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Analytic Geometry

## A Note on small deformations of balanced manifolds

*Une Note sur les petites déformations des variétés équilibrées*Jixiang Fu<sup>a</sup>, Shing-Tung Yau<sup>b</sup><sup>a</sup> Institute of Mathematics, Fudan University, Shanghai 200433, China<sup>b</sup> Department of Mathematics, Harvard University, Cambridge, MA 02138, USA

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## ABSTRACT

In this Note we prove that, under a weaker condition than the  $\partial\bar{\partial}$ -lemma, the existence of balanced metrics is preserved under small deformations. This weaker condition is satisfied on the twistor space over a compact self-dual four manifold.

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## R É S U M É

Dans cette Note nous montrons que, sous une condition plus faible que le lemme du  $\partial\bar{\partial}$ , l'existence de métriques équilibrées est préservée par des petites déformations. Cette condition affaiblie est satisfaite par l'espace des twisteurs sur une variété différentielle de dimension 4, compacte et auto-duale.

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## 1. Introduction

Let  $(X, J)$  be a compact complex  $n$ -dimensional manifold. Let  $g$  be a hermitian metric on  $X$  and  $\omega$  be the hermitian form associated to  $g$ . If  $d\omega^{n-1} = 0$ , then  $g$  or  $\omega$  is called a *balanced* metric. A complex manifold is called a balanced manifold if it admits a balanced metric. The balanced metric was first studied thoroughly by Michelsohn [9]. She especially proved that there exists an intrinsic characterization of compact complex manifolds with balanced metrics by means of positive currents.

**Theorem 1.** (See [9].) *A compact complex manifold  $X$  admits a balanced metric if and only if any positive current  $T$  on  $X$  of degree  $(1, 1)$  must be zero if it is the component of a boundary (i.e., if there exists a current  $S$  such that  $T = \partial\bar{S} + \bar{\partial}S$ ).*

Using this characterization, L. Alessandrini and G. Bassanelli proved the invariance of the existence of balanced metrics under modifications.

**Theorem 2.** (See [2,3].) *Let  $f : \bar{X} \rightarrow X$  be a proper modification of a compact complex manifold  $X$ . Then  $X$  admits a balanced metric if and only if  $\bar{X}$  admits a balanced metric.*

Another important question on balanced metrics involves the deformation invariance. In the following, we will assume that  $\{X_t \mid t \in \Delta(\epsilon)\}$  is a complex analytic family of compact complex manifolds. Here  $\Delta(\epsilon) = \{t \in \mathbb{C} \mid |t| < \epsilon\}$ . In this Note,

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we only consider the small deformation, that is, we may as well assume, if necessary,  $\epsilon$  will be small enough. It is well known [10] that any small deformation of a compact Kähler manifold is Kähler, that is, if  $X_0$  is Kähler, then  $X_t$  is also Kähler for small enough  $t$ . On the other hand, the existence of a balanced metric is not preserved under small deformations. Alessandrini and Bassanelli observed in [1] that the small deformation of the Iwasawa manifold constructed by Nakamura gives such an example. However, the second author's former student C.-C. Wu in her thesis proved that in case the complex manifold satisfies the  $\partial\bar{\partial}$ -lemma, the balanced condition is preserved under small deformations. Actually she proved

**Theorem 3.** (See [12].) *If  $X_0$  satisfies the  $\partial\bar{\partial}$ -lemma, then  $X_t$  also satisfies the  $\partial\bar{\partial}$ -lemma for small enough  $t$ .*

**Theorem 4.** (See [12].) *If  $X_0$  satisfies the  $\partial\bar{\partial}$ -lemma and admits a balanced metric, then  $X_t$  also admits a balanced metric for sufficiently small  $t$ .*

A complex manifold satisfies the  $\partial\bar{\partial}$ -lemma if for its every differential form  $\alpha$  such that  $\partial\alpha = 0$  and  $\bar{\partial}\alpha = 0$ , and such that  $\alpha = d\gamma$  for some differential form  $\gamma$ , there is some differential form  $\beta$  such that  $\alpha = i\partial\bar{\partial}\beta$ . There are several equivalent versions of the  $\partial\bar{\partial}$ -lemma, see [4]. In this Note, we consider the question of how to weaken the  $\partial\bar{\partial}$ -lemma condition such that the existence of balanced metrics is still preserved under small deformations. First we give

**Definition 5.** A compact complex  $n$ -dimensional manifold satisfies the  $(n-1, n)$ -th weak  $\partial\bar{\partial}$ -lemma if for its any real  $(n-1, n-1)$ -form  $\varphi$  such that  $\partial\varphi$  is a  $\partial$ -exact form, there exists an  $(n-2, n-1)$ -form  $\psi$  such that

$$\bar{\partial}\varphi = i\partial\bar{\partial}\psi. \quad (1)$$

Certainly the condition  $(n-1, n)$ -th weak  $\partial\bar{\partial}$ -lemma is weaker than the  $\partial\bar{\partial}$ -lemma. Our main result in this Note is

**Theorem 6.** *Let  $\{X_t \mid t \in \Delta(\epsilon)\}$  be a complex analytic family of compact complex  $n$ -dimensional manifolds. Suppose  $X_t$  satisfies the  $(n-1, n)$ -th weak  $\partial\bar{\partial}$ -lemma for small  $t \neq 0$ . If  $X_0$  admits a balanced metric, then for sufficiently small  $t$ ,  $X_t$  also admits a balanced metric.*

So when  $X_t$  for  $t \neq 0$  satisfies the  $\partial\bar{\partial}$ -lemma, then the existence of balanced metrics is preserved under small deformations. This result at present is mildly stronger than Theorem 4 since we don't know whether the  $\partial\bar{\partial}$ -lemma property is also closed under small deformations.

We should check that the small deformation of the Iwasawa manifold mentioned above does not satisfy the condition in Theorem 6. Actually if we use the notations in [1], we find that on  $X_t$  when  $t \neq 0$ ,

$$\bar{\partial}_t(\phi_{1,t} \wedge \bar{\phi}_{1,t} \wedge \phi_{3,t} \wedge \bar{\phi}_{3,t}) = t\partial_t(\phi_{3,t} \wedge \bar{\phi}_{1,t} \wedge \bar{\phi}_{2,t} \wedge \bar{\phi}_{3,t})$$

cannot be written as a  $\partial_t\bar{\partial}_t$ -exact form.

An application of the main result is

**Corollary 7.** *Let  $\{X_t \mid t \in \Delta(\epsilon)\}$  be a complex analytic family of compact complex  $n$ -dimensional manifolds. Suppose the Dolbeault cohomology group  $H^{2,0}(X_t, \mathbb{C}) = 0$  for small  $t \neq 0$ . If  $X_0$  admits a balanced metric, then there exists a balanced metric on  $X_t$  for sufficiently small  $t$ .*

**Proof.** We assume that  $t \neq 0$ . By Serre duality, we have  $H^{n-2,n}(X_t, \mathbb{C}) = 0$ . Now let  $\varphi_t$  be a real  $(n-1, n-1)$ -form on  $X_t$  such that  $\bar{\partial}_t\varphi_t$  is a  $\partial_t$ -exact form, i.e., there exists an  $(n-2, n)$ -form  $\eta_t$  such that  $\bar{\partial}_t\varphi_t = \partial_t\eta_t$ . Since  $\bar{\partial}_t\eta_t = 0$  and  $H^{n-2,n}(X_t, \mathbb{C}) = 0$ , there exists an  $(n-2, n-1)$ -form  $\psi_t$  such that  $\eta_t = i\bar{\partial}_t\psi_t$ . Therefore  $\bar{\partial}_t\varphi_t = i\partial_t\bar{\partial}_t\psi_t$ . So  $X_t$  satisfies the  $(n-1, n)$ -th weak  $\partial\bar{\partial}$ -lemma.  $\square$

**Corollary 8.** *Let  $\{X_t \mid t \in \Delta(\epsilon)\}$  be a complex analytic family of compact complex  $n$ -dimensional manifolds. If  $X_0$  admits a balanced metric and  $H^{2,0}(X_0, \mathbb{C}) = 0$ , then there exists a balanced metric on  $X_t$  for small enough  $t$ .*

**Proof.** Since the function  $h^{p,q}(t) = \dim H^{p,q}(X_t, \mathbb{C})$  is upper semicontinuous in  $t$ ,  $h^{2,0}(t) \leq h^{2,0}(0) = 0$  for small  $t$ . So  $H^{2,0}(X_t, \mathbb{C}) = 0$ . Then the corollary follows.  $\square$

It is well known that the twistor space  $Z$  associated to a compact self-dual four manifold is a complex manifold and the natural metric on it is a balanced metric (cf. [9,7]). Moreover, M. Eastwood and M. Singer in [5] observed  $H^{2,0}(Z, \mathbb{C}) = 0$ . So above corollary implies

**Corollary 9.** *Let  $Z$  be the twistor space associated to a compact self-dual four manifold. Then any small deformation of  $Z$  admits a balanced metric.*

Up to now, the global deformation stability of existence of balanced metrics is unclear (cf. [13,11]). We shall use above corollary to study this question.

**2. The proof of the main theorem**

Let  $X_0$  be a compact complex manifold with a balanced metric  $\omega$ . Let  $\pi : \mathcal{X} \rightarrow \Delta(\epsilon)$  be a small deformation of  $X_0$ . That is,  $\pi : \mathcal{X} \rightarrow \Delta(\epsilon)$  is a holomorphic map with maximal rank so that  $\pi$  is proper and each fiber  $X_t = \pi^{-1}(t)$  has the structure of a complex manifold which varies analytically with  $t$  (cf. [10]). Let  $\phi_t : X_t \rightarrow X_0$  be a diffeomorphism which varies smoothly with  $t$  and  $\phi_0$  is the identity map. Since  $\omega^{n-1}$  is a real  $d$ -closed  $(n-1, n-1)$ -form on  $X$ ,  $\Omega_t = \phi_t^* \omega^{n-1}$  is a real  $d$ -closed  $(2n-2)$ -form on  $X_t$ . We decompose  $\Omega_t$  as

$$\Omega_t = \Omega_t^{n-2,n} + \Omega_t^{n-1,n-1} + \Omega_t^{n,n-2}.$$

Then we have the following facts:

- (1)  $\Omega_t^{n-1,n-1}$  is real and approaches  $\omega^{n-1}$  as  $t \rightarrow 0$ ;
- (2)  $\Omega_t^{n-2,n} = \Omega_t^{n,n-2}$  and  $\Omega_t^{n-2,n}$  approaches zero as  $t \rightarrow 0$ ;
- (3) Since  $\Omega_t$  is  $d$ -closed, we have

$$\bar{\partial}_t \Omega_t^{n-1,n-1} + \partial_t \Omega_t^{n-2,n} = 0.$$

According to (1),  $\Omega_t^{n-1,n-1}$  is strictly positive definite for sufficiently small  $t$ . Then there exists a hermitian metric  $\omega_t$  on  $X_t$  such that  $\omega_t^{n-1} = \Omega_t^{n-1,n-1}$ . This observation is due to Michelsohn [9]. Clearly  $\omega_0 = \omega$  and  $\omega_t$  approaches  $\omega$  smoothly and uniformly as  $t \rightarrow 0$ . We use this metric  $\omega_t$  as the background metric on  $X_t$ .

If  $X_t$  satisfies the  $(n-1, n)$ -th weak  $\partial\bar{\partial}$ -lemma, then (3) implies that there exists an  $(n-2, n-1)$ -form  $\Psi_t$  on  $X_t$  such that

$$i\partial_t \bar{\partial}_t \Psi_t = \bar{\partial}_t \Omega_t^{n-1,n-1} = -\partial_t \Omega_t^{n-2,n}. \tag{2}$$

We can choose  $\Psi_t$  such that

$$\Psi_t \perp_{\omega_t} \ker(i\partial_t \bar{\partial}_t).$$

We let

$$\tilde{\Omega}_t = \Omega_t^{n-1,n-1} + i\partial_t \Psi_t - i\bar{\partial}_t \bar{\Psi}_t.$$

Then  $\tilde{\Omega}_t$  is a  $d$ -closed real  $(n-1, n-1)$ -form. If we can prove  $\tilde{\Omega}_t$  is strictly positive definite for sufficiently small  $t$ , then there exists a hermitian metric  $\tilde{\omega}_t$  such that  $\tilde{\omega}_t^{n-1} = \tilde{\Omega}_t$  and  $d\tilde{\omega}_t^{n-1} = 0$ . That is, we have gotten a balanced metric  $\tilde{\omega}_t$  on  $X_t$  as desired.

So we only need to prove the positivity of  $\tilde{\Omega}_t$ . To this end we introduce the Kodaira–Spencer operator  $E_t$  on  $\Lambda^{n-1,n}(X_t)$ , i.e., on the set of all  $(n-1, n)$ -forms on  $X_t$

$$E_t = \partial_t \bar{\partial}_t \bar{\partial}_t^* \partial_t^* + \partial_t^* \bar{\partial}_t \bar{\partial}_t^* \partial_t + \partial_t^* \partial_t,$$

where the Hodge-star operator  $*$ , which we have dropped the subscript  $t$ , is defined by the metric  $\omega_t$ . We consider the equation

$$E_t(\gamma_t) = -\partial_t \Omega_t^{n-2,n} (= \bar{\partial}_t \Omega_t^{n-1,n-1}). \tag{3}$$

We first check that  $\partial_t \Omega_t^{n-2,n} \perp_{\omega_t} \ker E_t$ . In [8], Kodaira and Spencer proved that  $E_t$  is self-adjoin, strongly elliptic of order 4, and a form  $\alpha_t \in \ker E_t$  if and only if

$$\partial_t \alpha_t = 0 \quad \text{and} \quad \bar{\partial}_t^* \partial_t^* \alpha_t = 0.$$

Then by (2), for any  $\alpha_t \in \ker E_t$ , we have

$$(-\partial_t \Omega_t^{n-2,n}, \alpha_t) = (i\partial_t \bar{\partial}_t \Psi_t, \alpha_t) = (i\Psi_t, \bar{\partial}_t^* \partial_t^* \alpha_t) = 0.$$

Therefore, by the theory of elliptic operators, there exists a unique smooth solution  $\gamma_t$  of Eq. (3) such that  $\gamma_t \perp_{\omega_t} \ker E_t$ .

As in [6], we can prove

$$\partial_t \gamma_t = 0 \quad \text{and} \quad i\Psi_t = \bar{\partial}_t^* \partial_t^* \gamma_t. \tag{4}$$

For readers' convenience, we prove these two identities here. From (2) and (3), we get  $E_t(\gamma_t) - i\partial_t \bar{\partial}_t \Psi_t = 0$ , which, from the definition of the operator  $E_t$ , is equivalent to

$$\partial_t \bar{\partial}_t (\bar{\partial}_t^* \partial_t^* \gamma_t - i\Psi_t) + \partial_t^* (\bar{\partial}_t \bar{\partial}_t^* + 1) \partial_t \gamma_t = 0.$$

By taking the  $L^2$ -norm of the left-hand side, we get

$$\partial_t \bar{\partial}_t (\bar{\partial}_t^* \partial_t^* \gamma_t - i\Psi_t) = 0 \quad \text{and} \quad \partial_t^* (\bar{\partial}_t \bar{\partial}_t^* + 1) \partial_t \gamma_t = 0. \quad (5)$$

On the other hand, for any  $\phi \in \ker(i\partial_t \bar{\partial}_t)$ , we have

$$(\bar{\partial}_t^* \partial_t^* \gamma_t, \phi) = (\gamma_t, \partial_t \bar{\partial}_t \phi) = 0.$$

Since  $\Psi_t \perp_{\omega_t} \ker(i\partial_t \bar{\partial}_t)$ ,

$$(\bar{\partial}_t^* \partial_t^* \gamma_t - i\Psi_t) \perp_{\omega_t} \ker(i\partial_t \bar{\partial}_t). \quad (6)$$

Combining (5) with (6), we obtain  $\bar{\partial}_t^* \partial_t^* \gamma_t - i\Psi_t = 0$ , which is the first identity in (4). The second in (4) follows from the second equality of (5), since

$$0 = \int_{X_t} \langle \partial_t^* (\bar{\partial}_t \bar{\partial}_t^* + 1) \partial_t \gamma_t, \gamma_t \rangle = \int_{X_t} (|\bar{\partial}_t^* \partial_t \gamma_t|^2 + |\partial_t \gamma_t|^2).$$

Now we can estimate  $\|i\partial_t \Psi_t\|_{C^0(\omega_t)}$  by elliptic estimates. Since the background metric  $\omega_t$  varies smoothly with  $t$  and approaches  $\omega_0$  as  $t \rightarrow 0$ , we have a uniform constant  $C$  such that

$$\|i\partial_t \Psi_t\|_{C^0(\omega_t)} \leq C \|\gamma_t\|_{C^3(\omega_t)} \leq C \|\partial_t \Omega_t^{n-2,n}\|_{C^{0,\alpha}(\omega_t)} \quad (7)$$

for some  $0 < \alpha < 1$ . Since also  $\Omega_t^{n-2,n}$  varies smoothly with  $t$  and approaches zero as  $t \rightarrow 0$  and the complex structure on  $X_t$  varies analytically with  $t$ ,  $\|\partial_t \Omega_t^{n-2,n}\|_{C^{0,\alpha}(\omega_t)}$  approaches zero uniformly as  $t \rightarrow 0$ . Thus  $\|i\partial_t \Psi_t\|_{C^0(\omega_t)}$  approaches zero as  $t \rightarrow 0$ . Therefore  $\bar{\partial}_t$  is strictly positive definite when  $t$  is small enough.

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