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Proof of the Positive Mass Theorem. II

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Abstract. The positive mass theorem states that for a nontrivial isolated physical system, the total energy, which includes contributions from both matter and gravitation is positive. This assertion was demonstrated in our previous paper in the important case when the space-time admits a maximal slice. Here this assumption is removed and the general theorem is demonstrated. Abstracts of the results of this paper appeared in [11] and [13].

Introduction

An initial data set for a space-time consists of a three-dimensional manifold N, a positive definite metric g_{ij}, a symmetric tensor p_{ij}, a local mass density μ, and a local current density J^i. The constraint equations which determine N to be a spacelike hypersurface in a space-time with second fundamental form p_{ij} are given by

$$\mu=\tfrac{1}{2}\left[R-\sum_{i,j}p^{ij}p_{ij}+\left(\sum_i p_i^i\right)^2\right]$$

$$J^i=\sum_j D_j\left[p^{ij}-\left(\sum_k p_k^k\right)g^{ij}\right],$$

where R is the scalar curvature of the metric g_{ij}. As usual, we assume that μ and J^i obey the dominant energy condition

$$\mu \geqq \left(\sum_i J^i J_i\right)^{1/2}.$$

An initial data set will be said to be asymptotically flat if for some compact set C, $N\backslash C$ consists of a finite number of components $N_1, \dots, N_p$ such that each N_i is diffeomorphic to the complement of a compact set in R^3. Under such diffeomorphisms, the metric tensor will be required to be written in the form

$$g_{ij}=\delta_{ij}+O(r^{-1})$$

and the scalar curvature of N will be assumed to be $O(r^{-4})$.

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With each N_k we associate a total mass M_k defined by the flux integral

$$M_k = \frac{1}{16\pi} \int_{\infty} \sum_{i,j} (g_{ij,j} - g_{jj,i}) d\sigma_i$$

which is the limit of surface integrals taken over large two spheres in N_k.

This number M_k is called the ADM mass of N_k (see Arnowitt, Deser, and Misner [1]). Classically it was assumed that the first term in the asymptotic expansion of g_{ij} is spherical. It was pointed out by York [11] that physically it is more desirable to relax this assumption to the one mentioned above. The method in this paper will work assuming only this general asymptotic condition of York.

In order for the total mass to be a conserved quantity, one assumes $p_{ij} = O(r^{-2})$ and $\sum_i p_{ii} = O(r^{-3})$.

In this formulation, the (generalized) positive mass theorem states that for an asymptotically flat initial data set, each end has nonnegative total mass. If one of the ends has zero total mass, the initial data set can be obtained from the metric tensor and the second fundamental form of a spacelike hypersurface in the Minkowski space-time. (In particular μ and J^i must be identically zero.)

We proved the positive mass theorem assuming the condition that $\sum_i p_i^i = 0$ in our previous paper. In this paper, we demonstrate the validity of the general theorem by reducing it to the previous case. It should be mentioned that the classical attempts in proving the positive mass theorem have been to treat the important case $\sum_i p_i^i = 0$ first and then reduce the general case to this case by asserting the existence of maximal slices (see, e.g. [2]). While we have similar steps, the basic ingredients are very different. For example, in the former method, it is necessary to prove that the space-time admits a slice with $\sum_i p_i^i = 0$. Not only is the existence of such a slice unknown, but also the space-time is expected to be more restrictive if such a slice does exist. Our approach can be described as follows.

We deform the metric g_{ij} and p_{ij} in two steps. In the first step, we consider the product manifold $N \times R$ with the product metric and extend p_{ij} trivially to be a tensor defined over $N \times R$. We want to find a hypersurface $\bar{N}$ in $N \times R$ which projects one to one onto N and whose mean curvature is the same as the trace of p_{ij} over $\bar{N}$. One of the motivations for considering such a hypersurface is that if N is a spacelike hypersurface in Minkowski space-time, the solution $\bar{N}$ can be identified with a linear slice of the Minkowski space-time. The second step is to observe that if such a hypersurface exists, the induced metric on this hypersurface can be deformed conformally to one with zero scalar curvature. If we can prove the existence of the hypersurface which is asymptotic to N in a suitable manner, we can prove that the total mass of N is the same as that of the hypersurface $\bar{N}$. We have then reduced the positive mass theorem to the case that we treated in our previous paper.

It happens that the hypersurface does not exist in general. Surprisingly its existence is closely related to the existence of apparent horizons in the initial data set (even if we assume the initial data set is nonsingular). The relation can be explained as follows. We perturb the equation that governs the hypersurface and prove that the perturbed equation admits an entire solution with the required asymptotic behavior. When the perturbation tends to zero, we prove that the hypersurfaces defined by the perturbed equations converge smoothly to a hypersurface. Although the hypersurface satisfies the required asymptotic conditions, it need not be a graph over N. The set over which it is not a graph has boundary consisting of spheres which are apparent horizons. By conformally closing these apparent horizons, we carry through the argument outlined above.

It should be pointed out that in a previous attempt by Jang to solve the positive mass theorem, the equation defining the above hypersurface was considered. However, our geometric interpretation of the equation and our way of using it are completely different from his. (He used a method outlined by Geroch which up to now has been unsuccessful in proving positivity of mass.) While Jang observes that the equation is not solvable in general, he provides no method to circumvent this situation. It should be emphasized that the major effort of this paper is to overcome this difficulty. For a historical account of the previous efforts to prove the mass theorem, see the references in [9]. We wish to point out that our method in this paper also works to prove the mass is positive for an initial data set with singularities, provided they are surrounded by apparent horizons.

For the reader's convenience, we suggest the reader to skip sections two and three for the first reading. They can read the first two paragraphs of pp. 238–240, statements of Propositions 1–3.

1. Statement of Results

As in the introduction, let N be an oriented asymptotically flat three dimensional manifold without boundary. Let ds^2 be a positive definite metric on N. Suppose that N is of smoothness class C^4, and that ds^2 is C^3. Assume that on each N_k there exist coordinates x^1, x^2, x^3 in which ds^2 has the expansion $ds^2=\sum_{i,j=1}^{3} g_{ij}dx^i dx^j$ with the g_{ij} satisfying the following inequalities for positive constants k_1, k_2, k_3

$$\begin{aligned} &g_{ij}=\delta_{ij}+b_{ij}, \quad |b_{ij}|\leqq k_1(1+r)^{-1},\\ &|\partial b_{ij}|\leqq k_2(1+r^2)^{-1}, \quad |\partial\partial b_{ij}|\leqq k_3(1+r^3)^{-1}, \end{aligned} \tag{1.1}$$

where $r^2=\sum_{i=1}^{3}(x^i)^2$ and ∂ is the Euclidean gradient. Note that (1.1) implies that the Christoffel symbols Γ^i_{jk} fall off as $O(r^{-2})$ and the curvature tensor as $O(r^{-3})$ as $r\to\infty$. We assume that the scalar curvature (Ricci scalar) R falls off like r^{-4}, i.e.,

$$|R|\leqq k_4(1+r^4)^{-1}, \quad |\partial R|\leqq k_5(1+r^5)^{-1} \tag{1.2}$$

for constants k_4, k_5.

We suppose also that on N we are given a symmetric two-tensor p_{ij} which on each N_k satisfy the inequalities

$$|p_{ij}|+r|\partial p_{ij}|+r^2|\partial\partial p_{ij}|\leqq k_6(1+r^2)^{-1} \tag{1.3}$$

for a constant k_6. We assume the trace of p_{ij} satisfies the faster falloff

$$\left|\sum_i p_{ii}\right|\leqq k_7(1+r^3)^{-1}. \tag{1.4}$$

As was mentioned in the introduction we will be assuming the dominant energy condition holds on N, i.e.,

$$\mu \geqq \left(\sum_i J^i J_i\right)^{1/2}. \tag{1.5}$$

We will refer to the triple (N, ds^2, p_{ij}) satisfying (1.1)–(1.5) as an *initial data set*. Note that we have weakened the asymptotic assumption on g_{ij} from that assumed in [9]. In [10] we have established the main result of [9] under this weaker assumption. We state our first theorem.

Theorem 1. *Let (N, ds^2, p_{ij}) be an initial data set. For $1 \leqq k \leqq p$, we have $M_k \geqq 0$.*

We will also prove that if some M_k is zero, the initial data set is trivial. For this we need to assume ds^2 is C^4 and expand (1.1) to include the following assumption

$$|\partial\partial\partial b_{ij}| + |\partial\partial\partial\partial b_{ij}| \leqq k_8(1+r^4)^{-1}. \tag{1.6}$$

Theorem 2. *If (N, ds^2, p_{ij}) is an initial data set satisfying* (1.6), *and $M_k = 0$ for some k, then (N, ds^2, p_{ij}) can be isometrically embedded into four dimensional Minkowski space $\mathbb{M}$ as a spacelike hypersurface so that ds^2 is the induced metric from $\mathbb{M}$ and p_{ij} is the second fundamental form. In particular N is topologically $\mathbb{R}^3$.*

2. The Basic Equation and Local Formulae

In this section we derive the basic formulae describing the local geometry of hypersurfaces in $N \times \mathbb{R}$. Suppose (N, ds^2, p_{ij}) is an initial data set as defined in Sect. 1. We form the Riemannian product $N \times \mathbb{R}$ with (positive definite) metric $ds^2 + dt^2$ where $t \in \mathbb{R}$ is a coordinate. We suppose that $\Sigma^3 \subseteq N \times \mathbb{R}$ is a smooth hypersurface, and let e_1, e_2, e_3, e_4 be a local orthonormal frame for Σ with e_4 normal to Σ and e_1, e_2, e_3 tangential. Let w_1, w_2, w_3, w_4 be the corresponding dual orthonormal coframe of one-forms. We may write the structural equations for $N \times \mathbb{R}$

$$dw_a = \sum_{b=1}^{4} w_{ab} \wedge w_b, \qquad w_{ab} + w_{ba} = 0, \tag{2.1}$$

$$dw_{ab} - \sum_{c=1}^{4} w_{ac} \wedge w_{cb} = -\tfrac{1}{2} \sum_{c,d=1}^{4} R_{abcd} w_c \wedge w_d, \tag{2.2}$$

where R_{abcd} is the curvature tensor of $N \times \mathbb{R}$. We adopt the convention that letters $a, b, c, \ldots$ run from 1 to 4 while the letters $i, j, k, \ldots$ denote indices between 1 and 3. We define the second fundamental form of Σ, which we denote $A = (h_{ij})_{1 \leqq i,j \leqq 3}$ by

$$w_{4i}|_\Sigma = \sum_j h_{ij} w_j, \qquad h_{ij} = h_{ji}, \tag{2.3}$$

where $(\cdot)|_\Sigma$ indicates restriction of a one-form to Σ. The mean curvature H of Σ is then given by $H = \sum_i h_{ii}$. Restricting (2.2) to Σ and using (2.3) we derive the curvature equation

$$\bar{R}_{ijk\ell} = R_{ijk\ell} + (h_{ik} h_{j\ell} - h_{i\ell} h_{jk}), \tag{2.4}$$

where $\bar{R}_{ijk\ell}$ denotes the *intrinsic* curvature of Σ. Applying the exterior derivative to (2.3) and using (2.2) we derive the Codazzi equation

$$\bar{D}_k h_{ij} - \bar{D}_j h_{ik} = R_{4ijk}, \tag{2.5}$$

where $\bar{D}$ is used to denote covariant differentiation with respect to the metric of Σ, and $\bar{D}_k h_{ij}$ is defined by

$$\sum_k \bar{D}_k h_{ij} w_k = dh_{ij} + \sum_k h_{ik} w_{kj} + \sum_h h_{kj} w_{ki}. \tag{2.6}$$

We now exploit the special structure of $N \times \mathbb{R}$. Let v be the downward unit parallel vector field tangent to the $\mathbb{R}$ factor, and consider the function $\langle e_4, v\rangle$ defined on Σ, where $\langle\cdot,\cdot\rangle$ is the inner product of $N \times \mathbb{R}$. For a smooth function φ on Σ, the covariant derivatives $\bar{D}_i\varphi$, $\bar{D}_i\bar{D}_j\varphi$, and the Laplacian $\Delta\varphi$ are given by

$$d\varphi = \sum_i (\bar{D}_i\varphi) w_i, \qquad d(\bar{D}_i\varphi) + \sum_j (\bar{D}_j\varphi) w_{ji} = \sum_j (\bar{D}_i\bar{D}_j\varphi) w_j$$

$$\Delta\varphi = \sum_i \bar{D}_i\bar{D}_i\varphi.$$

We calculate $\Delta\langle e_4, v\rangle$ by observing that $v = \sum_a \langle v, e_a\rangle e_a$ is parallel, so the covariant derivative $D_a v$ in $N \times \mathbb{R}$ is

$$0 = \sum_b (D_b v)_a w_b = d\langle v, e_a\rangle + \sum_b \langle v, e_b\rangle w_{ba}. \tag{2.7}$$

Using (2.3) we then get

$$d\langle e_4, v\rangle = -\sum_i \langle v, e_i\rangle w_{i4} = \sum_{i,j} h_{ij}\langle v, e_i\rangle w_j.$$

Thus by (2.6) and (2.7) we have

$$\bar{D}_i\bar{D}_j\langle v, e_4\rangle = \sum_k (\bar{D}_i h_{jk})\langle v, e_k\rangle - \sum_k h_{jk} h_{ik}\langle v, e_4\rangle.$$

Taking the trace and using (2.5) we get

$$\Delta\langle v, e_4\rangle = \sum_{i,k} R_{4iki}\langle v, e_k\rangle + \sum_k (\bar{D}_k H)\langle v, e_k\rangle - \left(\sum_{i,k} h_{ik}^2\right)\langle v, e_4\rangle. \tag{2.8}$$

We will need to compute $\bar{D}_\ell\bar{D}_k h_{ij}$, so we define

$$\begin{aligned}\sum_\ell (\bar{D}_\ell\bar{D}_k h_{ij}) w_\ell = d(\bar{D}_k h_{ij}) &+ \sum_\ell (\bar{D}_\ell h_{ij}) w_{\ell k}\\ &+ \sum_\ell (\bar{D}_k h_{\ell j}) w_{\ell i} + \sum_\ell (\bar{D}_k h_{i\ell}) w_{\ell j}.\end{aligned} \tag{2.9}$$

Applying the exterior derivative to (2.6) we then have

$$\begin{aligned}\sum_{k,\ell} (\bar{D}_\ell\bar{D}_k h_{ij}) w_\ell \wedge w_k = &-\tfrac{1}{2}\sum_{k,\ell,m} h_{kj}\bar{R}_{ki\ell m} w_\ell \wedge w_m\\ &-\tfrac{1}{2}\sum_{k,\ell,m} h_{ik}\bar{R}_{kj\ell m} w_\ell \wedge w_m.\end{aligned}$$

Equating coefficients then gives

$$\bar{D}_\ell \bar{D}_k h_{ij} - \bar{D}_k \bar{D}_\ell h_{ij} = -\sum_m h_{mj} \bar{R}_{mi\ell k} - \sum_m h_{im} \bar{R}_{mj\ell k}. \tag{2.10}$$

We wish to calculate $\Delta h_{ij} = \sum_k \bar{D}_k \bar{D}_k h_{ij}$ in terms of the mean curvature H, so we use (2.5)

$$\Delta h_{ij} = \sum_k \bar{D}_k \bar{D}_j h_{ik} + \sum_k \bar{D}_k R_{4ijk}, \tag{2.11}$$

where $\bar{D}_\ell R_{4ijk}$ is defined by

$$\sum_\ell \bar{D}_\ell R_{4ijk} w_\ell = dR_{4ijk} + \sum_\ell R_{4\ell jk} w_{\ell i} + \sum_\ell R_{4i\ell k} w_{\ell j} + \sum_\ell R_{4ij\ell} w_{\ell k}.$$

We may express this in terms of $D_\ell R_{4ijk}$ by using (2.3)

$$\bar{D}_\ell R_{4ijk} = D_\ell R_{4ijk} - R_{4i4k} h_{\ell j} - R_{4ij4} h_{k\ell} + \sum_m R_{mijk} h_{m\ell}. \tag{2.12}$$

We now use (2.10) in (2.11) to get

$$\Delta h_{ij} = \sum_k \bar{D}_j \bar{D}_k h_{ik} + \sum_k \bar{D}_k R_{4ijk} - \sum_{m,k} h_{mk} \bar{R}_{mikj} - \sum_{m,k} h_{im} \bar{R}_{mkkj}.$$

Finally, we apply (2.5) once more, together with the symmetry of (h_{ij}) to obtain

$$\begin{aligned} \Delta h_{ij} = \bar{D}_i \bar{D}_j H &+ \sum_k \bar{D}_k R_{4ijk} - \sum_{m,k} h_{mk} \bar{R}_{mikj} \\ &- \sum_{m,k} h_{im} \bar{R}_{mkkj} + \sum_k \bar{D}_j R_{4kik}. \end{aligned}$$

Using (2.4) and (2.12) we finally have

$$\begin{aligned} \Delta h_{ij} = \bar{D}_i \bar{D}_j H &- \Big(\sum_{m,k} h_{mk}^2\Big) h_{ij} + H \sum_m h_{im} h_{mj} \\ &- 2\sum_{m,k} h_{mk} R_{mikj} - \sum_{m,k} h_{im} R_{mkkj} \\ &+ \sum_k D_k R_{4ijk} + \sum_k D_j R_{4kik} \\ &- \sum_k R_{4i4k} h_{jk} - H R_{4ij4} - \sum_k R_{4k4k} h_{ij} \\ &- \sum_k R_{4ki4} h_{jk} + \sum_{m,k} R_{mkik} h_{mj}. \end{aligned}$$

We are not especially interested in the particular form for this equation, but we want estimates independent of Σ, so we note that we have the matrix inequality

$$\Delta h_{ij} \geqq \bar{D}_i \bar{D}_j H - \Big(\sum_{m,k} h_{mk}^2\Big) h_{ij} + H \sum_m h_{im} h_{mj} - c_1(|A|+1)\delta_{ij},$$

where c_1 depends only on k_1, k_2, k_3 (not on Σ). We are using $|A|^2 = \sum_{i,j} h_{ij}^2$. We now calculate $\Delta|A|^2$ as follows:

$$\tfrac{1}{2}\Delta|A|^2 = \sum_{i,j} h_{ij} \Delta h_{ij} + \sum_{i,j,k} (\bar{D}_k h_{ij})^2.$$

Therefore, we have

$$\tfrac{1}{2}\Delta|A|^2 \geqq \sum_{i,j,k} (\bar{D}_k h_{ij})^2 - |A|^4 - |H|\,|A|^3 + \sum_{i,j} h_{ij}\bar{D}_i\bar{D}_j H - c_2(|A|^2+1) \tag{2.13}$$

for a constant c_2. Since $\frac{1}{2}\Delta|A|^2 = |A|\Delta|A| + |\bar{D}|A||^2$, we get

$$|A|\Delta|A| \geqq \left(\sum_{i,j,k} (\bar{D}_k h_{ij})^2 - |\bar{D}|A||^2\right) - |A|^4 - |H|\,|A|^3 + \sum_{i,j} h_{ij}\bar{D}_i\bar{D}_j H - c_2(|A|^2+1). \tag{2.14}$$

We now record the following observation of [8]. We may write the first term T on the right of (2.14) as

$$T = \sum_{i,j,k} (\bar{D}_k h_{ij})^2 - |A|^{-2}\sum_k \left(\sum_{i,j} h_{ij}\bar{D}_k h_{ij}\right)^2.$$

This implies that

$$|A|^2 T = \tfrac{1}{2}\sum_{i,j,k,\ell,m} (h_{ij}\bar{D}_k h_{\ell m} - h_{\ell m}\bar{D}_k h_{ij})^2.$$

Setting $k=i$ and $m=j$ in the sum implies

$$\begin{aligned}|A|^2 T &\geqq \tfrac{1}{2}\sum_{i,j,\ell} (h_{ij}\bar{D}_i h_{\ell j} - h_{\ell j}\bar{D}_i h_{ij})^2 \\ &\geqq \tfrac{1}{18}\sum_\ell \left(\sum_{i,j} h_{ij}\bar{D}_i h_{\ell j} - \sum_{i,j} h_{\ell j}\bar{D}_i h_{ij}\right)^2,\end{aligned} \tag{2.15}$$

where we have used the Schwarz inequality. By (2.5),

$$\sum_{i,j} h_{\ell j}\bar{D}_i h_{ij} = \sum_j h_{\ell j}\bar{D}_j H + \sum_{i,j} h_{\ell j} R_{4iji}$$

$$\sum_{i,j} h_{ij}\bar{D}_i h_{\ell j} = \sum_{i,j} h_{ij}\bar{D}_\ell h_{ij} + \sum_{i,j} h_{ij} R_{4j\ell i}.$$

Putting these into (2.15) and using the inequality $(a-b)^2 \geqq \frac{1}{2}a^2 - b^2$ we get

$$|A|^2 T \geqq \tfrac{1}{36}\left(\sum_{i,j} h_{ij}\bar{D}_\ell h_{ij}\right)^2 - c_3|\bar{D}H|^2|A|^2 - c_3|A|^2.$$

This implies that

$$T \geqq \tfrac{1}{37}\sum_{i,j,k} (\bar{D}_k h_{ij})^2 - \frac{36c_3}{37}|\bar{D}H|^2 - \frac{36c_3}{37}.$$

Combining this with (2.14) then gives

$$|A|\Delta|A| \geqq \tfrac{1}{37}\sum_{i,j,k} (\bar{D}_k h_{ij})^2 - |A|^4 - |H|\,|A|^3 + \sum_{i,j} h_{ij}\bar{D}_i\bar{D}_j H - c_4|\bar{D}H|^2 - c_4(|A|^2+1). \tag{2.16}$$

Inequality (2.16) will be important for the estimates of the next section.

For the remainder of this section we specialize to the case when Σ is the graph of a function f defined on N. In this case we may extend our orthonormal frame e_1, e_2, e_3, e_4 to $N \times \mathbb{R}$ in such a way as to be parallel along the $\mathbb{R}$ factor. We also suppose that the given data, p_{ij}, μ, and J are extended parallel along the $\mathbb{R}$ factor. We assume that e_4 is taken to be the downward unit normal to Σ so that $\langle v, e_4\rangle > 0$ everywhere on Σ. Thus the following hold on $N \times \mathbb{R}$.

$$e_4 = (1+|Df|^2)^{-1/2}(+Df - v)$$

$$R = \sum_{a,b} R_{abab}$$

$$2\mu = R - \sum_{a,b} p_{ab}^2 + \Big(\sum_q p_{aa}\Big)^2$$

$$J_b = \sum_a D_a p_{ab} - \sum_a D_b p_{aa},$$

where R_{abcd} is the curvature tensor of $N \times \mathbb{R}$. Since e_1, e_2, e_3, e_4 is now extended in a natural way to all of $N \times \mathbb{R}$, we introduce the following notation [cf. (2.3)]

$$w_{4i} = \sum_j h_{ij} w_j + h_{i4} w_4 . \tag{2.17}$$

This defines $\sum_i h_{i4} w_i$ as a one-form on Σ. We wish to refine (2.8) in our setting. First note that since $N \times \mathbb{R}$ is given the product metric, and H is constant along the $\mathbb{R}$ factor, we have

$$0 = \sum_k R_{4iki}\langle v, e_k\rangle + R_{4i4i}\langle v, e_4\rangle$$

$$0 = \sum_k (D_k H)\langle v, e_k\rangle + (e_4 H)\langle v, e_4\rangle,$$

where $e_4 H$ is the directional derivative of H in direction e_4. Putting these into (2.8) then gives

$$\Delta\langle v, e_4\rangle = \Big(-\sum_i R_{4i4i} - e_4 H - |A|^2\Big)\langle v, e_4\rangle . \tag{2.18}$$

We now notice that

$$R = 2\sum_i R_{4i4i} + \sum_{i,j} R_{ijij},$$

so by (2.4) we have

$$R = 2\sum_i R_{4i4i} + \bar{R} - H^2 + |A|^2,$$

where $\bar{R}$ is the intrinsic scalar curvature of Σ. Thus by the definition of μ we have

$$\sum_i R_{4i4i} = \mu + \tfrac{1}{2}\Big(-\bar{R} + \sum_{a,b} p_{ab}^2 - \Big(\sum_a p_{aa}\Big)^2 - |A|^2 + H^2\Big). \tag{2.19}$$

We will also need to have an expression for $e_4\Big(\sum_i p_{ii}\Big)$ in terms of J, so we notice that

$$\sum_i D_4 p_{ii} = \sum_i D_i p_{i4} - J_4, \tag{2.20}$$

and we have

$$\sum_a D_a p_{ii} w_a = dp_{ii} + 2\sum_j p_{ji} w_{ji} + 2p_{i4} w_{4i} .$$

Summing on i and equating coefficients of w_4 we have by (2.17) and the symmetry of p_{ij}

$$\sum_i D_4 p_{ii} = e_4\left(\sum_i p_{ii}\right) + 2\sum_i p_{i4} h_{i4} . \tag{2.21}$$

We also have

$$\sum_a (D_a p_{i4}) w_a = dp_{i4} + \sum_a p_{a4} w_{ai} + \sum_a p_{ia} w_{a4}$$

which gives

$$D_i p_{i4} = e_i(p_{i4}) + \sum_j p_{j4} w_{ji}(e_i) + p_{44} h_{ii} - \sum_{i,j} p_{ij} h_{ij} .$$

Summing on i and using the definition of $\bar{D}$ we have

$$\sum_i D_i P_{i4} = \sum_i \bar{D}_i p_{i4} + p_{44} H - \sum_{i,j} p_{ij} h_{ij} .$$

Combining this with (2.20) and (2.21) implies

$$\begin{aligned} e_4\left(\sum_i p_{ii}\right) = \sum_i \bar{D}_i p_{i4} - J_4 + p_{44} H \\ - \sum_{i,j} p_{ij} h_{ij} - 2\sum_i p_{i4} h_{i4} . \end{aligned} \tag{2.22}$$

We now combine (2.18), (2.19), and (2.22)

$$\begin{aligned} 2\langle v, e_4\rangle^{-1} \Delta \langle v, e_4\rangle = \bar{R} &- \sum_{i,j} (h_{ij} - p_{ij})^2 - 2\sum_i p_{i4}^2 \\ &+ 4\sum_i p_{i4} h_{i4} - 2\sum_i \bar{D}_i p_{i4} + \left(\sum_i p_{ii}\right)^2 \\ &- H^2 + 2p_{44}\left(\sum_i p_{ii} - H\right) + 2e_4\left(\sum_i p_{ii} - H\right) \\ &- 2(\mu - J_4) . \end{aligned} \tag{2.23}$$

We now observe that since e_4 has been extended to be parallel along v we have by (2.17)

$$\begin{aligned} 0 &= \sum_i \langle v, e_i\rangle D_i e_4 + \langle v, e_4\rangle D_{e_4} e_4 \\ &= \sum_{i,j} \langle v, e_i\rangle h_{ij} e_j + \langle v, e_4\rangle \sum_j h_{j4} e_j . \end{aligned}$$

Since $D_j \langle v, e_4\rangle = \sum_i \langle v, e_i\rangle h_{ij}$, we have

$$h_{j4} = -\langle v, e^4\rangle^{-1} \bar{D}_j \langle v, e_4\rangle = -\bar{D}_j (\log \langle v, e_4\rangle) . \tag{2.24}$$

Hence if we compute $\Delta \log \langle v, e_4\rangle$ we have

$$\Delta \log \langle v, e_4\rangle = -\sum_i \bar{D}_i h_{i4} = \langle v, e_4\rangle^{-1} \Delta \langle v, e_4\rangle - \sum_i h_{i4}^2 .$$

Putting this into (2.23) and using the energy condition (1.5) we have

$$
\begin{aligned}
0 \leqq 2(\mu-|J|) \leqq \bar{R} & -\sum_{i,j}(h_{ij}-p_{ij})^2-2\sum_i(h_{i4}-p_{i4})^2 \\
& +2\sum_i \bar{D}_i(h_{i4}-p_{i4})+\Big(\sum_i p_{ii}\Big)^2-H^2 \\
& +2p_{44}\Big(\sum_i p_{ii}-H\Big)+2e_4\Big(\sum_i p_{ii}-H\Big). \qquad (2.25)
\end{aligned}
$$

We now introduce the equation which Σ will be required to satisfy. It is an equation proposed by Jang [5]. We will study the solutions of this equation later in this paper. The equation is

$$
H=\sum_i p_{ii}. \qquad (2.26)
$$

More explicitly, if Σ is the graph of a function f, it is the equation

$$
(1+Df^2)^{-1/2}\sum_{i,j}\bar{g}^{ij}D_iD_jf=\sum_{i,j}\bar{g}^{ij}p_{ij}, \qquad (2.27)
$$

where $\bar{g}_{ij}$ is the induced metric on Σ

$$
\begin{aligned}
\bar{g}_{ij} &= g_{ij}+f_{x^i}f_{x^j} \\
\bar{g}^{ij} &= g^{ij}-\frac{f^if^j}{1+|Df|^2} \\
f^i &= \sum_j g^{ij}f_{x^j}.
\end{aligned}
$$

Geometrically (2.27) says that we prescribe the mean curvature at each point of Σ to be equal to the trace of the restriction of p_{ij} (extended to $N\times\mathbb{R}$) to Σ. We will study solutions of (2.27) having the asymptotic behavior

$$
|f|=O(r^{-1/2}),\quad |\partial f|=O(r^{-3/2}),\quad |\partial\partial f|=O(r^{-5/2}),\quad |\partial\partial\partial f|=O(r^{-7/2}) \qquad (2.28)
$$

at each inifinity of N.

The inequality (2.25) is closely related to Eq. (2.27). In fact, (2.27) expresses the fact that $H-\sum_i p_{ii}$ does not change along vertical lines, so that $v\Big(H-\sum_i p_{ii}\Big)=0$. Assuming Σ satisfies (2.27), by (2.25) we have

$$
\begin{aligned}
0 \leqq 2(\mu-|J|) \leqq \bar{R} & -\sum_{i,j}(h_{ij}-p_{ij})^2-2\sum_i(h_{i4}-p_{i4})^2 \\
& +2\sum_i D_i(h_{i4}-p_{i4}). \qquad (2.29)
\end{aligned}
$$

It will afford us some convenience in the proof of Theorem 1 to assume strict inequality in (1.5). We prove a simple perturbation result which allows us to do so.

Lemma 1. *Let (N, ds^2, p_{ij}) be an initial data set. Given a number $\varepsilon>0$, there is a function $\varphi>0$ on N satisfying*

$$
\varphi=1+\frac{A_k}{r}+O(r^{-2}),\qquad |\partial\varphi|=O(r^{-2}),\qquad |\partial\partial\varphi|=O(r^{-3})
$$

on N_k with $|A_k|<\varepsilon$ so that $(N, \varphi^4 ds^2, \varphi^2 p_{ij})$ is an initial data set with mass density $\bar{\mu}$ and current density $\bar{J}$ satisfying $\bar{\mu}>|\bar{J}|$.

Proof. If $\varphi>0$ is a function on N, then we can compute

$$\bar{\mu}=\varphi^{-4}(\mu-4\varphi^{-1}\Delta\varphi)$$
$$|\bar{J}|=\varphi^{-4}\Big(\sum_{i,j} g^{ij}K_iK_j\Big)^{1/2},$$

where $K_i=J_i+4\sum_{\ell}\varphi^{-1}\varphi^{\ell}p_{i\ell}$. Thus if we let

$$T\varphi=\Delta\varphi+\tfrac{1}{4}\varphi\left[\Big(\sum_{i,j} g^{ij}K_iK_j\Big)^{1/2}-\mu\right],$$

we have $T1=|J|-\mu\leqq 0$, and

$$T\varphi=\tfrac{1}{4}\varphi^5(|\bar{J}|-\bar{\mu}).$$

The linearization of $T\varphi$ at $\varphi=1$ is given by

$$\Delta\eta+\tfrac{1}{4}(|J|-\mu)\eta+\sum_{i,\ell}\eta^{\ell}\frac{J^i p_{i\ell}}{|J|}$$

which is an isomorphism on suitable spaces, so by the implicit function theorem we can find φ near 1 so that $T\varphi<0$, hence $\bar{\mu}>|\bar{J}|$. [For example, one exhausts N by compact subdomains Ω and solves the inequality $T\varphi=f<0$ on Ω with $\varphi=1$ on $\partial\Omega$. Once one solves this equation, one can see easily that φ converges to the require solution when Ω tends to N. The existence on compact subdomains follows by applying the implicit functions to the map $T: H^2(\Omega)\to L^2(\Omega)$.] The asymptotic conditions for φ are easily shown.

3. The a Priori Estimates

In this section we prove the estimates which are needed to show existence of solutions to (2.27). We concentrate first on the local interior estimates, and then we construct suitable "barrier" functions [see (3.20)] to control the behavior of solutions at infinity.

We study a slightly more general equation then (2.27). Let $F(x)$ be a given C^2 function on N and suppose μ_1, μ_2, and μ_3 are constants so that

$$\sup_N |F|\leqq\mu_1, \quad \sup_N |DF|\leqq\mu_2, \quad \sup_N |DDF|\leqq\mu_3. \tag{3.1}$$

Suppose f is a given C^3 solution of

$$\sum_{i,j=1}^{3}\left(g^{ij}-\frac{f^if^j}{1+|Df|^2}\right)\left(\frac{D_iD_jf}{(1+|Df|^2)^{1/2}}-p_{ij}\right)=F. \tag{3.2}$$

We propose to derive suitable estimates on f and its derivatives in terms of μ_1, μ_2, and μ_3. We let $c_1, c_2, \ldots$ throughout this section be constants depending only on (N, g_{ij}, p_{ij}) *and* μ_1, μ_2, μ_3. *We will not explicitly denote the dependence on* μ_1, μ_2, μ_3.

We will use the notation of Sect. 2 for the graph of f. We first observe that by (2.4), (3.1), and (3.2) we have

$$|\bar{R}+|A|^2|\leqq c_1\,,$$

so inequality (2.25) implies

$$|A|^2+\sum_i(h_{i4}-p_{i4})^2\leqq\sum_i\bar{D}_i(h_{i4}-p_{i4})+c_2(|A|+1)\,.$$

Multiplying this inequality by φ^2 where φ has compact support on the graph Σ of f, and integrating by parts, we find

$$\int_\Sigma|A|^2\varphi^2\sqrt{\bar{g}}dx+\int_\Sigma\sum_i(h_{i4}-p_{i4})^2\sqrt{\bar{g}}dx$$
$$\leqq-2\int_\Sigma\varphi\sum_i(\bar{D}_i\varphi)(h_{i4}-p_{i4})\sqrt{\bar{g}}dx+c_2\int_\Sigma(|A|+1)\varphi^2\sqrt{\bar{g}}dx\,.$$

Using the inequality $2ab\leqq a^2+b^2$, we get

$$\int_\Sigma|A|^2\varphi^2\sqrt{\bar{g}}dx\leqq\int_\Sigma|\bar{D}\varphi|^2\sqrt{\bar{g}}dx+c_2\int_\Sigma(|A|+1)\varphi^2\sqrt{\bar{g}}dx \tag{3.3}$$

for any φ with compact support on Σ. We now replace φ in (3.3) by the function $|A|\cdot\varphi$ to obtain

$$\int_\Sigma|A|^4\varphi^2\sqrt{\bar{g}}dx\leqq\int_\Sigma|\bar{D}|A|\varphi|^2\sqrt{\bar{g}}dx+c_2\int_\Sigma(|A|^3+|A|^2)\varphi^2\sqrt{\bar{g}}dx\,. \tag{3.4}$$

Expanding, and integrating by parts, the first term on the right becomes

$$\int_\Sigma(|A|^2|\bar{D}\varphi|^2+2\varphi|A|\langle\bar{D}\varphi,\bar{D}|A|\rangle+\varphi^2|\bar{D}|A||^2)\sqrt{\bar{g}}dx$$
$$=\int_\Sigma|A|^2|\bar{D}\varphi|^2\sqrt{\bar{g}}dx-\tfrac{1}{2}\int_\Sigma\varphi^2\Delta|A|^2\sqrt{\bar{g}}dx+|\varphi^2|\bar{D}|A||^2\sqrt{\bar{g}}dx$$
$$=\int_\Sigma|A|^2|\bar{D}\varphi|^2\sqrt{\bar{g}}dx-\int_\Sigma\varphi^2|A|\Delta|A|\sqrt{\bar{g}}dx\,.$$

Putting this into (3.4) then gives

$$\int_\Sigma\varphi^2|A|(\Delta|A)+|A|^3)\sqrt{\bar{g}}dx\leqq\int_\Sigma|A|^2|\bar{D}\varphi|^2\sqrt{\bar{g}}dx+c_3\int_\Sigma(|A|^3+1)\varphi^2\sqrt{\bar{g}}dx\,,$$

where we have absorbed $|A|^2$ into $|A|^3+1$. We now use (2.16) to get

$$\int_\Sigma\sum_{i,j,k}(\bar{D}_kh_{ij})^2\varphi^2\sqrt{\bar{g}}dx\leqq c_4\int_\Sigma|A|^2|\bar{D}\varphi|^2\sqrt{\bar{g}}dx$$
$$-c_4\int_\Sigma\underset{i,j}{S}\,h_{ij}\bar{D}_i\bar{D}_jH\varphi^2\sqrt{\bar{g}}dx+c_4\int_\Sigma|\bar{D}H|^2\varphi^2\sqrt{\bar{g}}dx$$
$$+c_4\int_\Sigma(|A|^3+1)\varphi^2\sqrt{\bar{g}}dx\,.$$

We integrate by parts the second term on the right and absorb to get

$$\int_\Sigma\sum_{i,j,k}(\bar{D}_kh_{ij})^2\varphi^2\sqrt{\bar{g}}dx\leqq c_5\int_\Sigma|A|^2|\bar{D}\varphi|^2\sqrt{\bar{g}}dx$$
$$+c_5\int_\Sigma|\bar{D}H|^2\varphi^2\sqrt{\bar{g}}dx+c_5\int_\Sigma(|A|^3+1)\varphi^2\sqrt{\bar{g}}dx\,.$$

We now get rid of the second term on the right by observing that (3.2) says $H=\sum_i p_{ii}+F$, and we have

$$\sum_a D_a p_{ii} w_a = dp_{ii} + 2\sum_j p_{ji} w_{ji} + 2p_{i4} w_{4i}\,,$$

so summing on i we get

$$\sum_i D_j p_{ii} = \bar{D}_j\left(\sum_i p_{ii}\right) + 2\sum_i p_{i4} h_{ij}$$

which implies $\left|\bar{D}\sum_i p_{ii}\right|^2 \leqq c(|A|^2+1)$, and hence by (3.1) and (3.5) we have

$$\int_\Sigma \sum_{i,j,k} (\bar{D}_k h_{ij})^2 \varphi^2 \sqrt{\bar{g}}\,dx \leqq c_6 \int_\Sigma |A|^2 |\bar{D}\varphi|^2 \sqrt{\bar{g}}\,dx + c_6 \int_\Sigma (|A|^3+1)\varphi^2 \sqrt{\bar{g}}\,dx\,. \tag{3.6}$$

We observe that (3.4) directly implies

$$\int_\Sigma |A|^4 \varphi^2 \sqrt{\bar{g}}\,dx \leqq 2\int_\Sigma \sum_{i,j,k} (\bar{D}_k h_{ij})^2 \varphi^2 \sqrt{\bar{g}}\,dx + 2\int_\Sigma |A|^2 |\bar{D}\varphi|^2 \sqrt{\bar{g}}\,dx + c_7 \int_\Sigma (|A|^3+1)\sqrt{\bar{g}}\,dx\,.$$

Combining this with (3.6) and absorbing the term involving $|A|^3$ back to the left we get

$$\int_\Sigma |A|^4 \varphi^2 \sqrt{\bar{g}}\,dx \leqq c_8 \int_\Sigma |A|^2 |\bar{D}\varphi|^2 \sqrt{\bar{g}}\,dx + c_8 \int_\Sigma \varphi^2 \sqrt{\bar{g}}\,dx\,.$$

Finally, we may replace φ by φ^2 and absorb to get

$$\int_\Sigma |A|^4 \varphi^4 \sqrt{\bar{g}}\,dx \leqq c_9 \int_\Sigma |\bar{D}\varphi|^4 \sqrt{\bar{g}}\,dx + c_9 \int_\Sigma \varphi^4 \sqrt{\bar{g}}\,dx \tag{3.7}$$

for any Lipschitz function φ with compact support on Σ.

We now choose ϱ_0 with $0<\varrho_0\leqq 1$ so that for any point $x_0\in N$, the geodesic exponential map is a diffeomorphism on the ball with center at x_0 of radius ϱ_0. That such ϱ_0 exists follows from the conditions (1.1). We let $B_\sigma^4(X_0)$ denote the geodesic ball in $N\times\mathbb{R}$ centered at a point $X_0\in N\times\mathbb{R}$. For any point $X_0=(x_0, f(x_0))$ in Σ, we will give estimates on $\Sigma\cap B_\sigma^4(X_0)$ for suitable $\sigma>0$. We first bound the volume of $\Sigma\cap B_\sigma^4(X_0)$ by observing that (3.2) implies

$$\mathrm{div}_{N\times\mathbb{R}}(e_4) = F + \sum_{i,j}\left(g^{ij} - \frac{f^i f^j}{1+|Df|^2}\right) p_{ij}\,,$$

so we apply the divergence theorem on the four dimensional volume $B_\sigma^4(X_0)\cap\{(x,x^4): x^4<f(x)\}$ to obtain

$$\mathrm{Vol}\left(\sum \cap B_\sigma^4(X_0)\right) \leqq c_{10}\sigma^3 \tag{3.8}$$

for any $\sigma\leqq\varrho_0$, $X_0\in\Sigma$. The results of Hoffman and Spruck [4], generalizing the methods of Michael and Simon [6], now show that there is a number ϱ_1 with

$0<\varrho_1 \leqq \varrho_0$ so that the Sobolov inequality holds on $\Sigma \cap B^4_{\varrho_1}(X_0)$. In particular, it is true that

$$\left(\int_\Sigma \varphi^6 \sqrt{\bar{g}} dx\right)^{1/3} \leqq c_{11} \int_\Sigma (|\bar{D}\varphi|^2 + H^2 \varphi^2) \sqrt{\bar{g}} dx$$

for any Lipschitz φ vanishing outside $\Sigma \cap B^4_{\varrho_1}(X_0)$. Since H^2 is bounded by (3.2), we may apply Hölder's inequality and (3.8) to prove

$$\left(\int_\Sigma \varphi^6 \sqrt{\bar{g}} dx\right)^{1/3} \leqq c_{11} \int_\Sigma |\bar{D}\varphi|^2 \sqrt{\bar{g}} dx + c_{12} \varrho_1^2 \left(\int_\Sigma \varphi^6 \sqrt{\bar{g}} dx\right)^{1/3} .$$

If we take ϱ_1 small enough that $c_{12}\varrho_1^2 \leqq \frac{1}{2}$, we get

$$\left(\int_\Sigma \varphi^6 \sqrt{\bar{g}} dx\right)^{1/3} \leqq c_{13} \int_\Sigma |\bar{D}\varphi|^2 \sqrt{\bar{g}} dx \tag{3.9}$$

for any Lipschitz φ with support of φ contained in $\Sigma \cap B^4_{\varrho_1}(X_0)$. We emphasize that both ϱ_1 and c_{13} are *independent of* X, Σ.

We let ϱ denote the geodesic distance function to X_0 in $N \times \mathbb{R}$, and observe that $|D\varrho| = 1$ and hence $|\bar{D}\varrho| \leqq 1$ on Σ. We choose φ in (3.7) to be a function of ϱ satisfying

$$\varphi = \begin{cases} 1 & \text{for} \quad \varrho \leqq \dfrac{\varrho_1}{2} \\ 0 & \text{for} \quad \varrho \geqq \varrho_1 \end{cases}, \quad |\bar{D}\varphi| \leqq 3\varrho_1^{-1}, \quad |\varphi| \leqq 1 .$$

With this choice of φ, (3.7) and (3.8) imply

$$\int_{\Sigma \cap B^4_{\frac{\varrho_1}{2}}(X_0)} |A|^4 \sqrt{\bar{g}} dx \leqq c_{14} . \tag{3.10}$$

Note that we are taking ϱ_1 to be *fixed*, so we have not bothered to explicitly denote the dependence of c_{14} on ϱ_1.

We now show that $|A|^2$ is pointwise bounded. To see this, let $u = |A|^2 + 1$, and observe that by (2.13), (3.1), and (3.2)

$$\Delta u \geqq -c_{15}(|A|^2+1)u + 2\sum_{i,j} h_{ij} \bar{D}_i \bar{D}_j H .$$

Multiplying both sides by a nonnegative function ζ vanishing outside $\Sigma \cap B^4_{\frac{\varrho_1}{2}}(X_0)$, and integrating by parts we get

$$\int_\Sigma \left[\langle \bar{D}\zeta, \bar{D}u\rangle - c_{15}(|A|^2+1)u\zeta - 2\sum_i \bar{D}_i\zeta \left(\sum_j h_{ij}\bar{D}_j H\right) - 2\sum_{i,j} \bar{D}_i h_{ij} \bar{D}_j H \zeta\right] \sqrt{\bar{g}} dx \leqq 0$$

for any such ζ. It follows from (2.5) that $\left|\sum_i \bar{D}_i h_{ij}\right| \leqq c(|\bar{D}H| + 1)$, and from the discussion preceding inequality (3.6) that $|\bar{D}H|^2 \leqq c(|A|^2+1)$. We therefore have the following inequality

$$\int_\Sigma \left[\langle \bar{D}\zeta, \bar{D}u\rangle + \sum_i (\bar{D}_i\zeta) b_i u + \zeta e u\right] \sqrt{\bar{g}} dx \leqq 0 \tag{3.11}$$

for each nonnegative ζ vanishing outside $\Sigma \cap B^4_{\frac{\varrho_1}{2}}(X_0)$, where the functions b_i, e are

$$b_i = -2u^{-1} \sum_j h_{ij} \bar{D}_j H$$

$$e = -c_{15}(|A|^2+1) - 2u^{-1} \sum_{i,j} \bar{D}_i h_{ij} \bar{D}_j H .$$

Since b_i and e satisfy

$$|b_i| \leqq c_{16}$$

$$|e| \leqq c_{16}(|A|^2+1) ,$$

by (3.8) and (3.10) we have

$$\sup_{\Sigma \cap B^4_{\frac{\varrho_1}{2}}(X_0)} \left(\sum_i |b_i|^2 \right) + \int_{\Sigma \cap B^4_{\frac{\varrho_1}{2}}(X_0)} |e|^2 \sqrt{\bar{g}}\, dx \leqq c_{17} . \tag{3.12}$$

A standard iteration technique (see [7, Theorem 5.3.1] now gives the mean value-type inequality

$$u(X_0) \leqq c_{18} \left(\int_{\Sigma \cap B^4_{\frac{\varrho_1}{2}}(X_0)} u^2 \sqrt{\bar{g}}\, dx^{\;1/2} \right) \tag{3.13}$$

for a constant c_{18}. Note that this iteration technique works because we have the Sobolev inequality (3.9), and we may use the distance function ϱ in place of standard Euclidean distance. Also, it is crucial that $|e|$ is bounded in

$$L^2 \left(\Sigma \cap B^4_{\frac{\varrho_1}{2}}(X_0) \right)$$

and $2 > \frac{1}{2} \dim \Sigma \ldots \frac{3}{2}$, so that the structural conditions [7, 5.1.3] are satisfied. It now follows from (3.8), (3.10), and (3.13) that $|A|^2(X_0)$ is bounded, so we have an extrinsic curvature bound

$$\sup_{\Sigma} |A|^2 \leqq c_{19} . \tag{3.14}$$

We summarize what we have proven in the following proposition.

Proposition 1. *Suppose f is a C^3 solution of* (3.2) *with function F satisfying* (3.1). *There is a constant c_{19} depending only on the initial data (N, g_{ij}, p_{ij}) and on μ_1, μ_2, μ_3 so that* (3.14) *holds.*

We discuss the consequences of this result. If $X_0 \in \Sigma$, we let (y^1, y^2, y^3, y^4) be normal coordinates in $N \times \mathbb{R}$ centered at X_0 so that the tangent space to Σ at X_0 is the $y^1 y^2 y^3$-space. Thus, if the metric $ds^2 + dt^2$ for $N \times \mathbb{R}$ is given by

$$ds^2 + dt^2 = \sum_{a,b} \hat{g}_{ab} dy^a dy^b ,$$

we have

$$\hat{g}_{ab}(0) = \delta_{ab} , \qquad \frac{\partial g_{ab}}{\partial y^c}(0) = 0$$

for $1 \leqq a, b, c \leqq 4$. In a neighborhood of X_0, Σ is given by the graph of a function $w(y)$, $y=(y^1, y^2, y^3)$ on the $y^1y^2y^3$-space. The equation (3.2) satisfied by Σ is

$$\sum_{a,b=1}^{4} \left(\hat{g}^{ab} - \frac{W^a W^b}{|DW|^2}\right)\left(\frac{D_a D_b W}{|DW|} - p_{ab}\right) = 0 ,$$

where $W(Y) = w(y) - y^4$, $Y = (y^1, y^2, y^3, y^4)$. This gives an equation for w of the form

$$\sum_{i,j=1}^{3} B_{ij}(y, w, \partial w) w_{y^i y^j} = C(y, w, \partial w) \tag{3.15}$$

for y near 0, where $B_{ij}(y, w, p)$ and $C(y, w, p)$ are smooth functions of their arguments, $\partial w = (w_{y^1}, w_{y^2}, w_{y^3})$ is the Euclidean gradient, and (B_{ij}) is positive definite with

$$B_{ij}(0,0,0) = \delta_{ij} , \qquad C(0,0,0) = 0 . \tag{3.16}$$

The length of the second fundamental form of Σ is given by

$$|A|^2 = \sum_{a,b,c,d=1}^{4} \left(\hat{g}^{ac} - \frac{W^a W^c}{|DW|^2}\right)\left(\hat{g}^{bd} - \frac{W^b W^d}{|DW|^2}\right)\left(\frac{D_a D_b W}{|DW|}\right)\left(\frac{D_c D_d W}{|DW|}\right).$$

From this expression, one sees that (3.14) implies

$$\sum_{i,j=1}^{3} (w_{y^i y^j})^2 \leqq c_{20} \left(1 + \sum_{i=1}^{3} (w_{y^i})^2\right)^3 \tag{3.17}$$

in a neighborhood of 0. We can now prove a gradient bound on w as follows. Given a Euclidean unit vector ξ in the $y^1y^2y^3$-space, and a radus $\bar{\varrho}$, we define $S_\xi(\bar{\varrho})$ by

$$S_\xi(\bar{\varrho}) = \max_{0 \leqq \varrho \leqq \bar{\varrho}} \sum_{i=1}^{3} [w_{y^i}(\varrho\xi)]^2 .$$

By the mean value theorem, (3.17), and the fact that $u_{y^i}(0) = 0$, we have for all small $\bar{\varrho}$

$$S_\xi(\bar{\varrho}) \leqq c_{21} (\bar{\varrho})^2 (1 + S_\xi(\bar{\varrho}))^{5/2} .$$

Elementary calculus now implies that there is a $\varrho_2 > 0$ (depending only on c_{21}) so that $S_\xi(\bar{\varrho})$ remains bounded for $0 < \bar{\varrho} \leqq \varrho_2$ (thus w is also defined on the ball of radius ϱ_2). Because of this and (3.17), we then have

$$\sup_{|y| \leqq \varrho_2} (|w(y)| + |\partial w(y)| + |\partial\partial w(y)|) \leqq c_{22} \tag{3.18}$$

for constants $\varrho_2 > 0$, c_{22} independent of Σ. We will want to improve (3.18) a little so we define for $0 < \alpha \leqq 1$, the Hölder norm on $\{|y| < \bar{\varrho}\}$ by

$$\|h\|_{\alpha, \bar{\varrho}} = \sup_{\substack{|y_1| < \bar{\varrho} \\ |y_2| < \bar{\varrho}}} |y_1 - y_2|^{-\alpha} |h(y_1) - h(y_2)| .$$

We can now prove

Proposition 2 (Local Parametric Estimate). *Under the hypotheses of Proposition* 1, *there is a* $\varrho_3 > 0$ *depending only on the initial data and* μ_1, μ_2, μ_3 *so that for any*

$X_0 \in \Sigma$, the local defining function w for Σ (as discussed above) is defined on $\{|y| \leqq \varrho_3\}$, and satisfies for any $\alpha \in (0,1)$

$$\sup_{|y| \leqq \varrho_3} (|w(y)| + |\partial w(y)| + |\partial\partial w(y)| + |\partial\partial\partial w(y)| + \|\partial\partial\partial w\|_{\alpha,\varrho_3}) \leqq c_{23}(\alpha),$$

where c_{23} depends only on α, the initial data, and μ_1, μ_2, μ_3. Moreover, we may require

$$\Sigma \cap B^4_{\frac{\varrho_3}{2}}(X_0) \subseteqq \{Y : y^4 = w(y)\}.$$

We also have the following Harnack-type inequalities

$$\sup_{\Sigma \cap B^4_{\frac{\varrho_3}{2}}(X_0)} \langle e_4, v\rangle \leqq c_{24} \inf_{\Sigma \cap B^4_{\frac{\varrho_3}{2}}(X_0)} \langle e_4, v\rangle$$

$$\sup_{\Sigma \cap B^4_{\frac{\varrho_3}{2}}(X_0)} |\bar{D} \log \langle e_4, v\rangle| \leqq c_{25}.$$

Proof. The estimate for $|\partial\partial\partial w|$ and $\|\partial\partial\partial w\|_{\alpha,\varrho_3}$ (for $\varrho_3 \leqq \frac{1}{2}\varrho_2$) follows from (3.15), (3.16), (3.18) and standard Schauder estimates for linear elliptic equations with Lipschitz coefficients (see [7, 5.5]). Because of this estimate, Eq. (2.18) represents a uniformly elliptic equation on $\{|y| \leqq \frac{1}{2}\varrho_2\}$, so the following Harnack inequality (see [7, 5.3]) holds

$$\sup_{|y| \leqq \varrho_3} \langle v, e_4\rangle (y, w(y)) \leqq c_{25} \inf_{|y| \leqq \varrho_3} \langle v, e_4\rangle (y, w(y))$$

for ϱ_3 small enough. It is also standard (see [7, 5.5]) that

$$\sup_{|y| \leqq \varrho_3} |\partial \langle v, e_4\rangle (y, w(y))| \leqq c_{26} \sup_{|y| \leqq 2\varrho_3} |\langle v, e_4\rangle (y, w(y))|.$$

Combining this with the Harnack inequality on $\{|y| \leqq 2\varrho_3\}$ we have

$$\sup_{|y| \leqq \varrho_3} |\bar{D}\langle v, e_4\rangle (y, w(y))| \leqq c_{27} \inf_{|y| \leqq 2\varrho_3} |\langle v, e_4\rangle (y, w(y))|$$

which implies the stated estimate on $|\bar{D} \log \langle v, e_4\rangle|$. Finally, we note that by (2.24)

$$|D_{e_4} e_4|^2 = \sum_{i=1}^{3} h_{i4}^2 = |\bar{D} \log \langle e_4, v\rangle|^2$$

on Σ. Also, $|A|^2 = \sum_i |D_{ei} e_4|^2$, so we have

$$\sum_{a=1}^{4} |D_{e_a} e_4|^2 \leqq c_{28}$$

on Σ, and hence on $N \times \mathbb{R}$. Recall that e_4 is extended to $N \times \mathbb{R}$ by parallel translation along vertical lines. From this it follows that we may take $\Sigma \cap B_{\frac{1}{2}\varrho_3}(X_0) \subseteqq \{Y : y^4 = w(y)\}$ since any adjacent components of $\Sigma \cap B_{\frac{1}{2}\varrho_3}(X_0)$ would necessarily have a normal vector e_4 bounded away from $e_4(X_0)$ hence for ϱ_3 small such a component could not exist. This completes the proof of Proposition 2.

Our next task is to discuss the behavior of f at each infinity of N. For this purpose, we add to our hypotheses (3.1), (3.2) the following assumption on F

$$\begin{aligned} &F(x) = tf(x) + G(x) && \text{on} \quad N \\ &|G(x)| \leqq \mu_4(1+r^3)^{-1}, \quad |\partial G(x)| \leqq \mu_5(1+r^4)^{-1} && \text{on} \quad N_k \end{aligned} \tag{3.19}$$

for each k, where $t\in[0,1]$. Assuming that $f(x)$ tends to zero on each N_k, we will give estimates on the fall-off of f and its derivatives. We first give a bound on f by constructing suitable "barrier" functions near each infinity. For $\Lambda>0$, $\beta\in(0,1)$, we define a function $\bar{f}(r)$ for $r\geqq\Lambda^{\frac{1}{\beta+1}}$ on each N_k by

$$\bar{f}(r)=\Lambda\int_r^\infty (s^{2\beta+2}-\Lambda^2)^{-1/2}ds\,. \tag{3.20}$$

The following properties of $\bar{f}$ are easily checked

$$\begin{gathered}0\leqq\bar{f}(r)\leqq c_{29}\Lambda r^{-\beta}\quad\text{for}\quad r\geqq\Lambda^{\frac{1}{\beta+1}}\,,\\ \frac{\partial}{\partial r}\bar{f}\left(\Lambda^{\frac{1}{\beta+1}}\right)=-\infty\,.\end{gathered}\tag{3.21}$$

The *Euclidean* mean curvature $\bar{H}^e$, (with respect to the downward normal), and square length $|\bar{A}^e|^2$ of the second fundamental form of the graph of $\bar{f}$ are given by

$$\begin{gathered}\bar{H}^e(x,\bar{f}(x))=-(1-\beta)\Lambda r^{-2-\beta}\\ |\bar{A}^e(x,\bar{f}(x))|^2=(\beta^2+2\beta+3)\Lambda^2r^{-4-2\beta}\,.\end{gathered}$$

We wish to compute the mean curvature $\bar{H}$ of the graph of $\bar{f}$ with respect to ds^2. Using (1.1), it is not difficult to see

$$\begin{aligned}\bar{H}(x,\bar{f}(x))\leqq\bar{H}^e(x,\bar{f}(x))&+c_{30}r^{-1}|\bar{A}^e(x,\bar{f}(x))|\\ &+c_{30}\frac{r^{-2}|\partial\bar{f}(x)|}{\sqrt{1+|\partial\bar{f}(x)|^2}}\end{aligned}$$

for $r\geqq\Lambda^{\frac{1}{\beta+1}}$ on each N_k. This implies

$$\bar{H}(x,\bar{f}(x))\leqq-(1-\beta)\Lambda r^{-2-\beta}+c_{31}\Lambda r^{-3-\beta} \tag{3.22}$$

for $r\geqq\Lambda^{\frac{1}{\beta+1}}$.

We will show that $\bar{f}$ is a supersolution of (3.2) for suitably large Λ. For this purpose, we estimate the trace of the restriction of p_{ab} to the graph of $\bar{f}$. Using (1.4) we have

$$\begin{aligned}|\bar{P}|=\left|\sum_{i,j}\left(g^{ij}-\frac{\bar{f}^i\bar{f}^j}{1+|D\bar{f}|^2}\right)p_{ij}\right|&\leqq c_{32}r^{-3}+\frac{c_{32}r^{-2}|\partial\bar{f}|^2}{1+|\partial\bar{f}|^2}\\ &\leqq c_{32}r^{-3}+c_{32}\Lambda r^{-3-\beta}\,,\end{aligned}$$

where we have denoted the trace of the restriction of p_{ab} to the graph of $\bar{f}$ by $\bar{P}$, so by (3.19) and (3.22)

$$\begin{aligned}\bar{H}-\bar{P}-G&\leqq-(1-\beta)\Lambda r^{-2-\beta}+c_{33}(r^{-3}+\Lambda r^{-3-\beta})\\ &\leqq-(1-\beta)\Lambda r^{-2-\beta}+c_{33}\left(\Lambda^{-\frac{1-\beta}{1+\beta}}+\Lambda^{\frac{\beta}{1+\beta}}\right)r^{-2-\beta}\,,\end{aligned}$$

where we have used $r\geqq\Lambda^{\frac{1}{\beta+1}}$ to get the last inequality. From here we see that if $\Lambda=\Lambda_\beta$ is chosen sufficiently large (depending on β as well as the other data), then

$$\bar{H}-\bar{P}<G \tag{3.23}$$

for $r \geqq \Lambda^{\frac{1}{\beta+1}}$ on each N_k. In a similar way we see that for Λ large we have

$$-\bar{H}-\bar{P}>G\,, \tag{3.24}$$

so that the function $-\bar{f}$ is a subsolution of (3.2). We can now estimate f and its derivatives near infinity.

Proposition 3. *Suppose f is a C^3 solution of* (3.2), *with function F satisfying* (3.1) *and* (3.19). *Suppose also that $\lim_{x\to\infty} f(x)=0$ for each N_k. For any $\beta\in(0,1)$, there is a constant $c_{33}=c_{33}(\beta)$ depending only on β, the initial data (N, g_{ij}, p_{ij}), and the constants $\mu_1, \mu_2, \mu_2, \mu_4, \mu_5$ so that*

$$|f(x)|+|x||\partial f(x)|+|x|^2|\partial\partial f(x)|+|x|^3|\partial\partial\partial f(x)| \leqq c_{33}(\beta)|x|^{-\beta}$$

for any $x\in N_k$, any k.

Proof. The estimate of $|f(x)|$ comes directly from the properties of $\bar{f}$. Indeed, for any positive number L we observe that (3.23) implies that $\bar{f}+L$ is also a supersolution since the equation $H-P=G$ is insensitive to translation in the vertical direction. Since f tends to zero at each infinity, we observe that for L sufficiently large we have $\bar{f}(x)+L>f(x)$ for each x with $r=|x|\geqq \Lambda^{\frac{1}{\beta+1}}$. Define L_0 by

$$L_0=\inf\{L : \bar{f}+L>f\}\,.$$

Then $L_0\geqq 0$, and we show that $L_0=0$. To see this, we suppose on the contrary that $L_0>0$. Since f tends to zero at each infinity, it follows that there is a point $x_0\in N$ with $|x_0|\geqq \Lambda^{\frac{1}{\beta+1}}$ such that $f_{\Lambda,\beta}(x_0)+L_0=f(x_0)$. We note that it is impossible that $|x_0|=\Lambda^{\frac{1}{\beta+1}}$ since $\bar{f}+L_0$ has infinite slope for such points by (3.21) and hence the inequality $\bar{f}+L_0\geqq f$ would be violated at points near x_0. Thus we have $|x_0|>\Lambda^{\frac{1}{\beta+1}}$ and the function $\bar{f}-f$ has a minimum at x_0 so we have

$$\frac{\partial\bar{f}}{\partial x_i}(x_0)=\frac{\partial f}{\partial x^i}(x_0),$$

$\left(\dfrac{\partial^2(\bar{f}-f)}{\partial x^i\partial x^j}\right)(x_0)$ is a nonnegative definite matrix.

It follows that

$$g^{ij}(x_0)-\frac{f^i(x_0)f^j(x_0)}{1+|Df(x_0)|^2}=g^{ij}(x_0)-\frac{\bar{f}^i(x_0)\bar{f}^j(x_0)}{1+|D\bar{f}(x_0)|^2}\,.$$

We denote this matrix by B^{ij}, and we see that by subtracting (3.2) from (3.23) we get

$$\sum_{i,j} B^{ij}\frac{\partial^2(\bar{f}-f)}{\partial x^i\partial x^j}(x_0)<-tf(x_0)\leqq 0\,.$$

Since B^{ij} is positive definite, this contradicts the nonnegativity of the matrix of second partial derivatives. Therefore $L_0=0$, and we have shown $f(x)\leqq\bar{f}(x)$ for

$|x| \geqq A^{\frac{1}{\beta+1}}$ which implies by (3.21) that $f \leqq c(\beta) r^{-\beta}$. A similar method using (3.24) shows $-f \leqq \bar{f}$ hence

$$|f(x)| \leqq c_{34}(\beta)|x|^{-\beta} \tag{3.25}$$

on each N_k.

It is now elementary from Proposition 2 and (3.25) that $|\partial f|$, $|\partial\partial f|$, and $|\partial\partial\partial f|$ are bounded near infinity. In fact, standard Schauder estimates (see [7, 5.5]) applied to (3.2) in the ball $U(x) = \{y : |y-x| < 1\}$ then give

$$|\partial f(x)| + |\partial\partial f(x)| + |\partial\partial\partial f| \leqq c_{35}(\beta)|x|^{-\beta} \tag{3.26}$$

on each N_k. We now view (3.2) as the following linear equation

$$\sum_{i,j} a_{ij}(x) \frac{\partial^2 f}{\partial x^i \partial x^j} + \sum_i b_i(x) \frac{\partial f}{\partial x^i} - tf = \hat{G}$$

$$a_{ij} = (1+|Df|^2)^{-1/2}\left(g^{ij} - \frac{f^i f^j}{1+|Df|^2}\right), \qquad b_i = -\sum_{k,j} a_{kj} \Gamma^i_{kj},$$

$$\hat{G} = G + \sum_{i,j}\left(g^{ij} - \frac{f^i f^j}{1+Df^2}\right) p_{ij}.$$

To improve the bounds (3.26) on the derivatives of f, we fix a point $x_0 \in N_k$ and define coordinates $\bar{x} = (x - x_0)/\sigma$, $\sigma = |x_0|/2$. In terms of $\bar{x}$, our equation becomes

$$\sum_{i,j} a_{ij}(\bar{x}) \frac{\partial^2 f}{\partial \bar{x}^i \partial \bar{x}^j} + \sum_i \sigma b_i(\bar{x}) \frac{\partial f}{\partial \bar{x}^i} - t\sigma^2 f = \sigma^2 \hat{G}(\bar{x}) \tag{3.27}$$

for $\bar{x} \in \bar{U}_1(0) = \{|\bar{x}| < 1\}$. It follows from (3.26) that the Hölder coefficient $\|\partial f\|_{\beta, \bar{U}_1(0)}$ satisfies

$$\|\partial f\|_{\beta, \bar{U}_1(0)} = \sup_{\substack{|\bar{x}|<1 \\ |\bar{y}|<1}} \frac{|\partial f(\bar{x}) - \partial f(\bar{y})|}{|\bar{x} - \bar{y}|^{\beta}} \leqq c_{36}(\beta).$$

Therefore, Eq. (3.27) is uniformly elliptic, and the coefficients satisfy [by (1.1), (1.3)]

$$\sum_{i,j} \|a_{ij}\|_{\beta, \bar{U}_1(0)} + \sum_i \|\sigma b_i\|_{\beta, \bar{U}_1(0)}$$
$$c_{37}(\beta) \sup_{x \in \bar{U}_1(0)} |\sigma^2 \hat{G}(\bar{x})| + \|\sigma^2 \hat{G}\|_{\beta, \bar{U}_1(0)} \leqq c_{38}(\beta)\sigma^{-\beta}.$$

Standard methods (see [7, 5.5]) then show

$$|\bar{\partial} f(\bar{x})| + |\bar{\partial}\bar{\partial} f(\bar{x})| \leqq c_{39}\left(\sup_{\bar{U}_1(0)} (|f| + \sigma^2 \hat{G}) + \|\sigma^2 \hat{G}\|_{\beta, \bar{U}_1(0)}\right).$$

for $\bar{x} \in \bar{U}_{1/2}(0)$. Writing this in terms of the original coordinates x and using (3.25)

$$|x_0||\partial f(x_0)| + |x_0|^2 |\partial\partial f(x_0)| \leqq c_{40}(\beta)|x_0|^{-\beta}.$$

[Note that in dealing with (3.27), we do not have a bound on $t\sigma^2$, the coefficient of f, but we are using the fact that $t\sigma^2 \geqq 0$ which makes the sign of this term helpful in deriving the estimates.] A similar method by differentiating the equation gives estimates for $|\partial\partial\partial f|$. This completes the proof of Proposition 3.

4. Proof of the Existence

In this section we prove existence of solutions of (2.27), asymptotic to zero at infinity, and defined on the exterior of a finite family of apparent horizons. We also study the asymptotic behavior of these solutions on the apparent horizons, showing that they are asymptotic to the cylinder in $N \times \mathbb{R}$ over the horizons.

To solve (2.27) we introduce an auxilliary equation for $s \in [0,1]$, $t \in [0,1]$.

$$H(f) - sP(f) = tf, \tag{4.1}$$

where $H(f)$, $P(f)$ are given by

$$H(f) = \sum_{i,j} \left(g^{ij} - \frac{f^i f^j}{1+|Df|^2} \right) \frac{D_i D_j f}{\sqrt{1+|Df|^2}},$$

$$P(f) = \sum_{i,j} \left(g^{ij} - \frac{f^i f^j}{1+|Df|^2} \right) p_{ij}.$$

We first solve (4.1) for $t > 0$, and then study the limit as $t \to 0$. We will look for solutions of (4.1) in a weighted Hölder space $B^{2,\beta}$ for any $\beta \in (0,1)$ defined in the following way. We let $\tau(x)$ be a weight function on N satisfying $\tau \geqq 1$ on N, and $\tau(x) = r(x)$ on each end N_k. We then define a norm

$$\begin{aligned} \|f\|_{2,\beta} = \sup_{x \in N} (&\tau^\beta(x)|f(x)| + \tau^{1+\beta}(x)|Df(x)| \\ &+ \tau^{2+\beta}(x)|DDf(x)| + \tau^{2+2\beta}(x)\|DDf\|_{\beta,x}), \end{aligned}$$

where $\|DDf\|_{\beta,x}$ denotes the Hölder coefficient in the ball $B_{\tau(x)/2}(x)$

$$\|DDf\|_{\beta,x} = \sup_{x_1, x_2 \in B_{\tau(x)/2}(x)} \frac{|DDf(x_1) - DDf(x_2)|}{d(x_1,x_2)^\beta},$$

where $d(x_1,x_2)$ is distance. We let $B^{2,\beta}$ be the Banach space of $C^{2,\beta}$ functions on N with finite $\|f\|_{2,\beta}$. We first solve (4.1) for $t > 0$. This turns out to be straightforward because in this case we can derive a priori bounds on f and $|Df|$. To see this note that we have

$$H(f) = \sum_i D_i \left(\frac{f^i}{\sqrt{1+|Df|^2}} \right).$$

Differentiating (4.1) in the direction of x^k, we have

$$\begin{aligned} &\sum_{i,j} D_i \left(\left(g^{ij} - \frac{f^i f^j}{1+|Df|^2} \right) \frac{D_j D_k f}{\sqrt{1+|Df|^2}} \right) - \sum_i \frac{f^i}{\sqrt{1+|Df|^2}} R_{ik} \\ &+ s \left[2 \sum_{i,j,\ell} \left(g^{ij} - \frac{f^i f^j}{1+|Df|^2} \right) \frac{(D_j D_k f)(f^\ell p_{i\ell})}{1+|Df|^2} + \sum_{i,j} \left(g^{ij} - \frac{f^i f^j}{1+|Df|^2} \right) D_k p_{ij} \right] = t D_k f, \end{aligned} \tag{4.2}$$

where R_{ik} is the Ricci tensor of N, arising from the commuting of covariant derivatives. This implies in particular that the function $u = |Df|^2$ satisfies an inequality of the form

$$\sum_{i,j} D_i(A^{ij} D_j u) + \sum_i B^i D_i u + c u^{1/2} \geqq tu, \tag{4.3}$$

where A^{ij} is positive definite, B^i, C are bounded on N (independent of s, t). If $f \in B^{2,\beta}$ satisfies (4.1), then we have the following bounds

$$\sup_N t|f| \leqq \mu_1, \quad \sup_N t|Df| \leqq \mu_2, \tag{4.4}$$

where μ_1, μ_2 are constants depending only on (N, g_{ij}, p_{ij}). To prove (4.4), we simply note that since f tends to zero at infinity, either $\sup_N f \leqq 0$, or f has an interior maximum point. Using (4.1) at this point we would have

$$\max_N tf \leqq c_1 .$$

Similarly we show $\max_N(-tf) \leqq c_2$, thus proving the first inequality of (4.4). The second comes from the fact that $u = |Df|^2$ tends to zero at infinity, so using (4.3) at its maximum point we find

$$\sup_N t|Df|^2 \leqq c_3 \sup_N |Df|$$

which gives the second part of (4.4). The following lemma can now be proved.

Lemma 2. *Suppose $t > 0$, and $f \in B^{2,\beta}$ satisfies* (4.1) *for some $\beta \in (0,1)$. Then there is a constant $c_4(\beta, t)$ depending on β, t as well as (N, g_{ij}, p_{ij}) so that $\|f\|_{2,\beta} \leqq c_4(\beta, t)$.*

Proof. This lemma is a straightforward consequence of (4.4). We note that since $|Df|$ is bounded, (4.1) and (4.2) are *uniformly* elliptic equations. In particular, standard estimates (see [7, 5.3]) applied to (4.2) imply a Hölder estimate on $D_k f$ with exponent $\alpha \in (0,1)$ for some α. Thus f has a $C^{1,\alpha}$ bound. This implies a bound on the Hölder modulus of continuity for the coefficients of (4.1), so we have (see [7, 5.5]) a $C^{2,\alpha}$ bound on f. In particular, we get Lipschitz bounds on the coefficients of (4.1), so we can bound the $C^{2,\beta}$ norm of f for *any* $\beta \in (0,1)$. The decay near infinity can be derived, for example, using the barrier method of Proposition 3. This completes the proof of Lemma 2.

We can now easily solve (4.1) for $t > 0$.

Lemma 3. *For $t > 0$, there exists a solution $f \in B^{2,\beta}$ of the equation $H(f) - P(f) = tf$.*

Proof. We use a standard continuity method. Let $S = \{s \in [0,1] : (4.1)$ has a solution $f_s \in B^{2,\beta}\}$. We will show that $S = [0,1]$ by noting first that $0 \in S$ since $f \equiv 0$ is a solution of $H(f) = tf$. We then show that S is both open and closed (hence $S = [0,1]$). The fact that S is closed follows from Lemma 2, since if $\{s_n\}$ is a sequence in S with $s_n \to s$, and f_{s_n} is a solution in $B^{2,\beta}$ of $H(f_{s_n}) - s_n P(f_{s_n}) = tf_{s_n}$, then by Lemma 2

$$\|f_{s_n}\|_{2,\beta} \leqq c_4(\beta, t) .$$

In particular, this bound is independent of n, so we can choose a subsequence of f_{s_n} converging uniformly along with its first and second derivatives on compact subsets of N to a limit f_s satisfying $H(f_s) - sP(f_s) = tf_s$. Moreover, $\|f_s\|_{2,\beta} \leqq c_4(\beta, t)$, so that $f_s \in B^{2,\beta}$ for any $\beta \in (0,1)$. Thus $s \in S$, and S is a closed subset of $[0,1]$.

To prove that S is an open subset of $[0,1]$, we use results for linear equations together with the implicit function theorem. Let $s_0 \in S$, and $f_0 \in B^{2,\beta}$ be a solution of $H(f_0) - s_0 P(f_0) = tf_0$. We will show that there is $\varepsilon_0 > 0$ so that if $s \in [0,1]$ and $|s - s_0| < \varepsilon_0$, then $s \in S$. We define a Banach space $B^{0,\beta}$ for $\beta \in (0,1)$ to be those Hölder continuous functions h on N so that the following norm is finite

$$\|h\|_{0,\beta} = \sup_{x \in N} \left(\tau(x)^{2+\beta}|h(x)| + \tau(x)^{2+2\beta}\|h\|_{\beta,x}\right),$$

where as before $\|\cdot\|_{\beta,x}$ denotes the Hölder coefficient taken on the ball $B_{\tau(x)/2}(x)$. We then observe that $T: B^{2,\beta} \times \mathbb{R} \to B^{0,\beta} \times \mathbb{R}$ defined by $T(f,s) = (H(f) - tf - sP(f), s)$ is a C^1 mapping and $T(f_0, s_0) = (0, s_0)$. The linearization of T at (f_0, s_0) is the operator $L_0 : B^{2,\beta} \times \mathbb{R} \to B^{0,\beta} \times \mathbb{R}$ given by $L_0(\eta, \tau) = (L_0^\tau(\eta), \tau)$ where

$$L_0^\tau(\eta) = \sum_{i,j} A^{ij} D_i D_j \eta + \sum_i B^i D_i \eta - t\eta - \tau P(f_0)$$

$$A^{ij} = (1 + |Df_0|^2)^{-1/2}\left(g^{ij} - \frac{f_0^i f_0^j}{1 + |Df_0|^2}\right)$$

$$B^i = \sum_j D_j A^{ij} + 2s_0 \sum_{j,k} (1 + |Df_0|^2)^{-1/2} A^{ik} f^j p_{jk}.$$

It is fairly elementary to show that L_0 is a linear isomorphism from $B^{2,\beta} \times \mathbb{R}$ to $B^{0,\beta} \times \mathbb{R}$. Applying the inverse function theorem for Banach space, we see that T maps a neighborhood of (f_0, s_0) onto a neighborhood of $(0, s_0)$. In particular, there is $\varepsilon_0 > 0$ so that $(0, s)$ is in the image of T for $|s - s_0| < \varepsilon_0$; i.e., there exists f_s satisfying $H(f_s) - sP(f_s) = tf_s$. This shows that S is an open subset of $[0,1]$, and completes the proof of Lemma 3.

We now study the limit of the solutions constructed in Lemma 3 as t tends to 0. For this purpose, the estimates of Lemma 2 give no information since the constants become large when t is near 0. In fact, it is not generally true that the solutions of the perturbed equation converge as t tends to zero. Instead we use the parametric estimates of Sect. 3 to analyze the limit.

Proposition 4. *There is a sequence $\{t_i\}$ converging to zero and open sets Ω_+, Ω_-, Ω_0 so that if f_i satisfies $H(f_i) - P(f_i) = t_i f_i$ we have:*

(1) *The sequence $\{f_i\}$ converges uniformly to $+\infty$ (respectively $-\infty$) on the set Ω_+ (respectively Ω_-), and $\{f_i\}$ converges to a smooth function f_0 on Ω_0 satisfying* (2.27) *on Ω_0, and* (2.28) *on each N_k.*

(2) *The sets Ω_+ and Ω_- have compact closure, and $N = \bar{\Omega}_+ \cup \bar{\Omega}_- \cup \bar{\Omega}_0$. Each boundary component Σ of Ω_+ (respectively Ω_-) is a smooth embedded two-sphere satisfying $H_\Sigma - \mathrm{Tr}_\Sigma(p_{ij}) = 0$ (respectively $H_\Sigma + \mathrm{Tr}_\Sigma(p_{ij}) = 0$) where H_Σ is the mean curvature of Σ taken with respect to the inward normal to Ω_+ (respectively Ω_-) and $\mathrm{Tr}_\Sigma(p_{ij})$ is the trace of the restriction of p_{ij} to Σ. Moreover, no two connected components of Ω_+ can share a common boundary.*

(3) *The graphs G_i of f_i converge smoothly to a properly embedded limit submanifold $M_0 \subseteq N \times \mathbb{R}$. Each connected component of M_0 is either a component of the graph of f_0, or the cylinder $\Sigma \times \mathbb{R} \subseteq N \times \mathbb{R}$ over a boundary component Σ of Ω_+ or Ω_-. Any two connected components of M_0 are separated by a positive distance.*

Remark. The two-spheres making up the boundary components of Ω_+ and Ω_- will be referred to as apparent horizons in N (see [3] for explanation).

Corollary 1. *If the initial data* (N, g_{ij}, p_{ij}) *contains no apparent horizons then* (2.27) *has a solution on N satisfying the asymptotic conditions* (2.28).

Proof of Proposition 4. The assertions of (3) are a direct consequence of Propositions 2 and 3, for by the local estimate of Proposition 2 we can find a sequence $\{t_i\}$ so that the G_i converge to a properly embedded limiting submanifold M_0. The fact that M_0 is nonempty, and is a graph near infinity satisfying (2.28) on each N_k then follows from Proposition 3. The Harnack inequalities of Proposition 2 immediately imply that any connected component of M_0 has everywhere finite slope and hence is a graph, or has everywhere infinite slope and hence is a cylinder $\Sigma \times \mathbb{R}$ over a compact surface $\Sigma \subseteqq N$. We will show that Σ is a two-sphere momentarily. We first note that the convergence of G_i to M_0 also determines Ω_+, Ω_-, Ω_0. Our other assertions are clear except for the analysis of the boundary components of Ω_+ and Ω_-.

We first analyze the boundary $\partial\Omega_0$ of Ω_0. In order to do this, we observe that the Eq. (2.27) is translation invariant in the sense that for any $a \in \mathbb{R}$, $f_0 - a$ is also a solution of (2.27) defined on Ω_0. Let $G_{0,a}$ denote the graph of $f_0 - a$, and note that by the estimates of Proposition 2 there is a sequence a_i tending to $+\infty$ so that the graphs G_{0,a_i} converge smoothly on compact subsets of $N \times \mathbb{R}$ to a limiting three dimensional submanifold of $N \times \mathbb{R}$. By the Harnack inequality of Proposition 2, each component of this limiting submanifold is a cylinder over a compact surface in N. We denote this limit by $\Sigma_+ \times \mathbb{R}$ where Σ_+ is a family of compact surfaces in N. It also follows from (2.27) that Σ_+ satisfies the equation $H_{\Sigma_+} - \mathrm{Tr}_{\Sigma_+}(p_{ij}) = 0$ where H_{Σ_+} is computed with respect to the normal pointing outward from Ω_0. We show that each component Σ of Σ_+ is a two-sphere by using (2.29) on G_{0,a_i}. We let φ be a smooth function of compact support on G_{0,a_i} and multiply (2.29) by φ^2 and integrate by parts as in the derivation of (3.3) to arrive at

$$\int_{G_{0,a_i}} ((-\bar{R}) + P)\varphi^2 \sqrt{\bar{g}}\,dx \leqq 2 \int_{G_{0,a_i}} |\bar{D}\varphi|^2 \sqrt{\bar{g}}\,dx,$$

where $P = 2(\mu - |J|)$ can be taken strictly positive by Lemma 1. It follows that for any φ with compact support on $\Sigma \times \mathbb{R}$ we have

$$\int_{-\infty}^{\infty} \left[\int_{\Sigma} ((-K) + P)\varphi^2\,d\sigma\right] dx^4 \leqq 2 \int_{-\infty}^{\infty} \left[\int_{\Sigma} |\nabla\varphi|^2 + \left(\frac{\partial\varphi}{\partial x4}\right)^2 d\sigma\right] dx, \tag{4.5}$$

where $d\sigma$ is the area elements of Σ, and K, ∇ are the intrinsic Gauss curvature of Σ and the covariant derivative operator of Σ. Let $\chi(x^4)$ be a function satisfying $\chi(x^4) = 1$ for $|x^4| \leqq T$, $\chi(x^4) = 0$ if $|x^4| \geqq T+1$, and $\left|\frac{\partial x}{\partial x^4}\right| \leqq 2$. Let ζ be any function on Σ, and choose $\varphi = \chi\zeta$ in (4.5) to obtain

$$\begin{aligned}&\left(\int_{\Sigma} (-K + P)\zeta^2\,d\sigma\right)\left(\int_{-\infty}^{\infty} \chi^2\,dx^4\right)\\ &\quad \leqq 2\left(\int_{\Sigma} |\nabla\zeta|^2\,d\sigma\right)\left(\int_{-\infty}^{\infty} \chi^2\,dx^4\right) + 16\int_{\Sigma} \zeta^2\,d\sigma.\end{aligned}$$

Dividing both sides by $\int\limits_{-\infty}^{\infty} \chi^2 dx^4$ and letting T tend to infinity we get

$$\int\limits_{\Sigma}(-K+P)\zeta^2 d\sigma \leqq 2\int\limits_{\Sigma}|\nabla\zeta|^2 d\sigma \tag{4.6}$$

for any smooth function ζ on Σ. Choosing $\zeta \equiv 1$, we get

$$\int\limits_{\Sigma} P d\sigma \leqq \int\limits_{\Sigma} K d\sigma .$$

Since P is positive, by the Gauss-Bonnet theorem we conclude that Σ is a two-sphere.

By similar reasoning we can choose a sequence a_i converging to $-\infty$ so that G_{0,a_i} converges to a cylinder $\Sigma_- \times \mathbb{R}$ where Σ_- is a collection of two-spheres Σ in N satisfying $H_\Sigma - \mathrm{Tr}_\Sigma(p_{ij})=0$ where H_Σ is computed with respect to the inward normal to Ω_0. The fact that the graph G_0 is properly embedded implies that $f_0(x)$ converges either to $+\infty$ or $-\infty$ as x tends to a boundary point of Ω_0. Using this fact, it is clear that $\partial\Omega_0 = \Sigma_+ \cup \Sigma_-$.

From the construction of M_0, it follows that any boundary point of Ω_+ or Ω_- which does not lie in $\partial\Omega_0$ must lie on a cylindrical component $\Sigma \times \mathbb{R}$ of M_0. For such a Σ, we can verify (4.6) by using (2.29) on the graphs G_i, so we conclude that such Σ are two-spheres satisfying the appropriate equations. This concludes the proof of Proposition 4.

We can derive a little more information about the behavior of f_0 near $\partial\Omega_0$ from the preceding result. In fact, if we let Σ be a boundary component of Ω_0, say for definiteness that f_0 tends to $+\infty$ near Σ. (A similar argument works if f_0 tends to $-\infty$.) If we let θ be a coordinate on the two dimensional sphere Σ, and $t \in \mathbb{R}$ be along the linear factor of $\Sigma \times \mathbb{R}$, then we can define a coordinate system on a neighborhood of $\Sigma \times \mathbb{R}$ in $N \times \mathbb{R}$ by taking the fourth coordinate ϱ to be the distance function to $\Sigma \times \mathbb{R}$, say $\varrho > 0$ in $\Omega_0 \times \mathbb{R}$. Let $\mathcal{O}$ be a small neighborhood of Σ in N such that the coordinates (θ, t, ϱ) are nonsingular on $\mathcal{O} \times \mathbb{R}$. It is a consequence of Proposition 4 that for $T > 0$ sufficiently large, the 3-dimensional manifold $G_0 \cap (\mathcal{O} \times (T, \infty))$ can be expressed by the equation $\varrho = g_0(\theta, t)$ for a smooth function g_0 on $\Sigma \times (T, \infty)$. Moreover, it follows that $\lim\limits_{t\to\infty} g_0(\theta, t) = 0$ uniformly for $\theta \in \Sigma$. Using this information and the equation that g_0 satisfies, it is easy to show that the derivatives of g_0 up to second order also tend to zero as t goes to infinity. We summarize this information.

Corollary 2. *If Σ is a boundary component of Ω_0 on which f_0 tends to $+\infty$ (respectively $-\infty$), then for T sufficiently large, the 3-manifold $G_0 \cap (\mathcal{O} \times (T, \infty))$ (respectively $G_0 \cap (\mathcal{O} \times (-\infty, -T))$) can be represented in the form $\varrho = g_0(\theta, t)$ for a smooth positive function g_0 defined on $\Sigma \times (T, \infty)$ (respectively $\Sigma \times (-\infty, -T)$). Moreover, given $\varepsilon > 0$, there is a number $T_\varepsilon \geqq T$ so that*

$$g_0(\theta, t) + |Dg_0(\theta, t)| + |DDg_0(\theta, t)| < \varepsilon$$

for all $\theta \in \Sigma$ and $t \geqq T_\varepsilon$ (respectively $t \leqq -T_\varepsilon$).

5. Proof of Theorem 1

We use the function f_0 constructed in the previous section to prove Theorem 1. We want to prove that $M_k \geqq 0$, so we consider only that component of Ω_0 which contains N_k. For simplicity we denote the corresponding component of G_0 also as G_0. Let φ be a bounded Lipschitz function on G_0 which tends to zero and is square integrable near $(\partial\Omega_0)\times\mathbb{R}$. Multiplying (2.29) by φ^2 and integrating by parts we have

$$\int_{G_0} (P-\bar{R})\varphi^2 \sqrt{\bar{g}}\,dx \leqq -2\int_{G_0} \varphi^2 \sum_i (h_{i4}-p_{i4})^2 \sqrt{\bar{g}}\,dx$$
$$-4\int_{G_0} \varphi \sum_i \varphi_i (h_{i4}-p_{i4}) \sqrt{\bar{g}}\,dx.$$

Note that no boundary terms appear in the above inequality because by (2.28) we have

$$|h_{i4}-p_{i4}| = O(r^{7/2}),$$

and $\varphi \to 0$ near $\partial\Omega_0 \times \mathbb{R}$ whereas by Proposition 2, $|h_{i4}|$ is bounded near $\partial\Omega_0 \times \mathbb{R}$. By the arithmetic-geometric mean inequality,

$$\left|4\varphi \sum_i \varphi_i (h_{i4}-p_{i4})\right| \leqq 2\varphi^2 \sum_i (h_{i4}-p_{i4})^2 + 2|\bar{D}\varphi|^2.$$

Combining these inequalities we have

$$\int_{G_0} (P-\bar{R})\varphi^2 \sqrt{\bar{g}}\,dx \leqq 2\int_{G_0} |\bar{D}\varphi|^2 \sqrt{\bar{g}}\,dx$$

for any bounded Lipschitz φ on G_0 tending to zero and square integrable near $(\partial\Omega_0)\times\mathbb{R}$. We next observe that by Corollary 2 we can deform G_0 slightly in $\mathcal{O}\times(T,\infty)$ or $\mathcal{O}\times(-\infty,-T)$ for each boundary component of Ω_0 so that G_0 *coincides* with $\Sigma\times\mathbb{R}$ in $\mathcal{O}\times(T,\infty)$ or $\mathcal{O}\times(-\infty,-T)$ and so that G_0 satisfies

$$-\int_{G_0} \bar{R}\varphi^2 \sqrt{\bar{g}}\,dx \leqq 3\int_{G_0} |\bar{D}\varphi|^2 \sqrt{\bar{g}}\,dx \tag{5.1}$$

for φ as above. Making G_0 equal to $(\partial\Omega_0)\times\mathbb{R}$ near infinity will, of course, destroy the Eq. (2.27) which G_0 satisfies, but we need only (5.1) to finish the proof, and this modification of G_0 will afford us technical convenience. We next remove all infinities of G_0 except that asymptotic to N_k. This can be done by a conformal change of metric. Let Σ be a component of $\partial\Omega_0$, and note that by inequality (4.6), the first eigenvalue λ_1 of the operator $\Delta - \frac{1}{8}K$ on Σ is strictly positive. Let ζ_1 be the first eigenfunction, say $\zeta_1(x)>0$ for $x\in\Sigma$. It follows that the functions $e^{\pm\sqrt{\lambda_1}t}\zeta_1(x)$ are solutions of $\Delta - \frac{1}{8}\bar{R}=0$ on $\Sigma\times\mathbb{R}$. Let $\mathcal{S}^+$ denote those components of $\partial\Omega_0$ on which f_0 has limit $+\infty$, and $\mathcal{S}^-$ those on which f_0 has limit $-\infty$. Let G_0^ℓ denote the infinity of G_0 asymptotic to N_ℓ, i.e., $G_0^\ell = G_0 \cap (N_\ell \times \mathbb{R})$. For each $\ell \neq k$, let ψ_ℓ be a positive solution of $\Delta - \frac{1}{8}\bar{R}=0$ on N_ℓ satisfying

$$\psi_\ell = \frac{A_\ell}{r} + O(r^{-2}) \quad \text{as} \quad r\to\infty.$$

Such solutions ψ_ℓ can be constructed easily because of (5.1). Let ψ be a positive smooth function on G_0 satisfying the following

$$\psi = \begin{cases} 1 & \text{on} \quad G_0^k \\ \psi_\ell & \text{on} \quad G_0^\ell, \quad \ell \neq k \\ e^{-\sqrt{\lambda_1}t}\zeta_1 & \text{on} \quad G_0 \cap (\Sigma \times \mathbb{R}) \quad \text{for} \quad \Sigma \in \mathscr{S}^+ \\ e^{+\sqrt{\lambda_1}t}\zeta_1 & \text{on} \quad G_0 \cap (\Sigma \times \mathbb{R}) \quad \text{for} \quad \Sigma \in \mathscr{S}^- . \end{cases}$$

Thus ψ tends to zero at each infinity except G_0^k. If $\overline{ds}^2$ denotes the induced metric on G_0, we define a new metric ds_0^2 by $ds_0^2 = \psi^4 \overline{ds}^2$. For $\ell \neq k$, it follows from (1.1) and (2.28)

$$\psi_\ell^4 \bar{g}_{ij} = \left(\frac{A_\ell}{r}\right)^4 (\delta_{ij} + O(r^{-1}))$$

on G_0^ℓ. If we set $y^i = A_\ell^2 \frac{x^i}{r^2}$, $\varrho = |y|$, and write ds_0^2 in terms of the y coordinate system we have

$$ds_0^2 = \sum_{i,j} (\delta_{ij} + O(\varrho)) dy^i dy^j \tag{5.2}$$

for ϱ near zero. On $G_0 \cap (\Sigma \times \mathbb{R})$ for $\Sigma \in \mathscr{S}^\pm$, we have the expression

$$ds_0^2 = \zeta_1^4(x) e^{\pm 4\sqrt{\lambda_1}t}(dt^2 + d\sigma^2)$$

as $t \to \pm\infty$ where $d\sigma^2$ is the metric of Σ. If we set $\varrho = (2\sqrt{\lambda_1})^{-1} e^{\pm 2\sqrt{\lambda_1}t}$, we then have

$$ds_0^2 = \zeta_1^4(x)\,(d\varrho^2 + 4\lambda_1 \varrho^2 d\sigma^2) \tag{5.3}$$

for ϱ near zero, $x \in \Sigma$. If we choose a diffeomorphism of Σ with the standard S^2 having metric $d\sigma_0^2$ and write the flat metric in the punctured ball as $d\varrho^2 + \varrho^2 d\sigma_0^2$, we see that the resulting diffeomorphism establishes a uniform equivalence of $G_0 \cap (\Sigma \times \mathbb{R})$ with the punctured ball, i.e., lengths are distorted by at most a fixed constant.

We see from (5.2) and (5.3) that it is possible to add a point to G_0 for each component of $\partial\Omega_0$ and for each G_0^ℓ, $\ell \neq k$ to form a new manifold (N_0, ds_0^2) having only one infinity $N_0^k = G_0^k$. If $\{P_1, \ldots, P_s\}$ are the points we added to G_0, it follows from our construction that the metric ds_0^2 is uniformly equivalent to a smooth metric in a neighborhood of each P_i, and that the scalar curvature R_0 vanishes identically for points close to each P_i. If ζ is a bounded Lipschitz function on N_0, the equation

$$R_0 = \psi^{-5}(\bar{R}\psi - 8\Delta\psi)$$

together with (5.1) for $\varphi = \psi\zeta$ implies

$$5 \int_{N_0} \psi^{-2} |D_0(\psi\zeta)|^2 dv_0 - \int_{N_0} R_0 \zeta^2 dv_0 \leqq 8 \int_{N_0} |D_0\zeta|^2 dv_0 , \tag{5.4}$$

where D_0, dv_0 are the covariant derivative and volume form of N_0. We will use (5.4) in the following lemma to construct a solution of $\Delta - \frac{1}{8}R_0$.

Lemma 4. *There is a positive function u on N_0 satisfying $\Delta u-\frac{1}{8}R_0u=0$ except at $\{P_1,\dots,P_s\}$. At each P_j, u is continuous, and u is weakly harmonic in a neighborhood of P_j. Moreover, u satisfies*

$$u=1+\frac{A_k}{r}+O(r^{-2}) \quad \text{as} \quad r\to\infty$$

on N_0^k where the number A_k is negative.

Proof. Let B_σ be the bounded region of N_0 determined by $\{r=\sigma\}$, and for σ large we can find a function v_σ satisfying

$$\begin{aligned} \Delta v_\sigma-\tfrac{1}{8}R_0v_\sigma&=\tfrac{1}{8}R_0 \quad \text{on} \quad B_\sigma\\ v_\sigma&=0 \qquad \text{on} \quad \partial B_\sigma. \end{aligned}$$

This follows because (5.4) implies that the homogeneous problem $\Delta u-\frac{1}{8}R_0u=0$ with zero boundary data has only the trivial solution. Moreover, v_σ is Hölder continuous and weakly harmonic near each P_j. Inequality (5.4) then implies

$$5\int_{B_\sigma}\psi^{-2}|D_0(\psi v_\sigma)|^2dv_0\leqq\int_{B_\sigma}|R_0|\,|v_\sigma|dv_0.$$

Since ψ is a bounded function, we thus have

$$\int_{B_\sigma}|D_0(\psi v_\sigma)|^2\leqq c\int_{B_\sigma}|R_0|\,|v_\sigma|dv_0.$$

By the Sobolov inequality we thus have

$$\left(\int_{B_\sigma}|\psi v_\sigma|^6dv_0\right)^{1/3}\leqq c\int_{B_\sigma}|R_0|\,|v_\sigma|dv_0.$$

Since R_0 vanishes in a neighborhood U of $\{P_1,\dots,P_s\}$, and ψ is bounded below on $N_0\sim U$, we thus have by the Hölder inequality

$$\left(\int_{B_\sigma\sim U}|v_\sigma|^6dv\right)^{1/3}\leqq c\left(\int_{N_0}|R_0|^{6/5}dv_0\right)^{5/6}\left(\int_{B_\sigma\sim U}|v_\sigma|^6dv_0\right)^{1/6}$$

which implies

$$\int_{B_\sigma\sim U}|v_\sigma|^6dv_0\leqq c$$

with c independent of σ. Standard theory then gives a uniform pointwise bound on $|v_\sigma|$ in $B_\sigma\sim U$. The Harnack inequality applied to $v_\sigma+1$ gives a uniform estimate of $|v_\sigma|$ in U. It is now straightforward (see [9, Lemma 3.2]) to prove convergence of $v_\sigma+1$ to a function u satisfying $\Delta u+\frac{1}{8}R_0u=0$ on N_0, $u=1+\frac{A_k}{r}+O(r^{-2})$ on N_0^k. The positivity of u follows by using $\zeta=\min\{u,0\}$ in (5.4) and applying Stokes theorem in a standard way. This implies $u\geqq 0$, and that $u>0$ follows from the Harnack inequality.

To show that $A_k<0$, we use $\zeta=u$ in (5.4) and integrate by parts to obtain

$$A_k\leqq-\frac{5}{32\pi}\int_{N_0}\psi^{-2}|D_0(\psi u)|^2dv_0. \tag{5.5}$$

Note that although u may not be Lipschitz near P_j, we can justify its use in (5.4) by Lipschitz approximation. This completes the proof of Lemma 4.

We can now complete the proof of Theorem 1. The metric $u^4 ds_0^2$ on N_0 has zero scalar curvature, and is asymptotically flat in the sense of (1.1). If the results of [9] and [10] were applicable we would conclude that M_k is nonnegative. But we have

$$M_k^0 = M_k + 2A_k \tag{5.6}$$

as can be seen from the definition of mass. Since $M_k^0 \geqq 0$ and $A_k < 0$, it would follow that $M_k > 0$. Note that we have been assuming $\mu > |J|$ to conclude $M_k > 0$. In light of Lemma 1 we would then have $M_k \geqq 0$ for an arbitrary initial data set.

It remains for us to justify the use of [9] and [10] to assert $M_k^0 \geqq 0$. The problem is that the metric $u^4 ds_0^2$ is not smooth at $\{P_1, \ldots, P_s\}$. We note, however, that since the Laplace operator is uniformly elliptic near each P_j, there exists a positive Green's function $G(p, q)$ asymptotic to zero on N_k. If we define ψ by

$\psi(\cdot) = \sum_{j=1}^{s} G(P_j, \cdot)$, then ψ satisfies

$$\begin{aligned} \Delta\psi &= 0 && \text{on} \quad N_0 \sim \{P_1, \ldots, P_s\} \\ \psi &= \frac{B_k}{2r} + O(r^{-2}) && \text{on} \quad N_k \\ c^{-1}|y|^{2-n} \leqq \psi(y) &\leqq c|y|^{2-n} && \text{for coordinates } y \text{ at } P_j. \end{aligned}$$

For any $\varepsilon > 0$, consider the metric $(1 + \varepsilon\psi)^4 u^4 ds_0^2$. This metric is now smooth with infinities at each P_j. It is easy to see that the results of [9] and [10] apply to show that the mass on N_k given by $M_k^0 + \varepsilon B_k$ is nonnegative. Since $\varepsilon > 0$ is arbitrarily small we have $M_k^0 \geqq 0$. This completes the proof of Theorem 1.

6. Proof of Theorem 2

In this section we prove Theorem 2 which states that if $M_k = 0$ for some k then the initial data set is trivial. We first note that by Lemma 1 we can find a sequence of initial data sets $N^{(\ell)}$ converging smoothly to N as $\ell \to \infty$ with mass $M_k^{(\ell)} \to 0$ for the kth end and with $N^{(\ell)}$ satisfying $\mu < |J|$ for each ℓ. Then we may apply the analysis of Sect. 4 to construct graphs $G_0^{(\ell)}$ satisfying (2.27). By the estimates of Propositions 2 and 3 we may assume that the $G_0^{(\ell)}$ converge smoothly to a properly embedded limiting submanifold having a component G_0 which contains a graph over N_k satisfying (2.27). We now examine the proof of Theorem 1. If we let $U^{(\ell)}$ be an exhaustion of $N \times \mathbb{R}$ by bounded open sets, then we can choose ψ_ℓ, the conformal factor of Sect. 5 so that $\psi_\ell = 1$ on $G_0^{(\ell)} \cap U_\ell$. It then follows from (5.6) and the final arguments of Sect. 5 that $M_k^{(\ell)} + A_k^{(\ell)} \geqq 0$. Hence by (5.5) and the fact that $M_k^{(\ell)} \to 0$ we have

$$\lim_{\ell \to \infty} \int_{G_0^{(\ell)} \cap U_\ell} |\bar{D} u_\ell|^2 \sqrt{\bar{g}}\, dx = 0 . \tag{6.1}$$

Since $G_0^{(\ell)}$ converge to G_0, it follows that u_ℓ converges to a smooth positive function u on G_0 satisfying $\Delta u - \frac{1}{8}\bar{R} u = 0$, $u \sim 1$ on N_k. Thus by (6.1) we have that $u \equiv 1$ on G_0, and hence the equation satisfied by u implies that $\bar{R} \equiv 0$.

Thus we may apply Theorem 2 of [9] (see also [10]) to assert that G_0 is isometric to the flat $\mathbb{R}^3$. In particular, N is diffeomorphic to $\mathbb{R}^3$ and the solution f of (2.27) exists on all of N and has flat graph G_0. Now the metric on G_0 has the form $\bar{g}_{ij}=g_{ij}+f_{x^i}f_{x^j}$, and since G_0 is $\mathbb{R}^3$, we can choose coordinates $\bar{x}=(\bar{x}^1,\bar{x}^2,\bar{x}^3)$ on G_0 so that $\bar{g}_{ij}=\delta_{ij}$. We thus have

$$g_{ij}=\delta_{ij}-f_{\bar{x}^i}f_{\bar{x}^j}.$$

This shows that if $(\bar{x}^1,\bar{x}^2,\bar{x}^3,\bar{x}^4)$ are coordinates in M^4, the Minkowski space with metric $\sum_{i=1}^{3}(d\bar{x}^i)^2-(d\bar{x}^4)^2$, then the mapping $N\to\mathbb{M}^4$ defined by $\bar{x}\to(\bar{x},\ f(\bar{x}))$ is an isometric embedding of N. The second fundamental form of this embedding is given by

$$\pi_{ij}=(1-|\bar{D}f|^2)^{-1/2}f_{\bar{x}^i\bar{x}^j}.$$

Note that $|\bar{D}f|^2<1$ because g_{ij} is positive definite. The corresponding expression for h_{ij}, the second fundamental form of G_0 in $N\times\mathbb{R}$ is

$$h_{ij}=(1+|Df|^2)^{1/2}f_{\bar{x}^i\bar{x}^j},$$

where $|Df|^2$ is taken with respect to ds^2. Direct calculation shows $1+|Df|^2=(1-|\bar{D}f|^2)^{-1}$ so that $h_{ij}=\pi_{ij}$. On the other hand, since $\bar{R}\equiv 0$ we can integrate (2.29) over G_0 and apply Stokes theorem to show $h_{ij}=p_{ij}$. Therefore, we have $\pi_{ij}=p_{ij}$ and we have shown that the initial data set (N,ds^2,p_{ij}) is embeddable in $\mathbb{M}^4$. This completes the proof of Theorem 2.

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