partial v} = 0. \quad (2.3)$$

Then we have the following.

LEMMA 9. *f is one-to-one on ∂D .*

Proof. This is a consequence of the theorems of Heinz–Hildebrandt [22]. Suppose that *f* is not one-to-one on ∂D . Then since *f* is monotonic on the boundary, we may assume that there exists a nontrivial arc σ on ∂D so that $f(\sigma)$ is a point. By the arguments of [22], we know that in a neighborhood of σ ,

$$|\Delta f| = \left| \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} \right| \leq \beta |\nabla f|^2 \quad (2.4)$$

for some positive constant β .

By choosing $f(\sigma)$ to be the origin of a local coordinate in M , we may also assume

$$f(\sigma) = 0. \quad (2.5)$$

The Hilfssatz of [21] shows that *f* is C^1 in a neighborhood of σ by shrinking σ a little bit. Moreover when *f* is not a constant vector near σ , one has the asymptotic representation near a point $w_0 \in \sigma$:

$$\frac{\partial f}{\partial u} - i \frac{\partial f}{\partial v} = \bar{a}(w - w_0)^l + o(|w - w_0|^l) \quad (2.6)$$

where l is a positive integer and \bar{a} is a nonzero constant vector.

If $\dot{\sigma}$ is the tangent vector of σ , then $f_*(\dot{\sigma}) = 0$. The conformality condition of *f* makes the image of the vector normal to $\dot{\sigma}$ have length zero also. Hence (2.6) gives a contradiction and Lemma 9 is proved.

§3. APPROXIMATING A SMOOTH METRIC BY REAL ANALYTIC METRICS

In this section, we reduce the embedding problem for a smooth manifold with Jordan curves by a real analytic manifold with real analytic curves. Hence we set up the following hypothesis which will be proved in the next sections.

Hypothesis H. Suppose M is a compact real analytic three-dimensional manifold with real analytic convex boundary. Suppose γ is an analytic Jordan curve in ∂M . Then any solution to Plateau's problem for γ is embedded.

THEOREM 2. *Let M be a convex three-dimensional manifold as defined in §1. Let σ be a Jordan curve in ∂M . Then σ bounds an embedded solution to Plateau's problem.*

Proof. Let $f: D \rightarrow M$ be a solution to Plateau's problem for γ . The proof of Theorem 1 shows that the interior of D is mapped by *f* into ∂M or it is mapped into \dot{M} . If the image of the interior of D is completely contained in ∂M , then clearly *f* is an embedding. (In this case, γ is homotopic to zero on a component of ∂M . By the classification of surfaces, γ must separate that component of ∂M into two regions Ω_1 and Ω_2 where Ω_1 is a disk. By using the open mapping theorem, one can prove that

either $f(D) = \Omega_1$, Ω_2 or $\Omega_1 \cup \Omega_2$. In the first two cases, the open mapping theorem also shows that $f(\tilde{D}) \cap \gamma = \emptyset$ and hence $f|_{\tilde{D}}$ is a covering projection. Since the area of the pulled back metric is finite, the number of the sheets for the covering is finite. As finite groups cannot act freely on the disk, the mapping f is in fact a homeomorphism. The case $f(D) = \Omega_1 \cup \Omega_2$ cannot happen because Ω_1 , which is a disk, has area strictly less than $\Omega_1 \cup \Omega_2$. Hence we may assume that $f(D)$ is not a subset of ∂M . We shall use the terminology that we used in Section 1 in the definition of convex manifold.

It is well-known [37, 38] that N admits a real analytic structure and a real analytic metric. In fact we can approximate the original smooth metric by real analytic metrics in C^∞ -norm. Call the original metric $\Sigma_{ij}g_{ij}(x)dx^i \otimes dx^j$ and a sequence of approximating real analytic metric $\Sigma_{ij}g_{ij}^n(x)dx^i \otimes dx^j$ so that g_{ij}^n tends to g_{ij} in the smooth norm when $n \rightarrow \infty$.

In our definition of M , we have a function g which defines M and which is convex in a neighborhood of $N \setminus M$ and a smooth function \bar{g} which is strictly convex in a neighborhood of $N \setminus M$. Let $\{\epsilon_n\}$ be a sequence of real numbers tending to zero such that the functions $g + \epsilon_n \bar{g}$ are strictly convex with respect to the metric $\Sigma_{ij}g_{ij}^n dx^i \otimes dx^j$ in a fixed neighborhood of $N \setminus M$. Using the heat kernel of the metric $\Sigma_{ij}g_{ij}^n dx^i \otimes dx^j$, we can approximate $g + \epsilon_n \bar{g}$ by a real analytic function g_n which is strictly convex with respect to $\Sigma_{ij}g_{ij}^n dx^i \otimes dx^j$ in a fixed neighborhood of ∂M . Clearly we can assume that $\sup_{x \in \partial M} |g_n(x)| \rightarrow 0$ as $n \rightarrow \infty$.

By Sard's theorem, we may choose a sequence of positive numbers $\delta_n \rightarrow 0$ such that $\Sigma_n = \{x \in M^3 | g_n(x) = -\sup_{x \in \partial M} |g_n(x)| - \delta_n\}$ is real analytic. The manifold $M_n = \{x \in M | g_n(x) \leq -\sup_{x \in \partial M} |g_n(x)| - \delta_n\}$ is then an analytic convex manifold with analytic boundary Σ_n .

Recall that in the definition of convex manifold M^3 , we have a bilipschitz homeomorphism $\varphi: \partial M \times (-1, 1)$ into a tubular neighborhood of ∂M^3 such that for all $x \in \partial M^3$ and $1 \geq t_2 \geq t_1 \geq -1$,

$$\varphi(x, 0) = x \quad (3.1)$$

$$-c(t_2 - t_1) \geq g[\varphi(x, t_2)] - g[\varphi(x, t_1)] \quad (3.2)$$

where c is a positive constant.

Since \bar{g} is smooth and since g_n approximates $g + \epsilon_n \bar{g}$ in smooth sense, we may assume that for all n , for all $x \in \partial M$ and $1/2 \geq t_2 \geq t_1 \geq -1/2$, we have

$$\frac{-c}{2}(t_2 - t_1) \geq g_n[\varphi(x, t_2)] - g_n[\varphi(x, t_1)]. \quad (3.3)$$

It follows from (3.3) that for all $x \in \partial M$,

$$g_n \left[\varphi \left(x, \frac{1}{2} \right) \right] \leq -\frac{c}{4} + \sup_{x \in \partial M} |g_n(x)| \quad (3.4)$$

$$g_n \left[\varphi \left(x, -\frac{1}{2} \right) \right] \geq \frac{c}{4} - \sup_{x \in \partial M} |g_n(x)|. \quad (3.5)$$

Hence when n is large enough, then line segment $\{\varphi(x, t) | -1/2 \leq t \leq 1/2\}$ intersects Σ_n at least at one point. The inequality (3.3) shows that the intersection is a single point. This provides a one-to-one correspondence from ∂M to Σ_n . To show that it is a continuous map, we note that the inverse map is given by the projection of $\varphi^{-1}(\Sigma_n)$ (in

$\partial M \times (-1, 1)$ onto ∂M . The inverse map is a homeomorphism and the theorem of invariance of domain shows that the above correspondence is continuous. Under this correspondence, the Jordan curve σ on ∂M is mapped to another Jordan curve $\tilde{\sigma}_n$ on Σ_n .

Being a Jordan curve on a Riemann surface, we can approximate $\tilde{\sigma}_n$ by a real analytic Jordan curve σ_n on Σ_n so that σ_n is uniformly close to $\tilde{\sigma}_n$ and the curves σ_n and $\tilde{\sigma}_n$ bound an annulus of arbitrary small area with respect to the induced metrics $\Sigma_{i,j} g_{ij}^n dx^i \otimes dx^j$ on Σ_n . (This can be done by the uniformization theorem, for example.)

Meanwhile, let us assume the curve σ is Lipschitz. Then we claim that σ and $\tilde{\sigma}_n$ can be bounded by an annulus of arbitrary small area. In fact, the annulus can be described by $\{\varphi(x, t) | x \in \sigma, 0 \leq t \leq t_x \text{ where } t_x \text{ is the time when the curve } \varphi_x(t) = \varphi(x, t) \text{ intersects } \Sigma_n\}$. This is a subset of the image of the set $\sigma \times (-1/2, 1/2)$ under the Lipschitz map φ . Since this latter surface has finite area with respect to the induced metric, it is clear that the above annulus has arbitrary small area (with respect to all metrics that we are considering) when the annulus shrinks down to σ .

Let A_n be the infimum of the areas (with respect to the metric $\Sigma_{i,j} g_{ij}^n dx^i \otimes dx^j$) of C^1 -disks in M_n with boundary given by σ_n . Then it is clear that $\lim_{n \rightarrow \infty} \sup A_n$ is not greater than the infimum of the area (with respect to the metric $\Sigma_{i,j} g_{ij} dx^i \otimes dx^j$) of C^1 -disks in M with boundary σ . Later on, we shall prove that by passing to a subsequence of σ_n , the above inequality is in fact an equality.

Let f_n be a solution to the Plateau's problem for σ_n in M_n . Then according to Hypothesis H , we know that as a map from the unit disk into M_n , f_n is an embedding. In order to prove that a subsequence of $\{f_n\}$ converges to a solution to the Plateau's problem for σ in M , we shall normalize f_n by the three-point condition. Namely, let $\{p_1^n\}$, $\{p_2^n\}$ and $\{p_3^n\}$ be a sequence of points in M so that $p_i^n \in \sigma_n$ for all n, i and so that the points $\lim_{n \rightarrow \infty} p_1^n = p_1$, $\lim_{n \rightarrow \infty} p_2^n = p_2$ and $\lim_{n \rightarrow \infty} p_3^n = p_3$ are distinct points in σ . By using a conformal reparametrization of the unit disk we may assume that $f_n(0) = p_1^n$, $f_n(\sqrt{-1}) = p_2^n$ and $f_n(-1) = p_3^n$ for all n .

Since $\lim_{n \rightarrow \infty} \sup A_n$ has an upper bound, there is a uniform upper bound of the areas of f_n . By the conformality of f_n , we have also a uniform upper bound for the energies of f_n . Since the metrics $\Sigma_{i,j} g_{ij}^n dx^i \otimes dx^j$ are uniformly equivalent to the metric $\Sigma_{i,j} g_{ij} dx^i \otimes dx^j$ by constants independent of n , we conclude that the energy of f_n is uniformly bounded from above if we define the energy in terms of the fixed metric $\Sigma_{i,j} g_{ij} dx^i \otimes dx^j$. The standard argument [9], using the three-point condition, then shows that $\{f_n | \partial D\}$ converges uniformly on ∂D .

We are going to prove the convergence of a subsequence of f_n on \bar{D} . From now on, we use $B(x, r)$ to denote the disk with center x and radius r . Let $x \in \bar{D}$ be a point so that $B(x, r) \subset D$. Then by the standard Lebesgue argument, there is a number $r^2 < r_n < r$ so that the following inequality holds

$$L[f_n(\partial B(x, r_n))] \leq K \left(\log \frac{1}{r} \right)^{-1} \quad (3.6)$$

where the l.h.s. is the length of the curve $f_n(\partial B(x, r_n))$ with respect to the metric $\Sigma_{i,j} g_{ij}^n dx^i \otimes dx^j$ and K is a constant independent of n .

Let ρ be a positive number so that any geodesic ball (of the metric $\Sigma_{i,j} g_{ij} dx^i \otimes dx^j$) with center in M and radius ρ is smooth and strictly convex with respect to all metric tensors $\Sigma_{i,j} g_{ij}^n dx^i \otimes dx^j$. Cover M by finite numbers of such balls B_1, \dots, B_l . We may assume that all the metrics $\Sigma_{i,j} g_{ij}^n dx^i \otimes dx^j$ are uniformly equivalent to each other by a constant independent of n . There is a positive number ϵ so that any geodesic ball of

the metric $\sum_{i,j} g_{ij}^n dx^i \otimes dx^j$ with radius less than ϵ must be a proper subset of some B_i and the distance of this geodesic ball to ∂B_i is greater than ϵ .

Choose r so that $K(\log(1/r))^{-1} \leq \epsilon$. Then according to the choice of ϵ and (3.6) the curve $f_n[\partial B(x, r_n)]$ is a proper subset of some B_i and the distance of $f_n[\partial B(x, r_n)]$ from ∂B_i is greater than ϵ . We claim that when r is small enough, $f_n[B(x, r_n)] \subseteq B_i$.

First of all, we prove that the energy of f_n over $B(x, r_n)$ is less than a $L[f_n(\partial B(x, r_n))]^2$ where a is a constant depending only on M . In fact, all of our metrics on B_i are uniformly equivalent and so we may assume that B_i is diffeomorphic to the euclidean unit ball where all the metrics are uniformly equivalent to the euclidean metric (with the same uniform constant). For each Jordan curve $f_n[\partial B(x, r_n)]$ in this unit ball, we can find a solution h_n to Plateau's problem with respect to the euclidean metric. It is well-known (see [9]) that the (euclidean) area of h_n is not greater than a quarter of the square of the (euclidean) length of $f_n[\partial B(x, r_n)]$. Since all our metrics are uniformly equivalent to the euclidean metric, we see that the area of h_n with respect to the metric $\sum_{i,j} g_{ij}^n dx^i \otimes dx^j$ is uniformly dominated by the square of the length of $f_n[\partial B(x, r_n)]$. As f_n is a solution to Plateau's problem for the metric $\sum_{i,j} g_{ij}^n dx^i \otimes dx^j$, the energy of the map $f_n|_{B(x, r_n)}$ is uniformly dominated by the square of the length of $f_n[\partial B(x, r_n)]$. In particular, the energy of f_n over $B(x, r_n)$ is less than $aK^2(\log(1/r))^{-2}$.

Now suppose $f_n[B(x, r_n)]$ is not a subset of B_i . Then there is a point $y \in f_n[B(x, r_n)] \cap \partial B_i$ whose distance to $f_n[\partial B(x, r_n)]$ is not less than ϵ . Then according to the lemma proved in the Appendix, the area of $f_n[B(x, r_n)]$ is greater than some positive constant depending only on M and ϵ . On the other hand, if we choose r small, we can make $aK^2(\log(1/r))^{-2}$ smaller than the above fixed constant. This contradiction shows that $f_n[B(x, r_n)] \subseteq B_i$.

Once we know that $f_n[B(x, r_n)] \subseteq B_i$, we can use the argument of Lemma 1 to show that $f_n[B(x, r_n)]$ is a subset of the convex hull of $f_n[\partial B(x, r_n)]$ in B_i . Since the convex hull of $f_n[\partial B(x, r_n)]$ is arbitrary small when r is small, the diameter of $f_n[B(x, r^2)]$ is uniformly small when r is small. Hence we have proved the equicontinuity of the family $\{f_n\}$ on compact subsets of D . Essentially the same argument shows that the family $\{f_n\}$ is equicontinuous in a neighborhood of ∂D . Hence a subsequence of $\{f_n\}$ converges uniformly on D to a continuous map $f: D \rightarrow M$ such that $f|_{\partial D} = \sigma$.

Let us now prove that a subsequence of $\{f_n\}$ converges in smooth norm on compact sets of D . Note that in the previous arguments, we know that the energy of f_n over $B(x, r^2)$ is uniformly small. Since the image of f_n over $B(x, r^2)$ is a subset of B_i which is identified with the euclidean unit ball under a diffeomorphism, we may consider f_n as a vector valued function. The equations of f_n have the form

$$\frac{\partial^2 f_n^i}{\partial x^i} + \frac{\partial^2 f_n^i}{\partial y^2} = - \sum_{j,k} \Gamma_{jk}^i \left\{ \frac{\partial f_n^j}{\partial x} \frac{\partial f_n^k}{\partial x} + \frac{\partial f_n^j}{\partial y} \frac{\partial f_n^k}{\partial y} \right\} \quad (3.7)$$

where Γ_{jk}^i is the Christoffel symbol for the metric tensor $\sum_{i,j} g_{ij}^n dx^i \otimes dx^j$.

We are going to use a well-known argument (see [46]) to find a uniform estimate of $|\nabla f_n|$ over compact subsets of D .

Let φ be any smooth function with compact support in $B(x, r^2)$. Then (3.7) shows that

$$|\Delta(\varphi f_n)| \leq c_1 |\nabla(\varphi f_n)| |\nabla f_n| + c_2 |\nabla f_n| + c_3 \quad (3.8)$$

where c_i are constants independent of n .

By the Sobolev inequality (see [27]),

$$\left(\int_{B(x, r^2)} |\nabla(\varphi f_n)|^4 \right)^{1/4} \leq c_4 \left(\int_{B(x, r^2)} |\nabla^2(\varphi f_n)|^{4/3} \right)^{3/4} \quad (3.9)$$

where c_4 is a constant independent of n .

On the other hand, (3.8) shows that

$$\left(\int_{B(x, r^2)} |\nabla(\varphi f_n)|^{4/3} \right)^{3/4} \leq c_5 \left(\int_{B(x, r^2)} |\nabla f_n|^2 \right)^{1/2} \left(\int_{B(x, r^2)} |\nabla^2(\varphi f_n)|^4 \right)^{1/4} + c_6 \int_{B(x, r^2)} |\nabla f_n|^2 + c_7 \quad (3.10)$$

where c_5 , c_6 and c_7 are constants independent of n .

By the well-known L_p -estimate (see [1]) we have another constant c_8 independent of n so that

$$\left(\int_{B(x, r^2)} |\nabla^2(\varphi f_n)|^{4/3} \right)^{3/4} \leq c_8 \left(\int_{B(x, r^2)} |\Delta(\varphi f_n)|^{4/3} \right)^{3/4}. \quad (3.11)$$

Putting (3.9)–(3.11) together we see that

$$\begin{aligned} & \left[1 - c_8 c_5 c_4 \left(\int_{B(x, r^2)} |\nabla f_n|^2 \right)^{1/2} \right] \left[\int_{B(x, r^2)} |\nabla^2(\varphi f_n)|^4 \right]^{3/4} \\ & \leq c_8 c_6 \int_{B(x, r^2)} |\nabla f_n|^2 + c_8 c_7. \end{aligned} \quad (3.12)$$

As $\int_{B(x, r^2)} |\nabla f_n|^2$ is uniformly equivalent to the energy of f_n over $B(x, r^2)$, it is uniformly small and (3.12) gives an estimate of $\int_{B(x, r^2)} |\nabla^2(\varphi f_n)|^{4/3}$.

By (3.9), we also have an estimate of $\int_{B(x, r^2)} |\nabla(\varphi f_n)|^4$. If we choose φ to be equal to one on $B(x, r^2/2)$, then we have found an estimate of $\int_{B(x, r^2/2)} |\nabla f_n|^4$. Inequality (3.8) shows that we have an estimate of $\int_{B(x, r^2/4)} |\Delta(\varphi f_n)|^2$ and hence an estimate of $\int_{B(x, r^2/4)} |\nabla^2(\varphi f_n)|^2$. Sobolev inequality again gives an estimate of $\int_{B(x, r^2/4)} |\nabla(\varphi f_n)|^p$ for all $p \geq 1$. Applying the previous argument again, we get an estimate of $\int_{B(x, r^2/8)} |\nabla^2 f_n|^p$ for all $p \geq 1$. Sobolev inequality then provides a uniform estimate of ∇f_n over $B(x, r^2/16)$.

Therefore the r.h.s. of (3.7) is uniformly bounded. The Schauder theory (or standard potential theory, see [35]) then gives $C^{1,\alpha}$ estimates of f_n for all $0 < \alpha < 1$. The r.h.s. is then C^α and we can iterate the argument to find smooth estimates of f_n up to any order. This proves our claim that a subsequence of $\{f_n\}$ converges in smooth norm on compact sets of D .

Therefore the limit f is smooth in the interior of D . It is conformal and satisfies the minimal surface equation. It is also the uniform limit of $\{f_n\}$ over D ; it must be monotonic on ∂D and Lemma 9 shows that $f|_{\partial D}$ is in fact a homeomorphism. The standard lower semicontinuity argument shows that the area of f is not greater than $\lim_{n \rightarrow \infty} \inf A_n$ where A_n is the area of f_n . Combining with our previous choice of σ_n , we see that f must be a solution to Plateau's problem for σ with area equal to $\lim_{n \rightarrow \infty} A_n$.

By Osserman[42] and Gulliver[15], we know that f is an immersion in D . We assert that f is in fact an embedding. Otherwise there are two distinct points $x, y \in D$ so that $f(x) = f(y)$. Since $f|_{\partial D}$ is a homeomorphism and f maps \bar{D} into the interior of M , we know that both x and y belong to \bar{D} . By Lemma 2, there are convex neighborhoods U and V of x and y respectively so that the restriction of f to either U

or V is an embedding and $f(U)$ is transversal to $f(V)$. Locally, we may choose a coordinate system (x^1, x^2, x^3) so that $f(x) = f(y) = 0$, $f(U)$ contains the unit disk in (x^1, x^2) plane and $f(V)$ contains the unit disk in (x^1, x^3) plane. Let z_1 and z_2 be two points in V so that $f(z_1)$ is a point on the positive x^3 -axis, $f(z_2)$ is a point on the negative x^3 -axis and the euclidean length of the curve $f_n(z_1, z_2)$ is less than 1. On the other hand, since f_n converges to f in the C^1 -sense, $f_n(U)$ contains an open set which is a graph over the disk with center zero and radius $1/2$, when n is large. We may assume that $f_n(z_1)$ is always above these graphs and $f_n(z_2)$ is always below these graphs. Any curve in the (x^1, x^2, x^3) space which joins $f_n(z_1)$ and $f_n(z_2)$ must have euclidean length greater than 1 if it avoids the disk with center zero and radius $1/2$. Therefore $f_n(z_1, z_2)$ must intersect $f_n(U)$ when n is large. This is a contradiction as f_n is an embedding. Finally, we have reached the conclusion that f is an embedded solution to Plateau's problem for σ . This proves the theorem assuming σ is Lipschitz.

To prove the theorem in general, it remains to prove that σ can be approximated on ∂M by a Lipschitz Jordan curve $\tilde{\sigma}$ so that σ and $\tilde{\sigma}$ bounds an annulus with small area. To produce $\tilde{\sigma}$, we recall, in the beginning of our approximation procedure, we have a homeomorphism from a real analytic surface Σ_n to ∂M which is Lipschitz. Consider σ to be a Jordan curve in Σ_n and then approximate it by a smooth Jordan curve on Σ_n so that they bound an annulus with small area. Since the homeomorphism is Lipschitz, the smooth curve is mapped to a Lipschitz curve which bounds with σ an annulus having small area. This finishes the proof of Theorem 2.

In the next section, we shall also prove the following hypothesis.

Hypothesis K. Let σ be a real analytic Jordan curve in a three-dimensional compact real analytic manifold M with real analytic metric and real analytic convex boundary. Let $f: D \rightarrow M$ be a Douglas-Morrey solution for σ . If f is an embedding in a neighborhood of ∂D and $f(x) \notin f(\partial D)$ for $x \in D$, then f is an embedding of D into M .

From this hypothesis, we prove the following.

THEOREM 3. Let σ be a C^2 -regular Jordan curve in a three-dimensional compact manifold M with convex boundary. Let $f: D \rightarrow M$ be a solution to Plateau's problem for σ . If f has no boundary branch point and if $f(x) \notin f(\partial D)$ for $x \in \mathring{D}$, then f is an embedding of D into M . In case σ is a curve in R^3 , we need only assume σ to be C^1 -regular.

Proof. According to Heinz-Hildebrandt[22], f is C^1 in a neighborhood of ∂D . The assumption that f has no branch point on the boundary makes sure that f is an immersion in a neighborhood of ∂D . On the other hand, Lemma 9 shows that $f|_{\partial D}$ is one-to-one. Therefore f is in fact an embedding in a neighborhood of ∂D .

We claim that for small $\epsilon > 0$, f is an embedding on the annulus $N_\epsilon = \{x \in D \mid 1 - \epsilon \leq |x| \leq 1\}$ and $f(N_\epsilon) \cap f\{x \mid |x| < 1 - \epsilon\} = \emptyset$. The first statement is clear. To prove the second statement, we assume the contrary statement and find a sequence of positive numbers $\epsilon_i \rightarrow 0$ so that for some sequences $\{x_i\}$ and $\{y_i\}$ in D , we have $|x_i| < 1 - \epsilon_i$, $|y_i| \geq 1 - \epsilon_i$ and $f(x_i) = f(y_i)$. Without loss of generality, we may assume that $\{x_i\}$ converges to a point $x_0 \in \partial D$. Then our assumption in the theorem shows that $\{y_i\}$ converges to x_0 also. Therefore eventually both x_i and y_i belong to the neighborhood of ∂D where f is an embedding. This is a contradiction and our claim is proved.

Let $S = \{x \mid |x| = 1 - \epsilon/2\}$ and $D_{\epsilon/2} = \{x \mid |x| < 1 - \epsilon/2\}$. Then $f(S)$ is a smooth regular Jordan curve in M so that for all $|x| < 1 - \epsilon/2$, $f(x) \notin f(S)$ and f is an embedding in a

neighborhood of S . Lemma 8 shows that $f(S)$ is a curve of uniqueness for the solution of Plateau's problem in M . We shall use these facts to show that f is an embedding.

Recall that in the definition of convex manifold M , M is a proper subdomain of another manifold N with strictly convex boundary. By shrinking N a little, we may assume that N has a real analytic structure and ∂N is real analytic. We approximate the original smooth metric by a sequence of real analytic metric $\{ds_n^2\}$ in smooth norm and approximate the smooth curve $f(S)$ by a sequence of real analytic curves C_n in smooth norm. For each C_n and ds_n^2 , we find a Douglas-Morrey solution f_n from $D_{\epsilon/2}$ into M so that $f_n|_S$ parametrizes C_n .

As in Theorem 2 we can show that a subsequence of $\{f_n\}$ converges uniformly on $D_{\epsilon/2}$. By taking a subsequence we may assume $\{f_n\}$ converges smoothly on $\overline{D_{\epsilon/2}}$ to a Douglas-Morrey solution \tilde{f} of $f(S)$ on $\overline{D_{\epsilon/2}}$. (Note that we have to use the three-point condition to prove the equicontinuity of $\{f_n\}$ on S .)

By Lemma 5, $\tilde{f}|_S$ is a homeomorphism. The smooth convergence of $\{f_n\}$ implies that \tilde{f} is smooth in $\overline{D_{\epsilon/2}}$. Furthermore the above three-point condition implies that we may assume that $f = \tilde{f}$ at three distinct points on S . On the other hand, it is clear that \tilde{f} is a solution to Plateau's problem for $\tilde{f}(S)$. Hence as $\tilde{f}(S)$ is a curve of uniqueness for Plateau's problem, the uniqueness of Lemma 8 shows that there is a conformal automorphism k on $D_{\epsilon/2}$ with $f(x) = \tilde{f}(k(x))$ for $x \in D_{\epsilon/2}$. The three-point condition then implies that $k(x) = x$ for all $x \in D_{\epsilon/2}$ and $f = \tilde{f}$.

Therefore we have proved that f_n converges to f smoothly on $\overline{D_{\epsilon/2}}$. We claim that for n large enough, f_n has no branch points on S and $f_n(D_{\epsilon/2}) \cap f_n(S) = \emptyset$. The first assertion follows from the smooth convergence of f_n to f on $\overline{D_{\epsilon/2}}$ and the fact that f has no branch point on $S = \partial D_{\epsilon/2}$. If the second assertion were wrong, we can find sequences $\{x_n\} \subset D_{\epsilon/2}$ and $\{y_n\} \subset S$ so that $f_n(x_n) = f_n(y_n)$ for all n . Since $f(D_{\epsilon/2}) \cap f(S) = \emptyset$ and f_n converges uniformly to f on $\overline{D_{\epsilon/2}}$, we may assume that both $\{x_n\}$ and $\{y_n\}$ converge to the same point $x \in \partial D_{\epsilon/2}$. Take a coordinate neighborhood in M^3 with center at $f(x)$ so that each f_n maps a fixed (independent of n) neighborhood of x into the coordinate neighborhood. Then let X_n be the constant vector field defined on D which is equal to $(y_n - x_n)/|y_n - x_n|$. By passing to a subsequence, we may assume that X_n converges to a unit vector field X . Let f_n^1 be the first component of f_n with respect to the coordinate chart. Then, by the mean valued theorem, $X_n(f_n^1)(\bar{x}_n) = 0$ for some point \bar{x}_n on the line segment joining x_n and y_n . The smooth convergence of f_n to f on $\overline{D_{\epsilon/2}}$ then implies that $X(f^1)(x) = 0$. Similarly we can prove that $X(f^2)(x) = X(f^3)(x) = 0$. Hence we have arrived at the contradiction that g has a branch point at x . (Notice that g is conformal so that the differential of g is zero at x .)

Once we know that f_n has no branch point on S and $f_n(D_{\epsilon/2}) \cap f_n(S) = \emptyset$, we also know that f_n is an embedding in a neighborhood of S and Hypothesis K says that f_n is an embedding globally on $\overline{D_{\epsilon/2}}$. The arguments of Theorem 2 can then be utilized to prove that f is also an embedding on $\overline{D_{\epsilon/2}}$. This finishes the proof of Theorem 3.

§4. DEHN'S LEMMA FOR ANALYTIC MANIFOLDS

In the previous section we reduced the problem of existence for an embedded solution to Plateau's problem for a Jordan curve on the boundary of a convex three-manifold to proving the existence of an embedded solution to Plateau's problem when the three-dimensional manifold and the Jordan curve are analytic (Hypotheses H and K of the last section).

The topological analysis for analytic manifolds is simpler for two reasons. The first reason is that a solution $f: D \rightarrow M^3$ is analytic. The triangulability theorem for semianalytic sets[30] then shows that f is a simplicial map with respect to some triangulations of D and M^3 . The second reason the analytic case is better topologic-

ally is that the image of D is embedded near the boundary of M . This second fact follows from the boundary regularity theorem in [16]. The next theorem shows that these two topological properties for f are sufficient to prove that f is an embedding.

In the proof of the following theorem we have closely followed the topological arguments of Whitehead and Shapiro[49] in their proof of Dehn's lemma. In particular, the tower construction of Papakyriakopoulos will be of utmost importance. The reader can also consult[20, 50] for further details on the purely topological results. The reader should note that our proof of Theorem 4 differs from the standard topological proof of Dehn's Lemma in two key ways; (1) some of our topological constructions differ because our mappings need not be in general position; (2) we use paste and glue arguments and the least area property to show that some usual singularities encountered in the proof of Dehn's Lemma never actually occur.

Suppose now that $f: M^2 \rightarrow M^3$ is a mapping from a surface into a three dimensional manifold. Then we define the self-intersection set of f to be $S(f) = \{x \in M^2 | \exists y \neq x \in M^2 \text{ with } f(x) = f(y)\}$.

THEOREM 4. *Suppose M^3 is a smooth three-dimensional manifold with possibly empty boundary. If $f: D \rightarrow M^3$ is a solution to Plateau's problem with the following properties:*

- (1) $S(f)$ is disjoint from ∂D .
- (2) f is simplicial with respect to some triangulations of D and M^3 .
- (3) The image of the interior of f is disjoint from the boundary of M^3 .

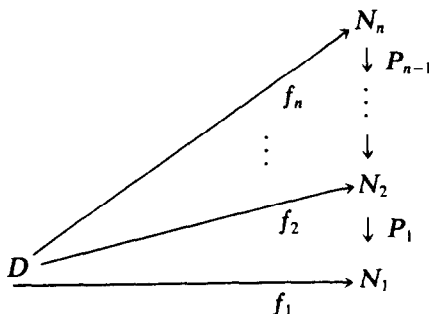
Then f is an embedding.

Proof. After suitably restricting the range space M^3 , we may assume that γ lies on the boundary of M^3 . We now begin the tower construction for the map $f: D \rightarrow M^3$.

Let N_1 be a regular neighborhood of $f(D)$. If the first homology group $H_1(N_1, \mathbb{Z}_2)$ is nonzero, then there exists a surjective homomorphism $\rho: \pi_1(N_1) \rightarrow \mathbb{Z}_2$. Since the kernel of ρ has index two in $\pi_1(N_1)$, there is a two sheeted covering space $p_1: \tilde{N}_1 \rightarrow N_1$ associated to this subgroup. After restricting the range space of f to N_1 , there is a new map $f_1: D \rightarrow N_1$. Let $\tilde{f}_1: D \rightarrow \tilde{N}_1$ be a lift of f_1 to the covering space \tilde{N}_1 . Then restricting the range space of \tilde{f}_1 to a regular neighborhood N_2 of $\tilde{f}_1(D)$, we get another map $f_2: D \rightarrow N_2$.

If $H_1(N_2, \mathbb{Z}_2)$ is nonzero, we can repeat the construction in the previous paragraph to get a 2-sheeted cover $P_2: \tilde{N}_2 \rightarrow N_2$ and a lift $\tilde{f}_2: D \rightarrow \tilde{N}_2$ of f_2 . After restricting the lift \tilde{f}_2 to a regular neighborhood of N_3 of $\tilde{f}_2(D)$, we get $f_3: D \rightarrow N_3$.

Repeating n -times, the construction outlined above yields a tower where $P_i: N_{i+1} \rightarrow N_i$ is the restriction of $P_i: \tilde{N}_i \rightarrow N_i$ to N_{i+1} .



Each N_i in the above tower is a Riemannian manifold with respect to the pulled back metric. Each of the lifts $f_i: D \rightarrow N_i$ is a solution to Plateau's problem for the Jordan curve $f_i(\partial D)$ with respect to this metric. (Otherwise there is an immersion $g: D \rightarrow N_i$ which is a candidate for Plateau's problem for $f_i(\partial D)$ and with respect to the metric pulled back to D , $\text{Area}(g)$ is less than $\text{Area}(f_i)$. We would then have $\text{Area}(P_1 \circ P_2 \circ \dots \circ P_{i-1} \circ g) = \text{Area}(g) < \text{Area}(f_i) = \text{Area}(f)$ which is impossible.)

We claim that for some $n = k$, the tower construction yields a N_k with $H_1(N_k, Z_2) = 0$. Also, the maps f_1, f_2, \dots, f_n can be made simultaneously simplicial with respect to a fixed triangulation of D that includes $S(f)$ as part of its 1-complex. To see this, choose triangulations T of D and K of M^3 for which f is simplicial. Then $f(D) = |L|$ for some subcomplex $L \subset K$. Let K' be the subdivision of K obtained by adding only the barycenters of simplices of $K - L$. As L is a subcomplex of K' we may iterate the construction to get K'' . Now $N(K, K'') = \{\bar{\sigma} \in K'' \mid \bar{\sigma} \cap L \neq \emptyset \text{ where } \bar{\sigma} \text{ is a closed simplex of } K'\}$ is a regular neighborhood of $f_1(D^2)$ in M^3 which we could take to be N_1 . The triangulation K'' restricts to a triangulation of N_1 , which we may lift to give a triangulation K_2 of \tilde{N}_1 . Then $\tilde{f}_1: D^2 \rightarrow \tilde{N}_1$ is a simplicial map from T to K_2 ; and we may iterate the construction to obtain f_1, f_2, \dots, f_n which are simultaneously simplicial with respect to the triangulation T of D^2 .

Let $X(f_i) = \{(\sigma, \tau) \in T \times T \mid \sigma, \tau \text{ are open simplices, } \sigma \neq \tau, \text{ and } f_i(\sigma) \cap f_i(\tau) \neq \emptyset\}$. Then the sets $X(f_i)$ are finite sets with $X(f_{j+1}) \subset X(f_j)$. We assert that in fact, $X(f_{j+1}) \neq X(f_j)$. To see this we first pick a base point $*$ for N_j which is contained in the subset $f_j(D)$ and we pick a base point $\tilde{*}$ for the covering space \tilde{N}_j which belongs to the subset $\tilde{f}_j(D) \cap P_j^{-1}(*)$. Since $i_* \pi_1(f_j(D), *) \rightarrow \pi_1(N_j, *)$ is an isomorphism, every element $[\alpha] \in \pi_1(N_j, *)$ is represented by a loop $\alpha: [0, 1] \rightarrow f_j(D)$ with $\alpha(0) = \alpha(1) = *$. If $X(f_{j+1}) = X(f_j)$, then the restriction map $P_j|_{\tilde{f}_j(D)}$ is one-to-one. This implies the loop α will lift to a map $\tilde{\alpha}: [0, 1] \rightarrow \tilde{N}_j$ with $\tilde{\alpha}(0) = \tilde{\alpha}(1) = \tilde{*}$. Hence, $P_j: \pi_1(\tilde{N}_j, \tilde{*}) \rightarrow \pi_1(N_j, *)$ is onto. However, this is impossible since by covering space theory, the subgroup $P_j(\pi_1(\tilde{N}_j, \tilde{*})) \subset \pi_1(N_j, *)$ has index two. This contradiction shows that the tower construction can not be continued to a height greater than the number of elements in $X(f_1)$. Therefore, if N_k is the space at the top of the tower, then $H_1(N_k, Z_2) = 0$. (Otherwise, we can construct another two-sheeted covering space.)

Suppose $f_k: D \rightarrow N_k$ is the lift of $f_1: D \rightarrow N_1$ to the top of the tower constructed above. Since the pairing between homology and cohomology with coefficients in a field is nondegenerate and $H_1(N_k, Z_2) = 0$, we have $H^1(N_k, Z_2) = 0$. Poincaré duality for manifolds with boundary shows that $H_2(N_k, \partial N_k, Z_2) = 0$. From the following part of the long exact sequence in homology for the pair $(N_k, \partial N_k)$

$$\rightarrow H_2(N_k, \partial N_k, Z_2) \rightarrow H_1(\partial N_k, Z_2) \rightarrow H_1(N_k, Z_2) \rightarrow,$$

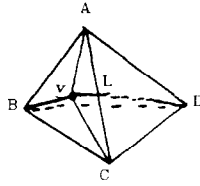
one computes that $H_1(\partial N_k, Z_2) = 0$. This shows that the first homology group with Z_2 -coefficients is zero for each boundary component of N_k . By the classification theorem for compact surfaces, each component of the boundary is a sphere.

We shall use the fact that the boundary of N_k consists entirely of spheres to show that $f_k: D \rightarrow N_k$ is an embedding. First note that since N_k is a simplicial regular neighborhood, there is, after a possible subdivision, a simplicial retraction $S: N_k \rightarrow f_k(D)$ whose restriction $R = S|_{\partial N_k}: \partial N_k \rightarrow f_k(D)$ has the property: R covers each open 2-simplex of $f_k(D)$ exactly two times and $R|_{(\partial N_k \setminus f_k(\partial D))}$ is locally one-to-one. The existence of such a retraction follows directly from the construction of the regular neighborhood and the collapsing properties of such a neighborhood onto an immersed codimension one simplicial submanifold whose boundary is the intersection of the

submanifold with the boundary of the ambient manifold. For the reader's convenience, we give a more detailed proof in the following paragraphs.

By definition of regular neighborhood (see [47] and especially pp. 7–8 of [20]), $f_k(D)$ is obtained from N_k by collapsing simplices with a free face. The process of collapsing N_k onto $f_k(D)$ can be carried out by sequentially collapsing a three-simplex σ with a free face Δ not contained in $f_k(D)$ and then collapsing the other free faces of σ which are not contained in $f_k(D)$. However, we shall collapse all free faces of σ which are not contained in $f_k(D)$ at the same time. There are three cases to be considered.

Case 1. Suppose σ is a three-simplex $[ABCD]$ with vertices A, B, C, D and exactly one free face $[ABC]$ not contained in $f_k(D)$. Then we let v be the barycenter of the free face $[ABC]$ and L the straight line joining v to the vertex D . The simplex σ can be collapsed by linearly projecting along L onto the other faces of σ .



Note that after the subdivision of σ obtained by adding the additional vertex v , the projection of σ along L is a simplicial map.

Case 2. Suppose $\sigma = [ABCD]$ has two free faces $[ABC]$ and $[ACD]$ not contained in $f_k(D)$. Then we let L be the straight line joining the barycenter of $[AC]$ with the barycenter of $[BD]$. The simplex σ can be collapsed by linearly projecting along L onto the other faces of σ .

Case 3. Suppose $\sigma = [ABCD]$ has three free faces $[ABC]$, $[ACD]$ and $[ABD]$ not contained in $f_k(D)$. Then we let L be the straight line joining the barycenter of $[BCD]$ with the vertex A . The simplex σ can be collapsed by linearly projecting along L onto the other faces of σ .

Suppose there are n three-simplices in N_k . After n sequential collapsings of these simplices, there is a piecewise linear map $R: \partial N_k \rightarrow f_k(D)$ which is simplicial after subdivision. In the collapsing process of each three-simplex σ , the free faces of σ not contained in $f_k(D)$ project in a one-to-one way onto the remaining complex. Thus $R|(\partial N_k - f_k(\partial D))$ is a locally one-to-one simplicial map. Since each open two-simplex Δ of $f_k(D)$ is a face of exactly two three-simplices and since $R|(\partial N_k - f_k(\partial D))$ is locally one-to-one, $R^{-1}(\Delta)$ consists of exactly two open two-simplices in ∂N_k . This completes the proof of the claim.

The Jordan curve $\gamma = f_k(\partial D)$ lies on some sphere S^2 in the boundary of N_k . By the Jordan curve Theorem, γ disconnects S^2 into two disks D_1 and D_2 . By our previous choice of a retraction, the map $R|(D_1 \cup D_2)$ covers no 2-simplex of $f_k(D)$ more than two times. Since area is carried only by the 2-simplices, $\text{Area}(R|D_1) + \text{Area}(R|D_2) \leq 2 \text{Area}(f_k)$, with strict inequality if some 2-simplex is not covered by one of the retracting maps $R|D_1$ or $R|D_2$. If the inequality is strict, we may assume that one of these maps, say $R|D_1$, has area strictly less than $\text{Area}(f_k)$. However, this is impossible since f_k is a solution to Plateau's problem for $f_{k-1}(\partial D)$. Therefore equality must hold,

$\text{Area}(R|D_1) = \text{Area}(R|D_2) = \text{Area}(f_k)$ and every 2-simplex of $f_k(D)$ is covered by $R|D_1$ or $R|D_2$.

Let σ be a 2-simplex with an edge E in $f_k(S(f_k))$. (By Lemmas 2 and 3, σ always exists.) Then by the above assertion, σ is covered by either $R|D_1$ or $R|D_2$, say $R|D_1$. Since f_k is real analytic, we may apply Lemma 2 to assume that after subdivision f_k is transverse to itself at points other than the vertices of the triangulation of $f_k(D)$.

Since R is the restriction of a retraction to ∂N_k , there will exist maps $R: \partial N_k \rightarrow (N_k - f_k(D))$ so that R_i is an embedding and R_i converges smoothly on each closed simplex to R . Now consider a 2-simplex σ_1 in $R^{-1}(\sigma)$ which is contained in D_1 . Let σ_2 be the 2-simplex in D_1 adjoining σ_1 along the edge $R^{-1}(E) \cap \sigma_1$. Then $\sigma_1 \cup \sigma_2$ forms a disk and we have exactly the situation described in the paragraph before Lemma 6. Hence $R|D_1$ has a folding curve along E and we can apply Lemma 7 to decrease the area of $R|D_1$. But the area can not be decreased because f_k is a solution to Plateau's problem for $f_k(\partial D)$. This contradiction shows that the lift $f_k: D \rightarrow N_k$ to the top of the tower must be an embedding.

Since $f_k: D \rightarrow N_k$ is an embedding, we have to show that $k = 1$ to complete the proof of the theorem. The following lemma on the topological properties of the singular set for a minimal immersion of a disk into a 3-manifold will be used to show that $k = 1$.

LEMMA 10. *Suppose $f: D \rightarrow M^3$ is a minimal immersion with $S(f) \neq \emptyset$ and $S(f) \cap \partial D = \emptyset$ and which is simplicial with respect to some triangulations of D and M^3 . Then there exists a Jordan curve γ_1 on D which bounds a subdisk D_1 with $\partial D_1 = D_1 \cap S(f)$.*

Proof. Since f is simplicial, the singular set is by definition a sub-complex of D . By the corollary to Lemma 3, $S(f)$ is a one-complex with every vertex in $S(f)$ joined by at least two edges in $S(f)$. A finite one-dimensional complex with these properties can be shown by an induction argument to have a simple closed curve in each path component. Thus the collection C of all Jordan curves in $S(f)$ is nonempty.

Now consider a Jordan curve γ in C with the following minimal property: γ is the boundary of a subdisk D_1 of D such that $(\text{int}(D_1)) \cap S(f)$ contains the smallest number of open one-simplexes of $S(f)$. Observe that any Jordan curve α which is different from γ and which is contained in $D_1 \cap S(f)$ will bound by the Jordan curve theorem a subdisk $D_2 \subset D_1$ such that $(\text{int}(D_2)) \cap S(f)$ has fewer open one-simplexes than $(\text{int}(D_1)) \cap S(f)$. Our minimality assumption on γ implies that such a Jordan curve α cannot exist. Lemma 10 will be proved by using this observation. There are two cases.

(i) Every closed one-simplex of $S(f)$ that is contained in $D_1 \cap S(f)$ and which intersects γ is a subset of γ . In this case, $(\text{int}(D_1)) \cap S(f) = \emptyset$. Otherwise it contains a path component of $S(f)$ which also contains a Jordan curve. This contradicts the above observation.

(ii) There is a one-simplex in $D_1 \cap S(f)$ which intersects γ at a point p and which is not contained in γ . Since we may assume that $\text{int}(D_1) \cap S(f)$ contains no closed Jordan curves, there exists a longest Jordan arc $\tau: [0, 1] \rightarrow D_1 \cap S(f)$ with $\tau(0) = p$ and $\tau((0, 1))$ contained in $\text{int}(D_1)$. Clearly $\tau(1)$ must be a vertex of $S(f)$. It must belong to γ because every vertex in $\text{int}(D_1) \cap S(f)$ is joined to an even number of edges of $S(f)$. The Jordan arc $\tau([0, 1])$ together with either of the arcs on γ joining $\tau(0)$ with $\tau(1)$

gives rise to a Jordan curve α in $D_1 \cap S(f)$ which is not equal to γ . The earlier observation shows α cannot exist. Hence case (ii) cannot occur.

Since one of the cases in the previous two paragraphs must occur, the lemma is proved.

Let us now assume that $k > 1$ and $S(f_{k-1})$ is non-empty. Note that $S(f_{k-1})$ consists entirely of double points which arise from identifying certain points of $f_{k-1}(D) \subset N_k \subset \tilde{N}_{k-1}$ with their images under the order two deck transformation $\sigma: \tilde{N}_{k-1} \rightarrow \tilde{N}_{k-1}$. The above lemma shows that there exists a parametrized Jordan curve $\gamma_1: S^1 \rightarrow D$ bounding a subdisk D_1 with $D_1 \cap S(f_{k-1}) = \partial D_1$. Since $S(f_{k-1})$ consists entirely of double points for the map f_{k-1} , there is a well defined double curve $\gamma_2: S^1 \rightarrow D$ corresponding to γ_1 . In other words, $f_k(\gamma_2) = \sigma(f_k(\gamma_1))$. As f_k is an embedding and $\gamma_2 = f_k^{-1} \circ \sigma \circ f_k(\gamma_1)$, γ_2 is a continuous Jordan curve.

The curve γ_2 bounds a subdisk D_2 of D . (We do not rule out the possibility that $D_1 = D_2$). Suppose that $\text{Area}(f_{k-1}(D_1)) \leq \text{Area}(f_{k-1}(D_2))$. Then we choose a diffeomorphism $h: D_2 \rightarrow D_1$ with $h(\alpha_2(t)) = \alpha_1(t)$. Now define a map $g: D \rightarrow N_{k-1}$ by

$$g(x) = \begin{cases} f_{k-1}(x) & \text{if } x \in (D - D_2) \\ f_{k-1} \circ h(x) & \text{if } x \in D_2 \end{cases}.$$

Note that g is a continuous piecewise smooth map with $\text{Area}(g) \leq \text{Area}(f_{k-1})$. If we can prove that g has a folding curve (see Lemma 7), then the area of g can be decreased which will contradict the least area property of f_{k-1} .

To check that g has a folding curve as defined in Section 2, we proceed as follows. Since $S(f_{k-1})$ is compact, Lemmas 2 and 3 show that $f_{k-1}: D \rightarrow N_{k-1}$ crosses itself transversely except at a finite number of points which are vertices in the triangulation of D . Pick a point $p \in \gamma_1(S^1)$ and $q = \sigma(p) \in \gamma_2(S^1)$ which correspond to a point of transverse self-intersection. As f_{k-1} is an immersion transverse to itself at $f_{k-1}(p)$, we may pick disk neighborhoods U_1 of p and U_2 of q so that $f_{k-1}(U_1)$ and $f_{k-1}(U_2)$ are embedded disks which intersect transversely along an arc $\alpha: [0, 1] \rightarrow N_k$.

For any point $x \in \partial D_1$ or ∂D_2 , let t_x and n_x be respectively the tangent vector and the outer normal vector of the oriented curves ∂D_1 or ∂D_2 at x . The transversality of $f_{k-1}|_{U_1}$ and $f_{k-1}|_{U_2}$ at $f_{k-1}(p) = f_{k-1}(q)$ implies that the plane spanned by $(f_{k-1})_*(t_p)$ and $(f_{k-1})_*(n_p)$ intersects transversely the plane spanned by $f_{k-1*}(t_q)$ and $f_{k-1*}(-n_q)$. But in our definition of g , $g_*(-n_p) = (f_{k-1})_*(-n_q)$, $g_*(n_p) = (f_{k-1})_*(n_q)$ and, $g_*(t_p) = (f_{k-1})_*(t_q)$. Therefore g has a folding curve along $g(\partial D_1)$ according to our definition. Since this is impossible by Lemma 7, the map f_{k-1} does not exist. This implies that $f: D \rightarrow M^3$ is an embedding which completes the proof of the theorem.

The next corollary is a simple consequence of the previous theorem and shows that Hypothesis H in the previous section holds.

Analytic Version of Dehn's Lemma. Suppose M^3 is a convex analytic three-dimensional manifold. Let γ be analytic Jordan curve on ∂M which is homotopically trivial in M . Then γ bounds an embedded solution to Plateau's problem and every solution to Plateau's problem for γ is embedded.

Proof. As in Theorem 2, we may assume that a solution f maps the interior of D into the interior of M . Since f has no boundary branch point [11], and $f|_{\partial D}$ is one-to-one, $f: D \rightarrow M$ is an immersion at the boundary and there is a neighborhood N of ∂D where $f|_N$ is an embedding. As $f(D - N)$ is compact and disjoint from the boundary of N , $f(D - N)$ also stays a positive distance $\epsilon > 0$ away from the boundary. This shows that $f(D)$ is an embedded surface near the boundary of M .

The theorems of Morrey[36] and Lewy[29] now apply to show that $f: D \rightarrow M$ is an analytic mapping. The triangulability theorem for analytic sets[30] shows f is simplicial with respect to some triangulations of D and M . Since we have now verified all the conditions of Theorem 3, the map f is an embedding.

Remark. The reader should note that we have actually proved a somewhat stronger version of Dehn's Lemma than that above. We have shown that whenever γ and M are analytic and $f: D \rightarrow M$ is a solution to Plateau's problem with $S(f) \cap \partial D$ empty, then f is an embedding. The main problem in generalizing this lemma to the smooth category is that we do not know if f is simplicial. For many reasons, including the problem of embedding, we make the next conjecture. A partial result in the direction of this conjecture can be found in the proof of Lemma 2.

CONJECTURE 1. *Suppose $F: \Omega \rightarrow M$ is a conformal harmonic mapping of a compact surface into a n -dimensional manifold. Suppose also that $S(f)$ is disjoint from $\partial\Omega$. Then f is simplicial with respect to some triangulations of Ω and M .*

§5. THE EMBEDDING THEOREM FOR PLANAR DOMAINS

In this section we shall generalize the embedding theorems of the previous section to include compact plane domains other than the disk. The proof of this more general case is similar to the proof for the disk. However, extra difficulties arise from the greater number of boundary components and the fact that Ω need not be simply connected. These additional problems restrict us to prove an embedding theorem only when the three manifold is orientable. We refer the reader to Section 1 for the related discussion on the existence of solutions to Plateau's problem for plane domains. Note that all propositions in §2 hold also for planar domains.

For notational convenience, we shall say that a continuous map g which maps a compact smooth surface Ω (possibly disconnected) into a three-dimensional manifold bounds a collection of mutually disjoint Jordan curves $\{\gamma_1, \dots, \gamma_n\}$ if $g|_{\partial\Omega}$ is a homeomorphism onto $\bigcap_{i=1}^n \gamma_i$.

THEOREM 5. *Let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ be a collection of disjoint unoriented Jordan curves on boundary of a three-dimensional orientable convex manifold M^3 . Suppose these Jordan curves bound a continuous mapping g from a smooth compact plane domain (possibly disconnected). Then there exists a branched minimal immersion from a smooth compact plane domain (possibly disconnected) into M^3 which bounds Γ and has least area among all such maps. Furthermore, all such least area maps must be embeddings.*

Remark. In the assumption of the area minimizing property of g , we allow the competing surfaces to have arbitrary orientations on the boundary. Further discussion on the assumptions in Theorem 5 will appear in a future paper.

Proof. The existence of a planar solution to Plateau's problem for $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$ follows easily from Theorem 1 in §1 (see also the proof of Theorem 2). That a solution $f: \Omega \rightarrow M$ is an immersion in the interior of Ω follows from the interior regularity results of Osserman[42] and Gulliver[15].

We first give a complete proof of the above theorem when M is analytic and the Jordan curves in Γ are analytic. In this case, a solution $f: \Omega \rightarrow M$ of least area is analytic by Morrey[36] and Lewy[29]. Since the image of an analytic mapping is

semianalytic, the triangulability of semianalytic sets (see [30]) shows that we may assume that $f: \Omega \rightarrow M$ is simplicial with respect to some triangulations of Ω and M .

By the discussion and the proof of Dehn's Lemma in the previous section, we may assume that $f: \Omega \rightarrow M$ is simplicial and that f is embedded near the boundary of M . After restricting the range of f to a regular neighborhood N_1 of $f(\Omega)$, we get a new map $f_1: \Omega \rightarrow N_1$. We will now construct a tower of partial covering spaces similar to the tower constructed in the proof of Theorem 4. However we need to be more careful in our choice of 2-sheeted covering spaces to insure that our mappings will lift. This problem arises because $H_1(\Omega, Z_2)$ is nonzero.

First, consider the collection $B = \{[\gamma_1], \dots, [\gamma_n]\}$ where $[\gamma_i]$ denotes the homology class of γ_i in $H_1(N_1, Z_2)$. If B does not span $H_1(N_1, Z_2)$, then there is a surjective homomorphism $\rho: H_1(N_1, Z_2) \rightarrow Z_2$ with $\rho([\gamma_i]) = 0$ for all $1 \leq i \leq n$. This homomorphism induces a surjective homomorphism $\bar{\rho}: \pi_1(N_1) \rightarrow Z_2$. Since the kernel of $\bar{\rho}$ has index two in $\pi_1(N_1)$, there is a two sheeted covering space $P_1: \tilde{N}_1 \rightarrow N_1$ associated to this subgroup. Since the map $f: \Omega \rightarrow N_1$ satisfies $f_{1*}(\pi_1(\Omega)) \subset P_{1*}(\pi_1(\tilde{N}_1)) = \text{Ker}(\bar{\rho})$, the lifting theorem for covering spaces implies that f_1 lifts to a mapping $\tilde{f}_1: \Omega \rightarrow \tilde{N}_1$. After restricting the range of \tilde{f}_1 to a regular neighborhood N_2 of $\tilde{f}_1(\Omega)$, we get a new map $f_2: \Omega \rightarrow N_2$.

Repeat the construction described in the previous paragraph over and over again until the boundary curves of the image of a lifted map $f_k: \Omega \rightarrow N_k$ generate $H_1(N_k, Z_2)$. The argument given in the proof of Theorem 4 shows that this can be achieved after taking a finite number of 2-sheeted covers.

We now show that the lift $f_k: \Omega \rightarrow N_k$ to the top of the tower is an embedding. The proof of this fact will follow from a close examination of the topology of the boundary of N_k and our assumptions on the area of f given in the statement of the theorem.

Since M is orientable, N_k is orientable. This implies that the boundary of N_k consists of a finite number of compact orientable surfaces. Our first problem is to calculate $H_1(\partial N_k, Z_2)$.

By abuse of notation, let $\gamma_1, \dots, \gamma_n$ denote the lifts by f_j of the boundary curves of Ω to N_j . The method of construction for our tower shows $H_1(N_k, Z_2)$ is spanned by elements in $B = \{[\gamma_1], \dots, [\gamma_n]\}$. After reordering, we may assume $[\gamma_1], [\gamma_2], \dots, [\gamma_m]$ form a basis for $H_1(N_k, Z_2)$ where $m \leq n$. Because the pairing between homology and cohomology with coefficients in a field is nondegenerate, $\dim_{Z_2}(H^1(N_k, Z_2)) = m$. Poincaré duality for manifolds with boundary shows that $\dim_{Z_2}(H_2(N_k, \partial N_k, Z_2)) = m$.

From the long exact sequence in homology for the pair $(N_k, \partial N_k)$, one computes directly that $\dim_{Z_2}(H_1(\partial N_k, Z_2)) \leq 2m$. Since the curves $\gamma_1, \gamma_2, \dots, \gamma_m$ are disjoint embedded Jordan curves which lie on compact orientable surfaces comprising the boundary of N_k , and represent independent homology classes of ∂N_k , we must have $\dim_{Z_2}(H_1(\partial N_k, Z_2)) = 2m$. Also, on each boundary surface of N_k having genus g there are exactly g of the circles $\gamma_1, \gamma_2, \dots, \gamma_m$. Otherwise the dimension of $H_1(\partial N_k, Z_2)$ would be greater than $2m$.

Given a compact orientable surface X of positive genus g and g disjoint Jordan curves $\alpha_1, \alpha_2, \dots, \alpha_g$ which represent independent classes in the first homology of X , $X - \bigcap_{i=1}^g \alpha_i$ is a planar domain. This fact can be proved by induction on the genus g and by applying the classification theorem for surfaces with boundary. We refer the reader to [31 or 32] for similar arguments.

From the discussion in the previous two paragraphs we may now conclude that $\partial N_k - (\bigcup_{i=1}^m \gamma_i)$ is a collection of spheres and planar surfaces. Therefore the path components of $\partial N_k - (\bigcup_{i=1}^m \gamma_i)$ will also consist of spheres and planar surfaces.

Consider a planar component ω of $\partial N_k - (\cup_{i=1}^n \gamma_i)$ as embedded in the disk D . We assume that ω is the interior of a compact planar domain $\bar{\omega}$ with r boundary circles $\alpha_1, \alpha_2, \dots, \alpha_r$ corresponding to the topological ends of ω . We may consider each curve α_i as arising from cutting a component of ∂N_k along one of the curves γ_j . We shall now show that each of the curves $\alpha_1, \alpha_2, \dots, \alpha_r$ arise from distinct curves in $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$.

Suppose that α_1 and α_2 are contained in $\bar{\omega}$ and correspond to the same curve, say γ_1 . If $p \in \alpha_1$ and $q \in \alpha_2$ correspond to the same point $s \in \gamma_1$, then there is a curve $\bar{\sigma}: [0, 1] \rightarrow \bar{\omega}$ with $\bar{\sigma}(0) = p$ and $\bar{\sigma}(1) = q$, and an associated curve $\sigma: [0, 1] \rightarrow \partial N_k$ with $\sigma(0) = \sigma(1) = s \in \gamma_1$. Furthermore, we may assume that σ intersects $\cup_{i=1}^n \gamma_i$ only at the point $s \in \gamma_1$.

If we consider $f_k(\Omega)$ as representing a class in $H_2(N_k, \partial N_k, Z_2)$, then intersection theory shows that $[\sigma] \cap [f_k(\Omega)] \neq 0 \in H_0(N_k, Z_2)$. Here $\cap: H_1(N_k, Z_2) \times H_2(N_k, \partial N_k, Z_2) \rightarrow Z_2$ is the intersection pairing on homology and $[f_k(\Omega)]$ is considered as a class in $H_2(N_k, \partial N_k, Z_2)$. This is impossible for the following reason. Since $f_k: \Omega \rightarrow N_k$ is a simplicial immersion, we may push $\gamma_1, \gamma_2, \dots, \gamma_n$ off the set $f_k(\Omega) \cup \partial N_k$. These new curves are homologous to the old curves. Hence $[\gamma_i] \cap [f_k(\Omega)] = 0 \in H_0(\partial N_k, Z_2)$ for $1 \leq i \leq n$. Since $[\gamma_1], [\gamma_2], \dots, [\gamma_n]$ span $H_1(N_k, Z_2)$, $[\sigma] = \sum_{i=1}^n a_i [\gamma_i]$, and $[\sigma] \cap [f_k(\Omega)] = \sum_{i=1}^n a_i [\gamma_i] \cap [f_k(\Omega)] = 0$. This contradiction implies the closure of any path component of $\partial N - \cup_{i=1}^n \gamma_i$ in ∂N_k is either a sphere or a planar surface bounded by distinct curves in $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$.

The above argument also shows that any smooth Jordan curve σ on a component of ∂M^3 which intersects Γ transversally must intersect Γ in even numbers of points. Using this fact, we are going to prove that if X is a component of ∂M with $X \cap \Gamma \neq \emptyset$, there exist two (possibly disconnected) planar domains Ω_1 and Ω_2 such that $\Omega_1 \cup \Omega_2 = X$ and $\Omega_1 \cap \Omega_2 = \Gamma$.

In fact, by the previous arguments, Γ cuts X into a collection of compact connected planar domains $\{P_1, \dots, P_l\}$. We are going to paint the P_i 's with either white or black colors. To start, we paint P_1 with the white color and the P_i 's adjacent to P_1 with the black color. Then we claim that no two black (connected) domains are adjacent to each other. Otherwise, we can construct a Jordan curve σ in P_1 union with these two black domains so that σ intersects transversally the common boundaries of these three domains at exactly three points. This contradicts the assertion in the last paragraph.

Now we paint those domains adjacent to the black domains with the white color. The previous argument shows that these new white regions are not adjacent to each other. Continuing in this process, we color all the P_i 's with black or white colors and no two adjacent P_i 's have the same color. Let Ω_1 be the union of those P_i 's with white color and Ω_2 be the union of those P_i 's with black color. Then it is clear that $\Omega_1 \cup \Omega_2 = X$ and $\Omega_1 \cap \Omega_2 = \Gamma$.

For each boundary component of ∂M which has nonempty intersection with Γ , we can pick similar domains. Let Ω'_1 be the union of all white domains and Ω'_2 be the union of all black domains. Then the boundary of both Ω'_1 and Ω'_2 is precisely $\cup_{i=1}^n \gamma_i$.

Since N_k is a simplicial regular neighborhood of $f_k(\Omega)$, there is a simplicial retraction $S: N_k \rightarrow f_k(\Omega)$ whose restriction $R: \partial N_k \rightarrow f_k(\Omega)$ to the boundary of N_k has the property that R covers each 2-simplex of $f_k(\Omega)$ exactly two times (see the proof of Theorem 4). As area is carried only by 2-simplices, $\text{Area}(R|\Omega'_1) + \text{Area}(R|\Omega'_2) \leq 2 \text{Area}(f_k)$. Since both $R|\Omega'_1$ and $R|\Omega'_2$ bound Γ , we also have $\text{Area}(R|\Omega'_1) \geq \text{Area}(f_k)$ and $\text{Area}(R|\Omega'_2) \geq \text{Area}(f_k)$. Hence $\text{Area}(R|\Omega'_1) = \text{Area}(R|\Omega'_2) = \text{Area}(f_k)$ and every 2-simplex of $f_k(\Omega)$ is covered by a 2-simplex of $R|\Omega'_1$ or $R|\Omega'_2$. As in the proof of

Theorem 4, this implies either $R|\Omega'_1$ or $R|\Omega'_2$ has a folding curve. Hence the area of either $R|\Omega'_1$ or $R|\Omega'_2$ can be decreased. This contradicts the least area property of f_k and completes the proof that f_k is an embedding.

Suppose now that $f: \Omega \rightarrow M^3$ is not an embedding so that $f_{k-1}: \Omega \rightarrow N_{k-1}$ exists. Recall that a connected planar surface has the topological property that every Jordan curve on the surface disconnects the surface into two-path components. After fixing an orientation on Ω , the argument given in the proof of Lemma 10 shows there exists a Jordan curve $\alpha_1: S^1 \rightarrow \Omega$ which bounds with some components of $\partial\Omega$ a closed connected subdomain Ω_1 in Ω with $\Omega_1 \cap S(f_{k-1}) = \alpha_1(S^1)$. As in the case when Ω is a disk, the double curve $\alpha_2: S^1 \rightarrow \Omega$ corresponding to α_1 is another Jordan curve in Ω . By our choice of α_1 , the Jordan curve α_2 will bound, with some components of $\partial\Omega$, a closed sub-domain Ω_2 of Ω whose interior is disjoint from Ω_1 .

Let X be the quotient space obtained by taking the disjoint union of Ω_1 , Ω_2 and $\Omega \setminus (\Omega_1 \cup \Omega_2)$ and then identifying the points $\alpha_1(t) \in \partial\Omega_1$ to $\alpha_2(t) \in \Omega \setminus (\Omega_1 \cup \Omega_2)$ and the points $\alpha_2(t) \in \partial\Omega_2$ to $\alpha_1(t) \in \Omega \setminus (\Omega_1 \cup \Omega_2)$. Since we are simply interchanging two planar subdomains, it is clear that X is still a planar domain.

Now define the map $g: X \rightarrow N_{k-1}$ by $g(\bar{p}) = f_{k-1}(p)$ where \bar{p} denotes the equivalence class of the point p . Since $f_{k-1}(\alpha_1(t)) = f_{k-1}(\alpha_2(t))$, it is easy to verify that g is a well defined, Lipschitz piecewise differentiable map. The map g still bounds Γ . Hence the least area property of f_{k-1} shows that $\text{Area}(g) \leq \text{Area}(f_{k-1})$. By using an argument in the proof of Theorem 4 and Lemma 2, we can show that g has a folding curve along the curve $g(\alpha_1(S^1)) \subset g(X)$. Hence Lemma 7 shows that the area of g can be decreased. This contradicts the least area property of f_{k-1} . Therefore f_{k-1} does not exist and f must be an embedding. This finishes the proof of Theorem 5 in the real analytic category.

If we merely assume that the manifold M is a general compact convex manifold, the approximation procedure of Theorem 2 shows that we can use the analytic version of Theorem 5 to find a least area embedded planar domain which bounds Γ .

If we combine the arguments of Theorem 2 and Theorem 3 together, we can deal with the following situation. Let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ be a collection of mutually disjoint Jordan curves on the boundary of a three-dimensional compact convex manifold M^3 . Let γ_{n+1} be another smooth Jordan curve in the interior of M^3 . Suppose f is a branched minimal immersion from a (possibly disconnected) smooth compact plane domain Ω into M which bounds $\Gamma \cup \{\gamma_{n+1}\}$ and has least area among all such maps. (Ω is supposed to vary too.) If $f(\bar{\Omega})$ does not intersect $\bigcup_{i=1}^{n+1} \gamma_i$ and f has no boundary branch point on γ_{n+1} , then f is an embedding.

With this remark, we can now prove Theorem 5 in its full generality. Let f be any least area map stated in the theorem and suppose for the moment that Ω is connected. Then as f is an immersion and $f(\bar{\Omega}) \subset M^3$, we can use Lemma 4'' and the corollary to Lemma 3 to find a closed disk D in $\bar{\Omega}$ where $f|_D$ is an embedding and $f(D) \cap f(\bar{\Omega} \setminus D) = \emptyset$. By taking $\gamma_{n+1} = f(\partial D)$ and $\Gamma = \{\gamma_1, \dots, \gamma_n\}$, we are exactly in the situation described by the last paragraph and so $f|_{\bar{\Omega} \setminus D}$ is an embedding. This easily shows that f is an embedding on every component of Ω .

It remains to show that if Ω_1 and Ω_2 are two distinct components of Ω , then $f(\Omega_1) \cap f(\Omega_2) = \emptyset$. In fact, since $\partial f(\Omega_1) \cap \partial f(\Omega_2) = \emptyset$, Lemma 2 shows that the self-intersection set of $f|\Omega_1 \cup \Omega_2$ is a finite one dimensional complex which is disjoint from $\partial(\Omega_1 \cup \Omega_2)$ and which consists at most of double points. As in the proof that f_{k-1} is an embedding, we can decrease area if the self-intersection set is nonempty. This contradiction shows that $f(\Omega_1) \cap f(\Omega_2) = \emptyset$ and finishes the proof of Theorem 5.

The following corollary of Theorem 5 is a new topological result and gives a

complete generalization of Dehn's lemma for maps of a planar domain into an oriented three dimensional manifold.

COROLLARY (Dehn's Lemma for Planar Domains). *Let $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$ be a collection of disjoint unoriented smooth Jordan curves on the boundary of a three dimensional orientable manifold M . Suppose these Jordan curves bound a continuous mapping g from a smooth compact plane domain (possibly disconnected). Then there exists a smooth embedding of a connected planar domain which bounds the Jordan curves in Γ .*

Proof. Without any loss of generality we may assume that M is compact. (For example, replace M by a regular neighborhood of the image of a map of a planar domain bounded by disjoint circles and enlarge it suitably so that one has a connected manifold.) Now put a strictly convex metric on the boundary of M and solve Plateau's problem as in Theorem 5. By Theorem 5 there is a least area embedding $f: \Omega \rightarrow M$ from a possibly disconnected planar domain Ω which bounds the Jordan curves in Γ .

Since M is path connected, we can take the internal connected sum of the various component of $f(\Omega')$ to acquire a path connected surface. Since we do not care about the orientations of the various components of $f(\Omega')$ there is no problem in taking connected sums. The resulting surface is another planar surface because the connected sum of two planar surfaces is again a planar surface. This completes the proof of the corollary.

Remark. (1) Note that the proof of Theorem 5 also shows the following more general situation holds: Suppose $f: \Omega \rightarrow M$ is a planar solution to Plateau's problem for a collection Γ of disjoint unoriented Jordan curves in M . If $f(\bar{\Omega})$ is contained in the interior of M and $\overline{S(f)} \cap \partial\Omega = \emptyset$, then f is an embedding.

It should be noted that if the boundary components of $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ give rise to $n - 1$ "independent" elements in $H_1(M, \mathbb{Z})$, then Γ can only bound a connected planar domain. In the case of the annulus, a proper subset of $\Gamma = \{\gamma_1, \gamma_2\}$ bounds a disconnected planar domain if and only if the Jordan curves in Γ are homotopically trivial in M . Thus, appropriate topological conditions imply the planar solution to Plateau's problem given in Theorem 6 is connected. In general, the geometric hypothesis stated in Theorem 1 always guarantees the connectedness of the solutions for arbitrary planar domain.

(2) It is simple to verify that the only place in the proof of Theorem 5 that we used M is orientable was to show that N_k at the top of the tower construction was orientable. The condition that M is orientable can be replaced by the condition that none of the loops in $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$ have nontrivial normal bundles on ∂M . In the case of the disks, this condition always holds and hence the theorem holds for the disk even if the three dimensional manifold is nonorientable.

§6. THE GENERAL CASE OF DEHN'S LEMMA

From the results in the previous section we know that every disk solution to Plateau's problem for a Jordan curve γ on the boundary of a convex manifold is embedded. We shall now show that the images of any two solutions to Plateau's problem for γ are equal or disjoint in the interior of the three-dimensional convex manifold. The reader may be interested in comparing this disjointness property with the well-known example in [28] of an extremal Jordan curve in \mathbb{R}^3 which is smooth

except at one point and which bounds an uncountable number of embedded minimal surfaces.

THEOREM 6 (General Dehn's Lemma). *Suppose M^3 is a three-dimensional convex manifold. If γ is a Jordan curve on the boundary which is contractible in M^3 , then*

- (1) *There exists a solution (with finite area) to Plateau's problem for γ .*
- (2) *Any solution to Plateau's problem for γ is embedded.*
- (3) *For any two solutions to Plateau's problem for γ , either they differ from each other by a conformal reparametrization of D or the images of them intersect only along γ .*

Proof. The first two statements were treated in Theorem 5. We need only to prove (3) here. Let $f_1, f_2: D \rightarrow M$ be two solutions to Plateau's problem for γ which do not differ from each other by a conformal reparametrization. Then we claim that $f_1(D) \cap f_2(D) = \gamma$. First we give a simple proof in the case γ is analytic and the metric on M^3 is analytic. In this case $f_1(D)$ and $f_2(D)$ are analytically embedded in M and the intersection sets $S(f_1, f_2) = \{x \in D \mid \exists y \in D \text{ with } f_1(x) = f_2(y) \text{ and } S(f_2, f_1) = \{y \in D \mid \exists x \in D \text{ with } f_2(x) = f_1(y)\}$ are analytic subsets of D .

Assume now that $f_1(D) \cap f_2(D) \neq \gamma$. Applying the technique used in the proof of Lemma 10, it is relatively easy to show that there is a closed Jordan curve $\gamma_1 \subset S(f_1, f_2)$ with $\gamma_1 \neq \partial D$. Let γ_2 be the related Jordan curve in $S(f_2, f_1)$. Let D_1 and D_2 be the subdisks of D bounded by γ_1 and γ_2 respectively. It is clear that $f_1|_{D_1}$ and $f_2|_{D_2}$ are both solutions to Plateau's problem for the Jordan curve $f_1(\gamma_1)$. Lemma 4 shows that $f_1(D_1) \neq f_2(D_2)$. However, this contradicts the fact that $f_1(\gamma)$ is a curve of uniqueness for Plateau's problem (see Lemma 8). This contradiction shows $f_1(D) \cap f_2(D) = \gamma$ as was to be proved.

We will now give a proof of (3) for an arbitrary Jordan curve on ∂M . As in the analytic case, the proof is based on substituting a subdisk of least area by another disk of least area which will eventually contradict the least area property for a solution to Plateau's problem. However, since the boundary behavior of the solution may be quite erratic near the boundary of M , there is no way to do the substitution directly. Hence we have to rely on approximation methods to carry out the proof. If either $f_1(D)$ or $f_2(D)$ are contained in ∂M , then the proof that $f_1(D) \cap f_2(D) = \gamma$ is clear. Hence we will assume that neither $f_1(D)$ nor $f_2(D)$ are contained in ∂M and that $f_1(D) \cap f_2(D) \neq \gamma$. By Lemma 2, there is an arc $k: [0, 1] \rightarrow \bar{M}$ which is an arc of transverse intersection of $f_1(D)$ and $f_2(D)$. By compactness of the embedded disks $f_1(D)$ and $f_2(D)$ we may assume that the intersection of a small ball B , centered at $\kappa(1/2)$, with each of the disks $f_1(D)$ and $f_2(D)$ are subdisks F and E respectively. We may also assume that E and F intersect transversely along κ with $\kappa(0), \kappa(1) \in \partial B$ and that the intersection of E and F on ∂B consist of smooth Jordan curves α and β respectively.

Let α_1, α_2 and β_1, β_2 be the subarcs of α and β respectively. They join one point of intersection of α and β to the other point of intersection. The arc κ divides E and F into closed subdisks E_1, E_2 and F_1, F_2 , respectively, with $\alpha_1, \alpha_2; \beta_1, \beta_2$ being part of the boundary of the respective subdisks.

Let $A_{ij} = \text{Area}(E_i \cup F_j)$, $B_{ij} = \text{Area}$ of a solution to Plateau's problem for the Jordan curve $\alpha_i \beta_j^{-1}$ and $\epsilon = \inf \{|A_{ij} - B_{ij}| \mid 1 \leq i, j \leq 2\}$. Since the disks E and F intersect transversely along κ , it is clear from Lemma 7 that $\epsilon > 0$.

Now construct a sequence of Jordan curves $\gamma_i: S^1 \rightarrow \partial M$ which converge uniformly to $\gamma: S^1 \rightarrow M$ and which are disjoint from γ . We will also assume that the area of the annulus of least area bounded by γ_i and γ is less than $\min(\epsilon/5, \text{Area}(S))$, and

the area of these annular regions converges to zero as i gets large. The assumption that the area is less than $\text{Area}(E)$ is to guarantee the existence of a least area annulus bounding α and γ_i . The existence follows from Theorem 1 by checking the required inequality on the areas directly.

Since the curves γ and α bound the unique annular solution $f_2|D - f_2^{-1}(\tilde{E})$ to Plateau's problem, the proof of Theorem 5 shows we may assume, after picking a subsequence, that there are annular solutions $F_i: \Omega \rightarrow M$ to Plateau's problem for γ_i and α which converge uniformly to the original unique embedded solution to Plateau's problem for the curve γ and α . Since the convergence of F_i is uniform in the C^∞ norm near the smooth curve α , we may assume that for large i , $F_i(\Omega)$ is embedded near α and is transverse to $f_1(D)$ near ∂B , and $F_i(\tilde{\Omega})$ is disjoint from B . The remark following Theorem 5 shows that the maps F_i are embedded for large i .

Fix a large i so that F_i has the above properties. Consider a continuous piecewise differentiable map $f_3: D \rightarrow M$ which is an embedding obtained by glueing the embedded annulus $F_i(\Omega)$ to the embedded disk E along the common boundary curve α . If the metric on M is analytic, then the intersection sets $S(f_1, f_3) = \{x \in D | \exists y \text{ with } f_1(x) = f_3(y)\}$ and $S(f_3, f_1) = \{x \in D | \exists y \text{ with } f_3(x) = f_1(y)\}$ will be finite 1-complexes having an even number of edges. (Since γ_i is disjoint from γ and $\gamma_i \subset \partial M$, $\gamma_i \cap f_1(D) = \emptyset$ and $S(f_1, f_3)$, $S(f_3, f_1)$ are disjoint from ∂D .) As the intersection is two-to-one, even if the metric is not analytic, Lemma 2 can be applied to show that $S(f_1, f_3)$ and $S(f_3, f_1)$ satisfy the same properties.

Suppose for the moment that there exists a Jordan curve δ_1 in $S(f_1, f_3)$ containing $f_1^{-1}(\kappa)$. Then let δ_2 be the corresponding Jordan curve in $S(f_3, f_1)$. The curves δ_1 and δ_2 bound subdisks D_1 and D_2 on D respectively. Since f_1 is a solution to Plateau's problem, $f_1(\delta_1)$ is a curve of uniqueness for Plateau's problem with $f_1|D_1$ being the unique solution. Therefore $\text{Area}(f_1|D_1) \leq \text{Area}(f_3|D_2)$.

By our choice of γ_i , the area of an annulus of least area between γ_i and γ is less than $\epsilon/5$. This gives the inequality $\text{Area}(f_3) \leq \text{Area}(f_1) + \epsilon/5$. Now replace the disk $f_3(D_2)$ by the disk $f_1(D_1)$ to obtain a continuous piecewise differentiable map $f_4: D \rightarrow M$ with $\text{Area}(f_4) \leq \text{Area}(f_3) \leq \text{Area}(f_1) + \epsilon/5$.

By our choice of ϵ , we may decrease the area of f_4 in B^3 by at least ϵ . This shows that there is a solution $f_5: D \rightarrow M$ to Plateau's problem for γ with $\text{Area}(f_5) \leq \text{Area}(f_1) - 4\epsilon/5$. Since the area of the annulus between γ_i and γ is less than $\epsilon/5$, there is another map $f_6: D \rightarrow M$ bounding γ with $\text{Area}(f_6) \leq \text{Area}(f_1) - 3\epsilon/5$. The existence of f_6 contradicts the least area property for f_1 . Therefore once we have proved the existence of the Jordan curve δ_1 in $S(f_1, f_3)$, the proof of Theorem 6 will be completed.

Consider the arc $\kappa = f_1^{-1}(\kappa)$ contained in $S(f_1, f_3)$ with end points p_1 and p_2 . We claim that there is a path in $X = (S(f_1, f_3) - \kappa) \cup \{p_1, p_2\}$ joining p_1 to p_2 . If p_1 and p_2 lie in different path components P_1 and P_2 respectively, then P_1 is a finite 1-complex with one vertex p_1 where an odd number of edges meet p_1 . An elementary induction on the number of edges in a finite 1-complex shows that a finite 1-complex cannot have an odd number of vertices where an odd number of edges meet. Since P_1 has one such vertex, we have a contradiction. Therefore there is a path joining p_1 to p_2 in X .

Since any shortest path σ joining p_1 to p_2 in X is embedded and disjoint from the interior of κ' , we can define δ_1 to be the composite path $\delta_1 = \sigma\kappa'$. This construction of σ_1 completes the proof of Theorem 6.

Remark. Theorem 6 can be suitably generalized to general plane domains. This will be discussed in a future paper.

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APPENDIX

LEMMA 1. Let M be a Riemannian manifold whose sectional curvature is bounded from above by a positive constant K . Let N be a minimal subvariety of M such that for some point $x \in N$, the distance (measured in M) of x from ∂M and ∂N is greater than $\epsilon > 0$. Then when δ is smaller than ϵ and $i(M)$, the radius of injectivity of M ; the area of $B(x, \delta) \cap N$ is greater than $n\Omega K^{-n} \int_0^\delta t^{-1} (\sin Kt)^n dt$ where $n = \dim N$ and $\Omega > 0$ depends only on n .

Proof. Let r be the distance function of M measured from x . Then when r is smaller than $i(M)$, one can prove that

$$|\Delta r^2| \geq 2nKr \cot(Kr) \quad (1.1)$$

where $n = \dim N$ and Δ is the Laplace–Beltrami operator of N . (see the arguments of [51] p. 243–243 and Bishop and Crittenden[7]). Integrating (1.1) over $B(x, t)$ and noting the fact that $|\nabla r| \leq 1$, we have

$$t \text{Area} [\partial(N \cap B(x, t))] \geq nK \int_{B(x, t)} r \cot(Kr). \quad (1.2)$$

Let $C(t) = \int_{B(x, t)} r \cot(Kr)$. Then by the co-area formula (13) and the fact that $|\nabla r| \leq 1$,

$$\frac{\partial C(t)}{\partial t} \geq t \cot(kt) \text{Area} [\partial(N \cap B(x, t))]. \quad (1.3)$$

Putting (1.2) and (1.3) together, we find

$$\frac{\partial C(t)}{\partial t} \geq nK \cot(Kt) C(t). \quad (1.4)$$

It is easy to verify that

$$\lim_{t \rightarrow 0} C(t) \sin(Kt)^{-n} = K^{-n-1} \Omega \quad (1.5)$$

where Ω some positive constant depending only on n .

It follows from (1.4) and (1.5) that

$$C(t) \geq K^{-n-1} \Omega (\sin Kt)^n \quad (1.6)$$

for all $t \leq \epsilon$ and $i(M)$.

Therefore (1.2) shows that

$$\text{Area} [\partial(N \cap B(x, t))] \geq n \Omega K^{-n} t^{-1} (\sin Kt)^n. \quad (1.7)$$

By the co-area formula again,

$$\begin{aligned} \text{Area} [N \cap B(x, t)] &\geq \int_0^t \text{Area} [\partial(N \cap B(x, \tau))] \\ &\geq n \Omega K^{-n} \int_0^t \tau^{-1} (\sin K\tau)^n d\tau. \end{aligned} \quad (1.8)$$

This completes the proof of Lemma 1.