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Complete manifolds with nonnegative scalar curvature and the positive action conjecture in general relativity

(minimal hypersurface)

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ABSTRACT We find some integrability conditions for low-dimensional manifolds to admit metrics with nonnegative scalar curvature. In particular, we solve the positive action conjecture in general relativity in the affirmative.

This is a continuation of our previous paper (1). We generalize most of the results there to higher-dimensional manifolds. Thus, the basic problem that we are considering is to find integrability conditions for manifolds to admit metrics with nonnegative scalar curvature.

For simplicity, we shall assume that all manifolds that we are going to consider are orientable. We define a class of compact manifolds inductively in the following manner. A three-dimensional manifold M is said to be of class C if $\pi_1(M)$ has no subgroup isomorphic to the fundamental group of a compact surface of genus ≥ 1. In general, we say that an n-dimensional compact manifold M with $n \geq 4$ is of class C if either the $\hat{A}$-genus of M (see refs. 2 and 3) is zero or, for every finite covering space of M, any codimensional one homology class can be represented by a compact hypersurface of class C.

The basic theorem that we want to announce here is the following:

THEOREM 1. *Let* M *be a compact manifold with dimension* ≤ 7 *and not of class C. Then, unless* M *is flat,* M *does not admit any metric with nonnegative scalar curvature.*

In order to show that there are a lot of manifolds that are not of class C, we mention the following class of manifolds. Suppose there is a sequence of manifolds $M_n, M_{n-1}, \ldots, M_4$ and M_3 so that each M_i has dimension i and there is a continuous map from M_i onto M_{i-1} that pulls the fundamental class $[M_{i-1}]$ back to be a nonzero class in $H^{i-1}(M_i,R)$. If M_3 admits an embedded incompressible surface of genus ≥ 1, then one can prove that each M_i is not of class C for $i \geq 3$. Hence we have the following:

COROLLARY. *The only metrics with nonnegative scalar curvature on the* n*-dimensional torus with* n ≤ 7 *are metrics with zero curvature.*

There is a generalization of *Theorem 1* to the noncompact case. We mention here a case that is called to be the positive action conjecture in general relativity. It has considerable importance in S. Hawking's theory of quantum gravity.

Let M be a noncompact n-dimensional manifold with finite number of ends so that each end is diffeomorphic to R^n minus a compact set. Suppose on each end, the metric tensor g_{ij} of M can be written as $(1 + m/r^{n-2})\delta_{ij}$ plus lower-order terms, in which r is the euclidean distance function. Then we call m to be the action of this end.

THEOREM 2. *Let* M *be an asymptotically flat manifold with dimension* ≤ 7. *Suppose the scalar curvature of* M *is nonnegative. Then the action of each end of* M *is nonnegative. It is zero for some end only if* M *is flat.*

The details of the proof of *Theorems 1* and *2* will appear elsewhere. The basic proof follows our previous note. We produce a minimal hypersurface of the manifold and try to produce a contradiction if the conclusion is not correct. However, some new ideas have to be introduced here because the dimension of our minimal hypersurface is greater than two. Indeed, one can make serious mistakes by claiming that our previous procedure in producing a minimal hypersurface works without essential change. This is because in applying the second variational formula

$$\int f^2[\mathrm{Ric}(n) + \|A\|^2] \leq \int |\nabla f|^2$$

for the minimal hypersurface H, the function f has to be restricted if one is not careful in producing the minimal hypersurface. This restriction on f is so severe that our previous argument will not work. As a matter of fact, we find out that, whether the action of the manifold is positive or not, we can always produce a minimal hypersurface as in our previous note when dim $M \geq 4$. This shows that any attempt without altering the plan of our previous note will end up in a failure. Hence in our proof, we are forced to use a special kind of variational approach to produce the minimal hypersurface. After that, we have to use nontrivial arguments to show that this minimal hypersurface is asymptotically flat with vanishing mass. As before, the second variational formula still shows that

$$\int (f^2R + |\nabla f|^2) > 0,$$

in which R is the scalar curvature of the hypersurface. Hence

$$\int (f^2R + 8|\nabla f|^2) > 0.$$

When our class of function f is large enough, we can use the above inequality to conformally deform the metric on the hypersurface to one with zero scalar curvature and negative mass.

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This violates our previous theorem (1). The dimensional restriction comes from the singularity of the minimal hypersurface. We believe that it is not essential.

Note Added in Proof. Concerning the existence of metrics with positive scalar curvature, we are able to prove that the connected sum of two manifolds with positive scalar curvature also admits such a metric. In fact, we can connect such manifolds along the sphere bundle of a higher codimensional submanifold. In particular, when we do surgery on a manifold with positive scalar curvature along a sphere with codimension ≤ 3, we still obtain a manifold with positive scalar curvature. In many cases, this last fact reduces the classification of manifolds of positive scalar curvature to a topological problem.

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