HEAT EQUATIONS ON MINIMAL SUBMANIFOLDS AND THEIR APPLICATIONS

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0. Introduction. Let M^n be a n-dimensional minimally immersed submanifold of $\bar{M}^{n+\ell}$, $\ell > 1$. Throughout this paper $\bar{M}^{n+\ell}$ is taken to be one of the simply connected space forms with curvature 1, 0, or -1, i.e. $\bar{M}^{n+\ell} = S^{n+\ell}$, $\mathbf{R}^{n+\ell}$, or $\mathbf{H}^{n+\ell}$. Given a point $p \in M$, let $r_p(x)$ be the distance function on $\bar{M}$, we denote the restriction of r_p to M as the extrinsic distance function on M. For any $a > 0$, we define the extrinsic ball centered at p with radius a by

$$D_p(a) = B_p(a) \cap M$$

where $B_p(a) = \{x \epsilon \bar{M} \mid r_p(x) \leq a\}$. Unless ambiguity arises, the subscript p will be subpressed.

Let $D \subset M$ be a compact domain. We consider the fundamental solutions of the heat equation (heat kernels), $H(x, y, t)$ and $K(x, y, t)$, for the Dirichlet and the Neumann boundary conditions respectively. They possess the properties:

(i) $\square_y H(x, y, t) = \square_y K(x, y, t) = 0$, for all $x, y \in D$ and $t \in [0, \infty)$.
(ii) $H(x, y, 0) = K(x, y, 0) = \delta_x$, for $x \in D$.
(iii) $H(x, z, t) = 0$, for $z \in \partial D$
$\partial K/\partial \nu_z(x, z, t) = 0$, for $z \in \partial D$.

where $\partial/\partial \nu_z$ stands for the differentiation in the z variable in the outward normal direction to ∂D.

The purpose of this paper is to develop comparison theorems for H and K.

THEOREM 1. *Let D be a compact domain in M. If $\bar{M}^{n+\ell} = \mathbf{R}^{n+\ell}$ (or $\mathbf{H}^{n+\ell}$), and for any $p \in D$ we define the extrinsic outer radius at p by*

$$a = \sup_{z \in D} r_p(z)$$

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then

$$H(p, y, t) \leq \bar{H}_a(r_p(y), t)$$

for all $y \in D$ *and* $t \in [0, \infty)$. *Here* $\bar{H}_a(r_p(y), t)$ *stands for the heat kernel with Dirichlet boundary condition on the ball centered at* 0 *with radius* a *in* $\mathbf{R}^n$ (*or* $\mathbf{H}^n$ *respectively*).

THEOREM 2. *Let* $D_p(a)$ *be the extrinsic ball centered at* p *with radius* a *in* M. *If* $\bar{M}^{n+\ell} = \mathbf{R}^{n+\ell}$ (*or* $\mathbf{H}^{n+\ell}$), *and if* $\bar{K}_a(r_p(y), t)$ *is the heat kernel with Neumann boundary condition on the ball centered at* 0 *with radius* a *in* $\mathbf{R}^n$ (*or* $\mathbf{H}^n$), *then*

$$K(p, y, t) \leq \bar{K}_a(r_p(y), t)$$

for all $y \in D_p(a)$ *and* $t \in [0, \infty)$.

THEOREM 3. *Let* $\bar{M}^{n+\ell} = S^{n+\ell}$. *Suppose* D *is a compact domain in* M. *For* $p \in D$ *if the outer radius at* p *is not greater than* $\pi/2$, *then*

$$H(p, y, t) \leq \bar{H}_a(r_p(y), t)$$

for all $y \in D$ *and* $t \in [0, \infty)$. $\bar{H}_a(r_p(y), t)$ *is the heat kernel with Dirichlet boundary condition on the ball of radius* a *in* S^n *centered at the northpole.*

THEOREM 4. *Suppose* $\bar{M}^{n+\ell} = S^{n+\ell}$. *Let* $D_p(a)$ *be any extrinsic ball of* M *with radius* $0 < a < \pi$. *If* $\bar{K}_a(r_p(y), t)$ *is the heat kernel with Neumann boundary condition on the ball of radius* a *in* S^n, *then*

$$K(p, y, t) \leq \bar{K}_a(r_p(y), t)$$

for all $y \in D_p(a)$ *and* $t \in [0, \infty)$.

THEOREM 5. *Let* M^n *be a compact manifold without boundary. Suppose* $M^n \hookrightarrow S^{n+\ell}$ *is a minimal immersion of* M *into* $S^{n+\ell}$. *If we denote the heat kernel on* M (*without boundary condition*) *by* $K(x, y, t)$ *and the heat kernel on* S^n *by* $\bar{K}(x, y, t)$, *then*

$$K(p, y, t) \leq \bar{K}(r_p(y), t)$$

for all $p, y \in M$ *and* $t \in [0, \infty)$.

Remark. In the above theorems, we simply transplanted the heat kernel of the model space into the domain in question. It is unambiguous to write the kernels on the model spaces, namely $\bar{H}$ and $\bar{K}$ for the ball of radius a centered at 0 in $\mathbf{R}^n$, $\mathbf{H}^n$, or S^n, as a function of r, because of the uniqueness of the kernels and the presence of the group of rotations as isometries.

In the first section of this work, we will prove Theorem 1 and 2, while Theorems 3 to 5 will be proved in Section 2. The last section consists of applications and consequences which follow Theorem 1-5.

One of the applications of Theorem and 3 are the mean-value inequalities for subharmonic functions defined on M (also see [10] and [11]). Lower bounds for the volume of the ball of radius a is obtained as a result. However in the case of when $\bar{M} = S^{n+\ell}$ and $a > \pi/2$, the usual technique does not give the mean-value inequality, but volume lower bound still follows by utilizing the Neumann heat kernel in Theorem 4.

Comparison theorems for the first eigenvalue of Dirichlet boundary problem on any compact domain D in $M \subseteq \mathbf{R}^{n+\ell}$ (or $\mathbf{H}^{n+\ell}$) are derived. Similar results also hold when $\bar{M} = S^{n+\ell}$, but we have to restrict ourself to domains which contain in a hemisphere of $S^{n+\ell}$. These comparisons are sharp and equality holds iff M is totally geodesic and D is an extrinsic ball. Theorems 1 and 3 also imply lower estimation for high eigenvalues of D. These estimates are up to a constant comparable to the H. Weyl formula.

The consequences of Theorem 5 are most interesting. By estimating the heat kernel for S^n carefully, we conclude that if the volume $V(M)$ of M is closed to the volume $V(S^n)$ of S^n, then M is totally geodesic. In fact, since the closeness requirement increases linearly with respect to the codimension ℓ, this implies that if $M \hookrightarrow S^{n+\ell}$ is of maximal dimension (M does not lie on any hyperplane of $\mathbf{R}^{n+1}$) then

$$V > \left[1 + \frac{2\ell + 1}{B_n}\right] V(S^n)$$

for some constant B_n which depends only on n. In general, we also observe that the number of components of M-H, where H is any hyperplane in $\mathbf{R}^{n+\ell+1}$ which passes through the origin, is bounded from above by a constant depending on $V(M)$. Lower bounds for all the eigenvalues of M are also established in terms of the ordered of the eigenvalue and $V(M)$.

1. $\bar{M}^{n+\ell} = \mathbf{R}^{n+\ell}$ or $\mathbf{H}^{n+\ell}$. In this section we will mainly deal with the cases when $\bar{M}^{n+\ell}$ equals $\mathbf{R}^{n+\ell}$ or $\mathbf{H}^{n+\ell}$. Before we prove Theorems 1 and 2, we will present the following proposition which can be found in standard references (also see [3] and [8]).

PROPOSITION 1. *Let M be a compact manifold with boundary ∂M. Suppose $p \in M$, and if $G(p, y, t)$ and $\bar{G}(p, y, t)$ are two C^2 functions defined on $M \times M \times [0, \infty)$ with the properties that:*

(i) $G(p, y, t) \geq 0$ *for all* $y \in M$, $t \in [0, \infty)$
(ii) $G(p, y, 0) = \bar{G}(p, y, 0) = \delta_p$
(iii) $\Box_y G(p, y, t) = 0$ *and* $\Box_y \bar{G}(p, y, t) \leq 0$ *for all* $y \in M$ *and* $t \in [0, \infty)$
(iv) (1) $G(p, z, t) = 0$ *and* $\bar{G}(p, z, t) \geq 0$ *for all* $z \in \partial M$, $t \in [0, \infty)$; *or* (2) $\partial G/\partial \nu_z(p, z, t) = \partial \bar{G}/\partial \nu_z(p, z, t) = 0$ *for all* $z \in \partial M$ *and* $t \in [0, \infty)$.

Then

$$G(p, y, t) \leq \bar{G}(p, y, t)$$

for all $y \in M$ and $t \in [0, \infty)$.

Proof. By property (ii),

$$\begin{aligned}
(1.1)\quad & \bar{G}(p, y, t) - G(p, y, t) \\
&= \int_0^t \frac{\partial}{\partial s} \int_M \bar{G}(p, z, s) G(y, z, t-s)\, dz\, ds \\
&\geq \int_0^t \int_M \Delta_z \bar{G}(p, z, s) G(y, z, t-s)\, dz\, ds \\
&\quad - \int_0^t \int_M \bar{G}(p, z, s) \Delta_z G(y, z, t-s)\, dz\, ds
\end{aligned}$$

(properties (i) and (iii))

$$\begin{aligned}
&= \int_0^t \int_{\partial M} \frac{\partial \bar{G}}{\partial \nu_z}(p, z, s) G(y, z, t-s)\, dz\, ds \\
&\quad - \int_0^t \int_{\partial M} \bar{G}(p, z, s) \frac{\partial G}{\partial \nu_z}(y, z, t-s)\, dz\, ds.
\end{aligned}$$

Clearly if either condition (1) or (2) of property (iv) holds, then the proposition follows.

We are ready to prove Theorems 1 and 2. Since the proof for the case $\bar{M} = \mathbf{H}^{n+\ell}$ is quite similar to that of $\mathbf{R}^{n+\ell}$, we will restrict ourself to $\bar{M} = \mathbf{R}^{n+\ell}$.

Proof of Theorem 1. $\bar{M} = \mathbf{R}^{n+\ell}$. In view of the above proposition, it suffices to check conditions (ii), (iii) and (iv) (1) for the transplanted heat kernel $\bar{H}(r(y), t)$. Due to the asymptotic expansion of $\bar{H}(r(y), t)$ for $r(y)$ near 0 as $t \to 0$

$$\bar{H}(r, t) \sim (4\pi t)^{-n/2} \exp\left(\frac{-r^2}{4\pi t}\right)[1 + a_1 t + a_2 t^2 + \cdots], \tag{1.2}$$

and also because of the fact that on M, the extrinsic distance function $r_p(y)$ is asymptotic to the intrinsic distance function, it is apparent that $\bar{H}(r, t) \sim \delta_p$ as $t \to 0$, hence condition (ii) is satisfied. Condition (iv) (1) follows from the fact that $\bar{H} \geq 0$ on $B_0(a) \subseteq \mathbf{R}^n$ and the definition of the outer radius a. We only need to check that

$$\Box_y \bar{H}(r(y), t) \leq 0 \tag{1.3}$$

We observe that by minimality of M in $\mathbf{R}^{n+\ell}$,

$$\Delta r_p^2(y) = 2n \quad \text{for all} \quad p, y \in M \tag{1.4}$$

Hence it is convenient to write $\bar{H}$ as a function of $s(y) = r^2(y)$. Computation shows that

$$\begin{aligned} \Delta_y \bar{H}(s, t) &= \bar{H}''|\nabla s|^2 + \bar{H}'\,\Delta s \\ &= \bar{H}''(4r^2|\nabla r|^2) + 2n\bar{H}' \end{aligned} \tag{1.5}$$

where $\bar{H}'$ and $\bar{H}''$ are the first and second derivatives of $\bar{H}$ with respect to the variable s. Since $|\nabla r| \leq 1$, clearly if $\bar{H}'' \geq 0$ then equation (1.4) becomes

$$\begin{aligned} \Delta_y \bar{H}(s, t) &\leq \bar{H}''(4s) + 2n\bar{H}'' \\ &= \bar{\Delta}\bar{H}(s, t) \end{aligned} \tag{1.6}$$

where $\bar{\Delta}$ is the Laplace operator on the ball $B_0(a)$ in $\mathbf{R}^n$. By the fact that $\bar{H}$ is the heat kernel of $B_0(a)$, this gives

$$\Box \bar{H}(s, t) \leq \bar{\Box} \bar{H}(s, t)$$
$$= 0.$$

To show $\bar{H}'' \geq 0$, we differentiate twice the equation

$$\bar{H}_t = \bar{\Delta} \bar{H} \tag{1.7}$$
$$= 4s\bar{H}'' + 2n\bar{H}''$$

and obtain

$$\bar{H}_t' = 4s\bar{H}^{(3)} + (2n + 4)\bar{H}'' \tag{1.8}$$

and

$$\bar{H}_t'' = 4s\bar{H}^{(4)} + 2(n + 4)\bar{H}^{(3)}. \tag{1.9}$$

The function $\bar{H}''$ hence satisfies a second ordered parabolic equation (1.8), and the maximum principle can be applied. The nonnegativity of $\bar{H}''$ will then follow from the nonnegativity of $\bar{H}''$ on the boundary of $0 \leq s \leq a$ and $t \in [0, \infty)$. When $t = 0$ and $s \neq 0$, $\bar{H}(s, 0) \equiv 0$, hence $\bar{H}''(s, 0) = 0$. At $s = a^2$, since $\bar{H}$ satisfies the Dirichlet boundary condition

$$\left.\frac{\partial \bar{H}}{\partial r}\right|_{r=a} = \left. 2r\bar{H}' \right|_{r=a} \leq 0 \tag{1.10}$$

and

$$\bar{H}_t|_{r=a} = 0. \tag{1.11}$$

Therefore from (1.6), we get

$$4a^2\bar{H}''|_{r=a} \geq 0. \tag{1.12}$$

For the case $s = 0$, we consider the equation

$$\bar{H}''(s(r), t) = \frac{\bar{\Delta}\bar{H}}{r^2} - \frac{n\bar{H}_r}{r^3} \tag{1.13}$$

$$= \frac{\bar{\Delta}\bar{H}}{r^2} + \frac{n|\bar{\nabla}\bar{H}|}{r^3}.$$

We will study the limit

$$\lim_{r\to 0}\left(\frac{\bar{\Delta}\bar{H}}{r^2} + \frac{n|\bar{\nabla}\bar{H}|}{r^3}\right).$$

However since $\bar{H}(0, y, t) = \Sigma\, e^{-\lambda_i t}\phi_i(0)\phi_i(y)$,

$$\bar{\Delta}\bar{H} + \frac{n|\bar{\nabla}\bar{H}|}{r} = \sum_{i=1}^{\infty} -\lambda_i e^{-\lambda_i t}\phi_i(0)\phi_i(y) + \left|\sum_{i=1}^{\infty}\frac{n}{r}e^{-\lambda_i t}\phi_i(0)\bar{\nabla}\phi_i(y)\right| \tag{1.14}$$

This reduces the questions to studying the behavior of each eigenspace at points near 0.

LEMMA 7. *Let M be a manifold with boundary. Suppose there exists a point $p \in M$, such that the metric of M is invariant under rotation around p, and M can be written as the ball of radius a around p with respect to the rotationally invariant metric. Then for each eigenspace*

$$E_\lambda = \{\phi\,|\,\bar{\Delta}\phi = -\lambda\phi\}$$

with eigenfunctions satisfying either the Dirichlet or Neumann boundary condition, either

(i) *$\phi(p) = 0$ for all $\phi \in E$ (up to scalar multiple)*

or

(ii) *there exists a unique (up to scalar multiple) $\phi \in E_\lambda$ which is rotationally symmetric i.e. ϕ can be written as a function of $r_p(y)$. In particular, if we insist that $\phi(p) \geq 0$ and $\int_M \phi^2 = 1$, then ϕ is unique in E_λ.*

Proof. Assuming conclusion (i) does not hold, we define the finite dimensional subspace $\bar{E}_\lambda$ of E_λ by

$$\bar{E}_\lambda = \{\phi \in E_\lambda | \phi(p) = 0\},$$

then $\bar{E}_\lambda \neq E_\lambda$. We claim that the orthogonal complement of $\bar{E}_\lambda$ in E_λ is one-dimensional. Indeed, if ϕ_1 and ϕ_2 are linearly independent in the orthogonal complement of $\bar{E}$, and if $\phi_1(p) = \alpha$ and $\phi_1(p) = \beta$, with $\alpha, \beta \neq 0$, then

$$\beta\phi_1(p) - \alpha\phi_2(p) = 0.$$

This shows $\beta\phi_1 - \alpha\phi_2 \in \bar{E}_\lambda$, which contradicts the fact that ϕ_1 and ϕ_2 are in the orthogonal complement of $\bar{E}_\lambda$. Clearly the uniqueness of the normalized ϕ which span the orthogonal space of $\bar{E}_\lambda$ in E_λ is the required rotationally symmetric eigenfunction in E_λ.

Remark. A similar version of Lemma 1 for homogeneous manifold can be found in [9]. The unique normalized function ϕ is usually known as the zonal function at p with respect to E_λ.

Returning to the proof of Theorem 1, in view of the lemma, Equation (1.13) can be written in the form

$$\bar{\Delta}\bar{H} + \frac{n|\bar{\nabla}\bar{H}|}{r} = \sum_\lambda e^{-\lambda t}\phi_\lambda(0)\left[|\bar{\nabla}\phi_\lambda(y)|\frac{n}{r} - \lambda\phi_\lambda(y)\right] \tag{1.15}$$

$$= \sum_\lambda e^{-\lambda t}\phi_\lambda(0)\left[-\frac{n}{r}\frac{\partial\phi_\lambda}{\partial r} - \lambda\phi_\lambda(y)\right]$$

where ϕ_λ with $\phi_\lambda(0) > 0$ is the rotational symmetric representative of E. The last equality follows from the fact that

$$\bar{\Delta}\phi_\lambda(p) = -\lambda\phi_\lambda(p) < 0$$

hence ϕ_λ has a local maximum at p. We will show that for each λ, ϕ_λ satisfies

$$\lim_{r\to 0}\left[-\frac{n}{r^3}\phi_r - \frac{\lambda}{r^2}\phi\right] \geq 0$$

where the subscript λ is being suppressed. Then Theorem 1 follows.

In fact, we show that the function

$$g = \frac{n\phi_r}{r} + \lambda\phi$$

vanishes to 2^{nd} order as $r \to 0$, hence it suffices to check that

$$\frac{\partial^2}{\partial r^2}\left[\frac{n\phi_r}{r} + \lambda\phi\right] \le 0.$$

By l'Hopital's rule

$$\lim_{r\to 0} \frac{n\phi_r}{r} = n\phi_{rr}|_{r=0}. \tag{1.16}$$

However at $r = 0$, one can easily check that

$$\begin{aligned} n\phi_{rr}|_{r=0} &= \bar{\Delta}\phi|_{r=0} \\ &= -\lambda\phi(0) \end{aligned} \tag{1.17}$$

for rotationally symmetric metric around 0, hence

$$\lim_{r\to 0}\left[\frac{n\phi_r}{r} + \lambda\phi\right] = 0.$$

Also

$$\frac{\partial g}{\partial r} = \frac{n\phi_{rr}}{r} - \frac{n\phi_r}{r^2} + \lambda\phi_r,$$

as $r \to 0$, this gives

$$\begin{aligned} \left.\frac{\partial g}{\partial r}\right|_{r=0} &= \lim_{r\to 0} \frac{n}{r}\left[\phi_{rr} - \frac{\phi_r}{r}\right] \\ &= \lim_{r\to 0} n\left[\phi_{rrr} - \frac{\phi_{rr}}{r} + \frac{\phi_r}{r^2}\right] \end{aligned} \tag{1.18}$$

but since

$$-\lambda\phi = \bar{\Delta}\phi \tag{1.19}$$

$$= \phi_{rr} + \frac{n-1}{r}\phi_r$$

after differentiating this yields

$$-\lambda\phi_r = \phi_{rrr} + \frac{n-1}{r}\left[\phi_{rr} - \frac{\phi_r}{r}\right]$$

As $r \to 0$, this implies

$$\phi_{rrr}|_{r=0} = \lim_{r=0} \frac{(n-1)}{r}\left[\frac{\phi_r}{r} - \phi_{rr}\right]$$

therefore substituting into (1.17), we have

$$\left.\frac{\partial g}{\partial r}\right|_{r=0} = n^2\left[\frac{\phi_r}{r^2} - \frac{\phi_{rr}}{r}\right]$$

$$= -n\left.\frac{\partial g}{\partial r}\right|_{r=0}$$

hence

$$\left.\frac{\partial g}{\partial r}\right|_{r=0} = 0$$

Finally,

$$\frac{\partial^2 g}{\partial r^2} = \lambda\phi_{rr} + \frac{n}{r}\left[\phi_{rrr} - \frac{\phi_{rr}}{r} + \frac{\phi_r}{r^2}\right]$$

$$- \frac{n}{r^2}\left[\phi_{rr} - \frac{\phi_r}{r}\right]$$

as $r \to 0$, we have

$$\left.\frac{\partial^2 g}{\partial r^2}\right|_{r=0} = -\frac{\lambda^2}{n}\phi(0) + \lim_{r\to 0}\frac{n}{r}\left[\phi_{rrr} - \frac{2\phi_{rr}}{r} + \frac{2\phi_r}{r^2}\right]. \tag{1.20}$$

The second term on the right hand side gives

$$n \lim_{r\to 0} \frac{1}{r}\left[\phi_{rrr} - \frac{2\phi_{rr}}{r} + \frac{2\phi_r}{r^2}\right]$$

$$= n \lim_{r\to 0}\left[\phi_{rrrr} - \frac{2\phi_{rrr}}{r} + \frac{4\phi_{rr}}{r^2} - \frac{4\phi_r}{r^3}\right].$$

On the other hand, differentiating Equation (1.17) twice yields

$$-\lambda\phi_{rr} = \phi_{rrrr} + \frac{n-1}{r}\left[\phi_{rrr} - \frac{2\phi_{rr}}{r} + \frac{2\phi_r}{r^2}\right].$$

Substituting into the above equation, then

$$\lim_{r\to 0} \frac{1}{r}\left[\phi_{rrr} - \frac{2\phi_{rr}}{r} + \frac{2\phi_r}{r^2}\right]$$

$$= \lim_{r\to 0}\left[-\lambda\phi_{rr} - \frac{n+1}{r}\left(\phi_{rrr} - \frac{2\phi_{rr}}{r} + \frac{2\phi_r}{r^2}\right)\right]$$

$$= \frac{\lambda^2}{n}\phi(0) - (n+1)\lim_{r\to 0}\frac{1}{r}\left[\phi_{rrr} - \frac{2\phi_{rr}}{r} + \frac{2\phi_r}{r^2}\right].$$

Therefore

$$(n+2)\lim_{r\to 0}\frac{1}{r}\left[\phi_{rrr} - \frac{2\phi_{rr}}{r} + \frac{2\phi_r}{r^2}\right] = \frac{\lambda^2}{n}\phi(0)$$

hence combining with Equation (1.19) gives

$$\left.\frac{\partial^2 g}{\partial r^2}\right|_{r=0} = -\frac{\lambda^2}{n}\phi(0) + \frac{\lambda^2}{n+2}\phi(0)$$

$$= -\frac{2\lambda^2}{n(n+2)}\phi(0) \leq 0.$$

as to be shown.

In the case where $\bar{M} = \mathbf{H}^{n+\ell}$, one simply writes $\bar{H}$ as a function of $s = \cosh r$ since $\Delta \cosh r = \bar{\Delta} \cosh r$ where $\bar{\Delta}$ is the Laplacian on $\mathbf{H}^n$. The computation on $\mathbf{H}^n$ will follow similarly to that of $\mathbf{R}^n$.

Proof of Theorem 2. Again we will consider the case when $\bar{M} = \mathbf{R}^{n+\ell}$ only. Similar to the proof of Theorem 1, we can reduce to check if $\bar{K}$ satisfies conditions (ii), (iii) and (iv) (2). Clearly condition (ii) follows from the remark in the previous proof. Consider

$$\begin{aligned}\frac{\partial \bar{K}}{\partial \nu_z}(r(z), t) &= \frac{\partial \bar{K}}{\partial r}(r(z), t) \cdot \frac{\partial r}{\partial \nu_z} \\ &= 0, \qquad \text{for} \quad z \in \partial D_p(a)\end{aligned}$$

since $\bar{K}$ is the Neumann heat kernel on $B_0(a)$ in $\mathbf{R}^n$. This gives (iv) (2).

To check condition (iii), we again write $\bar{K}$ as a function of $s = r^2$. Equation (1.6) asserts that if $\bar{K}'' \geq 0$, then Theorem 2 follows. Once again we differentiate Equation (1.7) twice and obtain (1.9)

$$\bar{K}_t'' = 4s\bar{K}^{(4)} + 2(n+4)\bar{K}^{(3)}.$$

In order to apply the maximum principle to $\bar{K}''$, we need to check that $\bar{K}'' \geq 0$ on the boundary of $0 \leq s \leq a^2$ and $0 \leq t < \infty$. When $s = 0$, or $t < \infty$. When $s = 0$, or $t = 0$, the proof of Theorem 1 is still valid, hence we will discuss the case when $s = a^2$.

It is known [3] that the heat kernels on a rotationally symmetric ball satisfies

$$\bar{K}' = 2s^{1/2}\bar{K}_r < 0 \tag{1.22}$$

for all $0 < s < a^2$ and $t \neq 0$. On the other hand by the Neumann boundary condition on $\bar{K}$, $\bar{K}' \equiv 0$ when $s = a^2$. Hence $\bar{K}'$ attains its maximum on $s = a^2$. Now consider equation on $a^2/2 \leq s \leq a^2$ and $0 \leq t < \infty$, we can apply the boundary maximum principle [17, pp. 170] and conclude that

$$\bar{K}''(a^2, t) > 0 \quad \text{for} \quad t \neq 0,$$

which proves the theorem.

2. $\bar{M}^{n+\ell} = S^{n+\ell}$. When $\bar{M}^{n+\ell}$ is $S^{n+\ell}$, one needs to write the kernels $\bar{H}$ and $\bar{K}$ as functions of $\cos r_p$ where r_p is the distance function from the northpole of S^n. In fact, for convenience sake, we will write $\bar{H}(s, t)$ and $\bar{K}(s, t)$ as functions of $s = 2(1 - \cos r)$ and t. We are now ready to prove Theorems 3–5.

Proof of Theorem 3. Clearly in view of Proposition 1 and the boundary conditions on H and $\bar{H}$, it suffices to check that

$$\Box \bar{H}(s, t) \leq 0.$$

However

$$\Delta \bar{H}(s, t) = \bar{H}''|\nabla s|^2 + \bar{H}'\Delta s \tag{2.1}$$

$$= \bar{H}''s(4 - s)|\nabla r|^2 - 2\bar{H}'(\Delta \cos r),$$

and since $\cos r$ is the restriction of the coordinate functions on the sphere,

$$\Delta \cos r = -n \cos r.$$

Hence

$$\Delta \bar{H}(s, t) = \bar{H}''s(4 - s)|\nabla r|^2 + 2n \cos r\bar{H}'.$$

If we can show that $\bar{H}'' \geq 0$, together with the fact that $|\nabla r|^2 \leq 1$, we have

$$\Delta \bar{H}(s, t) \leq s(4 - s)\bar{H}'' + n(2 - s)\bar{H}'$$

$$= \bar{\Delta}\bar{H}(s, t)$$

$$= \bar{H}_t(s, t)$$

where $\bar{\Delta}$ = Laplace operator on S^n. We differentiate twice the equation

$$\bar{H}_t = \bar{\Delta}\bar{H} \tag{2.2}$$

$$= s(4 - s)\bar{H}'' + n(2 - s)\bar{H}'$$

and apply the maximum principle to $\bar{H}''$, since it satisfies

$$\bar{H}_t'' = s(4-s)\bar{H}^{(4)} + (n+4)(2-s)\bar{H}''' - 2(n+1)\bar{H}''. \tag{2.3}$$

Similar to Theorem 1,

$$\bar{H}'' = 0 \quad \text{on} \quad \{(s,t)\,|\, s \neq 0,\, t = 0\}$$

and $\bar{H}'' > 0$ at $(s = 2(1-\cos a), t)$ for $t > 0$, because $\bar{H}_t \equiv 0$ at $s = 2(1-\cos a)$ and $0 > \bar{H}_r = \bar{H}'2\sin r$ (see [3]) for $r < a \leq \pi/2$.

To prove $\bar{H}''(0,t) \geq 0$, we consider the equation

$$\bar{H}'' = \frac{\bar{\Delta}\bar{H} - n(2-s)\bar{H}'}{s(4-s)} \tag{2.4}$$

which follows from (2.2). Using the eigenfunctions expansion for $\bar{H}$ and applying Lemma 1, we get

$$4\bar{H}'' = \sum_{\lambda} e^{-\lambda t}\phi_\lambda(0)\left(-\frac{\lambda\phi_\lambda}{\sin^2 r} - \frac{n\cos r(\partial\phi_\lambda/\partial r)}{\sin^3 r}\right) \tag{2.5}$$

where $\phi_\lambda(0) > 0$ and all ϕ_λ's are rotationally symmetric. Suppressing the subscript λ, we claim that the function

$$g = \lambda\phi + \frac{n\cos r\phi_r}{\sin r}$$

vanishes in 2^{nd} order and $\partial^2 g/\partial r^2$ is nonpositive as $r \to 0$. The theorem will then follow.

In fact

$$\begin{aligned} \lim_{r\to 0} g(r) &= \lambda\phi(0) + n\lim_{r\to 0}\frac{\cos r\phi_r}{\sin r} \\ &= \lambda\phi(0) + n\lim_{r\to 0}\frac{\cos r\phi_{rr} - \sin r\phi_r}{\cos r} \end{aligned} \tag{2.6}$$

by l'Hopital's rule.

On the other hand, since

$$\lim_{r\to 0} n\phi_{rr} = \Delta\phi(0)$$

$$= -\lambda\phi(0)$$

and $\phi_r \to 0$ as $r \to 0$,

$$\lim_{r\to 0} g(r) = 0.$$

Also

$$\frac{\partial g}{\partial r} = \lambda\phi_r + n\left[\frac{\cos r\phi_{rr}}{\sin r} - \frac{\phi_r}{\sin^2 r}\right], \tag{2.7}$$

as $r \to 0$ this becomes

$$\lim_{r\to 0}\frac{\partial g}{\partial r} = n\lim_{r\to 0}\left[\frac{\cos r\phi_{rr}}{\sin r} - \frac{\phi_r}{\sin^2 r}\right]$$

$$= n\lim_{r\to 0}\left[\phi_{rrr} - \frac{\phi_{rr}\cos r}{\sin r} + \frac{\phi_r}{\sin^2 r}\right]$$

$$= n\lim_{r\to 0}\phi_{rrr} - \lim_{r\to 0}\frac{\partial g}{\partial r}$$

where the second equality is obtained by l'Hopital's rule. On the other hand, differentiating the equation

$$-\lambda\phi = \Delta\phi = \phi_{rr} + \frac{(n-1)\cos r\phi_r}{\sin r} \tag{2.9}$$

with respect to r yields

$$-\lambda\phi_r = \phi_{rrr} + (n-1)\left[\frac{\cos r\phi_{rr}}{\sin r} - \frac{\phi_r}{\sin^2 r}\right].$$

Taking limit as $r \to 0$, we have

$$\lim_{r \to 0} \phi_{rrr} = -\frac{(n-1)}{n} \lim_{r \to 0} \frac{\partial g}{\partial r}.$$

Combining with (2.8) yields

$$\tag{2.10} \begin{aligned} \lim_{r \to 0} \frac{\partial g}{\partial r} &= \lim_{r \to 0} \phi_{rrr} \\ &= \lim_{r \to 0} \left(\frac{\cos r \phi_{rr}}{\sin r} - \frac{\phi_r}{\sin^2 r} \right) \\ &= 0. \end{aligned}$$

Finally, we consider the function

$$\frac{\partial^2 g}{\partial r^2} = \lambda \phi_{rr} + n \left(\frac{\cos r \phi_{rrr}}{\sin r} - \frac{2\phi_{rr}}{\sin^2 r} + \frac{2\phi_r \cos r}{\sin^3 r} \right),$$

as $r \to 0$, this gives

$$\tag{2.11} \begin{aligned} \lim_{r \to 0} \frac{\partial^2 g}{\partial r^2} &= -\frac{\lambda^2}{n} \phi(0) \\ &\quad + n \lim_{r \to 0} \left(\frac{\cos r \phi_{rrr}}{\sin r} - \frac{2\phi_{rr}}{\sin^2 r} + \frac{2\phi_r \cos r}{\sin^3 r} \right) \\ &= -\frac{\lambda^2}{n} \phi(0) + n \lim_{r \to 0} \left[\frac{\partial^4 \phi}{\partial r^4} - \frac{2\phi_{rrr}}{\sin r \cos r} \right. \\ &\quad \left. + 4 \frac{\phi_{rr}}{\sin^2 r} - 4 \frac{\cos r \phi_r}{\sin^3 r} - 2 \frac{\phi_r}{\sin r \cos r} \right] \\ &= -\frac{\lambda^2}{n} \phi(0) + 2\lambda \phi(0) + n \lim_{r \to 0} \frac{\partial^4 \phi}{\partial r^4} \\ &\quad - \frac{2\lambda^2}{n} \phi(0) - 2 \lim_{r \to 0} \frac{\partial^2 g}{\partial r^2}. \end{aligned}$$

Therefore

$$3 \lim_{r\to 0} \frac{\partial^2 g}{\partial r^2} = -\left(\frac{3\lambda^2}{n} - 2\lambda\right)\phi(0) + n \lim_{r\to 0} \frac{\partial^4 \phi}{\partial r^4}. \tag{2.12}$$

However differentiating (2.9) twice yields

$$-\lambda\phi_{rr} = \frac{\partial^4 \phi}{\partial r^4} + (n-1)\left[\frac{\cos r\phi_{rrr}}{\sin r} - \frac{2\phi_{rr}}{\sin^2 r} + \frac{2\cos r\phi_r}{\sin^3 r}\right],$$

hence

$$\frac{\lambda^2}{n}\phi(0) = \lim_{r\to 0}\frac{\partial^4\phi}{\partial r^4} + \frac{(n-1)}{n}\left[\lim_{r\to 0}\frac{\partial^2 g}{\partial r^2} + \frac{\lambda^2}{n}\phi(0)\right].$$

This together with (2.12) gives

$$\lim_{r\to 0}\frac{\partial^2 g}{\partial r^2} = -\frac{2\lambda(\lambda - n)}{n(n+2)}\phi(0). \tag{2.13}$$

But on a ball centered at p with radius $a \le \pi/2$ in S^n, the eigenvalues for Dirichlet boundary conditions are known [13] to be no less than n, hence the right hand side of (2.13) is nonpositive which is to be proved.

Proof of Theorem 4. It is obvious to see that the proof of Theorem 2 together with the computations in the proof of Theorem 3 implies Theorem 4. In fact, we only need to check that the eigenvalues on $B_p(a)$, $a \le \pi$, which appear in the expansion (2.5) with Neumann boundary condition satisfied

$$\lambda \ge n.$$

Lemma 1 asserts that we may assume the eigenfunctions are rotationally symmetric, in particular, they take constant value on $\partial B_p(a)$. Theorem 4 will indeed follow from the next lemma.

LEMMA 2. *Let M be a compact manifold with boundary ∂M. Suppose the Ricci curvature of M is bounded below by $(n-1)K > 0$. If any eigenfunction ϕ with eigenvalue λ satisfies*

$$\phi|_{\partial M} = \text{constant}$$

and

$$\left.\frac{\partial \phi}{\partial \nu}\right|_{\partial M} = 0,$$

then

$$\lambda \geq n.$$

In particular, if $M = B_p(a)$ with a rotationally symmetric metric and if ϕ is a rotationally symmetric eigenfunction, then $\lambda \geq n$.

Proof. Assume $\lambda < n$, we consider the function

$$h = |\nabla \phi|^2 + \frac{\lambda}{n}\phi^2.$$

Then

$$\begin{aligned}\frac{1}{2}\Delta h &= \sum_{i,j} \phi_j \phi_{jii} + \sum_{i,j} \phi_{ji}^2 + \frac{\lambda}{n}|\nabla\phi|^2 - \frac{\lambda^2}{n}\phi^2 \\ &\geq -\lambda|\nabla\phi|^2 + (n-1)|\nabla\phi|^2 + \frac{(\Delta\phi)^2}{n} + \frac{\lambda}{n}|\nabla\phi|^2 - \frac{\lambda^2}{n}\phi^2 \\ &= \left((n-1) - \frac{n-1}{n}\lambda\right)|\nabla\phi|^2 \\ &\geq 0\end{aligned}$$

which implies the maximum of h occurs on ∂M, say x_0. By the Hopf maximum principle, we have

$$\begin{aligned}0 < \frac{\partial h}{\partial \nu}(x_0) &= 2\left(\sum_j \phi_j \phi_{j\nu}(x_0) + \frac{\lambda}{n}\phi\phi_\nu(x_0)\right) \\ &= 2\sum_j \phi_j\phi_{j\nu}|_{x_0}\end{aligned}$$

However $\phi|_{\partial M} =$ constant and $\phi_\nu|_{\partial M} = 0$ means $\phi_j(x_0) = 0$ for all j. This contradicts the assumption.

Proof of Theorem 5. Since the heat kernel for the compact manifold S^n is the same as the kernel with Neumann condition $B_p(\pi)$ (see [3]), by the proof of Theorem 4 this implies $\bar{K}'' \geq 0$, hence Proposition 1 can be applied. However if the image of M in $S^{n+\ell}$ does not contain the antipodal point p' of p, then the transplanted function $\bar{K}$ may not be C^2 in M. We will show that this does not create any problem for the purpose of applying Proposition 1.

Let $a = \sup_{x \in M} r_p(x)$ be the extrinsic diameter of M at p. Clearly $M \subseteq B_p(a) =$ the ball of radius a centered at p in $S^{n+\ell}$. Equation (1.1) of Proposition 1 asserts that for any $y \in B_p(a-\epsilon) \cap M = D(a-\epsilon)$.

$$(2.14)\quad \bar{K}(r_p(y), t) - K(p, y, t) = \int_0^t \int_{\partial D(a-\epsilon)} \frac{\partial \bar{K}}{\partial \nu_z}(r_p(z), s) K(y, z, t-s)\, dz\, ds + \int_0^t \int_{\partial D(a-\epsilon)} \bar{K}(r_p(z), s) \frac{\partial K}{\partial \nu_z}(y, z, t-s)\, dz\, ds$$

where we use $D(a-\epsilon)$ instead of M. We will show that the right hand side of (2.14) converges to zero as $\epsilon \to 0$. Since for any $z \in \partial D(a)$ is a supremum point for r_p,

$$\frac{\partial \bar{K}}{\partial \nu_z}(r_p(z), s) = \bar{K}_r(r_p(z), s) \cdot \frac{\partial r_p}{\partial \nu_z}$$

tends to 0 as $\epsilon \to 0$. Therefore the first term on the right hand side of (2.14) vanishes. On $\partial D(a-\epsilon)$, the transplanted function $\bar{K}$ takes the constant value $\bar{K}(a-\epsilon, s)$, hence the second term

$$\begin{aligned}\int_0^t \int_{\partial D(a-\epsilon)} \bar{K}(r_p(z), s) \frac{\partial K}{\partial \nu_z}(y, z, t-s)\, dz\, ds &= \int_0^t \bar{K}(a-\epsilon, s) \int_{\partial D(a-\epsilon)} \frac{\partial K}{\partial \nu_z}(y, z, t-s)\, dz\, ds \\ &= \int_0^t \bar{K}(a-\epsilon, s) \int_{D(a-\epsilon)} \Delta_z K(y, z, t-s)\, dz\, ds\end{aligned}$$

which tends to

$$\int_0^t \bar{K}(a, s) \int_M \Delta_z K(y, z, t - s) dz\, ds = 0$$

by the compactness of M. Since y is an arbitrary point in $D(a - \epsilon)$ for any ϵ, this proves the assertion for any $y \notin \partial D(a)$. However by continuity of $\bar{K}$ and K on M, the theorem follows.

3. Applications. We will discuss some of the applications of the comparison theorems (1 to 5). Some of the consequences are the mean-value inequalities which was proved by Michael and Simon [10] for the case of M in $\mathbf{R}^{n+\ell}$, and proved by Mori [11] for the case when M is in $S^{n+\ell}$.

COROLLARY 1. *Let M^n be a minimally immersed submanifold in $\bar{M}^{n+\ell} = \mathbf{R}^{n+\ell}$, $H^{n+\ell}$, or $S^{n+\ell}$. Suppose f is a nonnegative subharmonic function defined on M. If $p \in M$ and $D_p(a) = B_p(a) \cap M$, then*

$$f(p) \leq C^{-1}(n, a) \int_{\partial D(a)} f(x) dx$$

where

$$c(n, a) = \begin{cases} n\omega_n (\sin a)^{n-1} & \text{if } \bar{M} = S^{n+\ell} \\ n\omega_n a^{n-1} & \text{if } \bar{M} = \mathbf{R}^{n+\ell} \\ n\omega_n (\sinh a)^{n-1} & \text{if } \bar{M} = \mathbf{H}^{n+\ell} \end{cases}$$

and ω_n = volume of the unit n-ball in $\mathbf{R}^n$. When $\bar{M} = S^{n+\ell}$, one has the restriction that $a \leq \pi/2$.

Proof. Let $G(x, y)$ denote the Green's function with Dirichlet boundary condition on $D(a)$. Then

$$\begin{aligned} f(p) &= -\int_{D(a)} G(p, y) \Delta f(y) dy \\ &\quad - \int_{\partial D(a)} \frac{\partial G}{\partial \nu_y}(p, y) f(y) dy \\ &\leq - \int_{\partial D(a)} \frac{\partial G}{\partial \nu_y}(p, y) f(y) dy \end{aligned}$$

by the nonnegativity of G and the subharmonicity of f. We claim that if $\bar{G}(r_p(y))$ denote the Green's function on $B(a)$ the ball centered at 0 (or $p =$ north pole) with radius a in $\mathbf{R}^n(\mathbf{H}^n$ or $S^n)$, then

$$\frac{\partial G}{\partial \nu_y}(p, y) \geq \frac{\partial \bar{G}}{\partial \nu_y}(p, y)$$

for $y \in \partial D(a)$. Indeed, since the Green's functions are given by

$$G(p, y) = \int_0^\infty H(p, y, t)dt$$

and

$$\bar{G}(r_p(y)) = \int_0^\infty \bar{H}(r_p(y), t)dt$$

and $\bar{H} \geq H$ with both

$$\begin{aligned} \bar{H}(r_p(y), t) &= H(p, y, t) \\ &= 0 \quad \text{for} \quad y \in \partial D(a), \end{aligned}$$

the claim follows. Together with (3.1), we have

$$\begin{aligned} f(p) &\leq -\int_{\partial D(a)} \frac{\partial \bar{G}}{\partial \nu_y}(p, y) f(y)dy \\ &= -\int_{\partial D(a)} \bar{G}_r(r(y)) \frac{\partial r}{\partial \nu_y} f(y)dy \\ &\leq -\int_{\partial D(a)} \bar{G}_r(r(y)) f(y)dy \end{aligned}$$

since $|\nabla r| \leq 1$ and $\bar{G}_r \leq 0$. By the fact that $\bar{G}$ is rotationally symmetric

$$f(p) \leq -\bar{G}_r(r(y)) \int_{\partial D(a)} f(y)dy,$$

and the constant $C(a, n)$ can be obtained by explicit computation of $\bar{G}_r$.

Remark. A consequence of Corollary 1 is the a estimate on the volume of $D(a)$ from below. Namely, if we let $f \equiv 1$, then Corollary 1 gives a lower bound on the area of $\partial D(r)$. Integrating the inequality from 0 to a yields: The volume of $D(a)$ is greater than or equal to the volume of $B(a)$ in $\mathbf{R}^n(\mathbf{H}^n$ or $S^n)$. In case of $\bar{M} = S^n$, a is restricted to be no greater than $\pi/2$.

COROLLARY 2. *Let M be a minimally immersed submanifold of $S^{n+\ell}$, then $D_p(a) = M \cap B_p(a)$ has volume no less than the ball $B_p^n(a)$ in S^n, for all $a \leq \pi$. When M is compact, the volume of M is no less than the volume of S^n.*

Proof. We consider the kernel $K(p, y, t)$ on $D_p(a)$. Since

$$K(p, y, t) = \Sigma\, e^{-\lambda_i t}\phi_i(p)\phi_i(y)$$

and

$$\bar{K}(r_p(y), t) = \Sigma\, e^{-\bar{\lambda}_i t}\bar{\phi}_i(p)\bar{\phi}_i(y)$$

where λ_i, ϕ_i and $\bar{\lambda}_i$, $\bar{\phi}_i$ are eigenvalues and eigenfunctions of $D_p(a)$ and $B_p(a)$ respectively. By Theorem 4, we have

$$\Sigma\, e^{-\lambda_i t}\phi_i(p)\phi_i(y) \leq \Sigma\, e^{-\bar{\lambda}_i t}\bar{\phi}_i(p)\bar{\phi}_i(y). \tag{3.3}$$

However, the first eigenvalue for Neumann boundary condition is zero with normalized eigenfunction $V^{-1/2}$, by taking $t \to \infty$ in (3.3) we obtain

$$\frac{1}{V(D(a))} \leq \frac{1}{V(B(a))},$$

which is to be shown.

COROLLARY 3. *Let $M \hookrightarrow \bar{M}$ be a minimal immersed submanifold. Suppose D is a C^2 compact domain in M. We define the outer radius of D by*

$$a = \inf_{p \in D} \sup_{z \in D} r_p(z).$$

If $\bar{M} = S^{n+\ell}$, we assume $a \leq \pi/2$, otherwise $0 < a < \infty$, then

$$\lambda_1(D) \geq \bar{\lambda}_1(B(a))$$

where $B(a)$ is the ball of radius a in $\mathbf{R}^n$ ($\mathbf{H}^n$ or S^n) respective to $\bar{M} = \mathbf{R}^{n+\ell}$ ($\mathbf{H}^{n+\ell}$ or $S^{n+\ell}$). Equality holds iff M is totally geodesic in $\bar{M}$ and $D = B(a)$.

Proof. By Theorem 1 and 3, it is clearly that if we compare the heat kernel H and $\bar{H}$ and let $t \to \infty$, we obtain a comparison for λ_1. However for the sake of the equality case, we will adopt another proof.

Let $p \in D$, be the point in D which realized as the center of $D = D_p(a)$. Suppose $\bar{\phi}(r)$ is the first eigenfunction on $B(a)$ in $\mathbf{R}^n$ ($\mathbf{H}^n$ or S^n). It is unambiguous to write $\bar{\phi}$ as a function of r because it is known that $\bar{\phi}$ is rotationally symmetric. We rewrite $\bar{\phi}$ as a function of $s = r^2$ ($s = \cosh r$ or $s = -\cos r$). Then

$$\begin{aligned}\Delta\bar{\phi} &= \bar{\phi}''|\nabla s|^2 + \bar{\phi}'\Delta s \\ &= \bar{\phi}''|\nabla s|^2 + \bar{\phi}'\bar{\Delta} s\end{aligned}$$

where $\bar{\phi}'$ and $\bar{\phi}''$ are first and second derivatives of $\bar{\phi}$ with respect to s, and $\bar{\Delta}$ is the Laplacian on $\mathbf{R}^n$ ($\mathbf{H}^n$ or S^n). Since

$$\begin{aligned}|\nabla s|^2 &= (2r|\nabla r|)^2(\sinh^2 r|\nabla r|^2 \text{ or } \sin^2 r|\nabla r|^2) \\ &\le |\bar{\nabla} s|^2,\end{aligned}$$

if $\bar{\phi}'' \geq 0$, then

$$\Delta\bar{\phi} \le \bar{\Delta}\bar{\phi} = -\bar{\lambda}_1\bar{\phi}.$$

For simplicity we will only demonstrate the case when $\bar{M} = \mathbf{R}^{n+\ell}$. The proofs of the other cases follow roughly along the same idea. Consider

$$\begin{aligned}-\lambda\bar{\phi} &= \bar{\Delta}\bar{\phi} \\ &= 4s\bar{\phi}'' + 2n\bar{\phi}'.\end{aligned} \tag{3.4}$$

To check $\phi'' > 0$ at $s = 0$, we look at the equation

$$\begin{aligned}\lim_{s\to 0} 4\bar{\phi}'' &= \lim_{s\to 0}\frac{-\lambda\bar{\phi} - 2n\bar{\phi}'}{s}. \\ &= \lim_{r\to 0}\frac{-\lambda\bar{\phi}}{r^2} - \frac{n\bar{\phi}_r}{r^3}.\end{aligned}$$

However by the proof of Theorem 1 this is

$$= \frac{2\lambda^2}{n(n+2)} \bar{\phi}(0)$$

$$> 0.$$

Now let s_0 be the first $s \geq 0$ such that $f''(s_0) = 0$. Because $f''(0) > 0$, this implies at s_0, the function f' has either a maximum or a point of inflection. On the other hand, if we differentiate Equation (3.4), we have

$$-\lambda\bar{\phi}' = 4s\bar{\phi}^{(3)} + 2(n+2)\bar{\phi}''.$$

At s_0, we see that

$$0 < -\lambda\bar{\phi}'(s_0) = 4s_0\bar{\phi}^{(3)}(s_0)$$

because

$$2r\bar{\phi}' = \bar{\phi}_r < 0$$

for the first eigenfunction $\bar{\phi}$. This is a contradiction. Therefore $\Delta\bar{\phi} \leq -\bar{\lambda}_1\bar{\phi}$.

To compare the first eigenvalues, first we consider the comparison

$$\lambda_1(D) \geq \lambda_1(D(a))$$

since $D(a) \supseteq D$. Equality holds iff

$$D = D(a),$$

hence we have reduced to the case when $D = D(a)$. An observation of Barta gives (see [4])

$$\begin{aligned}
\lambda_1(D(a)) &= -\frac{\Delta\phi}{\phi}(x) \\
&= -\frac{\Delta\bar{\phi}}{\bar{\phi}}(x) + \left.\frac{\phi\Delta\bar{\phi} - \bar{\phi}\Delta\phi}{\phi\bar{\phi}}\right|_x \\
&\geq \bar{\lambda}_1 + \left.\frac{\phi\Delta\bar{\phi} - \bar{\phi}\Delta\phi}{\phi\bar{\phi}}\right|_x .
\end{aligned}$$

However since both ϕ and $\bar{\phi}$ are positive and also

$$\int_{D(a)} (\phi \Delta \bar{\phi} - \bar{\phi} \Delta \phi) = 0$$

by the boundary conditions on ϕ and $\bar{\phi}$, the function $\phi \Delta \bar{\phi} - \bar{\phi} \Delta \phi$ must change sign. Therefore

$$\begin{aligned} \lambda_1(D(a)) &\geq \bar{\lambda}_1 + \sup_{x \in D(a)} \left[\frac{\phi \Delta \bar{\phi} - \bar{\phi} \Delta \phi}{\phi \bar{\phi}} \right] \\ &\geq \bar{\lambda}_1 \end{aligned}$$

as asserted. Equality holds iff

$$\begin{aligned} \phi \Delta \bar{\phi} &= \bar{\phi} \Delta \phi \\ &= -\lambda_1(D(a)) \phi \bar{\phi} \\ &= -\bar{\lambda}_1 \phi \bar{\phi}. \end{aligned}$$

On the other hand, the proof of

$$\Delta \bar{\phi} \leq -\bar{\lambda}_1 \bar{\phi}$$

implies that equality holds iff

$$|\nabla r| = 1,$$

which is equivalent to the condition that $D(a)$ is a minimal cone. However by the assumption that M is C^2, this implies $D(a) = B(a)$. Analytic continuation then asserts that M is totally geodesic.

COROLLARY 4. *Let $M \hookrightarrow \bar{M}$ be a minimally immersed submanifold of $\bar{M} = \mathbf{R}^{n+\ell}$ or $\mathbf{H}^{n+\ell}$. Suppose D is a compact domain in M, and if $\{\lambda_k\}_{k=1}^{\infty}$ are the eigenvalues of Δ on D with Dirichlet boundary condition, then*

$$\lambda_k^{n/2} \geq \frac{(4\pi)^{n/2}}{e} \cdot \frac{k}{V(D)}$$

for all $k \geq 1$.

Proof. By Theorem 1, the heat kernel on D satisfies

$$H(x, x, t) \le \bar{H}_a(0, 0, t)$$

where

$$a = \sup_{z \in D} r_x(z) \quad \text{for any} \quad x \in D.$$

However it is known that [7]

$$\bar{H}_a(0, 0, t) \le (4\pi t)^{-n/2}.$$

This together with the eigenfunction expansion of $H(x, x, t)$ yield

$$\Sigma\, e^{-\lambda_i t} \le V(D)(4\pi t)^{-n/2}.$$

Taking $t = 1/\lambda_k$ and using the fact that $\lambda_i \le \lambda_k$ for $i \le k$, we have

$$\begin{aligned} ke^{-1} &\le \sum_{i=1}^{k} e^{-\lambda_i/\lambda_k} \\ &\le V(D)\left(\frac{\lambda_k}{4\pi}\right)^{n/2} \end{aligned}$$

which proves the Corollary.

We will consider the case when M^n is a minimally immersed submanifold of $S^{n+\ell}$. In order to draw conclusions from Theorem 5, we need to estimate the trace of the heat kernel on S^n.

LEMMA 3. *Let $\bar{K}(x, y, t)$ be the heat kernel on S^n, then there exists a constant C_n depending only on n with*

$$C_n \le \frac{n^{n/2} e\Gamma(n/2, 1)}{2}$$

such that

$$\begin{aligned} \operatorname{Tr} \bar{K}(t) &= \int_{S^n} \bar{K}(x, x, t)dx \\ &\le 1 + (n+1)e^{-nt} + C_n t^{-1} e^{-nt} \end{aligned}$$

for all $t \ge 1$.

Proof. It is known that [1] the eigenvalues of S^n are given by

$$\bar{\lambda}_k = k(n + k - 1)$$

with multiplicities

$$m(k) = \frac{(n + k - 2)(n + k - 3) \cdots (n + 1)n}{k}(n + 2k - 1).$$

Therefore

$$\sum_{i=0}^{\infty} e^{-\bar{\lambda}_i t} = 1 + (n + 1)e^{-nt} + \sum_{i=2}^{\infty} m(k)e^{-\bar{\lambda}_k t}.$$

Now we claim that $m(k) \leq (n/2)k^{n-2}(n + 2k - 1)$. We will show this by induction on k. Clearly the claim is true for $k = 2$. For general $k > 2$, we consider

$$(3.7) \qquad \frac{m(k)}{(n + 2k - 1)} = \frac{n + k - 2}{k}\left(\frac{m(k - 1)}{n + 2k - 1}\right)$$

$$\leq \frac{n + k - 2}{k}\left(\frac{n(k - 1)^{n-2}}{2}\right)$$

by induction hypothesis. We need to show that

$$(3.8) \qquad (n + k - 2)(k - 1)^{n-2} \leq k^{n-1}, \qquad \text{for all } k.$$

When $n = 2$, this obviously holds. We will show the validity of (3.8) by induction on n. Differentiating (3.8) as a function of k, we obtain for the left hand side

$$(3.9) \qquad (n - 2)(k - 1)^{n-3}(n + k - 2) + (k - 1)^{n-2}$$

To see this we observe that the left hand side of (3.8) can be written as

$$(3.9) \quad ((n - 1) + (k - 1))(k - 1)^{n-2} = (k - 1)^{n-1} + (n - 1)(k - 1)^{n-2}$$

While the right hand side of (3.8) is

$$(3.10) \quad k^{n-1} = ((k - 1) + 1)^{n-1} > (k - 1)^{n-1} + (n - 1)(k - 1)^{n-2}.$$

$$= (n + k - 3)(k - 1)^{n-3}(n - 1)$$

and for the right hand side we have

$$(n-1)k^{n-2}. \tag{3.10}$$

Clearly as a function of k, (3.8) holds at $k=1$, but (3.9), (3.10) and induction hypothesis on n shows that the derivative of the function on the left hand side of (3.8) is no greater than the derivative of the function on the right hand side. This shows (3.8) is valid, hence proves our claim. Therefore

$$\begin{aligned}\sum_{k=2}^{\infty} m(k)e^{-\bar\lambda_k t} &\le \frac{n}{2}\sum_{k=2}^{\infty} k^{n-2}(n+2k-1)e^{-\bar\lambda_k t}\\ &\le \frac{n}{2}\sum_{k=2}^{\infty} (\bar\lambda_k)^{(n-2)/2}(n+2k-1)e^{-\bar\lambda_k t}.\end{aligned}$$

However for $t \ge 1$, one checks easily that

$$(\bar\lambda_k)^{(n-2)/2}(n+2k-1)e^{-\bar\lambda_k t}$$

is a decreasing function of $k \ge 1$, hence by the integral test, the right hand side of (3.11) is less than

$$\frac{n}{2}\int_1^{\infty} [x^2+(n-1)x]^{(n-2)/2}(2x+n-1)e^{-[x^2+(n-1)x]t}\,dx$$

for $t \ge 1$. Substituting $u = x^2+(n-1)x$, we have

$$\begin{aligned}\sum_{k=2}^{\infty} m(k)e^{-\bar\lambda_k t} &\le \frac{n}{2}\int_n^{\infty} u^{(n-2)/2}e^{-ut}\,du\\ &= \frac{n}{2}t^{-n/2}\int_{nt}^{\infty} u^{(n-2)/2}e^{-u}\,du\\ &\le \frac{n}{2}t^{-n/2}e\cdot\Gamma\left(\frac{n}{2},1\right)e^{-nt}(nt)^{(n-2)/2}\\ &= \frac{n^{n/2}e\Gamma(n/2,1)}{2}e^{-nt}t^{-1}\end{aligned}$$

for $t \ge 1$.

It is known that an immersion of M^n into $S^{n+\ell}$ is minimal iff $M \subset \bar{M}$

$$\Delta\phi = -n\phi$$

for any coordinate function ϕ in $\mathbf{R}^{n+\ell+1}$. In particular, if M^n is a compact minimal submanifold of $S^{n+\ell}$ then $n = \lambda_k$ is the k^{th} eigenfunction of M. In addition, if the immersion of M into $S^{n+\ell}$ is a maximal dimension, i.e. M does not contain in any hyperplane through the origin of $\mathbf{R}^{n+\ell+1}$, then the multiplicity of $n = \lambda_k$ is at least $n + \ell + 1$.

THEOREM 6. *Let $M^n \to S^{n+\ell}$ be a minimal immersion of the compact manifold M of maximal dimension. Suppose the spectrum of M is ordered by magnitude and if $n = \lambda_k$, then*

$$k \leq \frac{V(M)}{V(S^n)}[e^t + n + 1 + nC_n t^{-1}] - e^t$$

for any $t \geq 0$.

Proof. By Theorem 5, the heat kernel $K(x, x, t) = \Sigma_{i=0}^{\infty} e^{-\lambda_i t}\phi_i^2(x)$ satisfies

$$\sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i^2(x) \leq \bar{K}(x, x, t).$$

However it is easy to see that $\bar{K}(x, x, t)$ is a constant function since S^n is a homogeneous manifold, therefore

$$\int_{S^n} \bar{K}(x, x, t) = V(S^n)\bar{K}(x, x, t). \tag{3.13}$$

Together with Lemma 3, this gives

$$\begin{aligned} \sum_{i=0}^{\infty} e^{-\lambda_i t} &\leq \int_M K(x, x, t)dx \\ &\leq \int_M \bar{K}(x, x, t)dx \\ &= \frac{V(M)}{V(S^n)}\bar{K}(x, x, t) \\ &\leq \frac{V(M)}{V(S^n)}[1 + (n + 1)e^{-nt} + C_n t^{-1} e^{-nt}]. \end{aligned} \tag{3.14}$$

Substituting $t/n = t/\lambda_k$ in terms of t into (3.14), we obtain

$$1 + \sum_{i=1}^{\infty} e^{-\lambda_i t/\lambda_k} \leq \frac{V(M)}{V(S^n)}[1 + (n+1)e^{-t} + nC_n t^{-1} e^{-t}]$$

for $t \geq 0$.

On the other hand since, $\lambda_i/\lambda_k \leq 1$ for $i \leq k$, we have

$$(3.15) \qquad 1 + ke^{-t} \leq \frac{V(M)}{V(S^n)}[1 + (n+1)e^{-t} + nC_n t^{-1} e^{-t}].$$

We can obtain the theorem by multiplying (3.15) by e^t.

COROLLARY 5. *Let M be a compact minimally immersed submanifold of $S^{n+\ell}$. Suppose the immersion is of maximal dimension, then there exists a positive constant*

$$B_n < 2n + 3 + 2\exp(2nC_n)$$

such that

$$V(M) > \left(1 + \frac{2\ell - 1}{B_n}\right) V(S^n)$$

where ℓ is the co-dimension of the immersion in $S^{n+\ell}$.

Proof. We simply consider Theorem 6. Since the immersion is maximal dimension, the multiplicity of n is at least $n + \ell + 1$, hence $k \geq n + \ell + 1$. Letting $t = 2nC_n$, we have

$$n + \ell + 1 < \frac{V(M)}{V(S^n)}\left(\frac{1}{2}B_n\right) - e^{2nC_n},$$

hence

$$\frac{1}{2}B_n + \ell - \frac{1}{2} < \frac{V(M)}{2V(S^n)} \cdot B_n.$$

Remark. The case when $n = 2$ and M is homeomorphic to S^2 has been studied in [2], where Calabi showed that $V(M)$ has to be an integral multiple of 2π, and the integer must not be less than $\frac{1}{4}[(n + \ell + 1)^2 - 1]$. In particular, $n + \ell + 1$ has to be an odd number.

The next corollary describes the way that a minimal submanifold immerse into $S^{n+\ell}$. It shows that any minimal immersion cannot be too pathological.

COROLLARY 6. *Let M be a compact manifold. Suppose $M^n \hookrightarrow S^{n+\ell}$ is a minimal immersion of maximal dimension. Then for any hyperplane H which passes through the origin of $\mathbf{R}^{n+\ell+1}$ cannot divide M into more than*

$$\left(\frac{V(M)}{V(S^n)} - 1\right)\frac{B_n}{2} - \ell + \frac{5}{2}$$

components. In particular, if the minimal cone $C(M)$ over M^n in $\mathbf{R}^{n+2}$ is area minimizing and if $\ell = 1$, then the number of components of $M - H$ is no more than

$$\left(\frac{V(S^{n+1})}{V(S^n)} - 1\right)\frac{B_n}{2} + \frac{3}{2}$$

Proof. Since the immersion is of maximal dimension, the multiplicity of n is at least $n + \ell + 1$. If we denote k to be the last $\lambda_k = n$, then applying Theorem 6 and setting $t = 2nC_n$, we have

$$k \leq \frac{V(M)}{V(S^n)}\left(\frac{B_n}{2}\right) - e^{2nC_n}.$$

However since $\lambda_{k-n-\ell} = \lambda_{k-n-\ell+1} = \cdots = \lambda_k = n$, by the Courant nodal domain theorem (see [6] and [5]), the number of nodal domains for the $(k - n - \ )^{th}$ eigenfunction cannot exceed

$$k - n - \ell + 1 \leq \frac{V(M)}{V(S^n)} \cdot \frac{B_n}{2} - \frac{B_n}{2} - \ell + \frac{5}{2}.$$

On the other hand, since $H \cap M$ are the nodal sets of the coordinate functions, the number of components of $M - H$ is the same as the number of nodal domains of the $(k - n - \ell)^{th}$ eigenfunction. This proves the first part of the corollary. The second part follows from the fact that area minimizing cones in $\mathbf{R}^{n+\ell+1}$ has their volume bounded above by

$$\frac{V(M^n)}{2} = V(C^{n+1}(M) \cap B_0^{n+2}(1))$$

$$\leq \frac{V(S^{n+1})}{2}$$

COROLLARY 7. *Let M be a compact minimal submanifold of $S^{n+\ell}$. The k^{th} nonzero eigenvalue of M satisfies*

$$\lambda_k^{n/2} \geq A_n\left(\frac{e+k}{V(M)} - \frac{1}{V(S^n)}\right)$$

where A_n is a computable constant depending only on n.

Proof. By Theorem 5,

$$1 + \sum_{i=1}^{\infty} e^{-\lambda_i t} \leq \frac{V(M)}{V(S^n)}\left(1 + \sum_{i=1}^{\infty} e^{-\bar{\lambda}_i t}\right)$$

where $\bar{\lambda}_i$ are eigenvalues of S^n. However it is well known that

$$\sum_{i=1}^{\infty} e^{-\bar{\lambda}_i t} \leq (\text{Constant})t^{-n/2}$$

where the constant of course depends on n. Hence, setting $t = 1/\lambda_k$,

$$1 + ke^{-1} \leq \frac{V(M)}{V(S^n)}(1 + (\text{Constant})\lambda_k^{n/2}).$$

The corollary follows by letting $A_n = V(S^n)/\text{Constant}$.

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REFERENCES

[1] M. Berger, P. Gauduchon, and E. Mazet, *Le Spectre d'une Variété Riemannienne*, Lecture Notes in Math., **194**, Springer, Berlin, 1971.

[2] E. Calabi, Minimal immersions of surfaces in euclidean spheres, *J. Diff. Geom.*, **1** (1967), 111-125.

[3] J. Cheeger and S. T. Yau, A lower bound for the heat kernel, *Comm. Pure Appl. Math.*, **34** (1981), 465-480.

[4] S. Y. Cheng, Eigenfunctions and eigenvalues of Laplacian, *Proc. Symp. Pure Math.*, **27** (1975), 185-193.

[5] ———, Eigenfunctions and nodal sets, *Comm. Math. Helv.*, **51** (1976), 43-55.

[6] R. Courant and D. Hilbert, *Methods of Mathematical Physics* 1, *Interscience.*

[7] H. Donnelly and P. Li, Lower bounds for the eigenvalues of negatively curves manifolds, *Math. Zeit.*, **172** (1980), 29-40.

[8] P. Li, Thesis (1979), U. C. Berkeley.

[9] ———, Eigenvalues estimates on homogeneous manifolds, *Comm. Math. Helv.*, **55** (1980), 347-363.

[10] J. H. Michael and L. Simon, Sobolev and mean-value inequalities on generalized submanifolds of $\mathbf{R}^n$, *Comm. Pure Appl. Math.*, **26** (1973), 361-379.

[11] H. Mori, Remarks on mean-value inequalities on m-dimensional submanifolds of S^n, *Kodai Math. J.*, **2** (1979), 243-253.

[12] M. H. Protter and H. E. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1967.

[13] R. C. Reilly, Applications of the Hessian operator in a Riemannian manifold, *Indiana U. Math. J.* 26, **3** (1977), 459-472.