

The Role of Partial Differential Equations in Differential Geometry

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In the study of geometric objects that arise naturally, the main tools are either groups or equations. In the first case, powerful algebraic methods are available and enable one to solve many deep problems. While algebraic methods are still important in the second case, analytic methods play a dominant role, especially when the defining equations are transcendental. Indeed, even in the situation where the geometric object is homogeneous or algebraic, analytic methods often lead to important contributions. In this talk, we shall discuss a class of problems in differential geometry and the analytic methods that are involved in solving such problems.

One of the main purposes of differential geometry is to understand how a surface (or a generalization of it) is curved, either intrinsically or extrinsically. Naturally, the problems that are involved in studying such an object cannot be linear. Since curvature is defined by differentiating certain quantities, the equations that arise are nonlinear differential equations. In studying curved space, one of the most important tools is the space of tangent vectors to the curved space. In the language of partial differential equations, the main tool to study nonlinear equations is the use of the linearized operators. Hence, even when we are facing nonlinear objects, the theory of linear operators is unavoidable. Needless to say, we are then left with the difficult problem of how precisely a linear operator approximates a nonlinear operator.

To illustrate the situation, we mention five important differential operators in differential geometry. The first one, which is probably the most important one, is the Laplace–Beltrami operator. If the metric tensor is given by $\sum_{i,j} g_{ij} dx^i \otimes dx^j$,

then the operator is given by

$$L(\varphi) = \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x^i} \left[g^{ij} \sqrt{g} \frac{\partial \varphi}{\partial x^j} \right]$$

where $g = \det(g_{ij})$ and (g^{ij}) is the inverse matrix of (g_{ij}) .

The second one is the minimal surface operator and is given by

$$L(\varphi) = \sum_i \frac{\partial}{\partial x^i} \left[(1 + |\nabla \varphi|^2)^{-1/2} \frac{\partial \varphi}{\partial x^i} \right]$$

where $|\nabla \varphi|^2 = \sum_i (\partial \varphi / \partial x^i)^2$.

The third one is the Monge–Ampère operator

$$L(\varphi) = \det \left(\frac{\partial^2 \varphi}{\partial x^i \partial x^j} \right).$$

The fourth one is the complex Monge–Ampère operator

$$L(\varphi) = \det \left(\frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} \right).$$

The fifth one is the Einstein field equation which is a nonlinear hyperbolic system. If $\sum_{i,j} g_{ij} dx^i dx^j$ is the Lorentz metric to be determined, then the operator involved in the Einstein field equation is

$$L(g_{ij}) = R_{ij} - (R/2) g_{ij}$$

where R_{ij} is the Ricci tensor and R is the scalar curvature of the Lorentz metric.

Both the Laplace–Beltrami operator and the minimal surface operator are elliptic. The (real) Monge–Ampère operator is elliptic only at those functions φ where φ is strictly convex and the complex Monge–Ampère operator is elliptic only at those functions φ where φ is strictly plurisubharmonic. All the above operators except the Laplace–Beltrami operator are nonlinear. However, a suitable interpretation shows that the linearized operators of the minimal surface operator and the Monge–Ampère operators are the Laplace–Beltrami operators of certain metrics.

To see how these operators arose in differential geometry, we will discuss one important problem here. Roughly speaking, this problem is to ask how a space is curved globally. In a little more precise form, it can be stated as follows. Given a manifold M , find a necessary and sufficient condition for M to admit a metric with certain curvature properties.

To set up the terminology, we remind the reader of some definitions. From the curvature tensor, one can extract the following quantities. Given a point in the manifold and a two dimensional plane in the tangent space at that point, we can form the *sectional curvature* of the manifold at this plane. Given a point and a tangent at

a point, we can form the *Ricci curvature* in this tangent direction by averaging all the sectional curvatures of the two dimensional tangent planes that contain this tangent. Given a point, we can form the *scalar curvature* at this point by simply averaging all the sectional curvatures at this point. It is clear from these definitions that the sectional curvatures give much more information than the others. For example, as the sectional curvature tells us how the manifold curves in every two plane, it gives good control of the behavior of the geodesics of the manifold. The latter depends on the theory of ordinary differential equations. However, in the other cases, the information about geodesics is much less and the theory of partial differential equations must be involved. Thus in this talk we will concentrate only on the scalar curvature and the Ricci curvature. We begin by discussing the general method of obtaining integrability conditions for the existence of metrics with certain curvature conditions.

1. Integrability conditions. The problem of finding complete integrability conditions for the global existence of metrics with certain curvature conditions is rather difficult. However, for a two dimensional surface, this has a satisfactory answer, thanks to the Gauss–Bonnet theorem for compact surfaces and to the Cohn–Vossen inequality for the complete open surfaces. (The recent works of Kazdan–Warner [32] gave more precise information on the behavior of the curvature function in two dimensional geometry.)

In higher dimension, the situation is much more complicated partly because the curvature is a tensor and partly because the link between topological invariants and geometric invariants is rather weak at this stage. We list here the major methods that were used to find integrability conditions.

1. Chern’s theory of representing Euler class, Pontryagin classes and Chern classes by curvature forms gives the most basic integrability conditions for general manifolds. The celebrated theorem of Atiyah–Singer can be considered as a glorified generalization. Some of their applications will be explained later.

2. Bochner’s method of proving vanishing theorems via Hodge theory will remain to be important for a long time. It led to the Kodaira vanishing theorem, L^2 methods in several complex variables, etc.

3. The variational method has been one of the most classical and most important methods in differential geometry. It includes variation of curves, surfaces, maps, etc.

Naturally, these do not exhaust all the methods. However, for all the results that we are going to discuss, they are obtained by suitable combination of the above three methods.

2. Scalar curvature. The simplest problem concerning the scalar curvature is to find those manifolds which admit a complete metric whose scalar curvature has the same sign.

A long time ago, Yamabe [52] was interested in deforming a metric conformally to one with constant scalar curvature. The equation that is involved in such a process

has the following form

$$\Delta u = \frac{(n-2)}{4(n-1)} Ru - \frac{(n-2)}{4(n-1)} \bar{R} u^{(n+2)/(n-2)}$$

where n is the dimension of the manifold, R and \bar{R} are scalar curvatures of the undeformed and deformed metrics respectively.

As was pointed out by Trudinger [49], Yamabe's method does not seem to work. After Trudinger, there were works by Aubin, Berger, Eliason, Kazdan–Warner, Nirenberg, Moser, etc. An easy consequence of these results is that every compact manifold with dimension greater than two admits a metric with negative scalar curvature. Greene and Wu [27], using another method, proved that every noncompact manifold admits a complete metric with negative scalar curvature. Hence we conclude that in higher dimensions, existence of complete metrics with negative scalar curvature poses no topological restriction on the manifold.

However, complete metrics with nonnegative scalar curvature do give topological information. The first result in this direction is due to Lichnerowicz [35] who proved that for a compact spin manifold with positive scalar curvature, there are no harmonic spinors. Applying the Atiyah–Singer index theorem, the Lichnerowicz vanishing theorem then proves that for a compact spin manifold with positive scalar curvature, the \hat{A} -genus is zero. By pursuing these arguments, Hitchin [30] observed that the (mod 2) KO -theory invariant introduced by Milnor is also zero for a compact spin manifold with positive scalar curvature. In particular, any exotic sphere which does not bound a spin manifold admits no metric with positive scalar curvature.

While mathematicians were working on problems related to scalar curvature, it turned out that physicists, from other points of view, were also interested in similar problems.

Let us describe this problem in general relativity in geometric terms. Suppose we are given a Lorentzian metric on a four dimensional manifold. Then under a fairly general condition, one expects to prove the existence of a maximal space-like hypersurface, i.e., a hypersurface which is locally stable under the deformation of the induced area. Usually, we assume that the Lorentzian metric satisfies the weak energy condition so that, by the Gauss curvature equation, the scalar curvature of the above mentioned maximal space-like hypersurface has non-negative scalar curvature.

Since the maximal space-like hypersurface is three dimensional, we are dealing with a three dimensional manifold with nonnegative scalar curvature. On the other hand, it is well known that three dimensional manifolds are parallelizable. Hence, most of the known topological invariants in higher dimension vanish and the consequences derived from the Lichnerowicz theorem and the Atiyah–Singer index theorem provided no information. On the other hand, the above mentioned problem in general relativity does provide us some guideline. It roughly states [26] that, for an isolated physical system, nonnegativity of local mass density implies the

nonnegativity of total mass. In mathematical terms, it may be described as follows. Let M be a three dimensional manifold with non-negative scalar curvature. (It is the maximal space-like hypersurface mentioned above.) Suppose M is diffeomorphic to R^3 (the situation described below can be generalized to other three dimensional manifolds) such that the metric has the form $(1+m/2r)^4 ds_0^2 + O(1/r^2)$ where ds_0^2 is the standard euclidean metric on R^3 , r is the distance from the origin and $O(1/r^2)$ is a tensor which vanishes along with its first two derivatives like $1/r^2$ when r tends to infinity. The number m is called the total mass of the manifold M . The positive mass conjecture in general relativity says that m is nonnegative and is zero iff the metric is euclidean. A special case of the conjecture says that if we have a metric of nonnegative scalar curvature defined on R^3 which is euclidean outside a compact set, then the metric is euclidean everywhere. This last statement has direct bearing to the questions that geometers are considering.

This positive mass conjecture was proved by R. Schoen and myself recently. (The best previous work on the conjecture was a local result due to Choquet-Bruhat and Marsden [21]). Our motivation and method comes out from an attempt to understand the topology of three dimensional manifolds with nonnegative scalar curvature. Because of the nature of the topology of three dimensional manifolds, it is important to understand the fundamental group. In this regard, we proved that if the fundamental group of the three dimensional manifold with nonnegative scalar curvature contains a subgroup which is isomorphic to the fundamental group of a compact surface with genus ≥ 1 , then the metric is a flat metric. The method of proving this theorem and the above mentioned mass conjecture comes out from the study of the minimal surface equation mentioned in the beginning. It describes a surface in M which locally has minimal area compared with nearby surfaces. The study of such objects has been one of the most important branches in nonlinear elliptic partial differential equations and calculus of variations. (It motivated a new important subject—geometric measure theory—about which Almgren will talk during this Congress.) The reason that it is useful in studying the topology of the manifold is that it tells us how the internal geometry of the manifold behaves. In two dimensions, we can control the topology of the minimal surface, thanks to the work of C. B. Morrey. In higher dimensions, this remains to be studied.

It would be nice to give a criterion for a manifold to admit a metric with positive scalar curvature. However, we do not have a good existence theorem yet. In this regard, we may mention a theorem of B. Lawson and the author [34]. We proved that if a manifold admits a differentiable nonabelian connected compact Lie group action, then the manifold admits a complete metric with positive scalar curvature. (Combining with the above mentioned theorem of Hitchin, we showed that exotic spheres do not admit effective $SU(2)$ action if they do not bound a spin manifold. This gives a theorem in topology and illustrates how curvature can be used to deal with topological problems.) As a generalization of the above work on three dimensional manifold, we mention the following problem. If a compact

manifold with nonnegative scalar curvature is covered by the euclidean space topologically, is it a flat manifold?¹

3. Ricci curvature. As in the case of scalar curvature, the simplest problem concerning the Ricci curvature is to find those manifolds which admit a complete metric whose Ricci curvature has the same sign. Since the Ricci curvature is given by a tensor and the integrability condition is stronger, the problem of existence is considerably harder. The known integrability conditions are not yet complete and we shall only mention a few here.

First of all, Bonnet's theorem tells us that for a compact manifold with positive sectional curvature, the fundamental group must be finite and this was later generalized by Myers [41] for positive Ricci curvature and by Cheeger and Gromoll [14] to the case where we only assume the Ricci curvature to be nonnegative. For a non-compact complete manifold with nonnegative Ricci curvature, there are also conditions on the fundamental group due to Milnor [36], Wolf [51], Schoen and Yau [46]. It seems that a complete manifold with positive Ricci curvature should have a finite fundamental group. But this has never been proved. Metrics with negative Ricci curvature seem to be even harder to understand. For example, in higher dimension, we do not even know whether spheres admit such a metric or not. Only recently, the author [55] was able to produce such a metric on a compact simply-connected manifold. It would be interesting to find some integrability conditions for the existence. It seems possible that for a manifold to admit a metric with negative Ricci curvature, it should admit no effective differentiable nonabelian connected compact Lie group action. It would also be interesting to see whether a compact manifold can admit both a metric with nonnegative scalar curvature and a metric with negative Ricci curvature.

Because of the interest in general relativity, metrics with constant Ricci curvature are of particular importance. For a long time, the only known examples were those manifolds that are acted transitively upon by a compact Lie group. The first necessary condition for the existence was found by M. Berger [6] who proved that for four dimensional Einstein manifolds, i.e., manifolds with constant Ricci curvature, the Euler number must be positive unless they are flat. This inequality of Berger was later generalized by Hitchin [31]. In all these theorems, Chern's representation of the topological invariants by curvature plays a very important role.

For quite a long time, there was no example of nonhomogeneous Einstein manifolds. In particular, it was not known whether there exists a non-flat compact Riemannian manifold with zero Ricci curvature. (This attracted people's attention because of its analogue with the situation in general relativity.) Partly motivated by this

¹ After the Congress, R. Schoen and the author were able to generalize our work on three dimensional manifolds to higher dimensional manifolds. This was also achieved by Gromov and Lawson about the same time. Our works also indicate the possibility of classifying compact simply connected manifolds with positive scalar curvature.

question, Calabi [9] proposed a way to study the Ricci tensor for some special class of manifolds. He observed that in the case of Kähler manifolds, the expression for the Ricci tensor is particularly simple. This observation was based on Chern's representation of the first Chern's class [17] by the curvature form and can be described as follows. Let $\sum_{i,j} g_{ij} dz^i \otimes d\bar{z}^j$ be a Kähler metric defined on a compact complex manifold. Then the (1,1) form

$$\frac{\sqrt{-1}}{2\pi} \sum_{r,s} \frac{\partial^2}{\partial z^r \partial \bar{z}^s} [\log \det (g_{ij})] dz^r \wedge d\bar{z}^s$$

is closed, globally defined on the manifold and represents the first Chern class. According to Chern [17], this (1,1) form is also the Ricci form of the Kähler metric. Hence for a (1,1) form to be the Ricci form of some Kähler metric, it must be closed and represents the first Chern class. What Calabi asked was whether this is the only integrability condition. This question stimulated a lot of interest partly because it could give a complete understanding of the Ricci tensor of a Kähler manifold and partly because it would create a lot of examples of compact manifolds with zero Ricci curvature. For example, the $K-3$ surface is a compact simply connected manifold with zero first Chern class. Calabi's conjecture immediately shows the existence of a Ricci flat metric on the $K-3$ surface. (The simple-connectivity of the $K-3$ surface guarantees that it does not admit any flat metric.) The equation that is needed to solve Calabi's conjecture has the following form

$$(*) \quad \det \left(g_{ij} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} \right) = e^F \det (g_{ij})$$

where φ is the unknown function and F is a smooth function so that $\int_M e^F$ is the volume of M .

Equation (*) is similar to the real Monge–Ampère operator and can be considered as the complex Monge–Ampère equation. In order to make (*) to be elliptic, we have to look for functions φ so that $(g_{ij} + \partial^2 \varphi / \partial z^i \partial \bar{z}^j)$ is a positive definite metric.

In order to understand the equation (*), Calabi [11] studied the equation $\det (\partial^2 \varphi / \partial x^i \partial x^j) = 1$ where φ is required to be convex. He tried to prove that if φ is defined over the entire euclidean space, then it is a quadratic polynomial. He generalized Jörgen's theorem [59] from two dimension to dimension ≤ 5 . The important ingredient in his paper is the introduction of the quantity $S = \sum \varphi^{ir} \varphi^{js} \varphi^{kt} \varphi_{ijk} \varphi_{rst}$ where (φ^{ij}) is the inverse matrix of (φ_{ij}) and φ_{ijk} is the third derivative of φ with respect to x^i, x^j and x^k . This quantity comes up naturally from affine geometry, a geometry where we want to study quantities invariant under the special linear group. Affine geometry is very natural in dealing with the Monge–Ampère equation because the Monge–Ampère operator is clearly invariant under the special linear group. Indeed, the graph defined by the solution of the equation $\det (\partial^2 \varphi / \partial x^i \partial x^j) = 1$ has a nice affine geometric meaning. It is called the improper affine sphere. The important contribution of Calabi is that he found a nice formula when the linearized

operator of $M(\varphi) = \det(\partial^2 \varphi / \partial x^i \partial \bar{x}^j)$ operates on the above quantity S . His formula enables one to estimate, in the interior of the domain, the third derivatives of the solution of the equation $\det(\partial^2 \varphi / \partial x^i \partial \bar{x}^j) = F(x, \varphi)$ assuming that we know the lower order estimates of φ . It turns out that a complex analogue of Calabi's third order quantity exists and that a nice formula (as was shown by Nirenberg) still holds.

In 1971, Pogorelov [45] was able to push Calabi's method to prove that in general, any convex entire solution of the equation $\det(\partial^2 \varphi / \partial x^i \partial \bar{x}^j) = 1$ is a quadratic polynomial. One of the main ingredients of Pogorelov was his interior estimate of the second derivatives of the equation $\det(\partial^2 \varphi / \partial x^i \partial \bar{x}^j) = F(x)$. Besides the interior estimate, Pogorelov used a lot of convex geometry to prove the completeness of the affine metric which was the major point left in Calabi's approach. Later, Calabi, Cheng, Nirenberg and the author were able to prove the completeness of a large class of affine metrics. These include also the hyperbolic affine sphere where the equation is given by $\det(\partial^2 \varphi / \partial x^i \partial \bar{x}^j) = (-1/\varphi)^{n+2}$. This last method does not depend on convex geometry. It has direct influence on our later work mentioned below.

Coming back to the equation (*), one notices that Calabi proved that if F is close enough to zero, (*) has a unique solution. Assuming a curvature condition on the Kähler manifold, Aubin [4] indicated a variational method to prove the existence of solution to (*). (It was conjectured, for example, that such a curvature condition would imply that the manifold is the complex projective space. This is not enough for our later applications in geometry. Furthermore, for the Monge-Ampère equation, variational methods are still rather difficult.) In 1976, the author [55], [56] was able to use the continuity method to prove that (*) has a unique solution without any additional assumption. As usual, the basic steps in the proof are giving the a priori estimates of (*) up to the third derivatives. The third order estimate is essentially a consequence of the fundamental contributions of Calabi. The second order estimate is motivated by Pogorelov's work in [45]. However, both these estimates depend on the estimate of $\sup |\varphi|$. This was not known for a long time and was the major difficulty in solving (*). In case the right hand of (*) has the form $e^{\varphi+F} \det(g_{i\bar{j}})$, an estimate of $\sup |\varphi|$ follows trivially from the maximum principle. In [56], the estimate of $\sup |\varphi|$ depends on a delicate and technically very complicated interplay of the maximum principle and the integration method. Later there was a slight simplification of this estimate due to Kazdan [60] and Bourguignon. As a consequence of the solution of (*) and its proof, one can deduce the existence of a (canonical) Kähler Einstein metric on a compact Kähler manifold with zero or negative first Chern class. (In the special case where the right-hand side of (*) is $e^{\varphi+F} \det(g_{i\bar{j}})$, Aubin [4b] independently announced and sketched a proof which depends on the variational method of his previous paper [4a].)

In a way, the solution of (*), which is commonly known as Calabi's conjecture, gives a complete understanding of the Ricci tensor for a compact Kähler manifold.

However, when one thinks deeper, a lot of problems still have to be done in this direction. One may mention that the solution of Calabi's conjectures gives quite a lot of unexpected application in algebraic geometry [55]. The most interesting one is perhaps the uniqueness of the complex structure of the complex projective plane. This comes out from the canonical metrics that we construct on the algebraic manifolds. These metrics generalize the Poincaré metric of algebraic curves. One expects that they will be useful in the moduli problem of algebraic geometry. Indeed, two years ago, the author was able to use the metric above to prove that if M is an algebraic manifold of dimension n whose canonical line bundle is ample, then $(-1)^n 2(n+1)c_2 c_1^{n-2} \geq (-1)^n n c_1^n$ and equality holds iff M is covered by the complex ball. (For two dimension algebraic surfaces, there were works of Van de Ven, Bogomolov and Miyaoka. It was Miyaoka who found the above precise inequality independently. However, up to now, their algebraic method cannot be generalized to higher dimension and cannot decide what happens when equality holds.) An easy consequence of the theorem is that there is only one Kähler structure on the complex projective space. The Kähler metric with nonnegative Ricci can also be used to deal with problems related to algebraic manifolds. Up to covering problems and the study of complex torus, one can reduce the study of Kähler manifolds with nonnegative first Chern class to the study of simply-connected Kähler manifold with nonnegative first Chern class. In case the Kähler manifold M has zero first Chern class, then one can prove that for any Kähler class ω in $H^{1,1}(M)$, $\omega^{n-2} \cup c_2(M) \geq 0$ and that equality holds only if M is covered by the torus. There are also interesting works of S. Kobayashi [58] who showed how to use the Einstein metric to obtain new vanishing theorems. Bourguignon and Koiso were also able to extend the work of Berger–Ebin [61] to study the deformation of Einstein metrics. They generalized the work of Calabi–Vesentini [63] to Kähler manifolds with negative curvature. Since Einstein metrics have nice curvature properties, it may also be used to strengthen the transcendental method of Griffiths in algebraic geometry.

By pushing more the method that the author used above, Cheng and the author were able to prove the existence of complete Kähler Einstein metrics on many non-compact complex manifolds. For example, if D is a divisor with normal crossings in a compact algebraic manifold M so that $c_1(M) - c_1([D]) < 0$ (see [29]), then we can prove the existence of such a metric on $M \setminus D$. We can also prove the existence of a complete Kähler Einstein metric on any bounded pseudoconvex domain with C^2 boundary in a Stein manifold. It may be interesting to know that it is an easy consequence of the Schwarz lemma given by the author [54] that there is at most one complete Kähler Einstein metric with Ricci curvature $= -1$ on any complex manifold. (This fact was also pointed out by H. Wu.) Therefore, even in the case of noncompact manifolds, complete Kähler Einstein metric is canonical and deserves more investigation.

Concerning the Kähler Einstein metric on a smooth bounded domain Ω in C^n , the equation that we propose to solve has the form $\det(\partial^2 u / \partial z^i \partial \bar{z}^j) = e^{(n+1)u}$

and u is required to tend to infinity on $\partial\Omega$. In order to understand the boundary behavior of the metric near $\partial\Omega$, it suffices to study the boundary behavior of the function $v=e^{-u}$.

The function v satisfies another equation of the Monge–Ampère type. This equation was studied by other people, especially C. Fefferman [24] who studied its relation with the asymptotic behavior of the Bergman kernel. A few years ago, he demonstrated how to find the asymptotic behavior of v assuming its existence. He expanded v in terms of power series expansion of the defining function of Ω . His expansion shows that log terms must occur after the $(n+1)^{\text{th}}$ stage of expansion where $n=\dim \Omega$. His recent deep work on computing the coefficient of the Bergman kernel expansion also shows the importance of this function v . Partly inspired by his work, Cheng and the author were able to demonstrate that the actual solution is $C^{n+3/2-\delta}(\bar{\Omega})$ where $\delta>0$ is an arbitrary small constant. The optimal case should be $n+2-\delta$ and we believe our method will give it after suitable modification. In any case, the information that we obtain is enough to give suitable description of the Kähler Einstein metric near $\partial\Omega$.

Finally, let us come to the question of the existence of complete Kähler metrics with zero Ricci curvature. These metrics have considerable interest in general relativity. There are more conditions for the existence of such metrics and the knowledge of them is far less complete than the previous case. We outline here questions that may lead to future progress.

The first question is: Does every four dimensional compact simply-connected Riemannian manifold with zero Ricci curvature admit a Kähler structure? According to an observation of Hitchin, this is true for the $K-3$ surfaces where the author has constructed Ricci flat Kähler metrics. (In fact, it is true if the compact manifold is a spin manifold with nonzero index.)

The second question is: Can every complete Kähler manifold with zero Ricci curvature be compactified in the complex analytic sense? The author [57] proved that such a manifold does not admit any bounded holomorphic function which gives an indication to support the truth of the statement.

The third question is: Suppose \tilde{M} is one of the compactifications of our manifold M . Does the anticanonical line bundle of \tilde{M} admit a holomorphic section which is zero precisely on $\tilde{M}\setminus M$? If the metric on M “grows only polynomially”, then one can indeed prove that the volume form of M gives rise to such a section. This is based on a theorem proved by Calabi and the author [53] that complete noncompact Riemannian manifold with nonnegative Ricci curvature has infinite volume.

In any case, the author is able to prove that, for a compact Kähler manifold \tilde{M} , if the anticanonical line bundle of \tilde{M} admits a holomorphic section with nonsingular zero locus, then the completion of the zero locus admits a complete Kähler metric with zero Ricci curvature. The assumption that the zero locus is nonsingular seems to be not necessary. In fact, for many negative holomorphic

vector bundles over a compact Kähler Einstein manifold whose Chern classes satisfy some relation, the total spaces admit complete Kähler metric with zero Ricci curvature. (For many special bundles, Calabi also discovered these metrics. For the cotangent bundle of CP^1 , it was discovered earlier by Eguchi–Hansen and Hitchin. They even know the metric explicitly.) In these cases, when we compactify the total space, the zero section of the anticanonical line bundle has multiplicity greater than one.

In question three, we request $\tilde{M} \setminus M$ to be a divisor because one can use the growth of the volume to prove that none of the components of $\tilde{M} \setminus M$ is a subvariety with co-dimension greater than one. A theorem of Cheeger–Gromoll [14] also shows that the divisor $\tilde{M} \setminus M$ is connected unless M is the product of C and other space. One can prove that the plurigenera of \tilde{M} is zero because the positivity of $P_m(M)$ for some $m > 0$ would imply the existence of a non-zero (n, n) form $\tilde{V} = (\sqrt{-1})^n f dz^1 \wedge \dots \wedge dz^n \wedge d\bar{z}^1 \wedge \dots \wedge d\bar{z}^n$ where $f \geq 0$ and $\log f$ is pluriharmonic at points where $f \neq 0$. If dV is the volume form of M , then \tilde{V}/dV defines a function which is L^1 -integrable on M . The condition on \tilde{V} and the fact that M has zero Ricci curvature then imply that \tilde{V}/dV is a constant [53]. As M has infinite volume, this constant must be zero. This is a contradiction.

Recall that it is a consequence of the Schwarz lemma proved in [54] that M and its universal cover admit no bounded holomorphic function. Specialized to two dimensional complex surfaces, one can then use the classification theory to conclude that \tilde{M} must be rational at least when M is simply connected. In any case, we hope the questions asked above will be answered in the near future. An affirmative answer will be very interesting even for complex surfaces.

4. Applications to partial differential equations. Up to now, it seems that we mainly use methods of partial differential equations to deal with problems in geometry. It turns out that the reverse procedure is also the case. Very often the geometric situation motivates the study of certain quantities in differential equations which turns out to be useful. This is true especially for the minimal surface equation and the Monge–Ampère equation. Indeed, one can use the metric mentioned above to treat the Dirichlet boundary valued problem for the Monge–Ampère equation. The procedure does not depend on the concept of generalized solution. For the real Monge–Ampère equation, there were works of Alexandrov [1] and Pogorelov [43]. Pogorelov [43] sketched a proof for the smoothness of the generalized solution in case the right-hand side is independent of the unknown. (In [16] Cheng and the author gave a detailed proof of the smoothness in the general case where the right-hand side depends on the unknown. By a different procedure, we were also able to take care of several essential points overlooked in [43].) For the complex Monge–Ampère equation, the best previously known result was due to Bedford and Taylor [5] who proved the existence of C^1 generalized solution. (Using a different method, Gaveau [62] was able to obtain a generalized solution similar to that of Bedford and Taylor.)

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