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On the Isoperimetric Inequality for Minimal Surfaces.

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For any compact minimal submanifold of dimension k in $\mathbb{R}^n$, it is known that theie exists a constant $\bar{C}_k$ depending only on k, such that

$$V(\partial M)^{k/k-1} \geqslant \bar{C}_k V(M)\,,$$

where $V(\partial M)$ and $V(M)$ are the $(k-1)$-dimensional and k-dimensional volumes of ∂M and M respectively. We refer to [6] for a more detailed reference on the inequality. An open question [6] is to determine the best possible value of $\bar{C}_k$. When M is a bounded domain in $\mathbb{R}^k \subseteq \mathbb{R}^n$, the sharp constant is given by

$$C_k = \frac{V(\partial D)^{k/k-1}}{V(D)}\,, \tag{1}$$

where D is the unit disk in $\mathbb{R}^k$. One speculates that C_k is indeed the sharp constant for general minimal submanifolds in $\mathbb{R}^n$.

In the case $k = 2$, $C_2 = 4\pi$, it was proved [1] (see [7]) that if Σ is a simplyconnected minimal surface in $\mathbb{R}^n$, then

$$l(\partial\Sigma)^2 \geqslant 4\pi A(\Sigma)\,, \tag{2}$$

where $l(\partial\Sigma)$ and $A(\Sigma)$ denote the length of $\partial\Sigma$ and the area of Σ respectively.

In 1975, Osserman-Schiffer [5] showed that (2) is valid with a strict inequality for doubly-connected minimal surfaces in $\mathbb{R}^3$. Feinberg [2] later generalized this to doubly-connected minimal surfaces in $\mathbb{R}^n$ for all n. So far, the sharp constant, (1), has been established for minimal surfaces with topological restrictions.

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The purpose of this article is to prove the isoperimentric inequality (2) for those minimal surfaces in $\mathbb{R}^n$ whose boundaries satisfy some connectedness assumption (see Theorem 1). This has the advantage that the topology of the minimal surface itself can be arbitrary. An immediate consequence of Theorem 1 is a generalization of the theorem of Osserman-Schiffer. In fact, Theorem 2 states that any minimal surface (not necessarily doubly-connected) in $\mathbb{R}^3$ whose boundary has at most two connected components must satisfy inequality (2).

Finally, in Theorem 3, we also generalize the non-existence theorem of Hildebrandt [3], Osserman [4], and Osserman-Schiffer [5] to higher co-dimension.

1. – Isoperimetric inequality.

DEFINITION. The boundary $\partial\Sigma$ of a surface Σ in $\mathbb{R}^n$ is weakly connected if there exists a rectangular coordinate system $\{x^\alpha\}_{\alpha=1}^n$ of $\mathbb{R}^n$, such that, for every affine hypersurface $H^{n-1} = \{x^\alpha = \text{const.}\}$ in $\mathbb{R}^n$, H does not separate $\partial\Sigma$. This means, if $H \cap \partial\Sigma = \phi$, then $\partial\Sigma$ must lie on one side of H.

In particular, if $\partial\Sigma$ is a connected set, then $\partial\Sigma$ is weakly connected.

THEOREM 1. Let Σ be a compact minimal surface in $\mathbb{R}^n$. If $\partial\Sigma$ is weakly connected, then

$$l(\partial\Sigma)^2 \geqslant 4\pi A(\Sigma) .$$

Moreover, equality holds iff Σ is a flat disk in some affine 2-plane of $\mathbb{R}^n$.

PROOF. Let us first prove the case when $\partial\Sigma$ is connected. By translation, we may assume that the center of mass of $\partial\Sigma$ is at the origin, i.e.,

$$\int_{\partial\Sigma} x^\alpha = 0 , \quad \text{for all } 1 \leqslant \alpha \leqslant n . \tag{3}$$

By the assumption on the connectedness of $\partial\Sigma$, any coordinate system $\{x^\alpha\}_{\alpha=1}^n$ satisfies the definition of weakly connectedness.

Let $X = (x^2, \ldots, x^n)$ be the position vector, then $|X|^2 = \sum_{\alpha=1}^n (x^\alpha)^2$ must satisfy

$$\Delta(|X|^2) = 4 , \tag{4}$$

due to the minimality assumption on Σ. Here Δ denotes the Laplacian on Σ with respect to the induced metric from $\mathbb{R}^n$. Integrating (4) over Σ, and applying the divergence theorem, we have

$$4A(\Sigma)=2\int_{\partial\Sigma}|X|\frac{\partial|X|}{\partial\nu}, \tag{5}$$

where $\partial/\partial\nu$ is the outward unit normal vector to $\partial\Sigma$ on Σ. Since $\partial|X|/\partial\nu\leqslant 1$, we have

$$2A(\Sigma)\leqslant\int_{\partial\Sigma}|X|\leqslant(\partial\Sigma)^{\frac{1}{2}}\int_{\partial\Sigma}(|X|^2)^{\frac{1}{2}}. \tag{6}$$

In order to estimate the right hand side of (6), we will estimate $\int_{\partial\Sigma}(x^\alpha)^2$ for each $1\leqslant\alpha\leqslant n$. By (3), the Poincaré inequality implies that

$$\int_{\partial\Sigma}(x^\alpha)^2\leqslant\frac{l(\partial\Sigma)^2}{4\pi^2}\int_{\partial\Sigma}\left(\frac{dx^\alpha}{ds}\right)^2, \tag{7}$$

where d/ds is differentiation with respect to arc-length. Combining with (6) yields

$$4\pi A(\Sigma)\leqslant l(\partial\Sigma)^{\frac{3}{2}}\left(\int_{\partial\Sigma}\left|\frac{dx}{ds}\right|^2\right)^{\frac{1}{2}}=l(\partial\Sigma)^2, \tag{8}$$

because (dX/ds) is just the unit tangent vector to $\partial\Sigma$.

Equality holds at (8), implies

$$\frac{\partial|X|}{\partial\nu}=1 \tag{9}$$

$$|X|=\text{constant}=R \tag{10}$$

and equality at (7). The latter implies that

$$x^\alpha=a_\alpha\sin\frac{2\pi s}{l(\partial\Sigma)}+b_\alpha\cos\frac{2\pi s}{l(\partial\Sigma)} \tag{11}$$

where a_α and b_α's are constants for all $1\leqslant\alpha\leqslant n$. By rotation, we may assume that

$$\begin{cases} X(0)=(R,0,0,\dots,0)\\ \dfrac{dX}{ds}(0)=(0,1,0,\dots,0), \end{cases} \tag{12}$$

because (10) implies that $\partial\Sigma$ lies on the sphere of radius R. Evaluating (11) at $s = 0$, we deduce that

$$b_1 = R\,, \qquad b_\alpha = 0 \qquad \text{for } 2 \leq \alpha \leq n$$

(13) and

$$a_2 = \frac{l(\partial\Sigma)}{2\pi}\,, \qquad a_\alpha = 0 \qquad \text{for } \alpha \neq 2\,.$$

On the other hand, summing over $1 \leq \alpha \leq n$ on (7), we derive

$$R^2 l(\partial\Sigma) = \int_{\partial\Sigma} |X|^2 = \left(\frac{l(\partial\Sigma)}{2\pi}\right)^2 l(\partial\Sigma)\,, \tag{14}$$

Hence

$$R = \frac{l(\partial\Sigma)}{2\pi}\,.$$

Combining with (13), (11) becomes

$$\begin{cases} x^1 = R\cos\left(\dfrac{s}{R}\right) \\ x^2 = R\sin\left(\dfrac{s}{R}\right) \end{cases} \tag{15}$$

and

$$x^\alpha \equiv 0 \qquad \text{for } 3 \leq \alpha \leq n\,.$$

This implies $\partial\Sigma$ is a circle on the x^1x^2-plane centered at the origin of radius R. Equation (9) shows that Σ is tangent to the x^1x^2-plane along $\partial\Sigma$. By the Hopf boundary lemma, this proves that Σ must be the disk spanning $\partial\Sigma$.

For the general case when $\partial\Sigma$ is not connected. Let $\partial\Sigma = \bigcup_{i=1}^{p} \sigma_i$, where σ_i's are connected closed curves. By the assumption on weakly connectedness, we may choose $\{x^\alpha\}_{\alpha=1}^n$ to be the appropriate coordinate system. For any fixed $1 \leq \alpha \leq n$, we claim that there exist translations A_i^α, $2 \leq i \leq p$, generated by vectors v_i^α perpendicular to $\partial/\partial x^\alpha$, such that the union of the set of translated curves $\{A_i^\alpha \sigma_i\}_{i=2}^p$ together with σ_1 form a connected set. We prove the claim by induction on the number of curves, p. When $p = 2$, we observe that since no planes of the form $x^\alpha =$ constant separates σ_1 and σ_2, this is equivalent to the fact that there exists a number x, such that the plane $\mathrm{H} = \{x^\alpha = x\}$ must intersect both σ_1 and σ_2. Let q_1 and q_2 be the points of intersection between H with σ_1 and σ_2 respectively.

Clearly one can translate q_2 along H to q_1. Denote this by A_2^α, and $\sigma_1 \cup A_2^\alpha \sigma_2$ is connected now. For general p, we consider the set of numbers defined by

$$y_i = \max \{x^\alpha|_{\sigma_i}\}\ .$$

Without loss of generality, we may assume $y_1 \leqslant y_2 \leqslant \ldots \leqslant y_p$. Now we claim that the set $\bigcup_{i=2}^{p} \sigma_i$ cannot be separated by hyperspaces of the form $\mathrm{H} = \{x^\alpha = \text{constant}\}$. If so, say $\mathrm{H} = \{x^\alpha = x\}$ separates $\bigcup_{i=2}^{p} \sigma_i$, then x must be in the range of $x^\alpha|_{\sigma_1}$. This is because $\bigcup_{i=1}^{p} \sigma_i$ cannot be separated hence $H \cap \sigma_1 \neq \emptyset$. On the other hand, since H separates $\bigcup_{i=2}^{p} \sigma_i$, this means there exists some σ_i, $2 \leqslant i \leqslant p$, lying on the left of H, hence $y_i < x \leqslant y_1$, for some $2 \leqslant i \leqslant p$, which is a contradiction. By induction, there exist translations, A_i^α, $3 \leqslant i \leqslant p$, perpendicular to $\partial/\partial x$ such that $\sigma = \sigma_2 \cup \left\{\bigcup_{i=3}^{p} A_i^\alpha \sigma_i\right\}$ is connected. However, $\bigcup_{i=1}^{p} \sigma_i$ is non-separable by $\mathrm{H} = \{x^\alpha = \text{constant}\}$ implies $\sigma_1 \cup \sigma$ is non-separable also. Hence, there exists a translation A^α perpendicular to $\partial/\partial x^\alpha$, such that $\sigma_1 \cup A_\sigma^\alpha$ is connected. The set $A = A_2, AA_3, AA_4, \ldots, AA_p$ gives the desired translations. Notice that since all translations are perpendicular to $\partial/\partial x^\alpha$, then

$$x^\alpha|_{\sigma_i} \equiv x^\alpha|_{A^\alpha \sigma_i}\ , \quad \text{for all } i\ . \tag{16}$$

By the connectedness of $\sigma^\alpha = \sigma_1 \cup A_2^\alpha \sigma_2 \cup \ldots \cup A_p^\alpha \sigma_p$: we can view σ^α as a Lipschitz curve in $\mathbb{R}^n$. Clearly

$$\int_{\sigma^\alpha} x^\alpha = \sum_{i=1}^{p} \int_{\sigma_i} x^\alpha = 0\ ,$$

hence the Poincaré inequality can be applied to yield

$$\sum_{i=1}^{p} \int_{\sigma_i} (x^\alpha)^2 = \int_{\sigma^\alpha} (x^\alpha)^2 \leqslant \frac{l(\partial\Sigma)^2}{4\pi^2} \int_{\sigma^\alpha} \left(\frac{dx^\alpha}{dx}\right)^2 = \frac{l(\partial\Sigma)^2}{4\pi^2} \sum_{i=1}^{p} \int_{\sigma_i} \left(\frac{dx^\alpha}{ds}\right)^2. \tag{17}$$

Summing over all $1 \leqslant \alpha \leqslant n$ and proceeding as the connected case we derived the inequality (8).

When equality occurs, we will show that $\partial\Sigma$ is actually connected, and hence by the previous argument it must be a circle and Σ must be a disk. To see this, we observe that (10) still holds on $\partial\Sigma$. In particular, we may assume that $X(0)$ is a point on σ_1, and (12) is valid. However, Poincaré inequality is now applied on σ^α instead of $\partial\Sigma$, therefore equation (11) only applies to the curve σ^α. On the other hand, since $X(0) \in \sigma_1$, and $\sigma^\alpha = \sigma_1 \cup \left\{\bigcup_{i=2}^{P} A_i^\alpha \sigma_i\right\}$, the argument concerning the coefficients a_α and b_α's is still valid. Equations (15) can still be concluded on each σ^α, hence on $\partial\Sigma$, by (17). This implies $\partial\Sigma$ is a circle, and the Theorem is proved.

THEOREM 2. Let Σ be a compact minimal surface in $\mathbb{R}^3$. If $\partial\Sigma$ consists of at most two components, then

$$l(\partial\Sigma)^2 \geqslant 4\pi A(\Sigma) .$$

Moreover, equality holds iff Σ is a flat disk in some affine 2-plane of $\mathbb{R}^3$.

PROOF. In view of Theorem 1, it suffices to prove that when $\partial\Sigma = \sigma_1 \cup \sigma_2$ has exactly two connected components and is not weakly connected, Σ must be disconnected into two components Σ_1 and Σ_2 with $\partial\Sigma_1 = \sigma_1$ and $\partial\Sigma_2 = \sigma_2$. Indeed, if this is the case, we simply apply Theorem 1 to Σ_1 and Σ_2 separately and derive

$$\begin{aligned} l(\partial\Sigma)^2 = &= \big(l(\sigma_1) + l(\sigma_2)\big)^2 \\ &> l(\sigma_1)^2 + l(\sigma_2)^2 \\ &\geqslant 4\pi\big(A(\Sigma_1) + A(\Sigma_2)\big) \\ &= 4\pi A(\Sigma) . \end{aligned}$$

In this case, equality will never be achieved for (2).

To prove the above assertion, we assume that $\partial\Sigma = \sigma_1 \cup \sigma_2$ is not weakly connected. This implies, there exists an affine plane P_1' in $\mathbb{R}^3$ separating σ_1 and σ_2. For any oriented affine 2-plane in $\mathbb{R}^3$ must be divided into two open half-spaces. Defining the sign of these half-spaces in the manner corresponding to the orientation of the 2-plane, we consider the sets S_i^+ (or S_i^-) as follows: a 2-plane P is said to be in S_i^+ (or S_i^-) for $i = 1$ or 2, if σ_i is contained in the positive (or negative) open half-space defined by P. Obviously, $P_1' \in S_1^+ \cap S_2^-$ for a fixed orientation of P_1'. However, by the compactness of $\partial\Sigma = \sigma_1 \cup \sigma_2$, $S_1^+ \cap S_2^+ \neq \emptyset$ and $S_2^- \cap S_1^- \neq \emptyset$. Hence $\partial S_1^+ \cap \partial S_2^- \neq \emptyset$, by virtue of the fact that both S_1^+ and S_2^- are connected sets. This gives a 2-plane in $\mathbb{R}^3$, P_1, which has the property that σ_1 (and σ_2) lies in the closed positive (respectively negative) half-space defined by P_1. Moreover, both the sets $\sigma_1 \cap P_1$ and $\sigma_2 \cap P_1$ are nonempty.

By the assumption that $\partial\Sigma$ is not weakly connected and since P_1 does not separate σ_1 and σ_2, there exists an affine 2-plane in $\mathbb{R}^3$, P_2', which is perpendicular to P_1 and separating σ_1 and σ_2. Let us define $\overline{S}$ to be the set of oriented affine 2-planes in $\mathbb{R}^3$ which are perpendicular to P_1. Setting $\overline{S}_i^+$ (or $\overline{S}_i^-$) to be $S_i^+ \cap \overline{S}$ (or $S_i^- \cap \overline{S}$), and as before, we conclude that $\partial\overline{S}_1^+ \cap \partial\overline{S}_2^- \neq \emptyset$. Hence, there exists an affine 2-plane, P_2, perpendicular to P_1, and having the property that σ_1 (and σ_2) lie in the closed positive (respectively negative) half-space defined by P_2 and both sets $\sigma_1 \cap P_2$ and $\sigma_2 \cap P_2$ are nonempty.

Arguing once more that P_1 and P_2 do not separate the σ_i's, there must be an affine 2-plane P_3 perpendicular to both P_1 and P_2. Moreover, P_3 must separate σ_1 and σ_2 by the assumption the $\partial\Sigma$ is not weakly connected. We defined a rectangular coordinate system xyz such that P_1, P_2 and P_3 are the xy, yz, and xz planes respectively. Clearly by the properties of the 2-planes P_i's, σ_1 and σ_2 are contained in the closed octant $\{x \geqslant 0, y \geqslant 0, z \geqslant 0\}$ and the closed octant $\{x \leqslant 0, y \leqslant 0, z \leqslant 0\}$ respectively. In particular, σ_1 is contained in the cone defined by $C_1 = \{X \in \mathbb{R}^3 | X \cdot V \geqslant |X|/\sqrt{3}$, where $V = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})\}$ and σ_2 is contained in the cone $C_2 = \{X \in \mathbb{R}^3 | X \leqslant -|X|/\sqrt{3}$, where $V = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})\}$. However, one verifies that the two cones C_i, $i = 1, 2$, are contained in the positive and negative cones defined by the catenoid obtained from rotating the catenary along the line given by V. In view of Theorem 6 in [4], the minimal surface Σ must be disconnected. This concludes our proof.

2. – Nonexistence.

Let $(x^1, \dots, x^n)$ be a rectangular coordinate system in $\mathbb{R}^n$. We consider the $(n-1)$-dimensional surface of revolution S_a obtained by rotating the catenary $x^{n-1} = a \cosh (x^n/a)$ around the x^n-axis. One readily computes that its principal curvatures are

$$(\cosh^{-1}(z/a), \underbrace{-\cosh^{-1}(z/a), -\cosh^{-1}(z/a), \dots, -\cosh^{-1}(z/a)}_{(n-2)\ \text{copies}})$$

with respect to the inward normal vector (i.e. the normal vector pointing towards the x^n-axis). The set of hypersurfaces $\{S_a\}_{a>0}$ defines a cone in $\mathbb{R}^n$ as in the case when $n = 3$ (see [4]). This cone (positive and negative halves) is given by

$$\mathcal{C} = \{(x^1, \dots, x^n) \in \mathbb{R}^n | (x^1)^2 + \dots + (x^{n-1})^2 < (x^n)^2 \sinh^2 \tau\} \tag{18}$$

where τ is the unique positive number satisfying $\cosh \tau - \tau \sinh \tau = 0$. If Σ is a compact connected minimal surface in $\mathbb{R}^n$ with boundary decomposed into $\partial\Sigma = \sigma_1 \cup \sigma_2$, where σ_1 and σ_2 (each could have more than one connected component) lie inside the positive and negative part of $\mathcal{C}$ respectively, then arguing as in [5], Σ must intersect one of the surfaces S_a tangentially. Moreover, Σ must lie in the interior (the part containing the x^n-axis) of S_a, except at those points of intersection. This violates the maximum principle since Σ is minimal and any 2-dimensional subspace of the tangent space of S_a must have nonpositive mean curvature. Hence Σ must be disconnected. This gives the following:

THEOREM 3. Let C^+ and C^- be the positive and negative halves of the cone in $\mathbb{R}^n$ defined by (18). Suppose Σ is a minimal surface spanning its boundary $\partial\Sigma = \sigma_1 \cup \sigma_2$. If $\sigma_1 \subset C^+$ and $\sigma_2 \subset C^-$, then Σ must be disconnected.

We remark that using similar arguments, one can use surfaces of revolution having principal curvatures of the form $(k\lambda, \underbrace{-\lambda, -\lambda, -\lambda, \ldots, -\lambda}_{(n-2)\text{ copies}})$ as barrier to yield nonexistence type theorems for $(k+1)$-dimensional minimal submanifolds in $\mathbb{R}^n$.

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