Proc. Natl. Acad. Sci. USA
Vol. 75, No. 6, p. 2567, June 1978
Mathematics

Incompressible minimal surfaces, three-dimensional manifolds with nonnegative scalar curvature, and the positive mass conjecture in general relativity

(Riemannian manifold/fundamental group/stable surfaces)

RICHARD SCHOEN AND SHING-TUNG YAU

Department of Mathematics, University of California, Berkeley, California 94720

Communicated by S. S. Chern, March 10, 1978

ABSTRACT **We study three-dimensional Riemannian manifolds with nonnegative scalar curvature. We find new topological obstruction for such manifolds. Our method turns out to be useful in studying the positive mass conjecture in general relativity.**

Let M be a complete three-dimensional Riemannian manifold with fundamental group $\pi_1(M)$. Let N be a compact two-dimensional surface with fundamental group $\pi_1(N)$. Suppose there is a homomorphism mapping $\pi_1(N)$ into $\pi_1(M)$. Then it is of interest to know whether one can find an immersion ρ from N into M whose induced map on $\pi_1(N)$ is the same as ρ and whose induced area is minimal among all such immersions. It is trivial to see that some conditions must be imposed on ρ and the structure of M. We announce the following theorem:

THEOREM 1. *Let* N *be a two-dimensional Riemannian manifold. Suppose the following conditions are valid:* (i) *No nontrivial element of* $\ker(\rho)$ *can be represented by a simple closed curve.* (ii) *There is a positive number* $c > 0$ *such that every closed curve representing a nontrivial free homotopy class of* $\rho(\pi_1(N))$ *has length greater than* c. (iii) *For some closed curve* σ *in* N, $\rho([\sigma])$ *is not freely homotopic to any curve in* $M\backslash K$ *for some* K, *a compact set in* M. *Then there is an immersion from* N *into* M *whose induced map on* $\pi_1(N)$ *is the same as* ρ *and whose induced area is minimal among all such immersions.*

Remarks: (*i*) When the dimension of M is greater than three, we can find a branched minimal immersion satisfying the same conclusions. (*ii*) When M is compact, the above conditions *ii* and *iii* are redundant. In this case, *Theorem 1* has been independently found by Sacks and Uhlenbeck (1).

Because the area of the minimal surface constructed in *Theorem 1* is stable under local deformation, we may compute its second variation and obtain some useful inequalities. Applying the Gauss–Bonnet theorem, we prove the following:

THEOREM 2. *Let* M *be a complete orientable three-dimensional manifold with positive scalar curvature. Let* G *be a subgroup of* $\pi_1(M)$. *Suppose there is a compact set* K *in* M *such that for every closed curve* σ *representing a nontrivial element in* G, σ *is not freely homotopic to a closed curve in* $M\backslash K$. *Then* G *cannot be the fundamental group of a compact two-dimensional surface.*

When M is compact, a somewhat stronger theorem can be proved.

THEOREM 3. *Let* M *be a compact orientable three-dimensional manifold. Suppose* $\pi_1 M$ *contains a subgroup isomorphic to the fundamental group of a compact surface with genus* ≥ 1. *Then if* M *has nonnegative scalar curvature,* M *is flat.*

COROLLARY 1. *The only metric on the three-dimensional torus with nonnegative scalar curvature is the flat metric.*

Remarks: (*i*) The truth of *Corollary 1* was questioned by Kazdan and Warner (2). Fischer and Marsden (3) proved that *Corollary 1* is true if the metric lies in a small neighborhood of the standard Euclidean metric. (*ii*) Because *Theorem 1* is true even if we do not assume N is compact, *Theorem 2* and *Theorem 3* can be suitably generalized.

It turns out that *Corollary 1* of *Theorem 3* has an intimate relationship with the positive mass conjecture in general relativity. (See ref. 4.) A special case of this conjecture is the following:

COROLLARY 2. *Let* ds^2 *be a metric on the three-dimensional Euclidean space that is Euclidean outside a compact set. Then if* ds^2 *has nonnegative scalar curvature everywhere,* ds^2 *is the standard Euclidean metric everywhere.*

We also have the following result for the general positive mass conjecture.

THEOREM 4. *For an asymptotically flat metric on* R^3 *with nonnegative scalar curvature, the total energy* (see ref. 4) *is nonnegative. Moreover, if the total energy is zero, the metric is flat.*

Details of the proof of these theorems will appear elsewhere. A more general case of *Theorem 4* and its related literature will be discussed later.

Remarks: Corollary 2 was also proved by Leita (5) in case the manifold is a nonparametric hypersurface in R which is flat outside a compact set.

This work was done with partial support of National Science Foundation Grant MCS74-23180.

1. Sacks, J. & Uhlenbeck, K. (1978) *Ann. Math.*, in press.
2. Kazdan J. & Warner F. (1975) *J. Differential Geometry* **10**, 113–134.
3. Fischer, A. & Marsden J. (1975) *Duke Math. Journal* **42**, 519–547.
4. Geroch, R., (1975) *Proc. Symp. Pure Math.* **27**, 401–414.
5. Leita, M. (1976) Dissertation (Massachusetts Institute of Technology, Cambridge, MA).

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