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Obstacle Problem for von Kármán Equations

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The free boundary value problem in obstacle problem for von Kármán equations is studied. By using the method of complementarity analysis, Rockafellar's theory of duality is generalized to the nonlinear variational problems and a complementarity theory of obstacle problem for von Kármán plates is established. We prove that the uniqueness and existence of solution directly depend on a complementary gap function. Moreover, a generalized dual extreme principle is established. We prove that the nonlinear primal variational inequality problem is eventually equivalent to a semi-quadratic dual optimization problem defined on a statically admissible space. This equivalence can be used to develop an effective numerical method for solving nonlinear free boundary value problems. © 1992 Academic Press, Inc.

1. INTRODUCTION

This paper is devoted to the study of the free boundary variational problems in obstacle problems pertaining to the theory of thin elastic plates. The model we consider here was proposed by von Kármán for plates undergoing large deflections relative to their thickness. During the past 20 years, there has been increased mathematical interest in studying the theory of convex analysis and variational inequalities, (see [3–7, 12, 13, 17, 19]). Various unilateral problems for von Kármán plates have already been treated both from the theoretical and numerical point of view, see Duvaut and Lions [2], Ohtake, Oden, and Kikuchi [16], Oden [15], and Bielski and Telega [1]. The dual variational inequality for geometrical linear plates was studied by J. Lovisek [14], using the Fenchel–Rockafellar transformation. However, in the finite deformation theory, to formulate the dual problems is extremely difficult. Due to the nonlinearity of the geometrical deformation operator, Rockafellar's theory of duality (see [4]) cannot be directly applied to construct the dual variational problem. So the complementarity problem in this field is still open.

Complementarity theory has become a rich source of inspiration in both mathematical and engineering sciences. Complementarity theory has been extended and generalized in various directions to study a wide class of

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problems arising in optimization and control, mechanics, operations research, fluid dynamics, economics, and transportation equilibrium. Recently, Rockafellar's theory of duality has been generalized to the nonlinear variational boundary value problems, see Gao and Strang [8]. By introducing a complementary gap function, we recover complementary variational principles in the equilibrium problems of mathematical physics. It is proved that in the finite deformation theory, the extremum properties of primal–dual variational problems directly depend on this gap function (see [9–11]). In the present paper, we will see that this gap function plays a key role in the analysis of obstacle problems for von Kármán plates. A generalized complementary extremum principle is established where the continuity of the field variable is weakened, and we prove that the nonlinear primal variational boundary value problem is actually equivalent to a quadratic dual optimization problem. This result can serve as the theoretical foundation for developing an effective numerical method.

2. Free Boundary Value Problem and Variational Inequality

Consider a plate Ω in possible contact with a rigid obstacle G. The points of the plate refer to a fixed, right-handed Cartesian coordinate system $0x_1x_2x_3$. The middle plane of the undeformed plate, which is assumed to have a constant thickness, coincides with the $0x_1x_2$-plane. The materials points $x = (x_1, x_2, 0)$ of the undeformed plate constitute an open, bounded, connected subset $\Omega \subset \mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$. We denote by $u = (u_1, u_2)$ the horizontal and by ξ the vertical displacements of points $x \in \Omega$. The plate is subjected to a distributed load $\mathbf{t} = (0, 0, f)$, $f = f(x) \in \mathscr{L}^2(\Omega)$, per unit area of the middle surface. Suppose that the shape of the obstacle G is given by a prescribed strictly concave function $\psi(x) \in \mathscr{H}^2(\Omega)$. Then the von Kármán plate in the obstacle problem gives rise to unilateral coercive free boundary value problem (FBVP for short).

Problem 1. Find u and ξ such that

$$K\Delta\Delta\xi - h(\sigma_{\alpha\beta}\xi_{,\beta})_{,\alpha} \geq f(x) \quad \text{in } \Omega; \tag{1}$$

$$\sigma_{\alpha\beta,\beta} = 0 \quad \text{in } \Omega; \tag{2}$$

$$\xi(x) \geq \psi(x) \quad \text{in } \Omega; \tag{3}$$

$$\left(K\Delta\Delta\xi - h(\sigma_{\alpha\beta}\xi_{,\beta})_{,\alpha} - f(x)\right)(\xi - \psi) = 0 \quad \text{in } \Omega; \tag{4}$$

$$\sigma_{\alpha\beta} = C_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}(u, \xi) = C_{\alpha\beta\gamma\delta}\left(\theta_{\gamma\delta}(u) + \tfrac{1}{2}\xi_{,\gamma}\xi_{,\delta}\right) \quad \text{in } \Omega; \tag{5}$$

$$\theta_{\alpha\beta}(u) = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \quad \text{in } \Omega; \tag{6}$$

$$\xi(x) = 0, \qquad u_\alpha = 0, \qquad \frac{\partial\xi}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{7}$$

Here h is the plate thickness, $K = Eh^3/12(1 - \nu^2)$ is the bending rigidity of the plate with E the modulus of elasticity and ν the Poisson ratio. $\sigma = \{\sigma_{\alpha\beta}\}$ (resp. $\varepsilon_{\alpha\beta}$) denotes the stress (resp. strain) tensor in the plane of the plate, and $C = \{C_{\alpha\beta\gamma\delta}\}$ is the corresponding elasticity tensor. In this paper, we assume that $C_{\alpha\beta\gamma\delta} \in \mathscr{L}^\infty(\Omega)$ and C satisfies the usual symmetry and ellipticity properties, i.e.,

$$C_{\alpha\beta\gamma\delta} = C_{\beta\alpha\gamma\delta} = C_{\gamma\delta\alpha\beta} \tag{8}$$

$$C_{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} \geq \alpha_0 \varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} \qquad \forall \varepsilon_{\alpha\beta} = \varepsilon_{\beta\alpha}, \tag{9}$$

where $\alpha_0 = \text{const} > 0$. For instance, we can take $\alpha_0 = E/(1 + \nu)$. For isotropic plate, C has the form

$$C_{\alpha\beta\lambda\mu} = \frac{E}{2(1 - \nu^2)}\big((1 - \nu)(\delta_{\alpha\lambda}\delta_{\beta\mu} + \delta_{\alpha\mu}\delta_{\beta\lambda}) + 2\nu\delta_{\alpha\beta}\delta_{\lambda\mu}\big). \tag{10}$$

Throughout this paper the Greek indices take values 1 and 2.

For any solution (u, ξ) of Problem 1, the contact region $Z \subset \Omega$ is defined as

$$Z := \{x \in \Omega | \xi(x) = \psi(x)\ \forall x \in \Omega\}.$$

Since the boundary ∂Z of this subdomain Z is not known before the contact problem is solved, ∂Z is called a free boundary.

The solution of Problem 1 is very difficult to find because of the high degree of nonlinearity. In this paper, we like to develop a complementarity method. We will see that this nonlinear free boundary value problem is equivalent to a semi-quadratic dual optimization problem. The following functional spaces are essential for studying the problem of variational inequality:

$$\mathscr{H}^1(\Omega) = \{v | v, v_{,x}, v_{,y} \in \mathscr{L}^2(\Omega)\},$$

$$\mathscr{H}_0^1(\Omega) = \{v | v \in \mathscr{H}^1(\Omega), v|_{\partial\Omega} = 0\},$$

$$\mathscr{H}^2(\Omega) = \{v | v, v_{,\alpha}, v_{,\alpha\beta} \in \mathscr{L}^2(\Omega)\},$$

$$\mathscr{H}_0^2(\Omega) = \left\{v | v \in \mathscr{H}^2(\Omega), v = \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega\right\}.$$

On the space $\mathscr{H}^1(\Omega)$, we introduce the standard bilinear form

$$\langle u, v\rangle := \int_\Omega uv\, d\Omega + \text{boundary terms}. \tag{11}$$

For a given obstacle $\psi(x) \in \mathscr{H}_0^2(\Omega)$, take the set

$$\mathscr{K} := \{v \in \mathscr{H}_0^2(\Omega) | v \geq \psi \text{ a.e. in } \Omega\} \tag{12}$$

which is closed and convex. Introducing the bilinear forms

$$a(\xi, z) := K\int_\Omega [(1-\nu)\xi_{,\alpha\beta} z_{,\gamma\delta} + \nu \Delta\xi \Delta z] \, d\Omega \qquad \left(0 < \nu < \frac{1}{2}\right), \tag{13}$$

$$b(\varepsilon, e) := \int_\Omega C_{\alpha\beta\gamma\delta} \varepsilon_{\alpha\beta} e_{\gamma\delta} \, d\Omega. \tag{14}$$

Let

$$\phi(\xi, z) := \{\xi_{,\alpha} z_{,\beta}\}, \qquad \phi(\xi, \xi) = \phi(\xi); \tag{15}$$

it is easy to prove that FBVP (Problem 1) is equivalent to

PROBLEM 2. Find $u \in [\mathscr{H}_0^1(\Omega)]^2$ and $\xi \in \mathscr{K}$ to satisfy the variational inequality

$$a(\xi, z-\xi) + hb(\varepsilon(u, \xi), \phi(\xi, z-\xi)) \geq (f, z-\xi) \qquad \forall z \in \mathscr{K}, \tag{16}$$

and the variational equality

$$b(\varepsilon(u, \xi), \theta(v-u)) = 0 \qquad \forall v \in [\mathscr{H}_0^1(\Omega)]^2. \tag{17}$$

Indeed we can write

$$\begin{aligned} &K\int_\Omega [(1-\nu)\xi_{,\alpha\beta}(z-\xi)_{,\gamma\delta} + \nu\Delta\xi\Delta(z-\xi)] \, d\Omega \\ &\quad + \int_\Omega h\sigma_{\alpha\beta}\xi_{,\alpha}(z-\xi)_{,\beta} \, d\Omega \\ &\quad \geq \int_\Omega f(z-\xi) \, d\Omega \qquad \forall z \in \mathscr{K} \end{aligned}$$

by using the Gauss–Green theorem; then we have

$$\begin{aligned} &\int_\Omega \left(K\Delta\Delta\xi - h(\sigma_{\alpha\beta}(u, \xi)\xi_{,\beta})_{,\alpha}\right)(z-\xi) \, d\Omega \\ &\quad \geq \int_\Gamma h\sigma_{\alpha\beta}\xi_{,\alpha} n_\beta (z-\xi) \, d\Gamma + \int_\Gamma Q(\xi)(z-\xi) \, d\Gamma \\ &\quad - \int_\Gamma M(\xi) \frac{\partial(z-\xi)}{\partial n} \, d\Gamma + \int_\Omega f(z-\xi) \, d\Omega, \end{aligned}$$

where $n = (n_1, n_2)$ denotes the outward normal unit vector on Γ and

$$M(\xi) = -K\left[\nu\Delta\xi + (1-\nu)\left(2n_1n_2\xi_{,12} + n_1^2\xi_{,11} + n_2^2\xi_{,22}\right)\right] \quad (18)$$

and

$$Q(\xi) = -K\left[\frac{\partial\Delta\xi}{\partial n} + (1-\nu)\frac{\partial}{\partial\tau}\left[n_1n_2(\xi_{,22} - \xi_{,11}) + \left(n_1^2 - n_2^2\right)\xi_{,12}\right]\right]. \quad (19)$$

Here τ denotes the unit vector tangential to Γ such that n, τ, and $0x_3$ form a right-handed system. M is the bending moment and Q the total shearing force on the plate boundary Γ. Whence, for any given $z \in \mathscr{H}_0^2$,

$$\int_\Omega \left(K\Delta\Delta\xi - h\left(\sigma_{\alpha\beta}(u,\xi)\xi_{,\beta}\right)_{,\alpha} - f\right)(z - \xi)\, d\Omega \geq 0. \quad (20)$$

Let $\phi \in \mathscr{H}_0^2(\Omega)$ be such that $\phi \geq 0$; then $z = \xi + \phi \in \mathscr{K}$, and hence

$$\int_\Omega \left(K\Delta\Delta\xi - h\left(\sigma_{\alpha\beta}(u,\xi)\xi_{,\beta}\right)_{,\alpha} - f\right)\phi\, d\Omega \geq 0$$

$$\Rightarrow K\Delta\Delta\xi - h\left(\sigma_{\alpha\beta}(u,\xi)\xi_{,\beta}\right)_{,\alpha} - f \geq 0.$$

Furthermore, we take

$$z = \psi \in \mathscr{K}$$

$$z = 2\xi - \psi = \xi - (\psi - \xi) \in \mathscr{K}.$$

We obtain

$$\left.\begin{aligned} &\int_\Omega \left(K\Delta\Delta\xi - h\left(\sigma_{\alpha\beta}(u,\xi)\xi_{,\beta}\right)_{,\alpha} - f\right)(\psi - \xi)\, d\Omega \geq 0 \\ &\int_\Omega \left(K\Delta\Delta\xi - h\left(\sigma_{\alpha\beta}(u,\xi)\xi_{,\beta}\right)_{,\alpha} - f\right)(\xi - \psi)\, d\Omega \geq 0 \end{aligned}\right\}$$

$$\Rightarrow \int_\Omega \left(K\Delta\Delta\xi - h\left(\sigma_{\alpha\beta}(u,\xi)\xi_{,\beta}\right)_{,\alpha} - f\right)(\xi - \psi)\, d\Omega = 0.$$

This implies

$$\left(K\Delta\Delta\xi - h\left(\sigma_{\alpha\beta}(u,\xi)\xi_{,\beta}\right)_{,\alpha} - f\right)(\xi - \psi) = 0 \qquad \text{over } \Omega.$$

In the same way, if we multiply Eq. (2) by $v_\beta - u_\beta$ and integrate over Ω, we obtain, by the Gauss–Green theorem, the relation

$$\int_\Omega \sigma_{\alpha\beta}\theta_{\alpha\beta}(v - u)\, d\Omega = \int_\Gamma \sigma_{\alpha\beta}n_\beta(v_\alpha - u_\beta)\, d\Gamma. \quad (21)$$

For any given $v \in [\mathscr{H}_0^1(\Omega)]^2$, this gives rise to the variational equation (17).

For any given $(v, z) \in [\mathscr{H}_0^1(\Omega)]^2 \times \mathscr{K}$, the total potential of the system is given by a functional $\Pi: [\mathscr{H}_0^1(\Omega)]^2 \times \mathscr{K} \to \mathbb{R}$

$$\Pi(v, z) = \tfrac{1}{2}a(z, z) + \tfrac{1}{2}hb(\varepsilon(v, z), \varepsilon(v, z)) - \langle f, z\rangle. \tag{22}$$

If $\psi \in \mathscr{H}_0^2(\Omega)$ we can write $\mathscr{K} = \psi + \mathscr{C}$, where

$$\mathscr{C} := \{w \in \mathscr{H}_0^2(\Omega) | w \geq 0 \text{ a.e. in } \Omega\} \tag{23}$$

is a positive cone with its vertex at the origin. Further, putting

$$\xi = \psi + w, \qquad w \in \mathscr{C}, \tag{24}$$

we have

LEMMA 1. *The solution (u, ξ) of Problem 1 fulfills the variational inequality*

$$\langle \delta\Pi(u, \xi), (v, w)\rangle \geq 0 \qquad \forall v \in [\mathscr{H}_0^1(\Omega)]^2, \forall w \in \mathscr{C}. \tag{25}$$

where $\delta\Pi(u, \xi)$ stands for the Gâteaux derivative of Π at (u, ξ).

Proof. By the definition of the Gâteaux differential, it is easy to prove that

$$\frac{1}{2}\delta a(\xi, \xi; w) = \lim_{t \to +0} \frac{a(\xi + tw, \xi + tw) - a(\xi, \xi)}{2t} = a(\xi, w),$$
$$\frac{1}{2}\delta b(\varepsilon(u, \xi), \varepsilon(u, \xi); w) = b(\varepsilon(u, \xi), \phi(\xi, w)),$$
$$\frac{1}{2}\delta b(\varepsilon(u, \xi), \varepsilon(u, \xi); v) = b(\varepsilon(u, \xi), \theta(v)).$$

Noting the variational equation (17),

$$b(\varepsilon(u, \xi), \theta(v)) = 0 \qquad \forall v \in \mathscr{H}_0^1(\Omega),$$

and applying (16), we have

$$\begin{aligned} \delta\Pi(u, \xi; v, w) &= \langle \delta\Pi(u, \xi); v, w\rangle \\ &= a(\xi, w) + hb(\varepsilon(u, \xi), \phi(\xi, w)) - \langle f, w\rangle \\ &\geq 0 \qquad \forall w \in \mathscr{C}, \end{aligned}$$

which shows that the variational inequality (25) is equivalent to Problem 2.

Q.E.D.

The variational inequality has been studied extensively by many authors (see [3, 17]). But in the geometrical nonlinear case, we will find that the variational inequality is not equivalent to the corresponding variational extreme problem.

3. The Primal Problem and the Variational Extreme Principle

The following variational boundary value problem is called the primal problem:

Problem 3 (P). Find $(u, \xi) \in [\mathscr{H}_0^2(\Omega)]^2 \times \mathscr{K}$ such that

$$\Pi(u, \xi) = \inf\left\{\Pi(v, z) | v \in \left[\mathscr{H}_0^2(\Omega)\right]^2, z \in \mathscr{K}\right\}. \tag{26}$$

Theorem 1. *Suppose* (u, ξ) *is the stationary point of* Π. *For any given* $z \in \mathscr{K}$, *if*

$$\frac{1}{2} hb(\varepsilon(u, \xi), \phi(z)) = \int_\Omega \frac{1}{2} h\sigma_{\alpha\beta} z_{,\alpha} z_{,\beta} d\Omega \geq 0 \qquad \forall z \in \mathscr{H}_0^2(\Omega), \tag{27}$$

then primal problem (P) *has at least one solution which fulfills the free boundary value problem* (*Problem* 1). *The problem has a unique solution if the inequality in* (27) *strictly holds.*

Proof. From the convexity of $a(z, z)$ and $b(\varepsilon, \varepsilon)$, we have

$$\begin{aligned} a(z, z) &+ hb(\varepsilon(v, z), \varepsilon(v, z)) - a(\xi, \xi) - hb(\varepsilon(u, \xi), \varepsilon(u, \xi)) \\ &\geq \langle \delta a(\xi, \xi), (z - \xi)\rangle + \langle h\delta b(\varepsilon(u, \xi), \varepsilon(u, \xi), (\varepsilon(v, z) - \varepsilon(u, \xi))\rangle \\ &\qquad \forall v \in \left[\mathscr{H}_0^1(\Omega)\right]^2, \forall z \in \mathscr{H}_0^2(\Omega). \end{aligned} \tag{28}$$

By the way, it is easy to prove that

$$\varepsilon(v, z) = \varepsilon(u, \xi) + \delta\varepsilon(u, \xi)(v - u)(z - \xi) + \tfrac{1}{2}\phi(z - \xi); \tag{29}$$

i.e.,

$$\begin{aligned} \varepsilon_{\alpha\beta}(v, z) = \varepsilon_{\alpha\beta}(u, \xi) &+ \tfrac{1}{2}\left[(v_\alpha - u_\alpha)_{,\beta} + (v_\beta - u_\beta)_{,\alpha}\right] \\ &+ \xi_{,\alpha}(z - \xi)_{,\beta} + \tfrac{1}{2}(z - \xi)_{,\alpha}(z - \xi)_{,\beta}. \end{aligned} \tag{30}$$

Substituting into (28), we obtain

$$\frac{1}{2}a(z,z) + \frac{1}{2}hb(\varepsilon(v,z),\varepsilon(v,z)) - \langle f,z\rangle - \frac{1}{2}a(\xi,\xi) - \frac{1}{2}hb(\varepsilon(u,\xi),\varepsilon(u,\xi)) + \langle f,\xi\rangle \geq \langle \delta\Pi(u,\xi),(v-u)(z-\xi)\rangle + \frac{1}{2}hb(\varepsilon(u,\xi),\phi(z-\xi)) \quad \forall v \in [\mathscr{H}_0^1(\Omega)]^2, \forall z \in \mathscr{H}_0^2(\Omega). \tag{31}$$

From Lemma 1 we know that if (u,ξ) is the solution of Problem 1,

$$\langle \delta\Pi(u,\xi),(v-u)(z-\xi)\rangle \geq 0 \qquad \forall v \in [\mathscr{H}_0^1(\Omega)]^2, \forall z \in \mathscr{K}.$$

So we have

$$\Pi(v,z) - \Pi(u,\xi) \geq \tfrac{1}{2}hb(\varepsilon(u,\xi),\phi(z-\xi)) \quad \forall v \in [\mathscr{H}_0^1(\Omega)]^2, \forall z \in \mathscr{K}. \tag{32}$$

Obviously, for any given $z \in \mathscr{H}_0^2(\Omega)$, if constraint (27) holds, we have

$$\Pi(v,z) - \Pi(u,\xi) \geq 0 \qquad \forall v \in [\mathscr{H}_0^1(\Omega)]^2, \forall z \in \mathscr{K}, \tag{33}$$

which shows that the solution of the variational system (26) minimizes the total potential functional Π over $[\mathscr{H}_0^1(\Omega)]^2 \times \mathscr{K}$.

Now we are going to prove the existence and uniqueness of the solution. Actually, according to the theory of convex analysis, inequality (33) shows that for any given $u \in [\mathscr{H}_0^1(\Omega)]^2$, if the constraint (27) holds, the total potential functional $\Pi\colon \mathscr{K} \to \mathbb{R}$ is convex. From the theory of convex analysis, see Ekeland and Temam [4], it is known that if the closed convex subset $\mathscr{K} \subset \mathscr{H}^2(\Omega)$ is bounded, to every plane displacement $u \in [\mathscr{H}_0^1(\Omega)]^2$ the variational problem

$$\{\inf \Pi(u,z)\ \forall z \in \mathscr{K}\} \tag{34}$$

has at least one solution $\xi \in \mathscr{K}$. Moreover, if the inequality in (27) strictly holds, the potential functional $\Pi\colon \mathscr{K} \to \mathbb{R}$ is strictly convex. In this case, the variational problem has a unique solution.

On the other hand, for a given $\xi \in \mathscr{K}$, the extreme condition for the variational problem

$$\left\{\inf \Pi(v,\xi)\ \forall v \in [\mathscr{H}_0^1(\Omega)]^2\right\} \tag{35}$$

gives rise to the variational equation:

$$b\left(\theta(u)+\tfrac{1}{2}\phi(\xi),\theta(v-u)\right)=0 \qquad \forall v\in\left[\mathscr{H}^1(\Omega)\right]^2. \tag{36}$$

From the elliptic condition (9) and Korn's inequality, we have

$$b(\theta(v),\theta(v)) \geq \int_\Omega \alpha_0\theta(v)_{\alpha\beta}\theta(v)_{\alpha\beta} \geq c\|v\|^2_{[\mathscr{H}^1(\Omega)]^2}$$
$$\forall v\in\left[\mathscr{H}^1(\Omega)\right]^2/\bar{\mathscr{R}},$$

where $\bar{\mathscr{R}}$ denotes the space of plane rigid-body displacements, i.e.,

$$\bar{\mathscr{R}}:=\left\{\bar{r}\in\left[\mathscr{H}^1(\Omega)\right]^2\middle|\theta(\bar{r})=0\right\}$$
$$=\left\{\bar{r}\in\left[\mathscr{H}^1(\Omega)\right]^1\middle|\bar{r}_1=a_1+bx_2,\bar{r}_2=a_2-bx_1,a_1,a_2,b\in\mathbb{R}\right\}.$$

This shows that $b(\theta(v),\theta(u))$ is a bilinear coercive form of the quotient space $[\mathscr{H}^1(\Omega)]^2/\bar{\mathscr{R}}$. According to the Lax–Milgram theorem, the variational equality (36) has for every ξ a solution $u(\xi)$, since the application $v\to b(\theta(v)+\frac{1}{2}\phi(\xi),\theta(v))$ is linear and continuous on $[\mathscr{H}_0^1(\Omega)]^2$. Q.E.D.

Theorem 1 shows that in the geometrical nonlinear cases, the variational inequality (25) is not equivalent to the variational extreme problem (26). The convexity of the total potential functional depends on the right sign of the bilinear form $b(\varepsilon(u,\xi),\phi(z))$. In the next section, we will see this function plays an important role in the analysis of geometrical nonlinear problems.

4. Dual Spaces and Operators Decomposition

Even with the use of tensorial notation, formulations for the nonlinear problem of variational inequality are rather complex and lengthy. To see the inner beauty of the symmetry intrinsic in the nonlinear phenomena, abstract operator notation has to be used to study the dual problem.

Let $\mathscr{U}$ be the general displacement space:

$$\mathscr{U}:=\left\{\mathbf{u}|\mathbf{u}=\{u,\xi\},u\in\left[\mathscr{H}_0^1(\Omega)\right]^2,\xi\in\mathscr{H}_0^2(\Omega)\right\}, \tag{37}$$

$\mathscr{E}$ the general strain space:

$$\mathscr{E}:=\left\{\mathbf{E}|\mathbf{E}=\{\varepsilon_{\alpha\beta},\kappa_{\alpha\beta}\},\varepsilon_{\alpha\beta}=\varepsilon_{\beta\alpha}\in\left[\mathscr{H}^1(\Omega)\right]^4,\right.$$
$$\left.\kappa_{\alpha\beta}=\kappa_{\beta\alpha}\in\left[\mathscr{H}^2(\Omega)\right]^4\right\}, \tag{38}$$

For a given displacement $\mathbf{u} = (u_\alpha, \xi) \in \mathscr{U}$, the deformation of the von Kármán plate can be described by the nonlinear differential operator: $\Lambda(\mathbf{u})$: $\mathscr{U} \to \mathscr{E}$:

$$\Lambda(\mathbf{u}) = \begin{Bmatrix} \Lambda_\varepsilon(\mathbf{u}) \\ \Lambda_\kappa \end{Bmatrix} = \begin{Bmatrix} \Lambda_\theta + \frac{1}{2}\Lambda_\phi(\mathbf{u}) \\ \Lambda_\kappa \end{Bmatrix}, \tag{39}$$

in which

$$\begin{aligned} &\Lambda_\varepsilon(\mathbf{u})\mathbf{v} = \left\{\tfrac{1}{2}\left(v_{\alpha,\beta} + v_{\beta,\alpha} + \xi_{,\alpha} z_{,\beta}\right)\right\}, \\ &\Lambda_\kappa \mathbf{v} = \left\{-z_{,\alpha\beta}\right\}, \\ &\Lambda_\theta \mathbf{v} = \left\{\tfrac{1}{2}\left(v_{\alpha,\beta} + v_{\beta,\alpha}\right)\right\}, \\ &\Lambda_\phi(\mathbf{u})\mathbf{v} = \left\{-\xi_{,\alpha} z_{,\beta}\right\}, \end{aligned} \qquad \forall \mathbf{v} = (v, z) \in \mathscr{U}.$$

In continuum mechanics, Λ is called finite deformation operator. The nonlinearity caused by this operator is called the geometrical nonlinearity. Then the strain-displacement relations can be written in the following form:

$$\mathbf{E} = \Lambda(\mathbf{u})\mathbf{u} \Leftrightarrow \begin{cases} \varepsilon_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha} + \xi_{,\alpha}\xi_{,\beta}\right) \\ \kappa_{\alpha\beta} = -\xi_{,\alpha\beta}. \end{cases} \tag{40}$$

The direction derivative of $\mathbf{E}(\mathbf{u})$ at $\mathbf{u}(u, \xi)$ in the direction $\mathbf{v}(v, w)$ is given by

$$\delta\mathbf{E}(\mathbf{u}; \mathbf{v}) = \lim_{t \to +0} \frac{\mathbf{E}(\mathbf{u} + t\mathbf{v}) - \mathbf{E}(\mathbf{u})}{t} = \Lambda_T(\mathbf{u})\mathbf{v}, \tag{41}$$

where Λ_T: $\mathscr{U} \to \mathscr{E}$ is the Gâteaux derivative of $\mathbf{E}(\mathbf{u})$ at $\mathbf{u}$:

$$\Lambda_T(\mathbf{u}) = \begin{Bmatrix} \Lambda_t(\mathbf{u}) \\ \Lambda_\kappa \end{Bmatrix}, \tag{42}$$

in which

$$\Lambda_t(\mathbf{u}) = \Lambda_\theta + \Lambda_\phi(\mathbf{u}). \tag{43}$$

The complementary operator of $\Lambda_T(\mathbf{u})$ is given by

$$\Lambda_N(\mathbf{u}) := \Lambda(\mathbf{u}) - \Lambda_T(\mathbf{u}) = -\tfrac{1}{2}\Lambda_\phi(\mathbf{u}). \tag{44}$$

We can see here that Λ_N is a symmetric and quadratic operator,

$$\Lambda_n(\mathbf{u})\mathbf{v} = \left\{-\tfrac{1}{2}\xi_{,\alpha} z_{,\beta}\right\}. \tag{45}$$

It plays an important role in the finite deformation theory [10, 11]. The conjugate variable of the general strain $\mathbf{E} \in \mathscr{E}$ is the general stress $\mathbf{S} = \{h\sigma, M\}$ which is given by physical equation

$$\mathbf{S} = \frac{\partial w(\mathbf{E})}{\partial \mathbf{E}} \Rightarrow \begin{cases} M_{\alpha\beta} = D_{\alpha\beta\lambda\mu}\kappa_{\lambda\mu} \\ \sigma_{\alpha\beta} = C_{\alpha\beta\lambda\mu}\varepsilon_{\lambda\mu}, \end{cases} \tag{46}$$

where w is the stored energy,

$$w(\mathbf{E}) = w(\varepsilon, \kappa) = \frac{1}{2}D_{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}\kappa_{\lambda\mu} + \frac{h}{2}C_{\alpha\beta\lambda\mu}\varepsilon_{\alpha\beta}\varepsilon_{\lambda\mu}; \tag{47}$$

$D = \{D_{\alpha\beta\lambda\mu}\} = (h^3/12)C$. It is obvious that for any given $\mathbf{E} \in \mathscr{E}$, w: $\mathscr{E} \to \mathbb{R}$ is convex and quadratic. Let $\mathscr{S}$ be the general stress space defined by

$$\mathscr{S} = \Big\{\mathbf{S} | \mathbf{S} = \{h\sigma_{\alpha\beta}, M_{\alpha\beta}\}, \sigma_{\alpha\beta} = \sigma_{\beta\alpha} \in \left[\mathscr{H}^{-1}(\Omega)\right]^4, \\ M_{\alpha\beta} = M_{\beta\alpha} \in \left[\mathscr{H}^{-2}(\Omega)\right]^4\Big\}. \tag{48}$$

The bilinear form $\langle \mathbf{S}, \mathbf{E} \rangle$: $\mathscr{S} \times \mathscr{E} \to \mathbb{R}$ puts $\mathscr{S}$ and $\mathscr{E}$ in duality:

$$\langle \mathbf{S}, \mathbf{E} \rangle = \int_\Omega \left[M_{\alpha\beta}\kappa_{\alpha\beta} + h\sigma_{\alpha\beta}\varepsilon_{\alpha\beta} \right] d\Omega. \tag{49}$$

By using the Gauss–Green theorem, we have

$$\begin{aligned} \langle \mathbf{S}, \Lambda(\mathbf{u})\mathbf{v} \rangle &= \langle \Lambda_T^*(\mathbf{u})\mathbf{S}, \mathbf{v} \rangle + \langle \Lambda_N^*(\mathbf{u})\mathbf{S}, \mathbf{v} \rangle \\ &= \langle \Lambda_T^*(\mathbf{u})\mathbf{S}, \mathbf{v} \rangle - G(\mathbf{v}, -\Lambda_N^*(\mathbf{u})\mathbf{S}), \end{aligned} \tag{50}$$

where Λ_T^* is the operator adjoint to Λ_T:

$$\Lambda_T^*(\mathbf{u}) = \{\Lambda_\theta^*, \Lambda_\xi^*(\mathbf{u})\} \tag{51}$$

$$\Lambda_\theta^* \mathbf{S} = \begin{cases} -\{h\sigma_{\alpha\beta,\beta}\} & \text{in } \Omega \\ \{h\sigma_{\alpha\beta}n_\beta\} & \text{on } \Gamma \end{cases} \tag{52}$$

$$\Lambda_\xi^* \mathbf{S} = \begin{cases} -M_{\alpha\beta,\alpha\beta} - h(\sigma_{\alpha\beta}\xi_{,\beta})_{,\alpha} & \text{in } \Omega \\ h\sigma_{\alpha\beta}\xi_{,\beta}n_\alpha + M_{\alpha\beta,\beta}n_\alpha + \dfrac{\partial}{\partial\tau}(M_{\alpha\beta}n_\alpha\tau_\beta) & \text{on } \Gamma, \\ M_{\alpha\beta}n_\alpha n_\beta & \text{on } \Gamma. \end{cases} \tag{53}$$

$\Lambda_N^*(\mathbf{u})$ is the conjugate operator adjoint to Λ_N:

$$\Lambda_N^*(\mathbf{u})\mathbf{S} = \begin{cases} \frac{h}{2}\left(\sigma_{\alpha\beta}\xi_{,\beta}\right)_{,\alpha} & \text{in } \Omega, \\ -\frac{h}{2}\sigma_{\alpha\beta}\xi_{,\beta}n_\alpha & \text{on } \Gamma. \end{cases} \tag{54}$$

$G(\mathbf{v}, -\Lambda_N^*(\mathbf{u})\mathbf{S})$ is the so-called gap energy function:

$$G(\mathbf{v}, -\Lambda_N^*(\mathbf{u})\mathbf{S}) = \langle \mathbf{v}, -\Lambda_N^*(\mathbf{u})\mathbf{S} \rangle = \langle -\Lambda_n(\mathbf{u})\mathbf{u}, \sigma \rangle = G(\Lambda_N(\mathbf{u})\mathbf{v}, \mathbf{S}). \tag{55}$$

For given $\mathbf{u} = (u, \xi)$, $\mathbf{v} = (v, z)$, and $\mathbf{S} = (h\sigma, M)$,

$$G(\mathbf{v}, -\Lambda_N^*(\mathbf{u})\mathbf{S}) = h\int_\Omega \frac{1}{2}\xi_{,\alpha}z_{,\beta}\sigma_{\alpha\beta}\, d\Omega. \tag{56}$$

Remark. For any given $\mathbf{v} = (v, z)$, the nonlinear operator $\Lambda(\mathbf{v})$ only depends on the vertical displacement z. So $\Lambda(\mathbf{v})$ can also be written as $\Lambda(z)$, the same as for Λ_T, Λ_N, and their conjugate operators.

Let $\mathscr{T}$ be the general force space:

$$\mathscr{T} = \left\{\mathbf{t} = (t, p) | t \in \left[\mathscr{H}^{-1}(\Omega)\right]^2, p \in \mathscr{H}^{-2}(\Omega)\right\}. \tag{57}$$

The operator $\Lambda_T^*(\mathbf{u})$: $\mathscr{S} \to \mathscr{T}$ is called the equilibrium operator. For the given external force $\bar{\mathbf{t}} = (0, 0, f)$, the equilibrium conditions can be written as

$$-\Lambda_T^*(\mathbf{u})\mathbf{S} + \bar{\mathbf{t}} = \begin{cases} \Lambda_\theta^*\mathbf{S} = 0, \\ -\Lambda_\xi^*(\mathbf{u})\mathbf{S} + f \le 0 \end{cases} \quad \text{in } \Omega. \tag{58}$$

Figure 1 shows the inner relations between spaces and operators. In geometrical linear case, $\Lambda_N(\mathbf{u}) = \Lambda_\phi(\mathbf{u}) = 0$, the equilibrium operator is

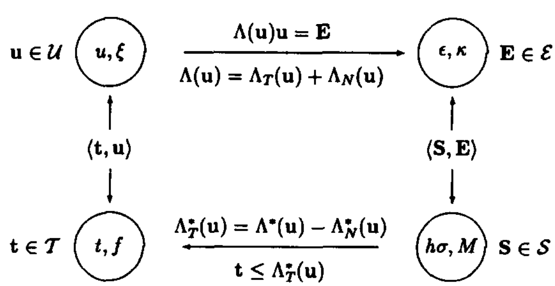

FIGURE 1

the conjugate operator Λ^* of the deformation operator $\Lambda = \{\Lambda_\theta, \Lambda_\kappa\}$. We can see here in geometrical nonlinear cases that the equilibrium operator is adjoint to the Gâteaux derivative of the deformation operator instead of itself. The symmetry between the primal system $S = \{\mathscr{U}, \mathscr{E}; \Lambda\}$ and the dual system $S^* = \{\mathscr{T}, \mathscr{S}; \Lambda_T^*\}$ is broken. In [8, 9] we can see this kind of broken symmetry was restored by the gap energy function $G(\mathbf{v}, -\Lambda_N^*(\mathbf{u})\mathbf{S})$.

5. The Dual Problem and Complementary Extremum Principles

To formulate the dual problem (P^*), we need to introduce the generalized primal function

$$\Phi(\mathbf{v}, \mathbf{E}) = W(\mathbf{E}) + F(\mathbf{v}), \tag{59}$$

where $W\colon \mathscr{E} \to \mathbb{R}$ is the internal energy,

$$W(\mathbf{E}) = \int_\Omega \frac{1}{2} D_{\alpha\beta\lambda\mu} \kappa_{\alpha\beta} \kappa_{\lambda\mu}(z)\, d\Omega + \int_\Omega \frac{h}{2} C_{\alpha\beta\lambda\mu} \varepsilon_{\alpha\beta} \varepsilon_{\lambda\mu}\, d\Omega; \tag{60}$$

$F\colon \mathscr{U} \to \mathbb{R}$ is the external energy done by the external force $\bar{\mathbf{t}} = (0, 0, f) \in \mathscr{T}$,

$$F(\mathbf{v}) = -\int_\Omega fz\, d\Omega + \Psi_{\mathscr{K}}(z) \qquad \forall \mathbf{v} = (v, z) \in \mathscr{U}, \tag{61}$$

in which, $\Psi_{\mathscr{K}}$ is the indicator function of the convex set $\mathscr{K}$:

$$\Psi_{\mathscr{K}}(z) = \begin{cases} 0 & \text{if } z \in \mathscr{K}, \\ +\infty & \text{otherwise}. \end{cases} \tag{62}$$

It is obvious that

$$\Pi(\mathbf{v}) = \Phi(\mathbf{v}, \Lambda(\mathbf{v})\mathbf{v}) \qquad \forall \mathbf{v} \in \mathscr{U} \cap \mathscr{K}. \tag{63}$$

We shall formulate the conjugate function Φ^* in the sense of Fenchel and Rockafeller:

$$\Phi^*(-\mathbf{t}, \mathbf{S}) = W^*(\mathbf{S}) + F^*(-\mathbf{t}). \tag{64}$$

Since $W\colon \mathscr{E} \to \mathbb{R}$ is convex and lower semicontinuous, it is easy to find its conjugate function by using the Legendre–Fenchel transformation,

$$\begin{aligned} W^*(\mathbf{S}) &= \sup_{\mathbf{E} \in \mathscr{E}} \{\langle \mathbf{E}, \mathbf{S} \rangle - W(\mathbf{E})\} \\ &= \int_\Omega \left[\frac{1}{2} D^{-1}_{\alpha\beta\lambda\mu} M_{\alpha\beta} M_{\lambda\mu} + \frac{h}{2} C^{-1}_{\alpha\beta\lambda\mu} \sigma_{\alpha\beta} \sigma_{\lambda\mu} \right] d\Omega \\ &= \int_\Omega w^*(\mathbf{S})\, d\Omega, \end{aligned} \tag{65}$$

where

$$w^*(\mathbf{S}) = \frac{1}{2} D^{-1}_{\alpha\beta\lambda\mu} M_{\alpha\beta} M_{\lambda\mu} + \frac{h}{2} C^{-1}_{\alpha\beta\lambda\mu} \sigma_{\alpha\beta} \sigma_{\lambda\mu}.$$

Obviously $W^*: \mathscr{S} \to \mathbb{R}$ is also convex and quadratic. According to the theory of convex analysis, we have the inverse form of the constitute equation

$$\mathbf{E} = \frac{\partial w^*(\mathbf{S})}{\partial \mathbf{S}}, \tag{66}$$

and the following relations are equivalent to each other:

$$\mathbf{S} = \frac{\partial w(\mathbf{E})}{\partial \mathbf{E}} \Leftrightarrow W(\mathbf{E}) + W^*(\mathbf{S}) = \langle \mathbf{E}, \mathbf{S} \rangle \Leftrightarrow \mathbf{E} = \frac{\partial w^*(\mathbf{S})}{\partial \mathbf{S}}.$$

From the point view of physics, W^* denotes the complementary internal energy.

In the same way, the conjugate function F^* of $F(\mathbf{u})$ should be the complementary external energy. For a given displacement $\mathbf{u} = (u, \xi)$, the conjugate variable of $\mathbf{u}$ is the external force $\mathbf{t} = (0, 0, f)$ which is determined by the equilibrium condition $\Lambda^*_T(\mathbf{u})\mathbf{S}$. So in terms of the general stress $\mathbf{S}$, the conjugate function of the external energy $F(\mathbf{v})$ can be given by the conjugate transformation:

$$F^*(-\Lambda^*_T(\mathbf{u})\mathbf{S}) = \sup_{\mathbf{v} \in \mathscr{U}} \left\{ \langle \mathbf{v}, -\Lambda^*_T(\mathbf{u})\mathbf{S} \rangle - F(\mathbf{v}) \right\}. \tag{67}$$

LEMMA 2. *For a given $\mathbf{u} \in \mathscr{U}$ and $\mathbf{S} \in \mathscr{S}$, we have*

$$F^*(-\Lambda^*_T(\mathbf{u})\mathbf{S}) = \langle \psi, -\Lambda^*_\xi(\mathbf{u})\mathbf{S} \rangle + \langle \psi, f \rangle + \Psi_{\mathscr{C}^*}(-\Lambda^*_\xi(\mathbf{u})\mathbf{S} + f) + \Psi_\Sigma(-\Lambda^*_\theta \mathbf{S}), \tag{68}$$

where $\Psi_{\mathscr{C}^*}$ is the indicator function of the polar cone $\mathscr{C}^*$,

$$\mathscr{C}^* = \left\{ w^* \in \mathscr{H}^{-2}(\Omega) \middle| \langle w^*, w \rangle \le 0 \; \forall w \in \mathscr{C} \right\}. \tag{69}$$

and Ψ_Σ is the indicator of the hyper-plane Σ,

$$\Sigma := \left\{ t \middle| t \in \left[\mathscr{H}^{-1}(\Omega) \right]^2, t = 0 \text{ in } \Omega \cup \partial\Omega \right\}. \tag{70}$$

Proof. For any given $\mathbf{v} = (v, z)$, we have

$$z \in \mathscr{K} \Leftrightarrow z = \psi + w \qquad \forall w \in \mathscr{C}.$$

From the definition of Legendre–Fenchel transformation, for a fixed $\mathbf{u} = (u, \xi)$, we have

$$F^*(-\Lambda_T^*(\mathbf{u})\mathbf{S}) = \sup_{\mathbf{v}\in\mathscr{U}} \{\langle \mathbf{v}, -\Lambda_T^*(\mathbf{u})\mathbf{S}\rangle - F(\mathbf{v})\}$$

$$= \sup_{w\in\mathscr{H}_0^2(\Omega)} \left\{\left\langle w, -\Lambda_\xi^*(\mathbf{u})\mathbf{S}\right\rangle + \langle w, f\rangle - \Psi_{\mathscr{K}}(\psi + w)\right\}$$

$$+\left\langle \psi, -\Lambda_\xi^*(\mathbf{u})\mathbf{S}\right\rangle + \langle \psi, f\rangle + \sup_{v\in[\mathscr{H}_0^1(\Omega)]^2} \{\langle v, -\Lambda_\theta^*\mathbf{S}\rangle\}$$

$$= \sup_{w\in\mathscr{C}} \left\{\left\langle w, -\Lambda_\xi^*(\mathbf{u})\mathbf{S} + f\right\rangle\right\} + \left\langle \psi, -\Lambda_\xi^*(\mathbf{u})\mathbf{S}\right\rangle + \langle \psi, f\rangle$$

$$+ \sup_{v\in[\mathscr{H}_0^1(\Omega)]^2} \{\langle v, -\Lambda_\theta^*\mathbf{S}\rangle\}.$$

However,

$$\sup_{w\in\mathscr{C}} \left\{\left\langle w, -\Lambda_\xi^*(\mathbf{u})\mathbf{S} + f\right\rangle\right\} = \Psi_{\mathscr{C}^*}\left(-\Lambda_\xi^*(\mathbf{u})\mathbf{S} + f\right)$$

$$= \begin{cases} 0 & \text{if } \Lambda_\xi^*(\mathbf{u}) - f \geq 0, \\ +\infty & \text{otherwise,} \end{cases}$$

because $\mathscr{C}$ is a positive cone. Moreover,

$$\sup_{v\in[\mathscr{H}_0^1(\Omega)]^2} \{\langle v, -\Lambda_\theta^*\mathbf{S}\rangle\} = \Psi_\Sigma(-\Lambda_\theta^*\mathbf{S}) = \begin{cases} 0 & \text{if } \Lambda_\theta^*\mathbf{S} = 0 \\ +\infty & \text{otherwise,} \end{cases}$$

So for any given $\mathbf{u} \in \mathscr{U}$, (68) is proved. Q.E.D.

Combining (65) and (68), the conjugate function Φ^* (64) of the total potential energy can be written as

$$\Phi^*(-\Lambda_T^*(\mathbf{u})\mathbf{S}, \mathbf{S}) = W^*(\mathbf{S}) + F^*(-\Lambda_T^*(\mathbf{u})\mathbf{S})$$

$$= \int_\Omega \left[\frac{1}{2} D_{\alpha\beta\lambda\mu}^{-1} M_{\alpha\beta} M_{\lambda\mu} + \frac{h}{2} C_{\alpha\beta\lambda\mu}^{-1} \sigma_{\alpha\beta}\sigma_{\lambda\mu}\right] d\Omega$$

$$+ \int_\Omega \psi\left[M_{\alpha\beta,\alpha\beta} + h\left(\sigma_{\alpha\beta}\xi_{,\beta}\right)_{,\alpha} + f\right] d\Omega$$

$$+ \Psi_{\mathscr{C}^*}\left(-\Lambda_\xi^*(\mathbf{u})\mathbf{S} + f\right) + \Psi_\Sigma(-\Lambda_\theta^*\mathbf{S}). \tag{71}$$

Unfortunately, although W^* and F^* are the complementary internal and external energies, respectively, the summation $\Phi^* = W^* + F^*$ is not the total complementary energy Π^* of the system (see [8, 9]). The difference between Π^* and Φ^* is none other than the gap energy function, i.e.,

$$\Pi^*(-\Lambda^*(\mathbf{u})\mathbf{S}) = \Phi^*(-\Lambda_T^*(\mathbf{u})\mathbf{S}, \mathbf{S}) + G(\mathbf{u}, -\Lambda_N^*(\mathbf{u})\mathbf{S}). \tag{72}$$

For a given $\mathbf{u} \in \mathscr{U} \cap \mathscr{K}, \Pi^*: \mathscr{S} \to \mathbb{R}$ is convex and quadratic. We will prove that for a stable system, the total complementary energy Π^* is greater than Φ^*, the summation of partial complementary energy functions. So the dual problem (P^*) can be described as the following.

PROBLEM 4 (P^*). For a given $\mathbf{u} \in \mathscr{U} \cap \mathscr{K}$, to find $\mathbf{S} = \{h\sigma_{\alpha\beta}, M_{\alpha\beta}\}$ such that

$$-\Pi^*(-\Lambda^*(\mathbf{u})\mathbf{S}) = \sup\{-\Pi^*(-\Lambda^*(\mathbf{u})\mathbf{T}) \mid \forall \mathbf{T} \in \mathscr{S}\}. \tag{73}$$

THEOREM 2. *For a given* $\mathbf{u} \in \mathscr{U} \cap \mathscr{K}$. *The dual problem* (P^*) (73) *has a unique solution.*

Proof. Suppose that $(\mathbf{u}, \mathbf{S})$ is the solution of FBVP. With $\mathbf{E}(\mathbf{u}) = \partial w^*/\partial \mathbf{S}$, the convexity of W^* gives

$$W^*(\mathbf{T}) - W^*(\mathbf{S}) \geq \langle \mathbf{E}, \mathbf{T} - \mathbf{S} \rangle \qquad \forall \mathbf{T} \in \mathscr{S}. \tag{74}$$

Substituting the geometrical relation $\mathbf{E} = \Lambda(\mathbf{u})\mathbf{u}$ and using the Gauss–Green theorem, we have

$$\begin{aligned} W^*(\mathbf{T}) - W^*(\mathbf{S}) &\geq \langle \Lambda_T(\mathbf{u})\mathbf{u}, (\mathbf{T} - \mathbf{S}) \rangle + \langle \Lambda_N(\mathbf{u})\mathbf{u}, \mathbf{T} - \mathbf{S}) \rangle \\ &= \langle \mathbf{u}, \Lambda_T^*(\mathbf{u})(\mathbf{T} - \mathbf{S}) \rangle - G(\mathbf{u}, \Lambda_N^*(\mathbf{u})\mathbf{T}) \\ &\quad + G(\mathbf{u}, \Lambda_N^*(\mathbf{u})\mathbf{S}) \qquad \forall \mathbf{T} \in \mathscr{S}, \end{aligned}$$

which means that

$$\begin{aligned} &\Pi^*(-\Lambda^*(\mathbf{u})\mathbf{T}) - \Pi^*(-\Lambda^*(\mathbf{u})\mathbf{S}) \\ &\quad \geq \langle \xi - \psi, \Lambda_\xi^*(\mathbf{u})(\mathbf{T} - \mathbf{S}) \rangle + \langle u, \Lambda_\theta^*(\mathbf{T} - \mathbf{S}) \rangle \\ &\qquad + \Psi_{\mathscr{C}^*}(-\Lambda_\xi^*(\mathbf{u})\mathbf{T} + f) + \Psi_\Sigma(-\Lambda_\theta^*\mathbf{T}) \\ &\quad \geq \langle \xi - \psi, \Lambda_\xi^*(\mathbf{u})\mathbf{T} - f \rangle \\ &\qquad + \langle u, \Lambda_\theta^*\mathbf{T} \rangle + \Psi_{\mathscr{C}^*}(-\Lambda_\xi^*(\mathbf{u})\mathbf{T} + f) \\ &\qquad + \Psi_\Sigma(-\Lambda_\theta^*\mathbf{T}) \qquad \forall \mathbf{T} \in \mathscr{S} \\ &\quad \geq 0 \qquad \forall \mathbf{T} \in \mathscr{S}, \end{aligned}$$

by the property of the indicator functions $\Psi_{\mathscr{C}^*}$ and Ψ_Σ. This shows that the solution of the FBVP minimizes the total complementary energy. Furthermore, for a given $\mathbf{u} \in \mathscr{U} \cap \mathscr{K}$, Π^* is strictly convex and quadratic, the theory of convex analysis ensures that the dual problem (73) has a unique solution. Q.E.D.

In the dual problem, not only the static quantities $\mathbf{S} = (h\sigma, M)$ are involved, but also the kinematic variable $\mathbf{u} \in \mathscr{U} \cap \mathscr{K}$. This fact is not surprising, since in the finite deformation theory, the kinematic field must be present in the equilibrium relation (1). In Theorem 2, the kinematic field $\mathbf{u}$ is taken from the solution of variational equation $\delta\Pi^*(-\Lambda^*(\mathbf{u})\mathbf{S}; \mathbf{v}, \mathbf{T}) = 0$ or the variational inequality of the primal problem. This is not easy work. To relax this constraint, we propose the generalized dual problem.

PROBLEM 5 (GP^*). Find ξ and $\mathbf{S}$ such that

$$-\Pi^*(-\Lambda^*(\xi)\mathbf{S}) = \sup\{-\Pi^*(-\Lambda^*(z)\mathbf{T}) \mid \forall z \in \mathscr{H}^2(\Omega), \mathbf{T} \in \mathscr{S}\}. \tag{75}$$

THEOREM 3. *For any given* $z \in \mathscr{H}^2(\Omega)$ *and* $\mathbf{T} = (h\tau, N) \in \mathscr{S}$, *if*

$$G(v, -\Lambda_N^*(v)\mathbf{T}) = h\int_\Omega \frac{1}{2}\tau_{\alpha\beta} z_{,\alpha} z_{,\beta}\, d\Omega \geq 0, \tag{76}$$

the generalized dual problem (75) *has at least one solution, which solves the FBVP* (*Problem* 1).

Proof. Suppose $(\xi, \mathbf{S})$ is one solution of FBVP. We will show that $(\xi, \mathbf{S})$ minimize Π^*. For a given $\mathbf{u} = (u, \xi)$, $\Lambda(\mathbf{u})\mathbf{u}$ is linear in u, quadratic in ξ, with $\mathbf{v} = (v, z)$ (actually only the vertical displacement z is need in the theorem), and $\delta\mathbf{v} = \mathbf{u} - \mathbf{v}$, we have from Taylor's theorem

$$\begin{aligned}\mathbf{E}(\mathbf{u}) &= \mathbf{E}(\mathbf{v}) + \delta\mathbf{E}(\mathbf{v})\delta\mathbf{v} + \tfrac{1}{2}\delta^2\mathbf{E}(\mathbf{v})\delta\mathbf{v}\delta\mathbf{v}\\ &= \Lambda_T(\mathbf{v})\mathbf{v} + \Lambda_N(\mathbf{v})\mathbf{v} + \Lambda_T(\mathbf{v})\delta\mathbf{v} - \Lambda_N(\delta\mathbf{v})\delta\mathbf{v}\\ &= \Lambda_T(\mathbf{v})\mathbf{u} + \Lambda_N(\mathbf{v})\mathbf{v} - \Lambda_N(\delta\mathbf{v})\delta\mathbf{v}.\end{aligned}$$

Substituting into Eq. (74) and using the Gauss–Green theorem, we have

$$\begin{aligned}&W^*(\mathbf{T}) - W^*(\mathbf{S})\\ &\quad\geq \langle \Lambda_T(\mathbf{v})\mathbf{u}, \mathbf{T}\rangle + \langle \Lambda_N(\mathbf{v})\mathbf{v}, \mathbf{T}\rangle - \langle \Lambda_N(\delta\mathbf{v})\delta\mathbf{v}, \mathbf{T}\rangle\\ &\qquad - \langle \Lambda_T(\mathbf{u})\mathbf{u}, \mathbf{S}\rangle - \langle \Lambda_N(\mathbf{u})\mathbf{u}, \mathbf{S}\rangle \qquad \forall\mathbf{T} \in \mathscr{S}, \forall z \in \mathscr{H}^2(\Omega)\\ &\quad= \langle \mathbf{u}, \Lambda_T^*(\mathbf{v})\mathbf{T}\rangle - \langle \mathbf{u}, \Lambda_T^*(\mathbf{u})\mathbf{S}\rangle - G(\mathbf{v}, -\Lambda_N^*(\mathbf{v})\mathbf{T})\\ &\qquad + G(\mathbf{u}, -\Lambda_N^*(\mathbf{u})\mathbf{S}) + G(\delta\mathbf{v}, -\Lambda_N^*(\delta\mathbf{v})\mathbf{T})\\ &\quad\geq \langle \xi, \Lambda_\xi^*(\mathbf{v})\mathbf{T} - f\rangle + \langle u, \Lambda_\theta^*\mathbf{T}\rangle - G(\mathbf{v}, -\Lambda_N^*(\mathbf{v})\mathbf{T})\\ &\qquad + G(\delta\mathbf{v}, -\Lambda_N^*(\delta\mathbf{v})\mathbf{T}) + G(\mathbf{u}, -\Lambda_N^*(\mathbf{u})\mathbf{S})\\ &\qquad\qquad\qquad \forall\mathbf{T} \in \mathscr{S}, \forall z \in \mathscr{H}^2(\Omega),\end{aligned}$$

which means that

$$\begin{aligned}
&\Pi^*(-\Lambda^*(\mathbf{v})\mathbf{T}) - \Pi^*(-\Lambda^*(\mathbf{u})\mathbf{S}) \\
&\quad \geq \left\langle \xi - \psi, \Lambda_\xi^*(\mathbf{v})\mathbf{T} - f\right\rangle + \langle u, \Lambda_\theta^*\mathbf{T}\rangle + G(\delta\mathbf{v}, -\Lambda_N^*(\delta\mathbf{v})\mathbf{T}) \\
&\qquad + \Psi_{\mathscr{C}^*}\left(-\Lambda_\xi^*(\mathbf{v})\mathbf{T} + f\right) + \Psi_\Sigma(-\Lambda_\theta^*\mathbf{T}) \\
&\hspace{7cm} \forall z \in \mathscr{H}^2(\Omega), \forall \mathbf{T} \in \mathscr{S} \\
&\quad = \begin{cases} +\infty & \text{if } \Lambda_\xi^*(\mathbf{v})\mathbf{T} - f < 0,\ \Lambda_\theta^*\mathbf{T} \neq 0, \\ \geq G(\delta\mathbf{v}, -\Lambda_N^*(\delta\mathbf{v})\mathbf{T}) & \text{otherwise.} \end{cases}
\end{aligned}$$

By assumption $G(\mathbf{v}, -\Lambda_N^*(\mathbf{v})\mathbf{T}) \geq 0$ for any statically admissible field $(z, \mathbf{T})$, we have

$$\Pi^*(-\Lambda^*(\mathbf{v})\mathbf{T}) - \Pi^*(-\Lambda^*(\mathbf{u})\mathbf{S}) \geq 0 \qquad \forall z \in \mathscr{H}^2(\Omega), \forall \mathbf{T} \in \mathscr{S}, \tag{77}$$

which means that the solution of FBVP $(\xi, \mathbf{S})$ minimizes Π^*. Q.E.D.

For a given $z \in \mathscr{H}^2(\Omega)$, $\Pi^*: \mathscr{S} \to \mathbb{R}$ is quadratic. On the other hand, for a given $\mathbf{T} \in \mathscr{S}$, $\Pi^*: \mathscr{H}^2(\Omega) \to \mathbb{R}$ is also quadratic. On the statically admissible space $\mathscr{C}^* \cap \Sigma$, Theorem 3 shows that the nonlinear primal variational problem (P) is actually equivalent to a semi-quadratic optimization problem:

$$\begin{aligned}
\sup\Big\{\Pi^*(-\Lambda^*(z)\mathbf{T}) \,\Big|\, &\forall z \in \mathscr{H}^2(\Omega), \forall \mathbf{T} \in \mathscr{S} \\
&\text{such that } -\Lambda_\xi^*(z)\mathbf{T} + f \in \mathscr{C}^*,\ -\Lambda_\theta^*\mathbf{T} \in \Sigma\Big\}.
\end{aligned}$$

Based on this theorem, we can develop an effective numerical method for this nonlinear free boundary value problem.

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