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Azumaya-type noncommutative spaces and morphisms therefrom: Polchinski's D-branes in string theory from Grothendieck's viewpoint

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Abstract

Grothendieck's equivalence of a commutative function ring and a local geometric space gives rise to the language of schemes and functor of points in 1960s that rewrote commutative algebraic geometry while Polchinski's identification/recognition in 1995 of D-branes – studied since the second half of 1980s as boundary conditions for open strings – as the source of Ramond-Ramond fields created by closed superstrings in the space-time rewrote string theory. In this work, we explain how a noncommutative version of Grothendieck's equivalence gives rise to a prototype intrinsic definition of D-branes that can reproduce the key, originally open-string-induced, properties of D-branes described in Polchinski's works. After the discussion of Azumaya-type noncommutative spaces and morphisms therefrom that form the algebro-geometric foundation of the current work, basic properties of D0-branes on a smooth curve/surface or a quasi-projective variety, the associated Chan-Paton modules, the Higgsing/un-Higgsing behavior – all under the current setting –, and their relation with Hilbert schemes and Chow varieties are given. When applied to the case of D0-branes on a (commutative) projective complex smooth surface, this gives also a picture in the current pure algebro-geometric setting that resembles gas of D0-branes in a work of Vafa. Related supplementary discussions/remarks are given in footnotes.

Key words: D-brane, Polchinski; noncommutative geometry, Azumaya, Grothendieck; D0-brane, moduli space.

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Chien-Hao Liu dedicates this work to his teacher Ann L. Willman,
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– the grace, courage, will power, and inner peace while in the turmoil of life –
throughout the treatment of her cancer.

0. Introduction and outline.

Introduction.

A *D-brane* (in full name: *Dirichlet brane* or *Dirichlet membrane*)[1] in string theory is by definition (i.e. by the very word 'Dirichlet') a boundary condition for the end-points of open strings. From the viewpoint of the field theory on the open-string world-sheet aspect, it is a boundary state in the $d = 2$ conformal field theory with boundary. From the viewpoint of open string target space(-time) M, it is a cycle or a union of submanifolds Z in M with a gauge bundle (on Z) that carries the Chan-Paton index for the end-points of open strings. For the second viewpoint, Polchinski recognized in 1995 in [Pol2] that a D-brane is indeed a source of the Ramond-Ramond fields on M created by the oscillations of closed superstrings in M. In particular, in a specific region of the Wilson's theory-space for D-branes, D-branes can be identified with the solitonic/black branes studied earlier[2] in supergravity and (target) space-time aspect of superstrings. This recognition is so fundamental that it gave rise to the second revolution of string theory. When M is compactified on a Calabi-Yau space Y, the preservation of supersymmetries in either the field theory on the open-string world-sheet or in the effective field theory after the compactification requires the D-brane to be supported on a union of Lagrangian submanifolds/subspaces or holomorphic cycles, (cf. [B-B-St], [H-I-V], and [O-O-Y]). When we focus only on the internal/compactified part of space-time, this gives us a preliminary mathematical definition of supersymmetric D-branes as a union of Lagrangian submanifolds with gauge bundles or a coherent (possibly torsion) sheaf on Y. While such definitions of D-branes is already very convenient in the study of superstring theory with branes and of stringy dualities, they are not adequate to serve as the intrinsic definition of D-branes as, among other issues, in general they cannot reproduce by themselves a key property of D-branes – the Higgsing/un-Higgsing behavior of D-branes – in its own mathematical framework in a natural way.

This subtlety actually does not seem to bother string theorists, likely for two reasons:

(1) The picture of supersymmetric D-branes as cycles in Y with a gauge bundle is generically correct/enough in the regime where branes are still branes.

(2) Under deformations of D-branes for which the mathematical picture in Item (1) is not complete enough to dictate the details, the very definition of D-branes as where open strings end tells us that we can look at the related open string theory, particularly its induced fields and their effective action on the brane, to determine what happens to the deformed D-branes.

Depending on one's taste/weight on such a subtlety, one is either satisfied with this picture or not. And if not, one is led to the following question:

- **Q. [D-brane]** *What is a D-brane intrinsically?*

[1] D-brane theory and open string theory are in a way counterpart to and interacting with each other. As a consequence, supersymmetric D-brane theory and open Gromov-Witten theory are closely related. In a train of communications with Duiliu-Emanuel Diaconescu [Dia] on a vanishing lemma in the last section of [L-Y3] and its comparison with [D-F], he drew our attention to the important distinction between pure open GW-invariants and open-string world-sheet instantons. The former depends only on the boundary condition set up on the stable maps by supersymmetric D-branes and a decoration on the brane (cf. [L-Y2: Sec. 7.2]) while the latter may interact via Wilson loops with the *general* gauge fields on the D-branes as well (cf. [Wi2: Sec. 4.2 and Sec. 4.4] and [D-F: introduction part of Sec. 3]). Thus, *D-brane theory and the field theory thereupon are a part in understanding open-string world-sheet instantons beyond the pure Gromov-Witten sector.* We attribute this footnote to him and thank him for the patient explanations of [D-F] to us.

[2] See [D-K-L] for a review and more references.

In other words, what is the intrinsic definition of D-branes so that by itself it can produce the properties of D-branes that are consistent with, governed by, or originally produced by open strings as well? This is the guiding question of the current work.

The answer to this question is indeed already suggested by string theorists: it is hinted already in the works (e.g. [Pol3]) of Polchinski and later put with even more weight by other string theorists[3] that D-branes have a close tie with noncommutative geometry. One cannot expect to have a good answer to Question [D-brane] without bringing appropriate noncommutative geometry into the intrinsic definition of D-branes. Indeed, Polchinski's *description of deformations of stacked D-branes* together with Grothendieck's *local equivalence of rings and spaces/geometries* and the notion of *functors of points* (Sec. 2.1) implies immediately (Sec. 2.2):

· **Polchinski-Grothendieck Ansatz [D-brane: noncommutativity].** *The world-volume of a D-brane carries a noncommutative structure locally associated to a function ring of the form $M_n(R)$, i.e., the $n \times n$ matrix-ring over a ring R for some $n \in \mathbb{Z}_{\geq 1}$.*

This brings us to a technical world in mathematics: noncommutative geometry. Due to the different languages used in differential geometry and in algebraic geometry for noncommutative geometry (though the philosophy to equate locally a space and a function ring in each category is in common), we focus now on supersymmetric D-branes of B-type, for which algebro-geometric language is appropriate.

From the basic properties of D-branes spelt out explicitly in the work of Polchinski, there are a special class of noncommutative spaces that are particularly related to D-branes, namely the Azumaya-type noncommutative spaces. These are the noncommutative spaces that locally have their function ring the matrix ring $M_n(R)$ over a commutative ring R. The ansatz of Grothendieck on the equivalence of a ring and a local geometry, when extended to the noncommutative case as well, enables us to directly look at rings themselves without having to deal with the technical subtle issue of the functorial construction of an associated space (i.e. a set of points with topology and other structures) to a ring as Grothendieck did in 1960s for commutative rings that rewrote commutative algebraic geometry. His ansatz of the contravariant equivalence of morphisms-between-spaces and morphisms-between-rings-locally, and the ansatz of composability, which says that the composition of morphisms $X \to Y$, $Y \to Z$ between spaces should be a morphism $X \to Z$, can then be used to give the notion of morphisms from an Azumaya-type noncommutative space to a (either commutative or noncommutative) space without having the spaces themselves. In this way, an Azumaya-type noncommutative space X can be phrased purely as a gluing system $\mathcal{R}$ of matrix rings and a morphism from X can be phrased purely as a gluing system of ring-homomorphisms to $\mathcal{R}$. A quasi-coherent sheaf on X is then a gluing system of modules over rings in $\mathcal{R}$. (Sec. 1.)

Once this language is formulated precisely, the following prototype definition of D-branes (of B-type and when a "brane" is still a brane) (Definition 2.2.3):

· **Definition [D-brane].** A *D-brane* is an Azumaya-type noncommutative space X with a fundamental module (i.e. the *Chan-Paton sheaf*) of its noncommutative structure sheaf. A *D-brane on an open-string target-space* Y is the image of a morphism from such an X to Y with the push-forward Chan-Paton sheaf.

alone gives a Higgsing/un-Higgsing property of D-branes in its own right that is consistent and originally deduced via open strings in the work of Polchinski; (Sec. 2.2 for highlights for general D-branes; Sec. 3.2 for the case of D0-branes; and Sec. 4.1 - 4.4 for D0-branes on a commutative

[3]See, for example, [Dou4] and [Dou5] of Douglas and [S-W2] of Seiberg and Witten for the development and more references up to 1999.

quasi-projective space). In particular, except that we have to stay on algebraic groups in the pure algebro-geometric setting, D0-branes in the current setting that move on a (commutative) smooth complex projective surface Y has the same Higgsing/un-Higgsing feature of gas of D0-branes in [Vafa1] of Vafa when we choose the morphims of the D0-brane to Y appropriately; (the last theme in Sec. 4.4). The anticipation (Sec. 4.5) that:

- **Anticipation [universal moduli space from D-branes].** *The moduli space of D-branes – or in general of D-branes coupled with NS-branes when defined correctly – on a target space should encompass simultaneously several standard moduli spaces in commutative geometry.*

is supported in the study of the moduli space of D0-branes; (Sec. 3 and Sec. 4.1- Sec. 4.4).

Finally, a word about reading the current work: Noncommutative geometry, in the language of either differential geometry or algebraic geometry, is a demanding topic and there is no way to bypass it. Readers who already know D-branes in the string-theoretic aspect from [Pol3] or [Pol4] are suggested to read Sec. 4.1 first to see how algebraic geometry in the line of Grothendieck is used to implement Polchinski's picture in a most elementary case: D0-branes on the complex line $\mathbb{C}$. Various general features of D-branes and their moduli space, following the above prototype definition, reveal themselves already in this example in a simplified form.

Remark 0.1 [*diverse D-"branes"*]. Mathematicians should be aware that there are numerous string theorists whose collective contribution shaped the understanding of D-branes nowadays, cf. the limited "short" list of stringy references of the current work, which have influenced us and became part of the background of the project. Their works led to diverse meanings/roles of D-branes in various physical contents. The current work addresses D-branes when they are "still branes", i.e. in the sense of [D-L-P], [P-C], [Pol2], [Pol3], [Pol4], and, e.g., [B-V-S1], [B-V-S2], [Vafa1], [Vafa2] that they are manifold/variety-type objects. The terms 'Polchinski's D-brane' and 'D0-brane gas' occasionally used in this work refer to [D-L-P], [Vafa1], and Polchinski's special contribution to this topic. Physicists use the same term 'D-branes' in the various different physical contents with good reasons, particularly from the aspect of stringy dualities. However, this is unfortunate/inconvenient for us as these other types of D-"branes" are no longer branes and have/involve very different mathematical contents/language as well. Lacking an official terminology, we use above-mentioned terms and terms like 'D-branes in the sense of Polchinski' to single out the particular meaning/type of D-branes studied in the above-quoted stringy works in the earlier years of D-branes for convenience.

Remark 0.2 [*other brane*]. It should be mentioned that, while D-branes have been a central object in string theory since 1995, there are other types of branes, (e.g.. NS-branes) in string theory as well that serve as the source for other types of fields created by closed strings in space-time; see [Pol4], [Jo], and [B-B-Sc] for a review. It is also worth noting that, since the work of Randall and Sundrum [R-S] in 1999, the use of branes has been extended outside of string theory and gives a new insight to the weakness of gravity in comparison with electro-magnetic, weak, and strong interactions in nature. That route hints at a connection of hyperbolic geometry and branes – a topic in its own right.

Convention. Standard notations, terminology, operations, facts in (1) (noncommutative; commutative) ring theory; (2) (commutative) algebraic geometry; (3) quantum field theory, supersymmetry; string theory can be found respectively in (1) [Jac]; [Mat]; (2) [Ha], [E-H]; (3) [I-Z], [P-S], [W-B]; [B-B-Sc], [G-S-W], [Jo], [Pol4], [Zw].

- Except the *zero-ring* 0, all *rings* or *algebras* (over an algebraically closed field) R in the general discussion of this work are *associative* with an *identity* 1 and are both *left- and* *right-Noetherian*. The term "*R-modules*", including "*ideals*" in R, means "*left* R-modules" (cf. *left* ideals in R) unless otherwise noted. $Z(R) :=$ the *center* of R. $M_n(R) :=$ the $n \times n$ *matrix ring* with entries in R.

· The term *field* has two completely different meanings: field in *quantum field theory* vs. field in the *theory of rings*.

· The *analytic space* $\mathbb{C}^n$, with the standard topology, of closed points in the affine space $\mathbb{A}^n$ over $\mathbb{C}$ is constantly denoted directly by $\mathbb{A}^n$. Similarly for $\mathbb{P}^n$ and other varieties. (In this work, we use the term 'varieties/schemes' mainly only to manifest/emphasize the fact that they arise from gluing of affine charts associated to rings.) In this way, $\mathbb{C}^n$ is kept to mean $\mathbb{C}^n$ as a $\mathbb{C}$- or $M_n(\mathbb{C})$-*module* as best possible. $\mathbb{C}^n$ as the n-th *product ring* of $\mathbb{C}$ will be denoted also by $\prod_n \mathbb{C}$.

· A representation (resp. commuting) scheme with the reduced scheme structure will be called *representation* (resp. *commuting*) *variety* for simplicity. Irreducibility is *not* implied here. (In fact, in general they are not irreducible.)

· Omitted subscripts (resp. superscripts) are indicated by $_\bullet$ (resp. $^\bullet$).

Outline.

1 Azumaya-type noncommutative spaces and morphisms therefrom.

We introduce in Sec. 1.1 a class of noncommutative spaces that are relevant to D-branes. Its foundation, central localizations of noncommutative rings, is given in Sec. 1.2. The ring-theoretic description of a space in Sec. 1.2 allows us to study as well the space of morphisms between noncommutative spaces without having to construct the noncommutative spaces.

1.1 Azumaya-type noncommutative spaces and morphisms therefrom.

Definition 1.1.1 [Azumaya-type noncommutative space]. An *Azumaya-type noncommutative space* is a triple $(X, \mathcal{O}_X, \mathcal{O}_X^{nc})$, where $(X, \mathcal{O}_X)$ is a (commutative Noetherian) scheme, as defined in [Ha], and $\mathcal{O}_X^{nc}$ is a coherent sheaf of noncommutative $\mathcal{O}_X$-algebras[4] on X that contains $\mathcal{O}_X$ by $1 \cdot \mathcal{O}_X$ in its center $\mathcal{Z}(\mathcal{O}_X^{nc})$. We will call $\mathcal{O}_X$ (resp. $\mathcal{O}_X^{nc}$) the *commutative* (resp. *noncommutative*) *structure sheaf* of X. A *strict morphism* from $(X, \mathcal{O}_X, \mathcal{O}_X^{nc})$ to $(Y, \mathcal{O}_Y, \mathcal{O}_Y^{nc})$ is a triple $(f, f^\sharp, f^{\sharp nc})$, where $(f : X \to Y\,,\, f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X)$ gives a morphism of schemes from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ and $f^{\sharp nc} : \mathcal{O}_Y^{nc} \to f_*\mathcal{O}_X^{nc}$ is a homomorphism of $\mathcal{O}_Y$-algebras that extends $f^\sharp$. A *general morphism* from $(X, \mathcal{O}_X, \mathcal{O}_X^{nc})$ to $(Y, \mathcal{O}_Y, \mathcal{O}_Y^{nc})$ consists of the following data:

- an inclusion pair $\mathcal{O}_X \subset \mathcal{A} \subset \mathcal{A}^{nc} \subset \mathcal{O}_X^{nc}$ of $\mathcal{O}_X$-subalgebras such that $\mathcal{A} \subset \mathcal{Z}(\mathcal{A}^{nc})$;
- a strict morphism $(f, f^\sharp, f^{\sharp nc})$ from $(X', \mathcal{O}_{X'}, \mathcal{O}_{X'}^{nc})$ to $(Y, \mathcal{O}_Y, \mathcal{O}_Y^{nc})$, where
  - $X' := \mathbf{Spec}\,\mathcal{A}$ is equipped with the tautological dominant finite morphism $X' \xrightarrow{\jmath} X$ of schemes,
  - $\mathcal{O}_{X'}^{nc}$ is the $\mathcal{O}_{X'}$-algebra on X' associated to $\mathcal{A}^{nc}$ as an $\mathcal{A}$-algebra.

A strict morphism is automatically a general morphism. A general morphism will also be called simply a *morphism*. Define $Mor(X, Y)$ to be the set of morphisms from X to Y. To simplify the notation, we will also denote $(X, \mathcal{O}_X, \mathcal{O}_X^{nc})$ collectively by X and both a strict morphism $(f, f^\sharp, f^{\sharp nc})$ and a general morphism $((\mathcal{A}, \mathcal{A}^{nc}), (f, f^\sharp, f^{\sharp nc}))$ collectively by $f : X \to Y$.

Definition/Example 1.1.2 [tautological morphism/surrogate]. With notations from Definition 1.1.1, the (strict) identity morphism $(X', \mathcal{O}_X, \mathcal{O}_X^{nc}) \to (X', \mathcal{O}_X, \mathcal{O}_X^{nc})$ defines a (general) morphism $X = (X, \mathcal{O}_X, \mathcal{O}_X^{nc}) \to X' = (X', \mathcal{O}_X, \mathcal{O}_X^{nc})$. Given X, we will call an $X \to X'$ arising this way a *tautological morphism* from X and X' an *surrogate* of X.

Example 1.1.3 [noncommutative point]. Let k be an algebraically closed field and $M_n(k)$ be the k-algebra of $n \times n$-matrices with entries in k. Then, $X = (Spec\, k, k, M_n(k)) =: Space\, M_n(k)$ defines an *Azumaya-type noncommutative point*. See Sec. 3.1 for more details.

Example 1.1.4 [morphism of commutative schemes]. An Azumaya-type noncommutative space $X = (X, \mathcal{O}_X, \mathcal{O}_X^{nc})$ is a commutative scheme if and only if $\mathcal{O}_X = \mathcal{O}^{nc}$. In this case, X has no surrogates except X itself and any morphism from X to $Y = (Y, \mathcal{O}_Y, \mathcal{O}_Y^{nc})$ is a strict morphism from X to Y. In particular, the natural inclusion $\mathcal{S}cheme \hookrightarrow \mathcal{A}zumaya\mathcal{S}pace$ of the category of commutative schemes into the category of Azumaya-type noncommutative spaces is fully faithful.

[4] The category of noncommutative algebras includes also commutative algebras. We will call a sheaf $\mathcal{G}$ of $\mathcal{O}_X$-algebras simply an $\mathcal{O}_X$-algebra. The *center* $\mathcal{Z}(\mathcal{G})$ of $\mathcal{G}$ is, by definition, the sheaf associated to the presheaf that assigns to each open set U of X the sub-$\mathcal{O}_X(U)$-algebra $Z(\mathcal{G}(U))$ of $\mathcal{G}(U)$.

The foundation of Definition 1.1.1 (i.e. of the sheaf $\mathcal{O}_X^{nc}$) is on central localizations of (noncommutative) rings. This will be discussed in Sec. 1.2. The following lemma follows immediately from the definition:

Lemma 1.1.5 [exhaustion]. *Let X and Y be Azumaya-type noncommutative spaces and X' be a surrogate of X. Then there is a canonical embedding $Mor(X',Y) \hookrightarrow Mor(X,Y)$.*

Remark 1.1.6 [noncommutative geometry]. Noncommutative algebraic geometry was developed with vigor by several schools of mathematicians immediately after Grothendieck's re-writing of commutative algebraic geometry in the 1960s. There are several classes of noncommutative spaces in existence; each is described in its own appropriate language. While many demanding fundamental issues have prevented it from reaching at the moment the same glory and a unified language as its commutative counterpart from Grothendieck's school, it is a constant growing subject. Readers are referred to, e.g. (in rough historical order) [Go], [vO-V], [A-Z], [J-V-V], [Ro1], [Ro2], [K-R1], [K-R2] from the algebraic aspect; [Co] from the analytic aspect; and [Man2], [Man3], [Kapr], [Lau], [leB1] from other aspects for details and more references.

Remark 1.1.7 [Azumaya-type noncommutative space]. The class of noncommutative spaces we define here, namely $(X,\mathcal{O}_X,\mathcal{O}_X^{nc})$, are chosen with D-branes in mind. While they may be thought of as noncommutative "clouds" (i.e. $\mathcal{O}_X^{nc}$) over (commutative) schemes (i.e. $(X,\mathcal{O}_X)$), the way we define a morphism from X to Y says that the main object of focus in the triple $(X,\mathcal{O}_X,\mathcal{O}_X^{nc})$ is $\mathcal{O}_X^{nc}$, rather than $(X,\mathcal{O}_X)$. This particular point is important in the realization of a D-brane of B-type as an Azumaya-type noncommutative space. We suggest readers to think of

$$(X,\mathcal{O}_X,\mathcal{O}_X^{nc}) \qquad \text{as} \qquad \begin{array}{l} \mathcal{O}_X^{nc}\,, \quad \textit{together with } \text{the system } L_{\mathcal{O}_X^{nc}} \text{ of sub-}\mathcal{O}_X\text{-algebra pairs:} \\ L_{\mathcal{O}_X^{nc}} \;=\; \left\{ (\mathcal{A},\mathcal{A}^{nc}) \;\middle|\; \begin{array}{l} \mathcal{O}_X \subset \mathcal{A} \subset \mathcal{A}^{nc} \subset \mathcal{O}_X^{nc}\,; \\ \mathcal{A},\,\mathcal{A}^{nc}\colon \text{sub-}\mathcal{O}_X\text{-algebras}\,;\ \mathcal{A} \subset \mathcal{Z}(\mathcal{A}^{nc}) \end{array} \right\}. \end{array}$$

I.e. $(X,\mathcal{O}_X,\mathcal{O}_X^{nc})$ together with the system $\{X \to X'\}_{X'}$ of surrogates in $\mathcal{A}zumaya\mathcal{S}pace$.

Example 1.1.8 [noncommutative point revisitd]. (Continuing Example 1.1.3.) A surrogate of the Azumaya-type noncommutative point $Space\,M_n(k)$ over k is given by a sub-k-algebra pair $k \subset C \subset R \subset M_n(k)$ with $C \subset Z(R)$. In particular, while $Space\,M_n(k)$ consists geometrically of only one point (i.e. $Spec\,k$), its surrogate $X' = (Spec\,C, C, R)$ can have more than one geometric points in $Spec\,C$. All these X''s should be thought of as part of the "geometry" of noncommutative point $Space\,M_n(k)$.

Definition 1.1.9 [left/right quasi-coherent/coherent sheaf]. A *left quasi-coherent sheaf* on $(X,\mathcal{O}_X,\mathcal{O}_X^{nc})$ is a sheaf of left $\mathcal{O}_X^{nc}$-modules that is quasi-coherent on $(X,\mathcal{O}_X)$. Similarly for the definitin of a *right quasi-coherent sheaf*, a *left coherent sheaf*, and a *right coherent sheaf* on $(X,\mathcal{O}_X,\mathcal{O}_X^{nc})$. A $\mathcal{O}_X^{nc}$*-module* is by convention a left $\mathcal{O}_X^{nc}$-module.

Hidden in the notion of $\mathcal{O}_X^{nc}$ in the tuple $(X,\mathcal{O}_X,\mathcal{O}_X^{nc})$ is the notion of central localizations, which we will discuss more thoroughly in Sec. 1.2.

1.2 A noncommutative space as a gluing system of rings.

The purely ring[5]-theoretic construction in this subsection enables us to talk about a "noncommutative scheme" without having to construct one[6]. The ring system to be defined is meant to carry the same information as the noncommutative scheme associated to $\mathcal{O}_X^{nc}$ in $X = (X, \mathcal{O}_X, \mathcal{O}_X^{nc})$ would. Such a description will later be used to study $Mor(X, Y)$. Behind the messy notations is the notion of Grothendieck-descent-data description of spaces/stacks/sheaves and morphisms between them.

Noncommutative localizations.

The notion of noncommutative localizations can be traced back to Ore in [Or1] and [Or2] in 1930s. Here we recall only definitions that will be needed later. See e.g. [Ga], [Goldm], [Jat], [St] for more details and thorough discussions.

A *Gabriel filter* on a ring R is a collection $\mathfrak{F}$ of ideals in R that satisfies[7]:

(1) if $I \in \mathfrak{F}$ and J is an ideal that contains I, then $J \in \mathfrak{F}$;

(2) if I, $J \in \mathfrak{F}$, then $I \cap J \in \mathfrak{F}$;

(3) if $I \in \mathfrak{F}$, then $(I : r) \in \mathfrak{F}$ for $r \in R$;

(4) if $I \in \mathfrak{F}$ and J is an ideal such that $(J : r) \in \mathfrak{F}$ for all $r \in I$, then $J \in \mathfrak{F}$.

Each Gabriel filter $\mathfrak{F}$ on R determines the subcategory $\mathcal{T}_{\mathfrak{F}}$ of $\mathfrak{F}$-*torsion* objects and the subcategory $\mathcal{F}_{\mathfrak{F}}$ of $\mathfrak{F}$-*torsion-free* objects in the category R-*Mod* of (left) R-modules. An object M in $\mathcal{T}_{\mathfrak{F}}$ is characterized by that each element m of M has its annihilator $Ann(m) \in \mathfrak{F}$; and an object N in $\mathcal{F}_{\mathfrak{F}}$ is characterized by that N contains no submodule in $\mathcal{T}_{\mathfrak{F}}$ except the zero submodule 0. Each object in $M \in R$-*Mod* fits into an exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$, where $t_{\mathfrak{F}}(M) := M' \in \mathcal{T}_{\mathfrak{F}}$ and $M'' \in \mathcal{F}_{\mathfrak{F}}$. In particular, $M \in \mathcal{T}_{\mathfrak{F}}$ (resp. $\mathcal{F}_{\mathfrak{F}}$) if and only if $t_{\mathfrak{F}}(M) = M$ (resp. $t_{\mathfrak{F}}(M) = 0$). The *localization* $M_{\mathfrak{F}}$ of $M \in R$-*Mod* with respect to $\mathfrak{F}$ is defined to be the $\mathfrak{F}$-*injective envelop* $E_{\mathfrak{F}}(M/t(M))$ of the $\mathfrak{F}$-torsion-free quotient module $M/t_{\mathfrak{F}}(M)$ of M. When $\mathfrak{F}$ is clear or omitted from the text, $E_{\mathfrak{F}}$, $\mathcal{F}_{\mathfrak{F}}$, $t_{\mathfrak{F}}$, $\mathcal{T}_{\mathfrak{F}}$, "$\mathfrak{F}$-torsion", and "$\mathfrak{F}$-torsion-free" will be denoted/called simply E, $\mathcal{F}$, t, $\mathcal{T}$, "*torsion*", and "*torsion-free*" respectively.

The following kind of localizations is closest to the localizations in the case of commutative rings. It is the one used in Definition 1.1.1 for $\mathcal{O}_X^{nc}$:

Definition 1.2.1 [central localization][8]. Given a ring R, a *central localization* of R is the localization $R_{\mathfrak{F}_S}$ of R with respect to the Gabriel filter $\mathfrak{F}_S$ associated to a multiplicatively closed subset S in the center $Z(R)$ of R.

[5]For non-algebraic-geometers: A *ring* R here is meant to be the *ring of functions* on a "space" X_R these functions are supposed to take as their defining domain, and a *ring-homomorphism* $R \rightarrow S$ is meant to be the *pulling-back* of functions on the underlying spaces when there is a map/morphism $X_S \rightarrow X_R$ between the spaces. Algebraic geometers have turn the picture of "space first, function-ring second" around to make the function-ring first and space – if functorially constructible at all – second. Indeed, physicists have already adopted such "function-ring first" philosophy (without knowing the "space") when studying supersymmetry and superfields on a superspace.

[6]The general functorial construction of noncommutative schemes that generalizes Grothendieck's school on commutative geometry is a subtle issue. See, e.g., [J-V-V: introduction] and Remark 1.1.6.

[7]Property (1) and Property (2) together define the notion of a *filter* of ideals in R; Property (3) and Property (4) together actually imply Property (1) and Property (2).

[8]Central localizations are particularly akin to Azumaya-type noncommutative spaces. It should be noted that most of the definitions, statements, and constructions we give based on central localizations *cannot* be taken directly for general localizations without additional works or modifications.

Explicitly, the Gabriel filter in the above definition is given by $\mathfrak{F}_S = \{I \,:\, \text{ideal of } R\,,\; I\cap S \neq \emptyset\}$ and the central localization is given by $R_{\mathfrak{F}_S} = [S^{-1}]R = R[S^{-1}] := (R\times S)/\sim$, where $(r_1,s_1)\sim(r_2,s_2)$ if and only if $s(r_1s_2 - r_2s_1) = 0$ for some $s\in S$.

Definition 1.2.2 [push-out, admissibility, and descent]. (1) Let $\varphi: R\rightarrow R'$ be a ring-homomorphism, $S\subset Z(R)$ be a multiplicatively closed subset in R such that $\varphi(S)\subset Z(R')$, and $\psi: R\rightarrow R_{\mathfrak{F}_S}$ be the central localization of R with respect to S. Then the central localization $\psi': R'\rightarrow R'_{\mathfrak{F}_{\varphi(S)}}$ of R' is called the *push-out* of ψ to R' via φ. (2) Given central localizations $\psi: R\rightarrow R_{\mathfrak{F}_S}$ and $\psi': R'\rightarrow R'_{\mathfrak{F}_{S'}}$, a ring-homomorphism $\varphi: R\rightarrow R'$ is called *admissible* to (S,S') if $\varphi(S)\subset S'$. For such φ, there is a canonical/unique ring-homomorphism $\varphi_{(S,S')}: R_{\mathfrak{F}_S}\rightarrow R'_{\mathfrak{F}_{S'}}$ that makes the following diagram commute:

$$
\begin{array}{ccc}
R & \stackrel{\varphi}{\longrightarrow} & R' \\
\psi\downarrow & & \downarrow\psi' \\
R_{\mathfrak{F}_S} & \stackrel{\varphi_{(S,S')}}{\longrightarrow} & R'_{\mathfrak{F}_{S'}}
\end{array} \quad .
$$

$\varphi_{(S,S')}$ is called the *descent* of φ under the central localizations.

Example 1.2.3 [2-step consecutive central localization]. Given central localizations $\psi_1: R\rightarrow R_1$ and $\psi_2: R\rightarrow R_2$ of R, one has the push-outs $\psi_{12}: R_1\rightarrow R_{12}$ and $\psi_{21}: R_2\rightarrow R_{21}$ of ψ_2 via ψ_1 and of ψ_1 via ψ_2 respectively. Then there is a canonical isomorphism $R_{12}\simeq R_{21}$ such that the following diagram

$$
\begin{array}{cccc}
R & \stackrel{\psi_2}{\longrightarrow} & & R_2 \\
\psi_1\downarrow & & & \downarrow\psi_{21} \\
R_1 & \stackrel{\psi_{12}}{\longrightarrow} & R_{12}\;\simeq & R_{21}
\end{array}
$$

commutes. Both the compositions $\psi_{12}\circ\psi_1: R\rightarrow R_{12}$ and $\psi_{21}\circ\psi_2: R\rightarrow R_{21}$ give central localizations of R. Such 2-step consecutive central localizations will appear in stating the cocycle conditions for the gluing of rings along their central localizations.

A ring-theoretic description of noncommutative spaces and their morphisms.

We give a description of a class of noncommutative spaces and their morphisms solely in terms of rings, ring-homomorphisms, and central localizations, *without* employing the notion of "points" and "topology" of a "space". This class contains the class of Azumaya-type noncommutative spaces introduced in Sec. 1.1 as a subclass.

Definition 1.2.4 [finite central cover of a ring]. Let A be a finite set and $\mathcal{U} := \{\varphi_\alpha: R\rightarrow R_\alpha\}_{\alpha\in A}$ be a finite collection of central localizations of R with respect to Gabriel filters $\mathfrak{F}_\alpha$, $\alpha\in A$, on R. We say that $\mathcal{U}$ is a *finite central cover* of R if $\sum_{\alpha\in A} I_\alpha = R$ for any tuple $(I_\alpha)_\alpha\in\prod_{\alpha\in A}\mathfrak{F}_\alpha$.

Definition 1.2.5 [gluing system of rings][9]. A (finite) *gluing system* of rings

$$\mathcal{R} \;=\; \Big(\{R_\alpha\}_{\alpha\in A} \rightrightarrows \{R_{\alpha_1\alpha_2}\}_{\alpha_1,\alpha_2\in A}\Big)$$

from central localizations consists of the following data:

[9]This is a Grothendieck's descent-data-of-objects description.

(1) [*local ring-charts*]

a finite collection $\{R_\alpha\}_{\alpha\in A}$ of rings; (A: the *index set* of $\mathcal{R}$)

(2) [*transition ring-homomorphisms*]

a finite central cover $\{R_{\alpha_1} \to R_{\alpha_1\alpha_2}\}_{\alpha_2\in A}$ for each R_{α_1} and a choice of ring-isomorphisms $\varphi_{\alpha_1\alpha_2} : R_{\alpha_1\alpha_2} \stackrel{\sim}{\to} R_{\alpha_2\alpha_1}$ for each $(\alpha_1,\alpha_2) \in A\times A$ such that $R_{\alpha\alpha} = R_\alpha$, $\varphi_{\alpha_1\alpha_2} = \varphi^{-1}_{\alpha_2\alpha_1}$, and $\varphi_{\alpha\alpha} = Id_{R_\alpha}$;

· [*cocycle conditions*]

the ring-homomorphism $R_{\alpha_1} \to R_{\alpha_1\alpha_2}$ pushes out the finite central cover $\{R_{\alpha_1} \to R_{\alpha_1\alpha_3}\}_{\alpha_3}$ of R_{α_1} to a finite central cover $\{R_{\alpha_1\alpha_2} \to R_{\alpha_1\alpha_2\alpha_3}\}_{\alpha_3}$ of $R_{\alpha_1\alpha_2}$ and one has the canonical isomorphisms $R_{\alpha_1\alpha_2\alpha_3} \simeq R_{\alpha_1\alpha_3\alpha_2}$ from the push-out diagrams; it is then required that the gluing ring-isomorphisms $R_{\alpha_1\alpha_2} \rightleftharpoons R_{\alpha_2\alpha_1}$ descend to ring-isomorphisms $R_{\alpha_1\alpha_2\alpha_2} \rightleftharpoons R_{\alpha_2\alpha_1\alpha_3}$ that make the following diagrams

$$\begin{array}{ccc} R_{\alpha_1\alpha_2} & \rightleftharpoons & R_{\alpha_2\alpha_1} \\ \downarrow & & \downarrow \\ R_{\alpha_1\alpha_2\alpha_3} & \rightleftharpoons & R_{\alpha_2\alpha_1\alpha_3} \end{array} \quad\text{and}\quad \begin{array}{ccccc} & & R_{\alpha_1\alpha_2\alpha_3} \simeq R_{\alpha_1\alpha_3\alpha_2} & & \\ & \swarrow\!\!\!\nearrow & & \nwarrow\!\!\!\searrow & \\ R_{\alpha_2\alpha_1\alpha_3} \simeq R_{\alpha_2\alpha_3\alpha_1} & & \rightleftharpoons & & R_{\alpha_3\alpha_2\alpha_1} \simeq R_{\alpha_3\alpha_1\alpha_2} \end{array}$$

commute. Note that under the requirement of the first diagram above the isomorphisms $R_{\alpha_1\alpha_2\alpha_3} \rightleftharpoons R_{\alpha_2\alpha_1\alpha_3}$, when exists, are unique.

We will write $R_\alpha \in \mathcal{R}$ to indicate that R_α is a ring-chart in the system $\mathcal{R}$. A (*finite central*) *refinement* of $\mathcal{R}$ is a gluing system $\mathcal{R}' = (\{R'_{\alpha'}\}_{\alpha'\in A'} \rightrightarrows \{R'_{\alpha'_1,\alpha'_2}\}_{\alpha'_1,\alpha'_2\in A'})$ of rings together with the following data:

· a surjective map $\tau : A' \to A$;

· a central localization ring-homomorphism $R_\alpha \to R'_{\alpha'}$ for each $\alpha \in A$ and $\alpha' \in \tau^{-1}(\alpha)$ such that

- for each $\alpha \in A$, $\{R_\alpha \to R'_{\alpha'}\}_{\alpha'\in\tau^{-1}(\alpha)}$ is a finite central cover of R_α;
- for all $(\alpha_1,\alpha_2) \in A\times A$ and $(\alpha'_1,\alpha'_2) \in \tau^{-1}(\alpha_1)\times\tau^{-1}(\alpha_2)$, $R_{\alpha_1} \to R'_{\alpha'_1}$ descends to $R_{\alpha_1\alpha_2} \to R'_{\alpha'_1\alpha'_2}$ and all the diagrams

$$\begin{array}{ccc} R_{\alpha_1\alpha_2} & \stackrel{\varphi_{\alpha_1\alpha_2}}{\longrightarrow} & R_{\alpha_2\alpha_1} \\ \downarrow & & \downarrow \\ R'_{\alpha'_1\alpha'_2} & \stackrel{\varphi'_{\alpha'_1\alpha'_2}}{\longrightarrow} & R'_{\alpha'_2\alpha'_1} \end{array}$$

commute.

We will denote $\mathcal{R}'$, together with this data of arrows from $\mathcal{R}$ to $\mathcal{R}'$, by $\mathcal{R}' \precsim \mathcal{R}$. Two gluing systems of rings $\mathcal{R}_1$ and $\mathcal{R}_2$ are said to be *equivalent*, in notation $\mathcal{R}_1 \sim \mathcal{R}_2$, if there exists a gluing system $\mathcal{R}_3$ such that both $\mathcal{R}_3 \precsim \mathcal{R}_1$ and $\mathcal{R}_3 \precsim \mathcal{R}_2$ exist/hold. The *equivalence class* of $\mathcal{R}$ under refinements is denoted by $[\mathcal{R}]$.

Definition 1.2.6 [gluing system of ring-homomorphisms][10]. A *gluing system of ring-homomorphisms* from a gluing system $\mathcal{R} = (\{R_\alpha\}_{\alpha\in A} \rightrightarrows \{R_{\alpha_1\alpha_2}\}_{\alpha_1,\alpha_2\in A})$ to another such system $\mathcal{S} = (\{S_\beta\}_{\beta\in B} \rightrightarrows \{S_{\beta_1\beta_2}\}_{\beta_1,\beta_2\in B})$ consists of the following data:

[10]This is a Grothendieck's descent-data-of-morphisms description.

- a map $\tau : B \to A$ on the index sets;
- [*ring-homomorphisms on ring-charts*]
  a collection $\{\varphi_\beta : R_{\tau(\beta)} \to S_\beta\}_{\beta\in B}$ of ring-homomorphisms such that
  - [*compatibility with localizations*]
    for all $\beta_1,\, \beta_2 \in B$, $\varphi_{\beta_1} : R_{\tau(\beta_1)} \to S_{\beta_1}$ is admissible and, hence, descends to a unique $\varphi_{\beta_1}|_{\beta_2} : R_{\tau(\beta_1)\tau(\beta_2)} \to S_{\beta_1\beta_2}$ that makes the diagram
$$\begin{array}{ccc} R_{\tau(\beta_1)} & \stackrel{\varphi_{\beta_1}}{\longrightarrow} & S_{\beta_1} \\ \downarrow & & \downarrow \\ R_{\tau(\beta_1)\tau(\beta_2)} & \stackrel{\varphi_{\beta_1}|_{\beta_2}}{\longrightarrow} & S_{\beta_1\beta_2} \end{array}$$
    commute, cf. Definition 1.2.2;
  - [*gluing conditions*]
    the diagrams
$$\begin{array}{ccc} S_{\beta_1\beta_2} & \rightleftharpoons & S_{\beta_2\beta_1} \\ {\scriptstyle \varphi_{\beta_1}|_{\beta_2}} \uparrow & & \uparrow {\scriptstyle \varphi_{\beta_2}|_{\beta_1}} \\ R_{\tau(\beta_1)\tau(\beta_2)} & \rightleftharpoons & R_{\tau(\beta_2)\tau(\beta_1)} \end{array}$$
    commute for all $(\beta_1, \beta_2) \in B \times B$.

We will call the system $\Phi := (\tau, \{\varphi_\beta\}_\beta)$ also a *morphism* from $\mathcal{R}$ to $\mathcal{S}$.

Example 1.2.7 [refinement as a morphism]. A refinement $\mathcal{R}' \precneqq \mathcal{R}$ contains a system $\Phi : \mathcal{R} \to \mathcal{R}'$ of ring-homomorphisms in its data. In particular, a central cover $\{R \to R_\alpha\}_\alpha$ of R gives rise to a morphism $\{R\} \to \{R_\alpha\}_\alpha$.

Ring-homomorphisms have the following affine-gluing property:

Lemma 1.2.8 [morphism: affine-gluing]. *Given finitely generated rings R and S, let $(\{S_\alpha\}_{\alpha\in A} \rightrightarrows \{S_{\alpha_1\alpha_2}\}_{\alpha_1,\alpha_2\in A})$ be a gluing system of rings associated to a finite central cover $\{S \to S_\alpha\}_{\alpha\in A}$ of S and $\Phi = \{\varphi_\alpha : R \to S_\alpha\}_{\alpha\in A}$ be a gluing system of ring-homomorphisms from R. Then, there exists a unique ring-homomorphism $\varphi : R \to S$ such that φ descends to Φ.*

We will call φ in the above lemma the *gluing* of the system Φ. A reverse of this lemma gives rise to the following definition:

Definition 1.2.9 [refinement of morphism]. Given a morphism $\Phi = (\tau, \{\varphi_\beta\}_\beta) : \mathcal{R} \to \mathcal{S}$ and a pair $(\mathcal{R}' \precneqq \mathcal{R}\,,\, \mathcal{S}' \precneqq \mathcal{S})$ of refinements, denote the index set of $\mathcal{R}$, $\mathcal{R}'$, $\mathcal{S}$, $\mathcal{S}'$ by A, A', B, B' respectively. Let $\tau : B \to A$ and $(\tau_{A',A} : A' \to A\,,\, \tau_{B',B} : B' \to B)$ be the maps on the index sets corresponding to Φ and the pair of refinements respectively. Then $(\mathcal{R}' \precneqq \mathcal{R}\,,\, \mathcal{S}' \precneqq \mathcal{S})$ is said to be Φ-*admissible* if, for all $\beta \in B$, φ_β is admissible with respect to the localizations maps in the system pair $(\mathcal{R}' \precneqq \mathcal{R}\,,\, \mathcal{S}' \precneqq \mathcal{S})$; cf. Definition 1.2.2. When this is the case, fix a $\tau' : B' \to A'$ so that the diagram
$$\begin{array}{ccc} A & \stackrel{\tau}{\longleftarrow} & B \\ {\scriptstyle \tau_{A',A}} \uparrow & & \uparrow {\scriptstyle \tau_{B',B}} \\ A' & \stackrel{\tau'}{\longleftarrow} & B' \end{array}$$

commute. Then Φ descends to a unique morphism $\Phi' = (\tau', \{\varphi'_{\beta'}\}_{\beta'}) : \mathcal{R}' \to \mathcal{S}'$, called a *refinement* of Φ with respect to $(\mathcal{R}' \stackrel{\leftarrow}{\precsim} \mathcal{R}\,,\, \mathcal{S}' \stackrel{\leftarrow}{\precsim} \mathcal{S})$.

Definition 1.2.10 [equivalence of morphisms]. Given equivalent ring-systems $\mathcal{R}_1 \sim \mathcal{R}_2$ and $\mathcal{S}_1 \sim \mathcal{S}_2$ and morphisms $\Phi_1 : \mathcal{R}_1 \to \mathcal{S}_1$ and $\Phi_2 : \mathcal{R}_2 \to \mathcal{S}_2$, we say that Φ_1 and Φ_2 are *equivalent*, in notation $\Phi_1 \sim \Phi_2$, if there exist common refinements $\mathcal{R}_1 \stackrel{\rightarrow}{\succsim} \mathcal{R}' \stackrel{\leftarrow}{\precsim} \mathcal{R}_2$ and $\mathcal{S}_1 \stackrel{\rightarrow}{\succsim} \mathcal{S}' \stackrel{\leftarrow}{\precsim} \mathcal{S}_2$ such that (1) $(\mathcal{R}' \stackrel{\leftarrow}{\precsim} \mathcal{R}_1, \mathcal{S}' \stackrel{\leftarrow}{\precsim} \mathcal{S}_1)$ and $(\mathcal{R}' \stackrel{\leftarrow}{\precsim} \mathcal{R}_2, \mathcal{S}' \stackrel{\leftarrow}{\precsim} \mathcal{S}_2)$ are Φ_1- and Φ_2-admissible respectively and (2) Φ_1 and Φ_2 can be descended to identical morphisms $\Phi'_1 = \Phi'_2 : \mathcal{R}' \to \mathcal{S}'$. The equivalence class of Φ will be denoted by $[\Phi]$. An element in $[\Phi]$ will be called a *representative* of $[\Phi]$.

Definition 1.2.11 [strict morphism on equivalence classes]. By a *strict morphism* from $[\mathcal{R}_0]$ to $[\mathcal{S}_0]$, we mean an equivalence class $[\Phi : \mathcal{R} \to \mathcal{S}]$, where $\mathcal{R} \in [\mathcal{R}_0]$ and $\mathcal{S} \in [\mathcal{S}_0]$.

By descending to a refinement $\mathcal{R}$ of $\mathcal{R}_0$ and taking the pre-composition with the localizations maps in $\mathcal{R} \stackrel{\leftarrow}{\precsim} \mathcal{R}_0$, one has the following lemma:

Lemma 1.2.12 [one-side refinement enough]. *A strict morphism from $[\mathcal{R}_0]$ to $[\mathcal{S}_0]$ can be represented by a $\Phi : \mathcal{R}_0 \to \mathcal{S}$, for some $\mathcal{S} \in [\mathcal{S}_0]$.*

Thus, in the discussion below, only the refinements on the $[\mathcal{S}_0]$-side are required.

Definition 1.2.13 [injective strict morphism]. A *injective strict morphism* $[\Phi_0] : [\mathcal{R}] \to [\mathcal{S}_0]$ is a strict morphism that can be represented by a $\Phi = (\tau, \{\varphi_\beta\}_\beta) : \mathcal{R} \to \mathcal{S}$, $\mathcal{S} \in [\mathcal{S}_0]$, such that (1) τ is surjective and (2) for each $R_\alpha \in \mathcal{R}$, there exists a $\beta \in \tau^{-1}(\alpha)$ such that $\varphi_\beta : R_\alpha \to S_\beta \in \mathcal{S}$ is a ring-monomorphism.

Definition 1.2.14 [(general) morphism]. A *general morphism* from $[\mathcal{R}]$ to $[\mathcal{S}_0]$ consists of the following data:

- an injective strict morphism $[\Phi_0] : [\mathcal{S}'_0] \to [\mathcal{S}_0]$,
- a strict morphism $[\Phi'_0] : [\mathcal{R}] \to [\mathcal{S}'_0]$.

We will denote the tuple $([\mathcal{S}'_0], [\Phi_0], [\Phi'_0])$ collectively by $[\Phi'_0]$ and a general morphism also by $[\Phi'_0] : [\mathcal{R}] \to [\mathcal{S}_0]$. A *representative* of $[\Phi'_0] : [\mathcal{R}] \to [\mathcal{S}_0]$ is given by a 3-*step ring-system-morphism diagram*

$$\mathcal{R} \xrightarrow{\Phi'} \mathcal{S}'' \stackrel{\leftarrow}{\precsim} \mathcal{S}' \xrightarrow{\Phi} \mathcal{S}$$

with $\mathcal{S} \in [\mathcal{S}_0]$; $\mathcal{S}'$, $\mathcal{S}'' \in [\mathcal{S}'_0]$; $\Phi \in [\Phi_0]$, and $\Phi' \in [\Phi'_0]$. A strict morphism is automatically a general morphism. A general morphism will also be called simply a *morphism*[11]. Define $Mor([\mathcal{R}], [\mathcal{S}_0])$ to be the set of morphisms from $[\mathcal{R}]$ to $[\mathcal{S}_0]$.

[11]For non-algebraic-geometers: A few words follow on why the *morphisms* in Sec. 1.1 and here are defined as they are. In the case of systems of *commutative rings*, "general morphism" is a redundant notion as the 3-step ring-system-morphism diagram $\mathcal{R} \xrightarrow{\Phi'} \mathcal{S}'' \stackrel{\leftarrow}{\precsim} \mathcal{S}' \xrightarrow{\Phi} \mathcal{S}$ can always be reduced to a 2-step diagram $\mathcal{R} \xrightarrow{\Phi} \mathcal{S}''' \stackrel{\leftarrow}{\precsim} \mathcal{S}$, which represents a strict morphism. In this case, $[\mathcal{R}]$ and $[\mathcal{S}]$ (resp. $\mathcal{R}$ and $\mathcal{S}$) are contravariantly associated to schemes (resp. atlases of affine charts on schemes). This reducibility from a 3-step diagram to a 2-step diagram no longer holds in general in the case of *noncommutative rings*, as the ring-homomorphisms φ_β on ring-charts are required to be admissible to the central localizations in the construction in order that gluings make sense and work. On the other hand, when we shrink the rings S_β and take only a system of their subrings $S'_{\beta'}$, the center can increase: $Z(S'_{\beta'}) \supset Z(S_\beta)$. Thus, a ring-homomorphism that is not admissible as a map to S_β but with the image contained in $S'_{\beta'}$ may become admissible as a map to $S'_{\beta'}$. In other words, the notion of general morphism partially takes care of the more subtle issue of a functorial construction of general localizations, allowing

Example 1.2.15 [non-strict morphism]. Let S be a subring of S_0 such that $Z(S) \supsetneqq Z(S_0)$ and $\Sigma = \{s_\beta\}_\beta$ be a finite subset in $Z(S) - Z(S_0)$ such that $S = \sum_{s_\beta \in \Sigma} s_\beta \cdot S$. Let $S \to S_\beta$ (resp. $S \to S_{\beta_1\beta_2}$) be the central localization with respect to s_β (resp. s_{β_1} and then s_{β_2}) , then $\{S \to S_\beta\}_\beta$ is a cover of S. Then the 3-step diagram

$$\begin{aligned}&\Big(\{S_\beta\}_\beta \;\rightrightarrows\; \{S_{\beta_1\beta_2}\}_{\beta_1,\beta_2}\Big)\\ &\qquad \xrightarrow{Id}\; \Big(\{S_\beta\}_\beta \;\rightrightarrows\; \{S_{\beta_1\beta_2}\}_{\beta_1,\beta_2}\Big) \;\;\precsim\;\; \Big(\{S\} \;\rightrightarrows\; \{S\}\Big) \;\longrightarrow\; \Big(\{S_0\} \;\rightrightarrows\; \{S_0\}\Big)\end{aligned}$$

represents a morphism $[Id] : (\{S_\beta\}_\beta \rightrightarrows \{S_{\beta_1\beta_2}\}_{\beta_1,\beta_2}) \to (\{S_0\} \rightrightarrows \{S_0\})$ that is not strict. See Sec. 4.2 for such examples with $S_0 = M_n(\mathbb{C})$.

Grothendieck Ansatz [ring vs. space]. *We shall hiddenly think of an equivalence class* $[\mathcal{R}]$ *of ring-systems as a "space" Space* $[\mathcal{R}]$ *with an equivalence class of atlases* $\{\mathit{Space}\, R_\alpha\}_\alpha$ *(with the gluing data from the arrows* $\{\mathit{Space}\, R_{\alpha_1\alpha_2}\}_{\alpha_1,\alpha_2} \rightrightarrows \{\mathit{Space}\, R_\alpha\}_\alpha$*), and a morphism* $[\mathcal{R}] \to [\mathcal{S}]$ *as a morphism* $\mathit{Space}\,[\mathcal{S}] \to \mathit{Space}\,[\mathcal{R}]$*. Cf. footnote 11.*

Remark 1.2.16 [*morphism vs. map*]. In defining a *morphism* in a category of noncommutative spaces, we mean to keep both the *domain* and the *target* of the morphism *fixed*. In terms of the ring-system language, this is reflected in the fact that a refinement of a ring-system $\mathcal{R}$ is another ring system $\mathcal{R}'$ *together with* a localization morphism $\mathcal{R} \to \mathcal{R}'$ and the fact that the trivial localization is the identity map (not just a ring-isomorphism). In contrast, later (Sec. 4) when we discuss the space of *maps* or of *D0-brane probes*, we remain to keep the target-space fixed but the *domain*-space will be taken as *not fixed*. The issue of automorphisms of the domain will then enter.

Remark 1.2.17 [*k-algebra*]. When all the rings $R_\alpha \in \mathcal{R}$ involved are k-algebras for a fixed ground field k, we will take as a convention that all the ring-homomorphisms involved are then required to be k-algebra-homomorphisms unless otherwise noted.

Remark 1.2.18 [*Azumaya-type noncommutative space*]. For an Azumaya-type noncommutative space $X = (X, \mathcal{O}_X, \mathcal{O}_X^{nc})$, an affine cover $\{U_\alpha\}_\alpha$ of $(X, \mathcal{O}_X)$ gives rise to a ring-system representation $\mathcal{R}_X$ of X defined by

$$\mathcal{R}_X \;=\; (\{R_\alpha\}_\alpha \;\rightrightarrows\; \{R_{\alpha_1\alpha_2}\}_{\alpha_1,\alpha_2}) \;:=\; (\{\mathcal{O}_X^{nc}(U_\alpha)\}_\alpha \;\rightrightarrows\; \{\mathcal{O}_X^{nc}(U_{\alpha_1} \cap U_{\alpha_2})\}_{\alpha_1,\alpha_2})\,.$$

---

us to stay in the much more tractable central localizations. This is not the whole story. In the correspondence of the category of *commutative rings* with the category of (commutative) *affine schemes*, one has the canonical identification: $\mathit{Mor}(R,S) = \mathit{Mor}(\mathit{Spec}\, S, \mathit{Spec}\, R)$ by construction. In the *noncommutative* case, the functorial construction of the operation "*Spec*" that associates to a ring a "space" is subtle. Indeed, what Grothendieck's school accomplished in the decade 1960s for commutative algebraic geometry is only partially realized through the work of several independent schools on noncommutative algebraic geometry in the four decades after then. There are several nonequivalent constructions/realization of the notion of "*Spec*", with each maintaining part of the equivalent characterizing properties of *Spec* in the commutative case, cf. sample references in Remark 1.1.6. In the current work, we take rings and ring-homomorphisms as more fundamental for "geometry" than the notion of "points" and "topologies". An injective strict morphism $[\Phi_0] : [\mathcal{S}_0'] \to [\mathcal{S}_0]$, is then meant to give a dominant morphism $\phi_0 : \mathit{Space}\,[\mathcal{S}_0] \to \mathit{Space}\,[\mathcal{S}_0']$, should the latter "spaces" be constructed functorially. Geometrically, a general morphism is then simply an ordinary morphism precomposed with a pinching and, hence, must be still an allowable morphism if the setting is natural. From these hidden words to the main text, one sees that we do want to include general morphisms to $\mathit{Mor}([\mathcal{R}], [\mathcal{S}_0])$ in any natural setting/definition. Surprisingly, these independent purely mathematical reasonings that attempt to extend Grothendieck's language of (commutative) algebraic geometry to the noncommutative case give rise to $\mathit{Mor}([\mathcal{R}], [\mathcal{S}])$ that is also required for modeling D-branes in string theory correctly!

Morphisms $X \to Y$ between Azumaya-type noncommutative spaces can be expressed contravariantly as morphisms $\mathcal{R}_Y \to \mathcal{R}_X$ of associated ring-systems. In particular, the notion of surrogates $X \to X'$ of X corresponds to the notion of injective strict morphisms $[\mathcal{R}'] \to [\mathcal{R}]$ into $[\mathcal{R}]$.

Before leaving this theme, we note that Lemma 1.2.8 and Lemma 1.2.12 together imply that:

Lemma 1.2.19 [local description of morphisms]. *Let R and S be rings. Then*

$$Mor\,(Space\,[\{S\}], Space\,[\{R\}]) \stackrel{\text{Grothendieck Ansatz}}{:=} Mor\,([\{R\}],[\{S\}]) \;\simeq\; Mor\,(R,S)$$

canonically, where $Mor\,(R,S)$ is the set of ring-homomorphisms from R to S.

2 D-branes from the viewpoint of Grothendieck.

2.1 The notion of a space(-time): functor of points vs. probes.

Space from a functor of points in algebraic geometry: a space without a space.

In the commutative case[12], let $\mathcal{S}cheme\,/S$ be the category of schemes over a base scheme S with a Grothendieck topology. A *functor of points* on $\mathcal{S}cheme\,/S$ is a presheaf $\mathcal{F}$ of sets on $\mathcal{S}cheme\,/S$. For example, take $S = Spec\,\mathbb{C}$, then a scheme $Y/\mathbb{C}$ determines an $\mathcal{F}_Y$ on $\mathcal{S}cheme\,/\mathbb{C}$ with $\mathcal{F}_Y(Z) := Mor_{\mathbb{C}-scheme}(Z,Y)$ for $Z \in \mathcal{S}cheme\,/\mathbb{C}$. In this case, Y can be recovered from $\mathcal{F}_Y$, cf. Yoneda lemma.

One can think of a functor of points $\mathcal{F}$ as a generalized space $\mathfrak{Y}_{\mathcal{F}}$ and $\mathcal{F}(Z)$ as the set $Mor\,(Z,\mathfrak{Y}_{\mathcal{F}})$ of Z-valued points on $\mathfrak{Y}_{\mathcal{F}}$. The construction of the moduli space that satisfies the functorial/universal property for a moduli problem leads one in general to such a generalized space. Encoded in the functor of points $\mathcal{F}$ on $\mathcal{S}cheme\,/S$ is the data of extension property of morphisms into $\mathfrak{Y}_{\mathcal{F}}$. In particular, $\mathcal{F}$ contains the information of tangent-obstruction structure of $\mathfrak{Y}_{\mathcal{F}}$ as well as of local properties like smoothness at a point (i.e. an element in, e.g., $\mathcal{F}(Spec\,\mathbb{C}) =: Mor\,(Spec\,\mathbb{C},\mathfrak{Y}_{\mathcal{F}})$) of $\mathfrak{F}_{\mathcal{F}}$. It is in this way that $\mathcal{F}$ describes the geometry of a "space" *without* giving the space beforehand, for example, as a point-set with a topology and other structures. Schemes, Deligne-Mumford stack (i.e. orbifolds), Artin stacks, and many moduli functors are all examples of functors of points.

There are diverse ways/versions to generalize the above to the noncommutative case. The particular one that is selected from Sec. 1.2 is to consider the category $\mathcal{R}ingSystem$ of gluing systems of rings with a Grothendieck topology defined by central covers, étale central covers, or fppf central covers. (The étale or fppf condition of a morphism can be defined purely ring-theoretically.) Note that, as we are dealing directly with rings, all the arrows in the commutative case above are reversed here. (However, if one wishes, one may write a ring system $\mathcal{R}$ by a formal symbol $Space\,\mathcal{R}$, meaning the associated space/geometry to $\mathcal{R}$, to preserve all the arrow directions.) A *functor of points* on $\mathcal{R}ingSystem$ is then a presheaf $\mathcal{F}$ of sets on $\mathcal{R}ingSystem$. Again, one can directly think of $\mathcal{F}$ as a generalized noncommutative space $\mathfrak{Y}_{\mathcal{F}}$. The data of extension properties of morphisms to $\mathfrak{Y}_{\mathcal{F}}$ is encoded in $\mathcal{F}$. Through this, $\mathcal{F}$ describes the geometry of a generalized noncommutative space $\mathfrak{Y}_{\mathcal{F}}$ without $\mathfrak{Y}_{\mathcal{F}}$ being given beforehand.

Space(-time) from probes in QFT/string theory: space(-time)s emerge from QFT.

[12] Unfamiliar readers are referred to [L-L-Y: Sec. 1] for a brief introduction of and literature guide for the notions of *Grothendieck topology*, *site*, and *stack*. All that is said here is standard from algebraic deformation theory.

There are two particular classes of quantum field theories (QFT's) that are directly relevant to the notion of target space(-time):

· *Nonlinear sigma models* are, by definition, quantum field theories whose field contents contain, among other fields, bosonic fields corresponding to maps from a domain (cf. world-volume of branes) to a target space(-time).

· In string theory, D0-brane physics is described by matrix theory. As the moduli space of a single D0-brane moving in a space(-time) is the space(-time) itself, the *moduli space of a single D0-brane* can be identified as the target space(-time).

These two concretely target-space(-time)-related situations can be hidden implicitly in a general quantum field theory that is seemingly irrelevant to a target space-time. Furthermore, depending on where we look at the theory in the related Wilson's theory-space[13], there can be more than one target space(-time)s hidden in one combinatorial class of quantum field theories. Even more, such target spaces can be taken either at the classical level – which usually involve only algebraic manipulations of the Lagrangian of the theory – or at the quantum level – which has to bring in the core techniques (and some arts as well) from quantum field theory. A quantum-corrected target space(-time) can be different from its associate classical target space(-time). The following three examples have been around for a while in the string-theory community:

Example 2.1.1 [gauged linear sigma model]. *Geometric phases* of a gauged linear sigma model are realized effectively by nonlinear sigma models. Birationally equivalent target spaces emerge. See [Wi1] and [M-P].

Example 2.1.2 [D0-brane probe of space(-time) and singularities]. A D0-brane moving in a singular space(-time) recognizes various (partial) resolutions of the singular space(-time) as the *moduli space of D0-branes* at different phases in the Wilson's theory-space of the $0 + 1$ dimensional matrix theory involved. Birationally equivalent smooth or partially resolved target spaces emerge from a single singular target space. See [D-G-M], [Do-M], and [G-L-R].

Example 2.1.3 [conformal field theory with boundary]. D0-branes are realized in a conformal field theory with boundary as a special class of boundary states. The *moduli space of* such *boundary states* gives rise to a target space(-time). See [M-M-S-S] and [S-S].

These examples suggest that quantum field theories, as probes to a target space(-time), can be more fundamental than the space(-time) itself. The latter may even lose its absolute meaning under dualities of quantum field theories, like what happens in mirror symmetry.

Functor of points vs. probes.

A comparison of these two notions is given below:

[13] See [L-Y1: appendix A.1] for highlights and a literature guide for mathematicians on this very important notion from quantum field theory. In particular, a *Wilson's theory-space* goes with *universal objects* over it that encode QFT contents, and a *duality* is a local isomorphism on Wilson's theory-space with these structures.

functor of points	*QFT's as probes*
· $\mathcal{S}cheme/S$	a category $\mathcal{B}rane$ of branes
· a functor of point $\mathcal{F}$ on $\mathcal{S}cheme/S$	a compatible system $\{\mathrm{QFT}_\Sigma\}_{\Sigma\in\mathcal{B}rane}$ of effective QFT on branes that have isomorphic target space(-time)s
· $\mathcal{F}(T)\,,\ T\in\mathcal{S}cheme/S$	bosonic fields on a brane that correspond to maps from the brane to a target space(-time)

Here, a 'brane' means the defining domain of a quantum field theory. For example, it can be the world-volume of a string, a D-brane, or an NS-brane. Note also that a functor of points $\mathcal{F}$ encodes the data of a space while an effective QFT from a QFT as a probe encodes more than just the information of the target space(-time).

2.2 D-branes as Azumaya-type noncommutative spaces.

Question: What is a D-brane intrinsically?

A *D-brane* (in full name: *Dirichlet brane* or *Dirichlet membrane*) in string theory is by definition (i.e. by the very word 'Dirichlet') a boundary condition for the end-points of open strings moving in a space-time. In the geometric/target-space-time aspect[14], one may start by thinking of the world-volume (cf. Remark/Definition 2.2.4) of a D-brane as an embedded submanifold $f: Z \hookrightarrow M$ in an open-string target space-time M such that:

- **[defining property of D-brane: D = Dirichlet]**
  The boundary of open-string world-sheets are mapped to $f(Z)$ in M.

Via this defining property, open strings induce then additional structures on Z, including a gauge field (from the vibrations of open-strings with end-points on $f(Z)$) and a Chan-Paton bundle (from the Chan-Paton index on the end-points of such an open string) on Z. Basic properties of D-branes under such a setting are given in [Pol3] and [Pol4].

To bring the relevant part of the work of Polchinski into the discussions and to enable a direct comparison/referral, let us introduce notations the-same-as/as-close-as-possible-to those in [Pol4: vol. I, Sec. 8.7]: let $\xi := (\xi^a)_a$ be local coordinates on Z and $X := (X^a; X^\mu)_{a,\mu}$ be local coordinates on M such that the embedding $f: Z \hookrightarrow M$ is locally expressed as

$$X \;=\; X(\xi) \;=\; (X^a(\xi); X^\mu(\xi))_{a,\mu} \;=\; (\xi^a, X^\mu(\xi))_{a,\mu}\,;$$

i.e., X^a's (resp. X^μ's) are local coordinates along (resp. transverse to) $f(Z)$ in M. This choice of local coordinates removes redundant degrees of freedom of the map f, and $X^\mu = X^\mu(\xi)$ can be regarded as (scalar) fields on Z that collectively describes the postions/shapes/fluctuations of Z in M locally. Here, both ξ^a's, X^a's, and X^μ's are $\mathbb{R}$-valued. The gauge field on Z is locally given by the connection 1-form $A = \sum_a A_a(\xi)d\xi^a$ of a $U(1)$-bundle on Z.

When n-many such D-branes Z are coincident, from the associated massless spectrum of (oriented) open strings with both end-points on $f(Z)$ one can draw the conclusion that

[14]It should be noted that there are also algebraic properties of D-branes realized as states or operators in a 2-dimensional conformal field theory with boundary. These algebraic properties from the open-string world-sheet perspective reflect the geometric properties of D-branes in the target space-time of strings. Our focus in this work is on the geometric aspect as given in [Pol3] and [Pol4].

(1) The gauge field $A = \sum_a A_a(\xi)d\xi^a$ on Z is enhanced to $u(n)$-valued.

(2) Each scalar field $X^\mu(\xi)$ on Z is also enhanced to matrix-valued, cf. footnote 17.

Property (1) says that there is now a $U(n)$-bundle on Z. But

· **Q.** *What is the meaning of Property (2)?*

For this, Polchinski remarks that:

· [*quote from* [Pol4: vol. I, Sec. 8.7, p.272]] "*For the collective coordinate X^μ, however, the meaning is mysterious: the collective coordinates for the embedding of n D-branes in space-time are now enlarged to $n \times n$ matrices. This 'noncommutative geometry' has proven to play a key role in the dynamics of D-branes, and there are conjectures that it is an important hint about the nature of space-time.*"

Particularly from the mathematical/geometric perspective, Property (2) of D-branes when they are coincident, the above question, and Polchinski's remark are more appropriately incorporated into the following guiding question:

· **Q. [D-brane]** *What is a D-brane intrinsically?*

In other words, what is the *intrinsic* definition of D-branes so that *by itself* it can produce the properties of D-branes (e.g. Property (1) and Property (2) above) that are consistent with, governed by, or originally produced by open strings as well?[15]

The noncommutativity ansatz: from Polchinski to Grothendieck.

To understand Property (2) of D-branes, one has two aspects that are dual to each other:

(A1) [*coordinate tuple as point*] A tuple $(\xi^a)_a$ (resp. $(X^a; X^\mu)_{a,\mu}$) represents a point on the world-volume Z of the D-brane (resp. on the target space-time M).

(A2) [*local coordinates as generating set of local functions*] Each local coordinate ξ^a of Z (resp. X^a, X^μ of M) is a local function on Z (resp. on M) and the local coordinates ξ^a's (resp. X^a's and X^μ's) together form a generating set of local functions on the world-volume Z of the D-brane (resp. on the target space-time M).

While Aspect (A1) leads one to the anticipation of a noncommutative space from a noncommutatization of the target space-time M when probed by coincident D-branes, Aspect (A2) of Grothendieck leads one to a different/dual[16] conclusion: a noncommutative space from a noncommutatization of the world-volume Z of coincident D-branes, as follows.

[15] Since the work of Ramond and of Neveu and Schwarz in 1971 that initiated string theory, there are by now at least three ways to enter superstring theory: Gate (1) the string-world-sheet/CFT way ($d = 1+1$ or $d = 2$ theory), Gate (2) the target-space-time/supergravity/soliton way ($d = 9+1$ or $d = 10+1$ theory), and Gate (3) the matrix-theory way ($d = 0+1$ theory). In Gate (1), after Wick-rotation, one can have Riemann surfaces, conformal field theories, moduli space of Riemann surfaces, ..., etc. before asking how strings move in a space-time. D-branes entered string theory in the second half of 1980s and took a central role after 1995 mainly from the development of Gate (2) during 1990 - 1995. In asking this question, we mean also to repeat Gate (1) but for D-branes instead of for strings. In other words, we are taking a "D-brane" as a fundamental object and asking, "What is (the definition of) a D-brane?", before addressing how they "move" in – i.e. are mapped into – a space-time.

[16] In what precise sense the noncommutativity of target space-time and the noncommutativity of world-volume of branes are dual to each other deserves more thoughts.

Denote by $\mathbb{R}\langle \xi^a \rangle_a$ (resp. $\mathbb{R}\langle X^a; X^\mu \rangle_{a,\mu}$) the local function ring on the associated local coordinate chart on Z (resp. on M). Then the embedding $f: Z \to M$, locally expressed as $X = X(\xi) = (X^a(\xi); X^\mu(\xi))_{a,\mu} = (\xi^a; X^\mu(\xi))$, is locally contravariantly equivalent to a ring-homomorphism

$$f^\sharp : \mathbb{R}\langle X^a; X^\mu \rangle_{a,\mu} \longrightarrow \mathbb{R}\langle \xi^a \rangle_a , \quad \text{generated by} \quad X^a \longmapsto \xi^a , \; X^\mu \longmapsto X^\mu(\xi) .$$

When n-many such D-branes are coincident, $X^\mu(\xi)$'s become $M_n(\mathbb{C})$-valued.[17] Thus, $f^\sharp$ is promoted to a new local ring-homomorphism:

$$\hat{f}^\sharp : \mathbb{R}\langle X^a; X^\mu \rangle_{a,\mu} \longrightarrow M_n(\mathbb{C}\langle \xi^a \rangle_a) , \quad \text{generated by} \quad X^a \longmapsto \xi^a \cdot \mathbf{1} , \; X^\mu \longmapsto X^\mu(\xi) .$$

Under Grothendieck's contravariant local equivalence of function rings and spaces, $\hat{f}^\sharp$ is equivalent to saying that we have now a map $\hat{f} : Z_{\text{noncommutative}} \to M$. Thus, the result of Polchinski re-read from the viewpoint of Grothendieck implies the following ansatz:

Polchinski-Grothendieck Ansatz [D-brane: noncommutativity]. *The world-volume of a D-brane carries a noncommutative structure locally associated to a function ring of the form* $M_n(R)$ *for some* $n \in \mathbb{Z}_{\geq 1}$ *and ring* R.[18]

This ansatz is further enforced if one recalls that scalar fields on the world-volume of a brane are supposed to come from elements in the function ring of that world-volume and the comparison of a functor of points vs. probes in Sec. 2.1.[19]

[17] Strictly as induced by open-strings, $X^\mu(\xi)'s$ are $u(n)$-valued for oriented open strings and either $so(n)$- or $sp(n/2)$-valued for unoriented open strings. Instead of any of these Lie algebras, here we directly think of $X^\mu(\xi)$ as $M_n(\mathbb{C})$-valued, where $M_n(\mathbb{C})$ is regarded not as a Lie algebra with a bracket (i.e. Lie product) but rather as an associative algebra (from the matrix multiplication) with an identity $\mathbf{1}$, for two reasons:

(1) These Lie algebras are not associative nor with an identity with respect to the Lie product. This makes the notion of localizations and covers, which are crucial in algebraic geometry for the local-to-global setup, difficult to implement. In view of noncommutative algebraic geometry over $\mathbb{C}$, it is more natural to think of $X^\mu(\xi)$'s as in a special class of $M_n(\mathbb{C})$-valued functions with $M_n(\mathbb{C})$ as an associative algebra with an identity. Any associative algebra defines also a tautological Lie algebra, with the Lie product $[x, y] := x \cdot y - y \cdot x$. One can use this to translate back to Lie algebras whenever needed.

(2) In seeking the intrinsic definition/structure of a D-brane (or D-brane world-volume), it is more natural to select the structures thereon as encompassing/universal as possible so that they contain all what different types of open strings can detect/see. Each specific sector of structures on D-brane world-volume seen by a particular type of open strings is then realized by a reduction from the universal structures on D-brane world-volume, as in the reductions of the structure group of principal GL_n fiber bundles.

Cf. footnote 21.

[18] On purely mathematical ground, the $M_n(R)$ in the ansatz can be generalized in some cases. For example, in the case that R is a Noetherian (commutative) integral domain, $M_n(R)$ can be replaced by the more general notion of an *R-order in a central simple Q_R-algebra*, where Q_R is the quotient field of R; cf. [Re].

[19] From C.-H.L: Several teachers and colleagues influenced my painfully slow realization/appreciation of this ansatz and its importance through the personal journey of string theory: *Orlando Alvarez* brought me to the beauty of string theory and T-duality at the dawn of its second revolution. *Rafael Nepomechie* shared with me his experience in the early days of higher-dimensional extended objects before they became dominating in "string theory". *Pei-Ming Ho* communicated the work [Ho-W] to me. The group meetings of the school of *Philip Candelas* and the insightful debates between *Jacques Distler* and *Vadim Kaplunovsky* promoted my understandings and kept me aware of subtleties as well. Teaching the late Professor *Raoul Bott* mirror symmetry, fall 2000, assigned by *Shing-Tung Yau* gave me a rare chance to slow down and to map out what I had still been ignorant of in the big picture. The heat and enthusiasm *Shiraz Minwalla* brought in to his various topic courses from field theory to strings, from phase structures in QFT to supersymmetry, $\cdots$ over the years helped me to access the mind of physicists at the frontier. *Shinobu Hosono* explained [H-S-T] to me in March 2002, in which the subtle issue of the multiplicity/wrapping of D-branes in the torsion-sheaf picture was brought out among other things.

Remark 2.2.1 [*D-brane and noncommutative geometry*]*.* The observation that D-brane should be related to noncommutative geometry was made soon after the second-revolution year 1995 of string theory; see [Dou4] and [Dou5] for a survey and, e.g., [Ho-W] for an earlier study and [Laz] for a more recent study in the differential/symplectic geometry category. Noncommutative structures on a D-brane itself and on a space-time are two related but separate issues, e.g. [Dou2], [C-H1], and [C-H2]. It is worth pointing out that, *from the viewpoint of Grothendieck, it is the noncommutative structure on the world-volume of a D-brane that comes first*. It is exactly because of such a structure on D-branes that a space-time may reveal its noncommutative nature when probed by a D-brane. Said algebro-geometrically in terms of function rings, since a ring-homomorphism from a noncommutative ring R to a commutative ring S must factor through a ring-homomorphism $R/[R,R] \to S$ from the commutatization $R/[R,R]$ of R, D-branes without a noncommutative structure thereon cannot probe/sense any noncommutativity, if any, of a space-time at all.

Remark 2.2.2 [*B-field and noncommutativity on D-brane*]*.* It is known that when the target space(-time) M has the B-field B turned on, the gauge theory on a D-brane world-volume Z can be expressed as a noncommutative gauge theory; (see [Ch-K] and [S-W2] for details and more references on this subject.) From the underlying formulation, this implies in particular that, in this case, the commutative product of a local function ring R on Z is deformed to a noncommutative $*$-product depending on B. When n-many D-branes Z coincide, these string-induced property on D-branes compared with our discussion above says that:

· If $B = 0$, then a local function ring on the world-volume of the coincident D-branes is of the form $M_n(R)$, where R is commutative.

· If $B \neq 0$, then a local function ring on the world-volume of the coincident D-branes can become $M_n(R_B)$, where R_B is a noncommutatization of R depending on/induced by B.

In this work, we ignore the effect of B-field.

The Polchinski-Grothendieck Ansatz for D-branes applies to both nonsupersymmetric and supersymmetric D-branes, and to both D-branes of A-type and D-branes of B-type (cf. [B-B-St], [H-I-V], and [O-O-Y]) *in the latter case.* Due to the different languages used in differential geometry and in algebraic geometry for noncommutative geometry (though the philosophy to equate a space and a function ring in each category is common), we will focus entirely on

Discussions with *Mihnea Popa*, spring 2002, and his joint Seminar on Derived Category with *Mircea Mustata*, fall 2002, influenced my mathematical understanding of D-branes of B-type. The semester-long communications with *Barton Zwiebach* on the draft of [Zw], spring 2003, improved my understanding of the physical fundamentals of string theory. *Paul Aspinwall* emphasized many subtleties of D-branes in his lectures at TASI 2003. The topic courses and talks of *Kentaro Hori*, *Andrew Strominger*, and *Cumrun Vafa* on string theory over the years printed in my mind various pictures of how, physicists think, D-branes should function. The daily summary of work to each other with *Ling-Miao Chou* over the years helped to clarify my thoughts. The vanishing lemma derived in [L-Y3] and its comparison with [D-F] led me to a train of discussions with *Duiliu-Emanuel Diaconescu*, December 2006, on the meaning of open-string world-sheet instantons in the open/closed string duality. These discussions propelled me to come back to re-think about D-brane theory as a companion theory to topological open strings and their instantons, particularly the virtual ones. Finally, it should be noted that, even with this ansatz, there are still other things missing mathematically to understand D-branes fully in a larger scope, cf. footnote 20.

Incidentally, while this work is under writing, *William Thurston* came to give a talk, May 2007, on the future of 3-dimensional geometry and topology after the justification of the geometrization conjecture of 3-manifolds. Hyperbolic geometry has now applications to cosmology and AdS/CFT correspondence. It is surprising how a change of course of life of a teacher can lead to a completely unexpected journey of his student. This detour is very demanding, yet only particularly lucky one is given a chance to it.

supersymmetric D-branes of B-type, for which algebro-geometric language is appropriate. The ansatz leads thus to a prototype[20] intrinsic definition of D-branes of B-type as follows:

Definition 2.2.3 [D-brane of B-type and Chan-Paton sheaf]. (1) A *D-brane of B-type* is an Azumaya-type noncommutative space $(X, \mathcal{O}_X, \mathcal{O}_X^{nc})$ over $\mathbb{C}$, together with a fundamental $\mathcal{O}_X^{nc}$-module $\mathcal{E}_X$. $\mathcal{E}_X$ is called the *Chan-Paton sheaf* on the D-brane X. We say that $\mathcal{E}_X$ has *rank* r if it has rank r as an $\mathcal{O}_X$-module. Note that $\mathcal{E}_X|_\eta \simeq \kappa_\eta^{n_1} \oplus \cdots \oplus \kappa_\eta^{n_s}$ at a generic point η of $(X, \mathcal{O}_X)$ with residue field κ_η if $\mathcal{O}_X^{nc}|_\eta / J(\mathcal{O}_X^{nc}|_\eta) \simeq M_{n_1}(\kappa_\eta) \times \cdots \times M_{n_s}(\kappa_\eta)$. Here $\mathcal{O}_X^{nc}|_\eta$ is the fiber of $\mathcal{O}_X^{nc}$ at η and $J(\,\cdot\,)$ is the Jacobson radical of $(\,\cdot\,)$. (2) A *D-brane* (of B-type) *in a target space* Y is a morphism $\Phi : X \to Y$. Here, Y can be a (commutative) scheme, an Azumaya-type noncommutative space, a noncommutative space represented by a ring-system, or whatever noncommutative space to which the notion of morphisms from X can be defined. The image Azumaya-type noncommutative space $\Phi(X)$ is called the *image D-brane* of X in Y. (3) The *Chan-Paton sheaf of a D-brane* $\Phi : X \to Y$ *on* Y is the push-forward $\Phi_*\mathcal{E}_X$ of $\mathcal{E}_X$, a coherent sheaf supported on $\Phi(X)$ in Y.

Remark/Definition 2.2.4 [D-brane vs. D-brane world-volume]. The world-volume of a D-brane is what a D-brane sweeps out in a space-time and, hence, has the extra time-dimension than the D-brane has. It has a Lorentzian structure by definition. The world-volume after Wick rotation is called a Euclidean D-brane world-volume, which has now a Riemannian structure. We will define a *Eulcidean D-brane world-volume of B-type* the same as in Definition 2.2.3 with 'D-brane' replaced by 'Euclidean D-brane world-volume'. Similarly, for a *Euclidean D-brane world-volume* (of B-type) *in a target space* Y and the *Chan-Paton sheaf* and *its push-forward* on Y. In general, we keep the word 'Euclidean' implicit and call it simply *D-brane world-volume* (of B-type) (resp. *D-brane world-volume* (of B-type) *in* Y). Readers should compare these simplified terminologies with the term 'world-sheet' in the commonly used statement by physicists: "The world-sheet of a string is a Riemann surface.", which takes the same interpretation implicitly.

How these two definitions fit in string theory and, by themselves, reproduce three key open-string-induced properties of D-branes can be summarized/highlighted as follows:

(1) [*interaction with open strings*]

- The Chan-Paton sheaf $\mathcal{E}_X$ should be identified with a singular coherent analytic sheaf on X with a (singular) connection A via a Kobayashi-Hitchin correspondence. An end-point of an open string in Y can then be coupled to the D-brane X via a morphism $\Phi : X \to Y$ and the connection A, regarded as on $\mathcal{E}_X$.

(2) [*source of Ramond-Ramond fields*]

[20] For non-string-theorists: There are two reasons we call this a "prototype" definition. The first one is mild: we focus only on the most essential fields on the brane and ignore the others. The second one is the true reason: the definition we give here reflects only what one should think mathematically about a D-brane *in a special region of the relevant Wilson's theory-space of string theory* (cf. [L-Y1: appendix A.1]) and, furthermore, we ignore also here the variation to the definition required to incorporate all forms of *D-brane bound states*. Once we move away from this region, what one should think of D-branes can become more complicated or even not that clear when trying to incorporate both mathematics and physics involved. However, since the mathematical definition given here naturally reproduces the key features of D-branes in its beginning years after Polchinski [Pol2], it is our strong belief that those more involved and languagewise more demanding features/descriptions of D-branes by string theorists in its growing years can finally be reached, beginning with the current prototype definition. While the detail of this advanced step remains challenging, there is definitely a related Floer-Gromov-Witten-type theory involved so that the coupling of D-branes and strings is always incorporated, cf. footnote 1.

· (Subject to that X here has to be interpreted as a Euclidean D-brane world-volume.) Identify $(X, \mathcal{O}_X)$ canonically with an analytic space X_{an} (with the structure sheaf $\mathcal{O}_{X_{an}}$ of analytic functions). A Ramond-Ramond field (i.e. a differential form) on Y can be pulled back and integrate over X_{an} via $\Phi : X \to Y$.

(3) [*Higgsing/un-Higgsing associated to un-stacking/stacking of D-brane*]

The Azumaya-type noncommutative structure $\mathcal{O}_X^{nc}$ on X makes the deformations of $\Phi : X \to Y$ locally matrix-valued, as in [Pol4]. It realizes the *Higgsing/un-Higgsing* behavior of the gauge theory on D-branes on Y via (a continuous family of) deformations of a morphism $\Phi : X \to Y$, as explained below:

(3.1) Associated to the (associative, unital) $\mathcal{O}_X$-algebra $\mathcal{O}_X^{nc}$ is the (non-associative, non-unital) Lie $\mathcal{O}_X$-algebra $\mathcal{O}_X^{nc,Lie} := (\mathcal{O}_X^{nc}, [\,\cdot\,,\,\cdot\,])$ with the commutator product $[s_1, s_2] := s_1 \cdot s_2 - s_2 \cdot s_1$ for local sections of $\mathcal{O}_X^{nc}$. A gauge theory on the D-brane X corresponds to a choice of a gauge sheaf $\mathcal{G}_X$ embedded in $\mathcal{O}_X^{nc,Lie}$. Here, a *gauge sheaf* is a sheaf of $\mathcal{O}_X$-Lie-algebras that generalizes the notion of the Lie-algebra bundle associated to the adjoint representation of the gauge group of a principal bundle.[21] This renders $\mathcal{E}_X$ a $\mathcal{G}_X$-module. Thus, it is enough to consider $\mathcal{O}_X^{nc}$ and $\mathcal{E}_X$ as an $\mathcal{O}_X^{nc}$-module.

(3.2) A D-brane $\Phi : X \to Y$ on Y determines a sheaf $\mathcal{O}_X \subset \mathcal{A}^{nc} \subset \mathcal{O}_X^{nc}$ of subalgebras of $\mathcal{O}_X^{nc}$, namely the image of the ring-system homomorphism $\mathcal{R}_Y \to \mathcal{R}_X$ that defines Φ. The associated *gauge symmetry on the D-brane on* Y is given by the sheaf $\mathcal{C}entralizer_{\mathcal{O}_X^{nc}}(\mathcal{A}^{nc})$ of *centralizer subalgebras* of $\mathcal{A}^{nc}$ in $\mathcal{O}_X^{nc}$. A continuous family $\Phi_t : X_t \to Y$ of deformations of the morphism $\Phi : X \to Y$ gives rise to a (not-necessarily flat) family $\mathcal{C}entralizer_{\mathcal{O}_{X_t}^{nc}}(\mathcal{A}_t^{nc})$ of sheaves of algebras. This realizes the Higgsing/un-Higgsing behavior of the gauge symmetry on D-branes on Y under deformations of D-branes on Y.[22]

[21] Readers may wonder why we do not take $\mathcal{O}_X^{nc,Lie}$ or $\mathcal{G}_X$ directly to define the noncommutative structure on X. There are two reasons: (1) The "geometry" (in the sense of "points" and "topology") associated to a non-associative, non-unital ring is less clear than that for an associative unital ring at the moment. (2) Since the function ring of local charts of the target space is associative and unital, if we use $\mathcal{O}_X^{nc,Lie}$ for X, we will have to consider ring-homomorphisms from an associative unital ring to a Lie ring. The only such ring-homomorphism is the zero-homomorphism. This renders such setting containing no contents as long as "probing a space(-time) via morphisms into it" is concerned. Cf. footnote 17.

[22] (1) A priori, one has a choice of whether or not the Higgsing/un-Higgsing of D-branes should be described as nearby points in the to-be-constructed moduli space of D-branes. For a fixed string target-space Y, the Wilson theory-space of "D-branes" in the region where they are still branes resembles the Wilson theory-space of a gauge system. With the type of the gauge system fixed, we have a continuum for the latter theory-space. The gauge group and hence the gauge bundle under Higgsing/un-Higgsing jump discontinuously but the situation is like that on the theory-space in Seiberg-Witten theory: there is a continuum as the theory-space. Another similar situation occurs in the geometric engineering of gauge theories, in which the compactification of a superstring theory on a degeneration family $\mathcal{X}$ of Calabi-Yau 3-spaces over a base B gives rise to a family $\{QFT_b\}_{b \in B}$ of $d = 4$ effective field theories, parameterized by B, whose gauge symmetry is enhanced at special locus of B that corresponds to singular fibers of $\mathcal{X}/B$. Mathematicians may also recall the moduli space $\mathcal{M}$ of coherent sheaves of a fixed Hilbert polynomial on a projective variety. Even when $\mathcal{M}$ is connected, the function on $\mathcal{M}$ that assigns to an $[\mathcal{F}] \in \mathcal{M}$ its sheaf-cohomology dimensions or Betti numbers is in general discontinuous. The upper-semicontinuity of such a function, in particular h^0 from the global section functor, on $\mathcal{M}$ can be taken as a resemblance of the phenomenon of enhancement of gauge symmetry due to additional zero/massless modes.

(2) It can happen that the "good part" of the (coarse) moduli space of objects of different nature admit canonical identifications. For example, the moduli space of maps, the moduli space of subschemes, and the moduli space of cycles canonically coincide when the maps are embeddings of reduced schemes with the trivial automorphism group. Ignoring the issue of automorphisms, it is the behavior under degenerations (i.e. moving away from such "good part" of the moduli space) that the nature of the objects we intend to parameter reveals itself. It is only

These highlights explain why we take Definition 2.2.3 as a prototype intrinsic definition for D-branes (or D-brane world-volumes) in the region of the theory-space where "branes are still branes". Details of the case of D0-branes are given in Sec. 3 and Sec. 4. The general higher-dimensional brane case can be thought of as sheafifying/smearing the discussion for D0-branes along a higher-dimensional cycle, chain, or more generally current in the sense of [G-H] or [Fe]; cf. [L-Y4].

Remark 2.2.5 [*other intrinsic definitions*]. There have been other working mathematical intrinsic definitions for D-branes by other authors aiming also to understanding D-branes (in the region of Wilson's theory-space where "branes are still branes"). For example, there were the interpretation of D-branes as *stable torsion sheaves*, given, e.g., in [H-S-T] in the algebro-geometric category from the viewpoint of BPS states and Gopakumar-Vafa invariants, and the notion of '*flat D-branes*', given in [B-M-R-S] in the smooth differential-geometric category from the viewpoint of K-theory. Each of these definitions singles out important key properties/features of D-branes in stringy literatures. In contrast, our prototype intrinsic definition of D-branes follows from the Grothendieck's viewpoint of Polchinski's work, phrased as the Polchinski-Grothendieck Ansatz for D-branes. This starting point is lower than these other existing intrinsic definitions and can reach up/be linked, for example, to [H-S-T] by considering D-brane images with the push-forward Chan-Paton sheaf on the target space and to [B-M-R-S] by considering formal linear combinations of D-branes with Chan-Paton sheaves and their equivalence classes in the K-group of the D-brane.

3 $\mathit{Mor}\,(\mathit{Space}\,M_n(\mathbb{C}), Y)$ as a coarse moduli space.

We realize in this section the space

$$\mathit{Mor}\,(\mathit{Space}\,M_n(\mathbb{C}), Y) \;:=\; \mathit{Mor}\,([\mathcal{R}], [\{M_n(\mathbb{C})\}]) \;=\; \mathit{Mor}\,(\mathcal{R}, [\{M_n(\mathbb{C})\}])$$

of morphisms from $\mathit{Space}\,M_n(\mathbb{C})$ to $Y = \mathit{Space}\,[\mathcal{R}] = \mathit{Space}\,([\{R_\gamma\}_{\gamma\in C} \rightrightarrows \{R_{\gamma_1\gamma_2}\}_{\gamma_1,\gamma_2\in C}])$ as a constructible set in a topological space from an adhesion of affine varieties/$\mathbb{C}$.

3.1 Central localizations of Artinian rings and their modules.

Recall first the *Structure Theorem of Artinian Rings*:

Theorem 3.1.1 [Artinian ring]. ([A-M], [A-N-T], and [Jat].)

(1) *Let R be an Artinian ring. The center $Z(R)$ of R is a commutative Artinian ring and hence has finitely-many maximal ideals. Let t be the number of maximal ideals in $Z(R)$.*

(2) *There exist a unique collection $\{e_1, \cdots, e_t\}$ of orthogonal primitive idempotents in $Z(R)$ such that $1 = e_1 + \cdots + e_t$ and that R is the direct sum of the two-sided ideals $R = Re_1 + \cdots + Re_t$. Up to permutations, the collection $\{Re_1, \cdots, Re_t\}$ is unique with respect to the following property:*

when the degeneration feature distinct for each moduli problem is captured in the setting may one now hope to have a correct description of the objects and hence their moduli space. Definition 2.2.3 is made with both (1) and (2) in mind.

· *$R = I_1 + \cdots + I_{t'}$, where I_i are two-sided ideals of R, $I_i \cdot I_j = 0$ for $i \neq j$, and each I_i is indecomposable in the sense that I_i cannot be decomposed as a direct sum $I_{i'} + I_{i''}$ with $I_{i'}$ and $I_{i''}$ non-zero two-sided ideals.*

Under such decomposition of R, each $R_i := Re_i$ is itself an Artinian ring with identity e_i and the decomposition $R = Re_1 + \cdots + Re_t$ can be written as the product of rings $R = R_1 \times \cdots \times R_t$. This decomposition restricts to a decomposition $Z(R) = Z(R_1) \times \cdots \times Z(R_t)$ with each $Z(R_i)$, $i = 1, \ldots, t$, an Artinian local ring.

(3) *Let $J(R)$ be the Jacobson radical of R. Then there is an orthogonal idempotent decomposition*

$$1 \;=\; \sum_{j_1=1}^{l_1} e_{1j_1} \;+\; \cdots \;+\; \sum_{j_s=1}^{l_t} e_{tj_t}$$

in R that refines the decomposition $1 = e_1 + \cdots + e_t$ in $Z(R)$, with $e_i = \sum_{j_i=1}^{l_i} e_{ij_i}$, such that the image $\bar{e}_{ij_i}$ of e_{ij_i} in $R/J(R)$ lies in $Z(R/J(R))$ and that

$$\bar{1} \;=\; \sum_{j_1=1}^{l_1} \bar{e}_{1j_1} \;+\; \cdots \;+\; \sum_{j_t=1}^{l_t} \bar{e}_{tj_t}$$

is an orthogonal primitive idempotent decomposition in $Z(R/J(R))$. Let

$$\mathfrak{m}_{ij_i} \;:=\; R\,(1 - e_{ij_i})\,R\,, \qquad \textit{for } 1 \leq i \leq t \textit{ and } 1 \leq j_i \leq l_i\,,$$

and $\mathit{Spec}\,R$ be the set of all prime ideals in R. Then all prime ideals in R are maximal ideals and

$$\mathit{Spec}\,R \;=\; \{\,\mathfrak{m}_{ij_i}\;:\; 1 \leq i \leq t\,,\, 1 \leq j_i \leq l_i\,\}\,.$$

(4) *Consider the directed graph Γ_R with the set of vertices $\mathit{Spec}\,R$ and a directed edge $\mathfrak{m}_{i_1j_{i_1}} \rightarrow \mathfrak{m}_{i_2j_{i_2}}$ for each pair $(e_{i_1j_{i_1}}, e_{i_2j_{i_2}})$ with $e_{i_1j_{i_1}}J(R)e_{i_2j_{i_2}} \neq 0$. Then Γ_R has exactly t-many connected components $\Gamma_R^{(i)}$, $i = 1, \cdots, t$, with the set of vertices of $\Gamma_R^{(i)}$ being $\{\mathfrak{m}_{ij_i} : 1 \leq j_i \leq l_i\}$. The two graphs $\Gamma_R^{(i)}$ and Γ_{R_i} are canonically isomorphic. In particular, each Γ_{R_i} is connected.*

(5) *By definition, $J(R) = \cap_{i=1}^{t} \cap_{j=1}^{l_i} \mathfrak{m}_{ij_i}$. The quotient $\mathfrak{m}_{ij_i}/J(R)$, with the induced addition and multiplication from those of R, is a simple ring and hence is isomorphic to a matrix ring $M_{n_{ij_i}}(k_{ij_i})$ for some skew-field k_{ij_i}. The decomposition $R = Re_1 + \cdots + Re_t$ restricts to a decomposition $J(R) = J(R)_1 + \cdots + J(R)_t$, which can be written canonically as $J(R) = J(R_1) \times \cdots \times J(R_t)$. With respect to this, one has isomorphisms*

$$R/J(R) \;\simeq\; \prod_{i=1}^{t} R_i/J(R_i) \;\simeq\; \prod_{i=1}^{t}\prod_{j_i=1}^{l_i} M_{n_{ij_i}}(k_{ij_i})\,.$$

Remark 3.1.2 [*quiver*]. The graph Γ_R associated to an Artinian ring R (as an R-module) in Theorem is an example of (various) quivers associated to an R-module. See Sec. 4.1 and footnote 36 for a theme in which we bring this in again.

The theorem gives a visualization of an Artinian algebra $R/\mathbb{C}$ (e.g. $M_n(\mathbb{C})$ and its subalgebras) as a noncommutative space of the form:

"a finite collection of commutative points (i.e. $Spec\, Z(R)$), with each point dominated/shadowed by a noncommutative cloud (i.e. $Z(R_i) \subset R_i$, where $R_i := Re_i$); associated to each noncommutative cloud (i.e. R_i) over a commutative point (i.e. $Spec\, Z(R_i)$) are a refined collection of commutative points (i.e. $Spec\,(\sum_{j=1}^{l_i} \mathbb{C} \cdot e_{ij})$) split off from and stacked over that point (more precisely, $Spec\, Z(R_i)_{\mathrm{red}}$) and are dominated/shadowed by that cloud (i.e. $\sum_{j=1}^{l_i} e_{ij} = e_i$ and $\mathbb{C} \cdot e_i \subset \sum_{j=1}^{l_i} \mathbb{C} \cdot e_{ij} \subset R_i$) and bound by directed bonds (i.e. $e_{ij_1} J(R_i) e_{ij_2}$ with the direction from e_{ij_1} to e_{ij_2}) created through that cloud (i.e. R_i)".

The following are immediate consequences of the theorem.

Lemma 3.1.3 [central non-zero-divisor invertible]. *Let R be an Artinian ring and $r \in Z(R)$ be a non-zero-divisor in R. Then r is invertible in R.*

Lemma 3.1.4 [direct-sum decomposition of module]. (Cf. *Peirce decomposition.*) *Let R be an Artinian ring and $R = Re_1 + \cdots + Re_t =: R_1 + \cdots + R_t$ be a decomposition of R as in Theorem 3.1.1 (2). Let M be an R-module. Then, $M = e_1 M + \cdots + e_t M =: M_1 + \cdots + M_t$ is a direct-sum decomposition of M such that $R_i M_i = M_i$ and $R_j M_i = 0$ for $j \neq i$. In particular, M_i is a R_i-module for $i = 1, \ldots, t$.*

Corollary 3.1.5 [localization = quotient]. (With notations from above.) *R_i is canonically isomorphic to both the quotient $R/(e_j : j \neq i) = R/(\sum_{j\neq i} e_j)$ of R and the localization $R[S_i^{-1}]$ of R, where S_i is the multiplicatively closed subset $\{1, e_i\}$. Similarly, M_i is canonically isomorphic to both the quotient $M/(\sum_{j \neq i} M_j)$ of M and the localization $M[S_i^{-1}]$ of M.*

Corollary 3.1.6 [localization: standard form]. (1) *Any nonzero central localization $R \to R'$ of an Artinian ring R is realized by inverting a finite multiplicatively closed subset $S \subset Z(R)$ that consists only of idempotents. I.e. $R' = R[S^{-1}]$ and $R' \to R$ is $R \to R[S^{-1}]$ for an aforementioned S.* (2) *Any central localization $f : R \to R'$ of an Artinian ring R is a quotient of R that admits a ring-set-homomorphism*[23] *$g : R' \to R$ such that $f \circ g = Id_{R'}$.* (3) *Fix a direct-sum decomposition $R = R_1 + \cdots + R_t$ from Theorem 3.1.1 (2). Then the localization $f : R \to R'$ in* (2) *is simply the projection of R onto the sum $R_{i_1} + \cdots + R_{i_{t'}}$ of some direct summands and $g : R' \to R$ in* (2) *can be taken to be the inclusion of $R_{i_1} + \cdots + R_{i_{t'}}$ into R.*

Proof. Let $R = R_1 + \cdots + R_t$ be a direct-sum decomposition of R from Theorem 3.1.1 (2). Then $S = S_1 + \cdots + S_t$, where $S_i := e_i S \subset Z(R_i)$, is a direct-sum decomposition of S and $R[S^{-1}] = R_1[S_1^{-1}] \times \cdots \times R_t[S_t^{-1}]$ canonically. This reduces the proof to the case that $t = 1$ in the decomposition of R (i.e. the case $Z(R)$ is an Artinian local ring).

When $Z(R)$ is an Artinian local ring, $R[S^{-1}] = 0$ if S contains an element in the maximal ideal of $Z(R)$, as such an element is nilpotent. Otherwise, all elements of S are not in the maximal ideal of $Z(R)$; then they are all invertible and, hence, $R[S^{-1}] = R$. In the former (resp. latter) case, we may replace S by $\{1, 0\}$ (resp. $\{1\}$). The corollary now follows.

□

Lemma 3.1.7 [localization in terms of generators of S]. *Let R be an Artinian ring and S be a multiplicatively closed subset in $Z(R)$, generated by*[24] *$\{s_1, \cdots, s_l\}$. Let n_0 be a positive integer such that every nilpotent element r of R satisfies $r^{n_0} = 0$. Then $R[S^{-1}] = R/\sum_{i=1}^{l} (s_i^{n_0})^{\perp}$, where $(\,\bullet\,)^{\perp} := \{r \in R : (\,\bullet\,) \cdot r = 0\}$.*

[23] See Definition 3.2.2.

[24] I.e. an element of S is either the identity 1 or a monomial of $s_1, \cdots, s_l$.

Proof. This follows immediately from Corollary 3.1.6. □

3.2 $Mor\,(Space\,M_n(\mathbb{C}), Y)$ as a coarse moduli space.

Definition 3.2.1 [ring-subset]. Let $R = (R, 0, 1, +, \cdot)$ be a ring, with the identity 1. An additive subgroup $R' \subset R$ is called a *ring-subset* of R if, in addition, (1) R' is closed under the multiplication $\cdot$ in R, and (2) there is an element $e \in R'$ such that $(R', 0, e, +, \cdot)$ is a ring with the identity e.

Definition 3.2.2 [ring-set-homomorphism]. Let R and S be rings with the identity 1_R and 1_S respectively. A map $\varphi : R \to S$ is called a *ring-set-homomorphism* if φ satisfies all the requirement for a ring-homomorphism *except* that it is not required that $\varphi(1_R) = 1_S$.

Note that e in Definition 3.2.1 is unique and satisfies $e^2 = e$.

Example 3.2.3 [ring-subset]. The image $\varphi(R)$ in Definition 3.2.2 is a ring-subset of S with the identity $\varphi(1_R)$. In particular, $\{0\} \subset R$ is the minimal ring-subset of R.

We will retain these terminologies for algebras and algebra-homomorphisms over a fixed ground field as well.

Surrogates of the Azumaya-type noncommutative point $Space\,M_n(\mathbb{C})$**.**

$M_n(\mathbb{C})$ is a simple ring in the sense that it is semi-simple as a left $M_n(\mathbb{C})$-module and has the only two-sided ideals the zero-ideal $(\mathbf{0})$ and itself $M_n(\mathbb{C})$. In particular, the only prime ideal of $M_n(\mathbb{C})$ is $(\mathbf{0})$ and the center $Z(M_n(\mathbb{C}))$ of $M_n(\mathbb{C})$ is given by $\mathbb{C} \cdot \mathbf{1}$. There are only two Gabriel filters on $M_n(\mathbb{C})$: $\mathfrak{F}_0$ that is generated by $(\mathbf{0})$ and is given by the set of all left ideals of $M_n(\mathbb{C})$ and $\mathfrak{F}_1 := \{M_n(\mathbb{C})\}$. The localization of $M_n(\mathbb{C})$ with respect to $\mathfrak{F}_0$ (resp. $\mathfrak{F}_1$) is the zero-ring 0 (resp. $M_n(\mathbb{C})$ itself). The former (resp. latter) covers the notion of the localization of $M_n(\mathbb{C})$ with respect to a non-invertible (resp. invertible) element. Thus, directly on $M_n(\mathbb{C})$, we see only a seemingly barren geometry. Things change when we bring in the notion of surrogates introduced in Sec. 1.1.

A surrogate of the Azumaya-type noncommutative point $Space\,M_n(\mathbb{C}) = (Spec\,\mathbb{C}, \mathbb{C}, M_n(\mathbb{C}))$ is given ring-theoretically by a subalgebra pair $\mathbb{C} \subset C \subset R \subset M_n(\mathbb{C})$ with $C \subset Z(R)$. It follows from Corollary 3.1.6 that a finite central cover of the sub-$\mathbb{C}$-algebra R of $M_n(\mathbb{C})$ can be described by a finite collection $\{(R_\alpha, e_\alpha)\}_{\alpha \in A}$ of ring-subsets of R (and hence of $M_n(\mathbb{C})$) that satisfies the following conditions:

(0) $R = \sum_{\alpha \in A} R_\alpha$.

(1) e_{α_1} commutes with elements of R_{α_2} for all $\alpha_1,\, \alpha_2 \in A$.

(2) $e_{\alpha_1} R_{\alpha_2} = e_{\alpha_2} R_{\alpha_1}$ for all $\alpha_1,\, \alpha_2 \in A$.

(3) $e_{\alpha_1} e_{\alpha_2} \in R_{\alpha_1}$ for all $\alpha_1,\, \alpha_2 \in A$.

(4) Fix a well-ordering of the index set A; then

$$1 \;=\; \sum_{\alpha} e_\alpha \;-\; \sum_{\alpha_1 < \alpha_2} e_{\alpha_1} e_{\alpha_2} \;+\; \sum_{\alpha_1 < \alpha_2 < \alpha_3} e_{\alpha_1} e_{\alpha_2} e_{\alpha_3} \;\pm\; \cdots \;+\; (-1)^{|A|+1} \sum_{\alpha_1 < \cdots < \alpha_{|A|}} e_{\alpha_1} \cdots e_{\alpha_{|A|}}\,.$$

Conditions (1), (2), and (3) imply that $e_{\alpha_2}R_{\alpha_1} = R_{\alpha_1} \cap R_{\alpha_2} = e_{\alpha_1}R_{\alpha_2}$, which is itself a ring with the identity $e_{\alpha_1}e_{\alpha_2}$. In particular, $(e_{\alpha_2}R_{\alpha_1}, e_{\alpha_2}e_{\alpha_1}) = ((e_{\alpha_1}e_{\alpha_2})R_{\alpha_1}, e_{\alpha_1}e_{\alpha_2})$ is a ring-subset of both rings $(R_{\alpha_1}, e_{\alpha_1})$ and $(R_{\alpha_2}, e_{\alpha_2})$. Condition (4) simplifies to $1 = \sum_\alpha e_\alpha$ when $R = \sum_\alpha R_\alpha$ is a direct sum.

Conversely, one has the following proposition:

Proposition 3.2.4 [subring in terms of a collection of ring-subsets]. (1) *Let* $\{(R_\alpha, e_\alpha)_{\alpha\in A}\}$ *be a finite collection of ring-subsets of* $M_n(\mathbb{C})$ *that satisfies Conditions* (1), (2), (3), *and* (4) *above. Then* $R := \sum_{\alpha\in A} R_\alpha$ *contains the identity* $\mathbf{1}$ *of* $M_n(\mathbb{C})$ *and is a sub-*$\mathbb{C}$*-algebra of* $M_n(\mathbb{C})$. (2) *There are tautological ring-homomorphisms* $R \to R_\alpha$, $\alpha \in A$, *that render the collection* $\{R \to R_\alpha\}_{\alpha\in A}$ *a finite central cover of* R.

Proof. Observe that for a ring-subset (P, e_P) and an idempotent e' of a ring Q that commutes with the elements in P, $(e'P, e'e_P)$ is another ring-subset of Q. In particular, elements in $e'P$ are closed under the multiplication in Q. Moreover, if, in addition, $e'e_p \in P$, then $P = (e_P - e'e_P)P + (e'e_P)P$ is an orthogonal direct-sum decomposition for P (when neither summand is zero). Using these observations, one can show that Properties (1), (2), and (3) imply that $R_{\alpha_1}R_{\alpha_2} \subset R_{\alpha_1} + R_{\alpha_2}$ for all $\alpha_1,\ \alpha_2 \in A$. This proves that $R := \sum_\alpha R_\alpha$ is closed under the multiplication in $M_n(\mathbb{C})$ as $R \cdot R \subset \sum_{\alpha_1,\alpha_2\in A}(R_{\alpha_1} + R_{\alpha_2}) = R$.

Now let

$$e \;:=\; \sum_\alpha e_\alpha \;-\; \sum_{\alpha_1<\alpha_2} e_{\alpha_1}e_{\alpha_2} \;+\; \sum_{\alpha_1<\alpha_2<\alpha_3} e_{\alpha_1}e_{\alpha_2}e_{\alpha_3} \;\pm\; \cdots \;+\; (-1)^{|A|+1} \sum_{\alpha_1<\,\cdots\,<\alpha_{|A|}} e_{\alpha_1}\,\cdots\, e_{\alpha_{|A|}}\,.$$

Then it follows from the above that $e \in R$. For $r \in R_{\alpha_i}$, $\alpha_i \in A$, one can check directly that $re = r$, using the property that $re_{\alpha_i} = r$ and the above defining expression of e. This implies that $er = r$ for every $r \in R$. It follows that (R, e) is a ring-subset of $M_n(\mathbb{C})$. The additional Condition (4), $e = 1$, implies then that R is a subalgebra of $M_n(\mathbb{C})$. This proves Statement (1).

Condition (1) implies that $\{e_\alpha\}_{\alpha\in A} \subset Z(R)$. For each $\alpha \in A$, the commutativity, idempotent property, and that both $e_\alpha R_\alpha = R_\alpha$ and $e_\alpha(e_\alpha R) = e_\alpha R$ hold imply that the orthogonal direct-sum decomposition $R = e_\alpha R + (1 - e_\alpha)R$ of R coincides with the decomposition $R = R_\alpha + e_\alpha^\perp$, where $e_\alpha^\perp := \{r \in R : e_\alpha r = 0\}$. This shows that the projection map $R \to R_\alpha$ from the above decomposition is identical with the central localization of R with respect to the multiplicatively closed subset $\{1, e_\alpha\}$. Furthermore, $\sum_{\alpha\in A} e_\alpha$ is invertible in $Z(R)$. Thus, $\{(R \to R_\alpha)\}_{\alpha\in A}$ is a central finite cover of R. This proves Statement (2).

$\square$

The space $\mathit{Mor}^{\text{ring-set}}(R, M_n(\mathbb{C}))$ **of ring-set-homomorphisms from** R **to** $M_n(\mathbb{C})$**.**

Let R be a finitely-presentable algebra over $\mathbb{C}$ and $\mathit{Mor}^{\text{ring-set}}(R, M_n(\mathbb{C}))$ be the set of ring-set-homomorphisms from R to $M_n(\mathbb{C})$. We will construct a topology on $\mathit{Mor}^{\text{ring-set}}(R, M_n(\mathbb{C}))$ in this theme.

Let

- $R = \langle g_0, g_1, \cdots, g_l\rangle/(r_1, \cdots, r_m)$ be a presentation of R as a quotient of the free unital associative $\mathbb{C}$-algebra $\langle g_0, g_1, \cdots, g_l\rangle$ generated by $g_0, g_1, \cdots, g_l$ by the two-sided ideal $(r_1, \cdots, r_m)$ generated by $\{r_i = r_i(g_0, \cdots, g_l) : i = 1, \ldots, m\}$. Here, for later use, we have the redundant generator $g_0 =$ the identity 1 and the redundant relators $g_0g_i = g_ig_0 = g_i$, $i = 0, 1, \ldots, l$, contained in the relator set $\{r_1, \cdots, r_m\}$.

- $\cdot$ $Gr^{(2)}(n; d, n-d) \simeq GL_n(\mathbb{C})/(GL_d(\mathbb{C}) \times GL_{n-d}(\mathbb{C}))$ be the Grassmannian manifold of ordered pairs (Π_1, Π_2) of $\mathbb{C}$-linear subspaces of $\mathbb{C}^n$ with $dim\,\Pi_1 = d$, $dim\,\Pi_2 = n-d$, and $\Pi_1 + \Pi_2 = \mathbb{C}^n$;

- $\cdot$ $\mathbf{1}_d$, $d = 0, \ldots, n$, be the diagonal matrix $Diag(1, \cdots, 1, 0, \cdots, 0)$ in $M_n(\mathbb{C})$ whose first d diagonal entries are 1 and the rest 0, (here, $\mathbf{1}_0$ = the zero-matrix 0 and $\mathbf{1}_n = \mathbf{1}$ by convention); and

- $\cdot$ '$m_1 \sim m_2$' means that m_1 and m_2 are in the same adjoint $GL_n(\mathbb{C})$-orbit in $M_n(\mathbb{C})$.

Let $Rep^{\text{ring-set}}(R, M_n(\mathbb{C}))$ be the subvariety of the affine space $\mathbb{A}^{n^2}_{(0)} \times \mathbb{A}^{n^2}_{(1)} \times \cdots \times \mathbb{A}^{n^2}_{(l)}$ (here $\mathbb{A}^{n^2}_{(i)}$ has the polynomial coordinate ring $\mathbb{C}[m_{i,jk} : 1 \le j, k \le n]$, $i = 0, \ldots, l$) determined[25] by the system of equations

$$r_1(M_0, M_1, \cdots, M_l) = \cdots = r_m(M_0, M_1, \cdots, M_l) = \text{the zero-matrix } 0 \in M_n(\mathbb{C}),$$

where $M_i = (m_{i,jk})_{jk}$. Note that this is like the ordinary representation variety of R in $M_n(\mathbb{C})$ *except* that it is not required that M_0 = the identity $\mathbf{1} \in M_n(\mathbb{C})$. For convenience, we will call the reduced affine scheme $Rep^{\text{ring-set}}(R, M_n(\mathbb{C}))$ the *representation variety* in our discussion. By construction, we have the Zariski topology on $Rep^{\text{ring-set}}(R, M_n(\mathbb{C}))$ and the analytic topology on the set $Rep^{\text{ring-set}}(R, M_n(\mathbb{C}))_{\mathbb{C}}$ of $\mathbb{C}$-points of $Rep^{\text{ring-set}}(R, M_n(\mathbb{C}))$. Regard $\mathbb{C}^n$ as the unique non-zero irreducible $M_n(\mathbb{C})$-module. Then, the correspondence $e \mapsto (e \cdot \mathbb{C}^n, e^{\perp})$, where $e^{\perp}$ here $= \{v \in \mathbb{C}^n : e \cdot v = 0\}$, gives rise to a (continuous) map from the set of idempotents $\sim \mathbf{1}_d$ in $M_n(\mathbb{C})$ to $Gr^{(2)}(n; d, n-d)$. It follows that the projection map $\pi_{(0)} : \mathbb{A}^{n^2}_{(0)} \times \mathbb{A}^{n^2}_{(1)} \times \cdots \times \mathbb{A}^{n^2}_{(l)} \to \mathbb{A}^{n^2}_{(0)}$ restricts to a map

$$\pi_{(0)} : Rep^{\text{ring-set}}(R, M_n(\mathbb{C})) \longrightarrow \amalg_{d=0}^{n} Gr^{(2)}(n; d, n-d).$$

Let

$$Rep^{\text{ring-set}}(R, M_n(\mathbb{C}))_{(d)} := \pi_{(0)}^{-1}(Gr^{(2)}(n; d, n-d)).$$

As a set, $Mor^{\text{ring-set}}(R, M_n(\mathbb{C})) = Rep^{\text{ring-set}}(R, M_n(\mathbb{C}))_{\mathbb{C}}$. This identification defines a preliminary analytic topology $\mathcal{T}_0$ on $Mor^{\text{ring-set}}(R, M_n(\mathbb{C})) = \amalg_{d=0}^{n} Mor^{\text{ring-set}}(R, M_n(\mathbb{C}))_{(d)}$ by bringing over the analytic topology on $Rep^{\text{ring-set}}(R, M_n(\mathbb{C}))_{\mathbb{C}}$. We then modify this preliminary analytic topology, following an analytic format of a *valuative criterion*, so that each $Mor^{\text{ring-set}}(R, M_n(\mathbb{C}))_{(d')}$, $d' < d$, adheres to $Mor^{\text{ring-set}}(R, M_n(\mathbb{C}))_{(d)}$ appropriately, for $d = 1, \cdots, n$, in the new topology. Let T be a (commutative, Noetherian) integral domain over $\mathbb{C}$ and $(D := (Spec\,T)_{\mathbb{C}}, p)$ be the associated analytic space together with a base $\mathbb{C}$-point p. Note that the residue field κ_p of T at p is canonically isomorphic to $\mathbb{C}$. Let T_p be the localization of T at p and Q_T be the field of fractions of T. Then, $T \subset T_p \subset Q_T$, T_p is a valuation ring of Q_T (regarded now as the field of fractions of T_p), and $M_n(T) \subset M_n(T_p) \subset M_n(Q_T)$.

Definition 3.2.5 [limit of family ring-set-homomorphisms]. Let $\phi : R \to M_n(Q_T)$ be a ring-set-homomorphism such that there exists a unique idempotent $e \in M_n(T_p)$ such that (1) $e \in Z(Im\,\phi)$; (2) $e \cdot \phi$ is a ring-set-homomorphism from R to T_p; in particular, $(e \cdot \phi)|_p : R \to M_n(\kappa_p) = M_n(\mathbb{C})$ makes sense; (3) $Im\,(e \cdot \phi)|_p$ is the unique maximum (with respect to inclusion) in the set of ring-subsets $Im\,(e' \cdot \phi)$ of $M_n(\kappa_p)$, where e' satisfies Condition (1) and Condition (2) above. For such a ϕ, we call $(e \cdot \phi)|_p$ the *limit* of ϕ over D at p.

[25] I.e. taking the reduced scheme associated to the possibly nonreduced subscheme described by the ideal generated by these equations.

Such a ϕ defines a rational map $\Phi_\phi : (D,p) \dashrightarrow Mor^{\text{ring-set}}(R, M_n(\mathbb{C}))^{\mathcal{T}_0}$ that is assigned the value $(e\cdot\phi)|_p$ at p.

Definition 3.2.6 [$Mor^{\text{ring-set}}(R, M_n(\mathbb{C}))$ **with analytic topology].** With the notations from above, let $\mathcal{T}$ be the weakest topology on $Mor^{\text{ring-set}}(R, M_n(\mathbb{C}))_{\mathbb{C}}$ such that

(1) the tautological inclusion $Mor^{\text{ring-set}}(R, M_n(\mathbb{C}))^{\mathcal{T}_0}_{(d)} \hookrightarrow Mor^{\text{ring-set}}(R, M_n(\mathbb{C}))^{\mathcal{T}}$ of sets is an embedding of topological spaces, for $d = 0, \cdots, n$, and that

(2) Φ_ϕ is continuous at p for all T, (D,p), and ϕ in Definition 3.2.5.[26]

$\mathcal{T}$ is called the *analytic topology* on $Mor^{\text{ring-set}}(R, M_n(\mathbb{C}))$.

Proposition 3.2.7 [independence of presentation]. *$(Mor^{\text{ring-set}}(R, M_n(\mathbb{C})), \mathcal{T})$ is independent of the choice of presentations of R in the construction.*

Proof. Associated to a new presentation

$$R \;=\; \langle g_0', g_1', \cdots, g_{l'}'\rangle/(r_1', \cdots, r_{m'}')$$

of R is a canonical ring-isomorphism

$$f^\sharp \;:\; \langle g_0', g_1', \cdots, g_{l'}'\rangle/(r_1', \cdots, r_{m'}') \;\xrightarrow{\sim}\; \langle g_0, g_1, \cdots, g_l\rangle/(r_1, \cdots, r_m)\,,$$

represented by a noncanonical ring-homomorphism $\tilde{f}^\sharp : \langle g_0', g_1', \cdots, g_{l'}'\rangle \to \langle g_0, g_1, \cdots, g_l\rangle$. $\tilde{f}^\sharp$ induces contravariantly a morphism

$$\tilde{f} \;:\; \mathbb{A}^{n^2}_{(0)} \times \mathbb{A}^{n^2}_{(1)} \times \cdots \times \mathbb{A}^{n^2}_{(l)} \;\longrightarrow\; \mathbb{A}^{n^2}_{(0)} \times \mathbb{A}^{n^2}_{(1)} \times \cdots \times \mathbb{A}^{n^2}_{(l')}$$

that restricts to a morphism

$$f \;:\; Rep^{\text{ring-set}}(R, M_n(\mathbb{C})) \;\longrightarrow\; Rep^{\text{ring-set}}(R, M_n(\mathbb{C}))'\,,$$

where $Rep^{\text{ring-set}}(R, M_n(\mathbb{C}))' \subset \mathbb{A}^{n^2}_{(0)} \times \mathbb{A}^{n^2}_{(1)} \times \cdots \times \mathbb{A}^{n^2}_{(l')}$ is the representation variety associated to the new presentation of R. Reverse this argument, now from $\langle g_0, g_1, \cdots, g_l\rangle/(r_1, \cdots, r_m)$ to $\langle g_0', g_1', \cdots, g_{l'}'\rangle/(r_1', \cdots, r_{m'}')$, implies that f is indeed an isomorphism.

Since a ring-isomorphism sends the identity to the identity, f restricts to isomorphisms $f_{(d)} : Rep^{\text{ring-set}}(R, M_n(\mathbb{C}))_{(d)} \xrightarrow{\sim} Rep^{\text{ring-set}}(R, M_n(\mathbb{C}))'_{(d)}$, for $d = 0, \cdots, n$. In other words, $f_{(d)} : Mor^{\text{ring-set}}(R, M_n(\mathbb{C}))^{\mathcal{T}_0}_{(d)} \xrightarrow{\sim} Mor^{\text{ring-set}}(R, M_n(\mathbb{C}))'^{\,\mathcal{T}_0'}_{(d)}$, for $d = 0, \cdots, n$. Furthermore, each valuative criterion setup $\Phi_\phi : (D,p) \dashrightarrow Mor^{\text{ring-set}}(R, M_n(\mathbb{C}))$ gives a valuative

[26] This is a valuative criterion. The meaning of this topology in terms of analytic geometry is as follows. Under deformations of a morphism from $Space\, M_n(\mathbb{C})$ to $Space\, R$, some connected components of the image points of $Space\, M_n(\mathbb{C})$ may move away toward the boundary at infinity of $Space\, R$ and disappear in the end. This corresponds to a drop from $M_0 \sim \mathbf{1}_d$ to some $M_0 \sim \mathbf{1}_{d'}$ with $d' < d$. When we consider only $Mor(Space\, M_n(\mathbb{C}), Space\, R)$ by itself, $M_0 \sim \mathbf{1}$ must always hold. However, when we consider $Mor(Space\, M_n(\mathbb{C}), Space\, R)$ that occurs as a subset in $Mor(Space\, M_n(\mathbb{C}), Space\, \mathcal{R})$ for a gluing system $\mathcal{R}$ of rings that contains R as a member, it can happen that some of the connected components of the image of a morphism $Space\, M_n(\mathbb{C}) \to Space\, \mathcal{R}$ is not contained in $Space\, R$. This explains geometrically why, in the equivalent ring-theoretic language, we enlarge here the class of maps from ring-homomorphisms to ring-set-homomorphisms. Furthermore, when the morphism deforms, the number of connected components in $Space\, R$ of the image of morphisms from $Space\, M_n(\mathbb{C})$ to $Space\, \mathcal{R}$ can change. The topology on $Mor^{\text{ring-set}}(R, M_n(\mathbb{C}))$ defined in Definition 3.2.6 ring-theoretically takes all these issues into account. Such treatment automatically comes up and is rerquired in building a (general) morphism from $[\mathcal{R}]$ to $[\{M_n(\mathbb{C})\}]$, following Definition 1.2.14.

criterion setup $\Phi_{\phi'} = f \circ \Phi_\phi : (D,p) \dashrightarrow \mathit{Mor}^{\text{ring-set}}(R, M_n(\mathbb{C}))'$ and vice versa. As we choose the topology $\mathcal{T}$ on $\mathit{Mor}^{\text{ring-set}}(R, M_n(\mathbb{C}))$ (resp. $\mathcal{T}'$ on $\mathit{Mor}^{\text{ring-set}}(R, M_n(\mathbb{C}))'$) to be the weakest topology that renders all inclusions $\mathit{Mor}^{\text{ring-set}}(R, M_n(\mathbb{C}))^{\mathcal{T}_0}_{(d)} \hookrightarrow \mathit{Mor}^{\text{ring-set}}(R, M_n(\mathbb{C}))$ (resp. $\mathit{Mor}^{\text{ring-set}}(R, M_n(\mathbb{C}))'^{\mathcal{T}_0}_{(d)} \hookrightarrow \mathit{Mor}^{\text{ring-set}}(R, M_n(\mathbb{C}))'$) embeddings of topological spaces and all Φ_ϕ's (resp. $\Phi_{\phi'}$'s) continuous, this implies that

$$f \;:\; (\mathit{Mor}^{\text{ring-set}}(R, M_n(\mathbb{C})), \mathcal{T}) \longrightarrow (\mathit{Mor}^{\text{ring-set}}(R, M_n(\mathbb{C}))', \mathcal{T}')$$

is an isomorphism. This completes the proof.

$\square$

By construction, there is a canonical (continuous) bijective embedding

$$\tau_{R,n} \;:\; \mathit{Rep}^{\text{ring-set}}(R, M_n(\mathbb{C}))_{\mathbb{C}} \longrightarrow \mathit{Mor}^{\text{ring-set}}(R, M_n(\mathbb{C}))\,.$$

Remark 3.2.8 [*moduli problem*]. By construction, $\mathit{Mor}^{\text{ring-set}}(R, M_n(\mathbb{C}))$ is a coarse moduli space of ring-set-homomorphisms from R to $M_n(\mathbb{C})$. Since a ring-set-homomorphism with a fixed domain and target does not have non-trivial automorphisms, it is instructive to think of $\mathit{Mor}^{\text{ring-set}}(R, M_n(\mathbb{C}))$ as representing the functor

$$\begin{array}{cccc} \mathcal{F} \;:\; & \left(\begin{array}{l}\text{(commutative) varieties}/\mathbb{C}\\ \text{with analytic topology}\end{array}\right)^{\circ} & \longrightarrow & (\,\text{sets}\,) \\ & V & \longmapsto & \mathit{Mor}_{\mathcal{O}_V\text{-}\mathit{Alg}}(\mathcal{O}_V \otimes R\,,\, \mathcal{O}_V \otimes M_n(\mathbb{C})) \end{array}$$

similar to a functor of points. Here, $(\cdots)^\circ$ is the category $(\cdots)$ with the arrows reversed.

$\mathit{Mor}(\mathit{Space}\, M_n(\mathbb{C}), Y)$ as a coarse moduli space.

Let Y be a noncommutative space presented as a gluing system of finitely-presentable rings $\mathcal{R} = (\{R_\alpha\}_{\alpha\in A} \rightrightarrows \{R_{\alpha_1\alpha_2}\}_{\alpha_1,\alpha_2\in A})$. We fix a well-ordering of the index set A for convenience. Denote the identity of R_α by 1_{R_α}. Assume that each central localization $\varphi_{\alpha_1\alpha_2} : R_{\alpha_1} \to R_{\alpha_1\alpha_2}$ is associated to a finitely-generated multiplicatively closed subset $S_{\alpha_1\alpha_2}$ in $Z(R_{\alpha_1})$.

Definition 3.2.9 [admissible tuple]. A tuple $(\varphi_\alpha : R_\alpha \to M_n(\mathbb{C}))_{\alpha\in A}$ of ring-set-homomorphisms to $M_n(\mathbb{C})$ is called *admissible* if it satisfies the following conditions:

(1) $\varphi_{\alpha_1}(1_{R_{\alpha_1}})$ commutes with elements of $\varphi_{\alpha_2}(R_{\alpha_2})$ for all α_1, $\alpha_2 \in A$.

(2) $\varphi_{\alpha_1}(1_{R_{\alpha_1}})\,\varphi_{\alpha_2}(R_{\alpha_2}) = \varphi_{\alpha_2}(1_{R_{\alpha_2}})\,\varphi_{\alpha_1}(R_{\alpha_1})$ for all α_1, $\alpha_2 \in A$.

(3) $\varphi_{\alpha_1}(1_{R_{\alpha_1}})\,\varphi_{\alpha_2}(1_{R_{\alpha_2}}) \in \varphi_{\alpha_1}(R_{\alpha_1})$ for all α_1, $\alpha_2 \in A$.

(4) Let $\mathbf{1}$ be the identity matrix in $M_n(\mathbb{C})$. Then

$$\begin{aligned} \mathbf{1} \;=\; \sum_\alpha \varphi_\alpha(1_{R_\alpha}) \;-\; & \sum_{\alpha_1<\alpha_2} \varphi_{\alpha_1}(1_{R_{\alpha_1}})\,\varphi_{\alpha_2}(1_{R_{\alpha_2}}) \\ & + \sum_{\alpha_1<\alpha_2<\alpha_3} \varphi_{\alpha_1}(1_{R_{\alpha_1}})\,\varphi_{\alpha_2}(1_{R_{\alpha_2}})\,\varphi_{\alpha_3}(1_{R_{\alpha_3}}) \\ & \pm\; \cdots \;+\; (-1)^{|A|+1} \sum_{\alpha_1<\cdots<\alpha_{|A|}} \varphi_{\alpha_1}(1_{R_{\alpha_1}})\,\cdots\,\varphi_{\alpha_{|A|}}(1_{R_{\alpha_{|A|}}})\,. \end{aligned}$$

(5) $\varphi_{\alpha_2}(1_{R_{\alpha_2}}) \cdot \big(\varphi_{\alpha_1}(s)^{\perp} \cap \varphi_{\alpha_1}(R_{\alpha_1})\big) = 0$, where $\varphi_{\alpha_1}(s)^{\perp} := \{m \in M_n(\mathbb{C}) : \varphi_{\alpha_1}(s) \cdot m = 0\}$, for all α_1, $\alpha_2 \in A$ and $s \in S_{\alpha_1\alpha_2}$. This condition is equivalent to the existence of push-out $\varphi_{\alpha_1}|_{\alpha_2}$ under localizations in the following commutative diagram:

$$
\begin{array}{ccc}
R_{\alpha_1} & \stackrel{\varphi_{\alpha_1}}{\longrightarrow} & \varphi_{\alpha_1}(R_{\alpha_1}) \\
\downarrow & & \downarrow \\
R_{\alpha_1\alpha_2} & \stackrel{\varphi_{\alpha_1}|_{\alpha_2}}{\longrightarrow} & \varphi_{\alpha_2}(1_{R_{\alpha_2}}) \cdot \varphi_{\alpha_1}(R_{\alpha_1})
\end{array} \quad .
$$

(6) $\varphi_{\alpha_1}|_{\alpha_2} = \varphi_{\alpha_2}|_{\alpha_1} \circ \varphi_{\alpha_1\alpha_2}$ for all α_1, $\alpha_2 \in A$.

The meaning of these conditions is given below.

- Conditions (1) - (4): The finite collection $\{(\varphi_\alpha(R_\alpha), e_\alpha := \varphi_\alpha(1_{R_\alpha}))\}_{\alpha\in A}$ of ring-subsets of $M_n(\mathbb{C})$ glue to $\sum_{\alpha\in A}\varphi_\alpha(R_\alpha)$ that is a subalgebra of $M_n(\mathbb{C})$. Cf. Proposition 3.2.4.
- Condition (5): Elements in $\varphi_{\alpha_1}(S_{\alpha_1\alpha_2})$ become invertible after being mapped to $e_{\alpha_2} \cdot \varphi_{\alpha_1}(R_{\alpha_1})$ and, hence, φ_{α_1} can be pushed out to a ring-homomorphism $\varphi_{\alpha_1}|_{\alpha_2}$ from $R_{\alpha_1\alpha_2}$ to the localization $e_{\alpha_2} \cdot \varphi_{\alpha_1}(R_{\alpha_1})$ of $\varphi_{\alpha_1}(R_{\alpha_1})$. Cf. Lemma 3.1.3.
- Condition (6): The gluing conditions on the tuple $\{\varphi_\alpha : R_\alpha \to M_n(\mathbb{C})\}_{\alpha\in A}$ as a system of ring-homomorphisms from $\mathcal{R}$ to $(\{\varphi_\alpha(R_\alpha)\}_{\alpha\in A} \rightrightarrows \{e_{\alpha_2}\cdot\varphi_{\alpha_1}(R_{\alpha_1})\}_{\alpha_1,\alpha_2\in A})$. Cf. Condition (2) above and Definition 1.2.6.

Thus, Conditions (1) - (6) are necessary conditions for the tuple $\{\varphi_\alpha\}_{\alpha\in A}$ to represent a morphism from $(\mathit{Spec}\,\mathbb{C}, \mathbb{C}, M_n(\mathbb{C}))$ to Y. It follows from Definition 1.2.14 that they are also sufficient and that such presentations are effective in the sense that different admissible tuples give different morphisms. This proves the following lemma:

Lemma 3.2.10 [admissible tuple = morphism]. *A tuple $\Phi = (\varphi_\alpha : R_\alpha \to M_n(\mathbb{C}))_{\alpha\in A}$ of ring-set-homomorphisms to $M_n(\mathbb{C})$ corresponds to a morphism from $\mathit{Space}\,M_n(\mathbb{C})$ to $Y = \mathit{Space}\,\mathcal{R}$ if and only if Φ is admissible. As sets, $\mathit{Mor}\,(\mathit{Space}\,M_n(\mathbb{C}), Y) = \{$ admissible tuples $\}$.*

Fix now the following data of presentations and representatives:

- [*ring chart*]
  a finite presentation for each ring-chart R_α in $\mathcal{R}$

  $$R_\alpha \;=\; \langle g_0^{(\alpha)}, g_1^{(\alpha)}, \cdots, g_{l^{(\alpha)}}^{(\alpha)}\rangle / (r_1^{(\alpha)}, \cdots, r_{m^{(\alpha)}}^{(\alpha)})\,,$$

  with the redundant generator $g_0^{(\alpha)} = 1_{R_\alpha}$ and the redundant relators

  $$g_0^{(\alpha)} g_i^{(\alpha)} \;=\; g_i^{(\alpha)} g_0^{(\alpha)} \;=\; g_i^{(\alpha)}\,,\;\; i \,=\, 0,\, 1,\, \ldots\,, l^{(\alpha)}\,,$$

  contained in the relator set $\{r_1^\alpha, \cdots, r_{m^\alpha}^\alpha\}$, as before;
- [*localization*]
  a lifting (as sets) $\tilde{S}_{\alpha_1\alpha_2}$ of $S_{\alpha_1\alpha_2}$ in $\langle g_0^{(\alpha_1)}, g_1^{(\alpha_1)}, \cdots, g_{l^{(\alpha_1)}}^{(\alpha_1)}\rangle$ for each $(\alpha_1, \alpha_2) \in A \times A$;

· [*transition data*]

a representative in the induced presentation of $R_{\alpha_2\alpha_1}$ for each $g_i^{(\alpha_1)}$, $i = 0, \ldots, l^{(\alpha_1)}$, $\tilde{s} \in \tilde{S}_{\alpha_1\alpha_2}$, and $(\alpha_1, \alpha_2) \in A \times A$,

$$(g_i^{(\alpha_1\alpha_2)}, s_i^{(\alpha_1\alpha_2)})\,,\;\; (g_{\tilde{s}}^{(\alpha_1\alpha_2)}, s_{\tilde{s}}^{(\alpha_1\alpha_2)}) \;\in\; \langle g_0^{(\alpha_2)}, g_1^{(\alpha_2)}, \cdots, g_{l^{(\alpha_2)}}^{(\alpha_2)} \rangle \times \tilde{S}_{\alpha_2\alpha_1}$$

so that $\varphi_{\alpha_1\alpha_2}(g_i^{(\alpha_1)}, 1_{R_{\alpha_1}}) = (g_i^{(\alpha_1\alpha_2)}, s_i^{(\alpha_1\alpha_2)})$ and $\varphi_{\alpha_1\alpha_2}(1_{R_{\alpha_1}}, \tilde{s}) = (g_{\tilde{s}}^{(\alpha_1\alpha_2)}, s_{\tilde{s}}^{(\alpha_1\alpha_2)})$. (Here, to simplify notations, we identify elements in a presentation of a ring with the corresponding elements in that ring.)

Let

· $\mathbb{A}^{n^2}_{(\alpha,\,i)}$, $\alpha \in A$, $i = 0, \ldots, l^{(\alpha)}$, be the affine space with the polynomial coordinate ring $\mathbb{C}[m_{i,jk}^{(\alpha)} : 1 \le j, k \le n]$;

· $\mathbf{A}_\alpha$ be the affine space $\mathbb{A}^{n^2}_{(\alpha,0)} \times \mathbb{A}^{n^2}_{(\alpha,1)} \times \cdots \times \mathbb{A}^{n^2}_{(\alpha,\,l^{(\alpha)})}$ and $\mathbf{A}$ be the affine space $\prod_{\alpha\in A} \mathbf{A}_\alpha = \mathbb{A}^{\sum_{\alpha\in A}(1+l^{(\alpha)})\,n^2}$;

· $R(\mathbf{A}_\alpha) := \mathcal{O}_{\mathbf{A}_\alpha}(\mathbf{A}_\alpha) = \otimes_{i=0}^{l^{(\alpha)}} \mathbb{C}[m_{i,jk}^{(\alpha)} : 1 \le j, k \le n]$ and $R(\mathbf{A}) := \mathcal{O}_{\mathbf{A}}(\mathbf{A}) = \otimes_{\alpha\in A} R(\mathbf{A}_\alpha)$;

· $\Psi_\alpha : R(\mathbf{A}) \otimes_{\mathbb{C}} \langle g_0^{(\alpha)}, g_1^{(\alpha)}, \cdots, g_{l^{(\alpha)}}^{\alpha} \rangle \to R(\mathbf{A}) \otimes_{\mathbb{C}} M_n(\mathbb{C}) = M_n(R(\mathbf{A}))$ be the tautological $R(\mathbf{A})$-algebra-homomorphism defined/generated by[27]

$$1 \otimes g_i^{(\alpha)} \;\longmapsto\; 1 \otimes \left(m_{i,jk}^{(\alpha)}\right)_{jk}$$

and $Im\,\Psi_\alpha$ be the image $R(\mathbf{A})$-submodule of Ψ_α in $M_n(R(\mathbf{A}))$;

· $E_{\mathbf{A}} = \mathbf{A} \times M_n(\mathbb{C})$ be the trivialized trivial vector bundle on $\mathbf{A}$ with fiber the $\mathbb{C}$-algebra $M_n(\mathbb{C})$; the associated sheaf of local sections of $E_{\mathbf{A}}$ is $\mathcal{O}_{\mathbf{A}} \otimes M_n(\mathbb{C})$; elements and sub-$R(\mathbf{A})$-modules in $\mathcal{O}_{\mathbf{A}} \otimes M_n(\mathbb{C})$ are canonically identified respectively with global sections and constructible sets in $E_{\mathbf{A}}$.

Define

$$Rep^{\,\text{ring-set}}(\mathcal{R}, M_n(\mathbb{C})) \;\subset\; \prod_{\alpha\in A} Mor^{\,\text{ring-set}}(R_\alpha, M_n(\mathbb{C}))$$

to be the locus in the indicated product space determined by the following system of constraints from the defining conditions of admissible tuples, via the canonical bijective embedding

$$\prod_{\alpha\in A} Mor^{\,\text{ring-set}}(R_\alpha, M_n(\mathbb{C})) \xleftarrow{\;\prod_{\alpha\in A}\tau_{R_\alpha,n}\;} \prod_{\alpha\in A} Rep^{\,\text{ring-set}}(R_\alpha, M_n(\mathbb{C}))_{\mathbb{C}} \;\subset\; \mathbf{A}\;:$$

(0.1) $$r_1^{(\alpha)}(M_{\alpha,0}, M_{\alpha,1}, \cdots, M_{\alpha,\,l^{(\alpha)}}) \;=\; \cdots \;=\; r_{m^{(\alpha)}}^{(\alpha)}(M_{\alpha,0}, M_{\alpha,1}, \cdots, M_{\alpha,\,l^{(\alpha)}})$$
$$=\; \text{the zero-matrix } 0 \,\in\, M_n(\mathbb{C}), \text{ where } M_{\alpha,\,i} = \left(m_{i,jk}^{(\alpha)}\right)_{jk}.$$

(1.1) $M_{\alpha_1,0}\, M_{\alpha_2,i} = M_{\alpha_2,i}\, M_{\alpha_1,0}$ for all α_1, $\alpha_2 \in A$, $i = 0, \ldots, l^{(\alpha_2)}$.

[27]Recall that the multiplication $\cdot$ in the tensor product $\mathbb{C}$-algebra $R \otimes_{\mathbb{C}} S$ of two $\mathbb{C}$-algebras R and S is $\mathbb{C}$-linearly generated by defining $(r_1 \otimes s_1) \cdot (r_2 \otimes s_2) = (r_1 r_2) \otimes (s_1 s_2)$.

(1.2) $\quad M_{\alpha_1,0}\, Im\,\Psi_{\alpha_2} = M_{\alpha_2,0}\, Im\,\Psi_{\alpha_1}$ for all $\alpha_1,\, \alpha_2 \in A$.

(1.3) $\quad M_{\alpha_1,0}\, M_{\alpha_2,0} \in Im\,\Psi_{\alpha_1}$ for all $\alpha_1,\, \alpha_2 \in A$.

(1.4) $\quad$ ($\mathbf{1} \in M_n(\mathbb{C})$ is the identity)

$$\mathbf{1} \;=\; \sum_{\alpha} M_{\alpha,0} \;-\; \sum_{\alpha_1<\alpha_2} M_{\alpha_1,0}\, M_{\alpha_2,0} \;+\; \sum_{\alpha_1<\alpha_2<\alpha_3} M_{\alpha_1,0}\, M_{\alpha_2,0}\, M_{\alpha_3,0} \\ \pm\; \cdots \;+\; (-1)^{|A|+1} \sum_{\alpha_1<\,\cdots\,<\alpha_{|A|}} M_{\alpha_1,0}\, \cdots\, M_{\alpha_{|A|,0}}\,.$$

(1.5) $\quad M_{\alpha_2,0}\cdot \left(\Psi_{\alpha_1}(\tilde{s})^{\perp}_{E_{\mathbf{A}}} \cap Im\,\Psi_{\alpha_1}\right) = 0$ for all $\alpha_1,\, \alpha_2 \in A$ and $\tilde{s} \in \tilde{S}_{\alpha_1\alpha_2}$. Here[28]

$$\Psi_{\alpha_1}(\tilde{s})^{\perp}_{E_{\mathbf{A}}} \;:=\; \{m \in E_{\mathbf{A}} : \Psi_{\alpha_1}(\tilde{s})\cdot m = 0\}\,.$$

(1.6)
$$\left(M_{\alpha_2,0}\, M_{\alpha_1,i}\right)\, \left(M_{\alpha_1,0}\; s_i^{(\alpha_1\alpha_2)}(M_{\alpha_2,0},\, \cdots,\, M_{\alpha_2,\,l^{(\alpha_2)}})\right) \\ =\; M_{\alpha_1,0}\; g_i^{(\alpha_1\alpha_2)}(M_{\alpha_2,0},\, \cdots,\, M_{\alpha_2,\,l^{(\alpha_2)}})$$

and

$$\left(M_{\alpha_2,0}\, M_{\alpha_1,0}\right)\, \left(M_{\alpha_1,0}\; s_{\tilde{s}}^{(\alpha_1\alpha_2)}(M_{\alpha_2,0},\, \cdots,\, M_{\alpha_2,\,l^{(\alpha_2)}})\right) \\ =\; \left(M_{\alpha_1,0}\; g_{\tilde{s}}^{(\alpha_1\alpha_2)}(M_{\alpha_2,0},\, \cdots,\, M_{\alpha_2,\,l^{(\alpha_2)}})\right)\, \left(M_{\alpha_2,0}\; \tilde{s}(M_{\alpha_1,0},\, \cdots,\, M_{\alpha_1,\,l^{(\alpha_1)}})\right)$$

for all $\alpha_1,\, \alpha_2 \in A$, $i = 0,\, \ldots\, , l^{(\alpha_1)}$, and $\tilde{s} \in \tilde{S}_{\alpha_1\alpha_2}$.

Proposition 3.2.11 [$Mor\,(Space\, M_n(\mathbb{C}), Y)$]**.** *$Mor\,(Space\, M_n(\mathbb{C}), Y)$ is given by a constructible set in the product space $\prod_{\alpha\in A} Mor^{\text{ring-set}}\,(R_\alpha, M_n(\mathbb{C}))$, independent of the data of presentation chosen in the construction.*

Proof. Conditions (0.1), (1.1), and (1.4) are closed conditions. Condition (1.6) can be restricted to a finite generating set of $S_{\alpha_1\alpha_2}$ and, hence, gives also a closed condition. Conditions (1.2), (1.3), and (1.5) involve image $R(\mathbf{A})$-submodules $Im\,\Psi_\bullet$ in $M_n(R(\mathbf{A}))$. Let $S^0_{\alpha_1\alpha_2}$ be a finite generating set of $S_{\alpha_1\alpha_2}$ and $\tilde{S}^0_{\alpha_1\alpha_2} \subset \tilde{S}_{\alpha_1\alpha_2}$ its corresponding lifting in $\langle g_0^{(\alpha_1)}, g_1^{(\alpha_1)},\, \cdots\, , g_{l^{(\alpha_1)}}^{(\alpha_1)}\rangle$. Then, it follows from Lemma 3.1.7 and the fact that every nilpotent element m of $M_n(\mathbb{C})$ satisfies $m^n = 0$ that the seemingly possibly-infinite system of constraints from Condition (1.5) can be replaced by the following finite system:

(1.5)$'$ $\quad M_{\alpha_2,0}\cdot \left(\Psi_{\alpha_1}(\tilde{s}^n)^{\perp}_{E_{\mathbf{A}}} \cap Im\,\Psi_{\alpha_1}\right) = 0$ for all $\alpha_1,\, \alpha_2 \in A$ and $\tilde{s} \in \tilde{S}^0_{\alpha_1\alpha_2}$.

Thus, the solution set to Conditions (1.2), (1.3), and (1.5) is described by a finite intersection of constructible sets on $\mathbf{A}$ described via determinantal varieties.

This shows that the solution set to the system of constraints from Condition (0.1) and Conditions (1.1) - (1.5) is a constructible set in $\prod_{\alpha\in A} Rep^{\text{ring-set}}\,(R_\alpha, M_n(\mathbb{C}))$ and, hence, in $\prod_{\alpha\in A} Mor^{\text{ring-set}}\,(R_\alpha, M_n(\mathbb{C}))$. That different choices of data of presentations give isomorphic solution sets (with the subset topology) follows the same discussion as that in the proof of

[28]Caution that $\Psi_{\alpha_1}(\tilde{s})^{\perp}_{E_{\mathbf{A}}}$ here is defined to be the union of fiberwise $\perp$ of $\Psi_{\alpha_1}(\tilde{s})$ in $E_{\mathbf{A}}$. In general, it is not a sub-$R(\mathbf{A})$-module of $M_n(R(\mathbf{A}))$.

Proposition 3.2.7. Since $Mor\,(Space\,M_n(\mathbb{C}), Y) = Rep^{\,\text{ring-set}}\,(\mathcal{R}, M_n(\mathbb{C}))$ as sets, this concludes the proof.

$\square$

We remark that from the proof above, the constructible set referred to in Proposition 3.2.11 is of algebraic kind. It is the set of $\mathbb{C}$-points (with the analytic topology) of a finite union of constructible sets in varieties/$\mathbb{C}$.

Finally, note that in discussing the space of morphisms from $Space\,M_n(\mathbb{C})$ to Y, both $Space\,M_n(\mathbb{C})$ and Y are thought of as fixed. The automorphism group of $M_n(\mathbb{C})$ as a $\mathbb{C}$-algebra is given by $GL_n(\mathbb{C})$ via the adjoint $GL_n(\mathbb{C})$-action on $M_n(\mathbb{C})$. This induces a $GL_n(\mathbb{C})$-action on $Mor\,(Space\,M_n(\mathbb{C}), Y)$.

Definition 3.2.12 [isomorphism between morphisms]. Two morphisms from $Space\,M_n(\mathbb{C})$ to Y are said to be *isomorphic*, in notation $\Phi_1 \sim \Phi_2$, if they are in the same $GL_n(\mathbb{C})$-orbit in $Mor\,(Space\,M_n(\mathbb{C}), Y)$. Define the *space* $Map\,(Space\,M_n(\mathbb{C}), Y)$ *of maps* from $Space\,M_n(\mathbb{C})$ to Y to be the quotient space $Mor\,(Space\,M_n(\mathbb{C}), Y)/\sim$ (with the quotient topology). It parameterizes isomorphism classes of morphisms from $Space\,M_n(\mathbb{C})$ to Y.

4 D0-branes on a commutative quasi-projective variety.

A D0-brane in the sense of Definition 2.2.3 is simply an Azumaya-type noncommutative point $Space\,M_n(\mathbb{C})$ (cf. Example 1.1.3 and Example 1.1.8) together with the irreducible $M_n(\mathbb{C})$-module $\mathbb{C}^n$ as the Chan-Paton space/module. A D0-brane on a target space Y is given by an isomorphism class of morphisms from $Space\,M_n(\mathbb{C})$ to Y. The moduli space of D0-branes on Y in this sense is given then by $Map\,((Space\,M_n(\mathbb{C}); \mathbb{C}^n), Y) = Map\,(Space\,M_n(\mathbb{C}), Y) = Mor\,(Space\,M_n(\mathbb{C}), Y)/\sim$. This moduli space for the case of Y being a (commutative) complex quasi-projective smooth curve/surface, or a variety is given in this section to illustrate Sec. 1 - Sec. 3. These examples already reveal simplified key features of D-branes that are fundamental for beyond. Details involving only linear algebras in, e.g., [Ho-K] or straightforward manipulations are omitted.

4.1 D0-branes on the complex affine line $\mathbb{A}^1$.

Various themes concerning D0-branes on $\mathbb{A}^1$ are given in this subsection to illustrate the far-reaching/power of the Polchinski-Grothendieck Ansatz for D-branes, in particular the reproduction of D-brane properties in the work of Polchinski. Same/Similar phenomena occur also for other targets in later subsections by same/similar reasons, which we then omit but focus mainly on the moduli problem. The general discussions in Sec. 1 - Sec. 3 are intentionally made explicit in this example. For that reason, some important algebro-geometric notions are slightly repeated in this subsection for concreteness.

The moduli space $Map\,((Space\,M_n(\mathbb{C}); \mathbb{C}^n)\,,\, \mathbb{A}^1)$ of D0-branes on $\mathbb{A}^1$.

Let $Y = \mathbb{A}^1 = Spec\,\mathbb{C}[y]$ be the affine line over $\mathbb{C}$. Then the Grothendieck Satz or Lemma 1.2.19 says that $Mor\,(Space\,M_n(\mathbb{C}), Y) = Mor\,(\mathbb{C}[y], M_n(\mathbb{C}))$. The corresponding $\mathbb{C}$-algebra representation variety $Rep\,(\mathbb{C}[y], M_n(\mathbb{C}))$ is given by $\mathbb{A}^{n^2}$ with a closed point represented by $m = (m_{ij})_{i,j} \in M_n(\mathbb{C})$ corresponding to the $\mathbb{C}$-algebra-homomorphism

$$\varphi_m : \mathbb{C}[y] \to M_n(\mathbb{C})\,, \ \text{generated by } 1 \mapsto \mathbf{1} \text{ and } y \mapsto m\,.$$

We will call the $GL_n(\mathbb{C})$-action on $Rep(\mathbb{C}[y], M_n(\mathbb{C}))$ by post-compositions with the conjugations on $M_n(\mathbb{C})$ still the *adjoint action*. It follows that

$$Map((Space\, M_n(\mathbb{C}); \mathbb{C}^n), \mathbb{A}^1) \;=\; Map(Space\, M_n(\mathbb{C}), \mathbb{A}^1) \;=\; Rep(\mathbb{C}[y], M_n(\mathbb{C}))/\!\sim,$$

the orbit-space[29] of the adjoint action with the quotient topology. This space is a connected non-Hausdorff topological space, well-understood in other contents from algebraic geometry and Lie groups and Lie algebras as follows.

Each adjoint-orbit O_{φ_m} is represented by a Jordan form J_m of m, unique up to permutations of diagonal blocks of J_m with distinct characteristic values. An adjoint-orbit on $Rep(\mathbb{C}[y], M_n(\mathbb{C}))$ is closed if and only of it is represented by φ_m associated to a diagonal matrix m. Given an orbit O_{φ_m}, let $\overline{O_{\varphi_m}}$ be the closure of O_{φ_m} in $\mathbb{A}^{n_2}$. It has the property that O_{φ_m} is an open dense subset in $\overline{O_{\varphi_m}}$ and that $\overline{O_{\varphi_m}}$ is a union of O_{φ_m} and finitely many lower-dimensional orbits, (e.g. [Stei]). Note that any two orbits $O_{\varphi_{m_1}}$ and $O_{\varphi_{m_2}}$ satisfy either $O_{\varphi_{m_1}} \cap \overline{O_{\varphi_{m_2}}} = \emptyset$ or $O_{\varphi_{m_1}} \subset \overline{O_{\varphi_{m_2}}}$.

Definition 4.1.1 [partial order on $Rep(\mathbb{C}[y], M_n(\mathbb{C}))/\sim$]. Define a *partial order* on the orbit-space $Rep(\mathbb{C}[y], M_n(\mathbb{C}))/\!\sim$ by setting $O_{\varphi_{m_1}} \prec O_{\varphi_{m_2}}$ if $O_{\varphi_{m_1}} \subset \overline{O_{\varphi_{m_2}}}$.

This partial order can be described in terms of Jordan forms, as follows.

Let $J_j^{(\lambda)} \in M_j(\mathbb{C})$ be the matrix

$$\begin{bmatrix} \lambda & & & 0 \\ 1 & \lambda & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda \end{bmatrix}_{j\times j}$$

A Jordan form J in $M_n(\mathbb{C})$ is a matrix of the following form

$$\begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k \end{bmatrix} \quad \text{with each } A_i \in M_{n_i}(\mathbb{C}) \text{ of the form } \quad \begin{bmatrix} J_{d_{i1}}^{(\lambda_i)} & & \\ & \ddots & \\ & & J_{d_{ik_i}}^{(\lambda_i)} \end{bmatrix}.$$

Here, omitted entries are all zero, $n_1 \geq \cdots \geq n_k > 0$, and $d_{i1} \geq \cdots \geq d_{ik_i} > 0$. We thus have a *double partition* of n by non-increasing positive integers:

$$\pi(n) \;:\; n \;=\; n_1 + \cdots + n_k\,; \quad \pi(n_i) \;:\; n_i \;=\; d_{i1} + \cdots + d_{ik_i}\,,\; i = 1\,, \ldots\,, k\,.$$

We will call this double partition the *type*, in notation $type(J)$, of J. Denote also the set of all such double partitions of n by $PP(n)$. Then the admissible permutations of the blocks $A_i, \cdots, A_k$ induces a finite group action on $PP(n)$. The quotient set is denoted by $PP(n)/\!\sim$. For a general $m \in M_n(\mathbb{C})$, define its type by $type(m) = type(J_m)$, which is uniquely defined after passing to $PP(n)/\!\sim$.

Definition 4.1.2 [partial order between Jordan forms]. *Given two Jordan forms J_1 and J_2, we say that $J_1 \prec J_2$ if the following two conditions are satisfied:*

(1) *J_1, J_2 have the same characteristic values $\lambda_1, \cdots, \lambda_k$ of the same multiplicities n_i for λ_i.*

(2) *Let $A_{1i}, A_{2i} \in M_{n_i}(\mathbb{C})$ be the diagonal blocks of J_1 and J_2 respectively that are associated to λ_i and $\mathbf{1}_{n_i}$ be the identity of $M_{n_i}(\mathbb{C})$. Then $rank((A_{1i} - \lambda_i \mathbf{1}_{n_i})^j) \leq rank((A_{2i} - \lambda_i \mathbf{1}_{n_i})^j)$ for all $j \in \mathbb{N}$.*

[29]We shall always think of such an orbit-space $M/\!\sim$ as an Artin stack with atlas M. When M is smooth, it is in this sense that we define a smooth map to $M/\!\sim$.

This defines a partial order $\prec$ on the set of Jordan matrices in $M_n(\mathbb{C})$ that is invariant under admissible permutations of diagonal blocks of distinct characteristic values.

Proposition 4.1.3 [partial order of orbits via Jordan forms]. ([M-T], [Ge], [Dj].)

$$O_{\varphi_{m_1}} \prec O_{\varphi_{m_2}} \text{ if and only if } J_{m_1} \prec J_{m_2}.$$

The following simplified/coarser partial order helps us to see things more directly.

Definition 4.1.4 [isotopic decay].[30] The composition of a sequence of operations of the form $J_j^{(\lambda)} \to Diag(J_{j_1}^{(\lambda)}, J_{j_2}^{(\lambda)})$ with $j = j_1 + j_2$, $j_1 \geq j_2$, will be called an *isotopic decay.*

Given two Jordan forms J_1 and J_2, define $J_1 \prec\prec J_2$ if J_1 is obtained from J_2 by a sequence of isotopic decays and an re-arrangement of the sub-blocks in each diagonal block associated to a characteristic value.

Lemma 4.1.5 [coarser partial order]. (1) *$O_{m_1} \prec O_{m_2}$ if $J_{m_1} \prec\prec J_{m_2}$.* (2) *$\prec$ and $\prec\prec$ generate the same equivalence relation, in notation $\approx$, on the set of Jordan forms.*

Remark 4.1.6 [orbit dimension drop under $\prec\prec$]. (E.g. [We]; also [Ge] or [Bas].) Let

$$\begin{array}{c} T^{(b_1,\,\cdots\,b_i)}_{i\times j} \\ (i\le j)\end{array} \;=\; \begin{bmatrix} b_1 & & & & & \\ b_2 & b_1 & & & & \\ & b_2 & \ddots & & & 0 \\ \vdots & \ddots & \ddots & b_1 & & \\ b_i & \cdots & & b_2 & b_1 & \end{bmatrix}_{i\times j}, \qquad \begin{array}{c} T^{(b_1,\,\cdots\,b_j)}_{i\times j} \\ (i\ge j)\end{array} \;=\; \begin{bmatrix} & & 0 & & \\ b_1 & & & & \\ b_2 & b_1 & & & \\ & b_2 & \ddots & & \\ \vdots & \ddots & \ddots & b_1 & \\ b_j & \cdots & & b_2 & b_1 \end{bmatrix}_{i\times j}.$$

Here, all omitted entries are zero. The centralizer of J, in the form given previously, consists of all matrices of the form

$$\begin{bmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_k \end{bmatrix}$$

with each $B_i \in M_{n_i}(\mathbb{C})$ of the block form $[B_{i,rs}]_{k_i\times k_i}$ where

$$B_{i,rs} \;=\; T^{(b_{i,rs;1},\,\cdots\,b_{i,rs;di_r})}_{di_r\times di_s} \text{ for } r\ge s, \quad B_{i,rs} \;=\; T^{(b_{i,rs;1},\,\cdots\,b_{i,rs;di_s})}_{di_r\times di_s} \text{ for } r<s.$$

(Again, omitted entries are all zero.) The dimension of the stabilizer of J, as given, is thus

$$\begin{aligned} n \;\le\; & dim_{\mathbb{C}} Stab\,(J) \\ =\; & \sum_{i=1}^{k} \Big((d_{i1} + \cdots + d_{ik_i}) + 2(d_{i2} + \cdots + d_{ik_i}) + \cdots + 2(d_{ik_i})\Big) \;\le\; n^2\,. \end{aligned}$$

Thus, for each $J_j^{(\lambda)} \to Diag(J_{j_1}^{(\lambda)}, J_{j_2}^{(\lambda)})$ with $j_1 \geq j_2$ the corresponding new adjoint-orbit drops the dimension by an integral amount $\geq j_2$.

Some properties of $Map\,((Space\, M_n(\mathbb{C}); \mathbb{C}^n), \mathbb{A}^1)$ are listed below:

[30] For topologists: Here the term "*isotopic*" comes from the notion of "isotope" in physics/chemitry, not topology. The reason why we choose this term is partially enlightened in footnote 35.

(1) The equivalence relation $\approx$ in Lemma 4.1.5 descends to an equivalence relation, still denoted by $\approx$, on the topological space $\mathit{Map}\,((\mathit{Space}\,M_n(\mathbb{C});\mathbb{C}^n),\mathbb{A}^1)$. The associated quotient space $\mathit{Map}\,((\mathit{Space}\,M_n(\mathbb{C});\mathbb{C}^n),\mathbb{A}^1)/\!\approx$ is the n-th symmetric product $S^n\mathbb{A}^n := (\mathbb{A}^1)^n/\mathit{Sym}_n \simeq$ [31] $\mathbb{A}^n$ of $\mathbb{A}^1$, where Sym_n is the permutation group of n letters. Each $\approx$-equivalence class of points on $\mathit{Map}\,((\mathit{Space}\,M_n(\mathbb{C});\mathbb{C}^n),\mathbb{A}^1)$ contains a unique maximal point and a unique minimal point with respect to $\prec$ on $\mathit{Map}\,((\mathit{Space}\,M_n(\mathbb{C});\mathbb{C}^n),\mathbb{A}^1)$. Any other point in the same class is sandwiched between the two by $\prec$.

(2) The types of Jordan forms give rise to a finite stratification $\{S_t\}_t$ of $\mathit{Map}\,((\mathit{Space}\,M_n(\mathbb{C});\mathbb{C}^n),\mathbb{A}^1)$. The stratum associated to the double partition

$$\pi(n)\;:\;n\;=\;n_1+\,\cdots\,+n_k\,;\quad \pi(n_i)\;:\;n_i\;=\;d_{i1}+\,\cdots\,+d_{ik_i}\,,\;i=1\,,\,\ldots\,,k\,,$$

of n is homeomorphic to $(\mathbb{C}^k-(\text{diagonal locus}))/\mathit{Sym}_k$. Here, 'diagonal locus' means the set of all points whose coordinates have some identical entries. The stratum $S_{(n=1+\,\cdots\,+1)}$ is open dense in $\mathit{Map}\,((\mathit{Space}\,M_n(\mathbb{C});\mathbb{C}^n),\mathbb{A}^1)$.

The Chan-Paton space/module on D0-branes on $\mathbb{A}^1$.

Let $m\in M_n(\mathbb{C})$ with the Jordan form as given above and $\langle\mathbf{1},m\rangle$ be the sub-algebra of $M_n(\mathbb{C})$ generated by $\mathbf{1}$ and m. $\langle\mathbf{1},m\rangle$ is commutative. The *c*haracteristic polynomial and the *min*imal polynomial of m are then respectively

$$f_m^c(\lambda)\;=\;(\lambda-\lambda_1)^{n_1}\,\cdots\,(\lambda-\lambda_k)^{n_k}\quad\text{and}\quad f_m^{\min}(\lambda)\;=\;(\lambda-\lambda_1)^{d_{11}}\,\cdots\,(\lambda-\lambda_k)^{d_{k1}}\,.$$

Lemma 4.1.7 [interpolation formula]. *Given $g(\lambda):=(\lambda-\lambda_1)^{d_1}\,\cdots\,(\lambda-\lambda_k)^{d_k}\in\mathbb{C}[\lambda]$ with the λ_i's distinct from each other, then the inverse of $g(\lambda)/(\lambda-\lambda_i)^{d_i}$ in $\mathbb{C}[\lambda]/((\lambda-\lambda_i)^{d_i})$ exists, for $i=1,\,\ldots\,,k$. Denote this inverse by $(1/g_{(i)})(\lambda)$, which is a polynomials of degree $\le d_i-1$. Let $d=d_1+\,\cdots\,+d_k$ and $f(\lambda)$ be a polynomial of degree $<d$. Then there exist unique polynomials $f_i(\lambda)$ with deg $f_i(\lambda)<d_i$ such that*

$$f(\lambda)\;=\;\sum_{i=1}^{k}\,f_i(\lambda)\cdot(1/g_{(i)})(\lambda)\cdot\frac{g(\lambda)}{(\lambda-\lambda_i)^{d_i}}\,.$$

Indeed, $f_i(\lambda)$ is the Taylor expansion of $f(\lambda)$ in $(\lambda-\lambda_i)$ up to (including) degree d_i-1.

It follows that, as a $\mathbb{C}$-algebra,

$$\begin{aligned}\langle\mathbf{1},m\rangle\;&\simeq\;\mathbb{C}[\lambda]/(f_m^{\min}(\lambda))\\ &=\;\sum_{i=1}^{k}\left((1/f_{m\,(i)}^{\min})(\lambda)\cdot\frac{f_m^{\min}(\lambda)}{(\lambda-\lambda_i)^{d_{i1}}}\right)\;\simeq\;\prod_{i=1}^{k}\left(\mathbb{C}[\lambda]/(\lambda-\lambda_i)^{d_{i1}}\right)\,.\end{aligned}$$

The sum in the above expression is a direct sum of orthogonal indecomposable ideals in $\mathbb{C}[\lambda]/(f_m^{\min}(\lambda))$ associated to the decomposition

$$1\;=\;\sum_{i=1}^{k}(1/f_{m\,(i)}^{\min})(\lambda)\cdot\frac{f_m^{\min}(\lambda)}{(\lambda-\lambda_i)^{d_{i1}}}$$

[31] However, caution that under this isomorphism that comes from the ring generated by elementary symmetric polynomials, the diagonal locus in $(\mathbb{A}^1)^n$ becomes a complicated discriminant locus in $\mathbb{A}^n$.

through the complete set of primitive orthogonal idempotents in $\mathbb{C}[\lambda]/(f_m^{\min}(\lambda))$. The length $l_{\langle \mathbf{1}, m\rangle}$ of $\langle \mathbf{1}, m\rangle$ is $deg\, f_m^{\min}(\lambda) = d_{11} + \cdots + d_{k1}$.

Let $\mathbb{C}^n$ be the unique non-zero irreducible representation of $M_n(\mathbb{C})$. Up to the $GL_n(\mathbb{C})$ adjoint action, we may assume that m is already a Jordan form $J = Diag\,(A_1, \cdots, A_k)$ given earlier. Let $\mathbf{1}_{(i)} = Diag\,(0, \cdots, 0, \mathbf{1}_{n_i}, 0, \cdots, 0)$, where $\mathbf{1}_{n_i}$ in the i-th position is the identity matrix $\in M_{n_i}(\mathbb{C})$ and the 0 in the j-th position are the zero-matrix $\in M_{n_j}(\mathbb{C})$ for $j = 1, \cdots, i-1, i+1, \cdots, k$. Then

$$\left((1/f_{J\,(i)}^{\min})(\lambda) \cdot \frac{f_J^{\min}(\lambda)}{(\lambda - \lambda_i)^{d_{i1}}} \right) \Bigg|_{\lambda = J} \;=\; \mathbf{1}_{(i)}\,.$$

This implies that $\mathbf{1}_{(i)} \in \langle \mathbf{1}, J\rangle$ for $i = 1, \ldots, k$ and that $\mathbf{1} = \mathbf{1}_{(1)} + \cdots + \mathbf{1}_{(k)}$ is an orthogonal primitive idempotent decomposition in $\langle \mathbf{1}, J\rangle$. The corresponding direct-sum decomposition, now as $\langle \mathbf{1}, J\rangle$-modules,

$$\mathbb{C}^n \;=\; \mathbf{1}_{(1)} \cdot \mathbb{C}^n \,+\, \cdots \,+\, \mathbf{1}_{(k)} \cdot \mathbb{C}^n \;=\; \mathbb{C}^{n_1} \,+\, \cdots \,+\, \mathbb{C}^{n_k} \;=:\; V_1 \,+\, \cdots \,+ V_k$$

is the same decomposition of $\mathbb{C}^n$ that renders J the given diagonal block form. As a $\langle \mathbf{1}, J\rangle$-module, V_i $(= \mathbb{C}^{n_i})$ decomposes into a direct sum $V_i = \mathbb{C}^{d_{i1}} + \cdots + \mathbb{C}^{d_{ik_i}} =: V_{i1} + \cdots + V_{ik_i}$ of indecomposable $\langle \mathbf{1}, J\rangle$-modules. *Spec* $\langle \mathbf{1}, J\rangle$ has k-many connected components, associated respectively to ideals $(\mathbf{1} - \mathbf{1}_{(i)})$ in $\langle \mathbf{1}, J\rangle$, $i = 1, \ldots, k$. One has that

$$\langle \mathbf{1}, J\rangle/(\mathbf{1} - \mathbf{1}_{(i)}) \;=\; \langle \mathbf{1}, J\rangle \cdot \mathbf{1}_{(i)} \;\simeq\; \langle \mathbf{1}_{n_i}, A_i\rangle \;\simeq\; \mathbb{C}[\lambda]/((\lambda - \lambda_i)^{n_i})$$

and that the annihilator $Ann(V_i)$ of V_i $(= \mathbb{C}^{n_i})$ as an $\langle \mathbf{1}, J\rangle$-module is $(\mathbf{1} - \mathbf{1}_{(i)})$. In terms of $\langle \mathbf{1}, J\rangle \simeq \prod_{i=1}^k \langle \mathbf{1}_{n_i}, A_i\rangle$, the $\langle \mathbf{1}, J\rangle$-modules V_i, $V_{i1}, \cdots, V_{ik_i}$ are also $\langle \mathbf{1}_{n_i}, A_i\rangle$-modules automatically.

The above algebraic statements correspond to the following geometric picture of Chan-Paton modules on the associated D0-branes on $\mathbb{A}^1$:

(1) Under Grothendieck Ansatz or Lemma 1.2.19, $\varphi_J : \mathbb{C}[y] \to M_n(\mathbb{C})$ gives(/is equivalent to) a morphism $\hat{\varphi}_J : Space\, M_n(\mathbb{C}) \to \mathbb{A}^1$ with the image subscheme $Im\,\hat{\varphi}_J \simeq Spec\,\langle \mathbf{1}, J\rangle$ associated to the ideal

$$Ker(\varphi_J) \;=\; (f_m^{\min}(y)) \;=\; \Big((y - \lambda_1)^{d_{i1}} \cdots (y - \lambda_k)^{d_{k1}} \Big)$$

in $\mathbb{C}[y]$. Thus, on $\mathbb{A}^1$ there are k-many (generally non-reduced) points located respectively at $y = \lambda_1, \cdots, \lambda_k$ (in the underlying complex plane $\mathbb{C}$ of $\mathbb{A}^1$) where D0-branes in Polchinski's sense may sit upon. These are the *D0-branes on* $\mathbb{A}^1$ *associated to* φ_J in the sense of Definition 2.2.3. From the discussion, for a general φ_m, they depend only on the minimal polynomial $f_m^{\min}(\lambda)$ of m.

(2) The push-forward[32] $\hat{\varphi}_{J*}\mathbb{C}^n = \sum_{i=1}^k \hat{\varphi}_{J*}V_i = \sum_{i=1}^k \sum_{j=1}^{k_i} \hat{\varphi}_{J*}V_{ij}$ is now an $\mathcal{O}_{Im\,\hat{\varphi}_J}$-module of length n. Decompose $Im\,\hat{\varphi}_J$ into a disjoint union $\amalg_{i=1}^k Z_i$, where Z_i is the subscheme of $\mathbb{A}^1$ associated to the ideal $((y - \lambda_i)^{d_{i1}})$. Then $\hat{\varphi}_{J*}V_i$ is supported on Z_i and, hence, is an $\mathcal{O}_{Z_i}$-module of length n_i. The decomposition $\hat{\varphi}_{J*}V_i = \sum_{j=1}^{k_i} \hat{\varphi}_{J*}V_{ij}$ is automatically a direct-sum decomposition as $\mathcal{O}_{Z_i}$-modules as well. Let $Z_i^{(l)}$, $l \le n_i$, be the subscheme of Z_i associated to the ideal $((y - \lambda_i)^l)$ in $\mathbb{C}[y]$. Note that $Z_i^{(l)}$ has length l and that $Z_i^{(1)}$ is

[32]For non-algebro-geometers: $\mathbb{C}^n$ as a $\langle \mathbf{1}, J\rangle$-module is now a $\mathbb{C}[y]$-module via φ_J, with annihilator $Ker(\varphi_J)$. Thus, though $\hat{\varphi}_J$ is not directly defined, $\hat{\varphi}_{J*}\mathbb{C}^n$ is well-defined. This is the Grothendieck Ansatz on quasi-coherent modules versus quasi-coherent sheaves, similar to that on rings versus spaces.

the $\mathbb{C}$-point in Z_i and $Z_i^{(n_i)} = Z_i$. Then, $\hat{\varphi}_{J*}V_{ij}$ is a rank-1 $\mathcal{O}_{Z_i}$-module of length d_{ij} and is supported on $Z_i^{(d_{ij})}$. As $\mathcal{O}_{Z_i}$-modules,

$$\hat{\varphi}_{J*}V_{i1} \;\simeq\; \mathcal{O}_{Z_i}$$

and

$$\hat{\varphi}_{J*}V_{ij} \;\simeq\; \text{the ideal } (y-\lambda_i)^{d_{i1}-d_{ij}} \cdot \mathcal{O}_{Z_i} \text{ of } \mathcal{O}_{Z_i} \;\simeq\; \text{the quotient } \mathcal{O}_{Z_i^{(d_{ij})}} \text{ of } \mathcal{O}_{Z_i}\,.$$

In our setting[33], we call $\hat{\varphi}_{J*}V_i$ the *Chan-Paton module on the D0-brane supported on* $Z_i \subset \mathbb{A}^1$ *associated to* φ_J. From the discussion, for a general φ_m, their isomorphism class depends only on both $f_m^{\min}(\lambda)$ and the type of m.

Comparison with Hilbert schemes and Chow varieties.

The Hilbert scheme $\mathit{Hilb}^{\,n}_{\mathbb{A}^1} =: (\mathbb{A}^1)^{[n]}$ of n points on $\mathbb{A}^1$ parameterizes 0-dimensional subschemes of length n on $\mathbb{A}^1$. Such a subscheme of $\mathbb{A}^1$ is given uniquely by an ideal $(f) \subset \mathbb{C}[y]$, where f is a monic polynomial of degree n. In terms of matrices, it is thus represented by an $m \in M_n(\mathbb{C})$ such that both the characteristic polynomial and the minimal polynomial of m are f.[34] Observe that the Jordan form of (omitted entries are zero; the multiplicity of $\lambda_i = n_i$)

$$J_+^{(\lambda_1,\,\cdots,\,\lambda_k)} \;:=\; \begin{bmatrix}
\lambda_1 &&&&&&&&&&&& \\
1 & \ddots &&&&&&&&&&& \\
& \ddots & \ddots &&&&&&&&&& \\
&& 1 & \lambda_1 &&&&&&&&& \\
&&& 1 & \lambda_2 &&&&&&&& \\
&&&& 1 & \ddots &&&&&&& \\
&&&&& \ddots & \ddots &&&&&& \\
&&&&&& 1 & \lambda_2 &&&&& \\
&&&&&&& \ddots & \ddots &&&& \\
&&&&&&&& 1 & \lambda_k &&& \\
&&&&&&&&& 1 & \ddots && \\
&&&&&&&&&& \ddots & \ddots & \\
&&&&&&&&&&& 1 & \lambda_k
\end{bmatrix},$$

where $(\lambda_1,\,\cdots,\,\lambda_k)$ is the n-tuple $(\lambda_1,\,\cdots,\,\lambda_1,\,\lambda_2,\,\cdots,\,\lambda_2,\,\cdots,\,\lambda_k,\,\cdots,\,\lambda_k)$ with the specified multiplicity n_i for λ_i, is $\mathit{Diag}\,(J_{n_1}^{(\lambda_1)},\,\cdots,\,J_{n_k}^{(\lambda_k)})$, up to a permutation of the blocks. Its characteristic polynomial and minimal polynomial are identical: $(y-\lambda_1)^{n_1}\,\cdots\,(y-\lambda_k)^{n_k}$. Let $\mathbb{C}^n$ parameterizes the ordered tuples of roots of monic polynomial of degree n, then the embedding

$$\mathbb{C}^n \;\hookrightarrow\; M_n(\mathbb{C})\,, \quad (\lambda_1,\,\cdots,\,\lambda_n) \;\mapsto\; J_+^{(\lambda_1,\,\cdots,\,\lambda_n)}$$

descends to an embedding

$$\Phi_{\mathit{Hilb}} : (\mathbb{A}^1)^{[n]} \;\longrightarrow\; \mathit{Map}\,((\mathit{Space}\,M_n(\mathbb{C});\mathbb{C}^n),\mathbb{A}^1)\,, \quad \textstyle\prod_{i=1}^n (y-\lambda_i) \;\longmapsto\; \varphi_{J_+^{(\lambda_1,\,\cdots,\,\lambda_n)}}\,.$$

[33]See footnote 35 for remarks on the original setting in string theory.

[34]In other words, m is a regular matrix in $M_n(\mathbb{C})$.

On the other hand, the Chow variety $Chow^{(n)}_{0,\mathbb{A}^1}$ of n points on $\mathbb{A}^1$ parameterizes 0-cycles of order n on $\mathbb{A}^1$ and is identical to the n-th symmetric product $S^n(\mathbb{A}^1)$ of $\mathbb{A}^1$. Such a 0-cycle on $\mathbb{A}^1$ happens to be represented uniquely by a monic polynomial in y of degree n as well. Thus there is a canonical isomorphism $(\mathbb{A}^1)^{[n]} \simeq S^n(\mathbb{A}^1)$. However, from the general ground of Chow groups, the support of a cycle is meant to be a reduced subscheme with each of its irreducible components marked with a multiplicity. Thus, in terms of matrices, it is represented by an $m \in M_n(\mathbb{C})$ such that the minimal polynomial of m has only simple roots. Such matrices are exactly the diagonalizable matrices. Again, let $\mathbb{C}^n$ parameterizes the ordered tuples of roots of monic polynomial of degree n, then it follows that the embedding

$$\mathbb{C}^n \hookrightarrow M_n(\mathbb{C})\,, \quad (\lambda_1, \cdots, \lambda_n) \mapsto Diag\,(\lambda_1, \cdots, \lambda_n)$$

descends to an embedding

$$\Phi_{Chow} : S^n(\mathbb{A}^1) \longrightarrow Map\,((Space\,M_n(\mathbb{C}); \mathbb{C}^n), \mathbb{A}^1)\,, \quad \prod_{i=1}^n (y - \lambda_i) \longmapsto \varphi_{Diag\,(\lambda_1, \cdots, \lambda_n)}\,.$$

In other words, $Im\,\Phi_{Hilb}$ parameterizes conjugacy classes of regular representations of $\mathbb{C}[y]$ in $M_n(\mathbb{C})$ while $Im\,\Phi_{Chow}$ parameterizes conjugacy classes of diagonal representations of $\mathbb{C}[y]$ in $M_n(\mathbb{C})$.

Note that, under the isomorphism $(\mathbb{A}^1)^{[n]} \simeq S^n(\mathbb{A}^1)$, Φ_{Hilb} and Φ_{Chow} coincide only on the open dense subset, points of which correspond to 0-dimensional reduced subschemes of length n on $\mathbb{A}^1$. For all p in the complement of this subset, $\Phi_{Chow}(p) \prec \Phi_{Hilb}(p)$ by an isotopic decay. In particular, $Map\,((Space\,M_n(\mathbb{C}); \mathbb{C}^n), \mathbb{A}^1)$ contains $(\mathbb{A}^1)^{[n]}$ and $S^n(\mathbb{A}^1)$ distinctly and, for $n \geq 3$, has more points than $\Phi_{Hilb}((\mathbb{A}^1)^{[n]}) \cup \Phi_{Chow}(S^n(\mathbb{A}^1))$. In the current case, it happens that Φ_{Hilb} and Φ_{Chow} give rise to

$$(\mathbb{A}^1)^{[n]} \xrightarrow{\sim} Map\,((Space\,M_n(\mathbb{C}); \mathbb{C}^n), \mathbb{A}^1)/\approx \xleftarrow{\sim} S^n(\mathbb{A}^1)\,.$$

This is only accidental and does not generalize to Y of dimension ≥ 2.

Note also that, for all p, the Chan-Paton module at $\Psi_{Hilb}(p)$ gives exactly the structure sheaf $\mathcal{O}_{Z_p}$ of the subscheme Z_p p represents while the Chan-Paton module at $\Psi_{Chow}(p)$ gives an association of $\mathbb{C}^{n_i}$ to each p_i (as an $\mathcal{O}_{p_i}(=\mathbb{C})$-module), for $p = \sum_{i=1}^k p_i$ as a 0-cycle. Thus, Chan-Paton spaces/modules in the sense of Definition 2.2.3 tells the difference of subschemes versus cycles as well.[35] This is a general feature.

[35]Some stringy comments follow. When generalized to higher-dimensional D-branes, these notions produce different notions of "wrappings" of a D-brane around a submanifold/subvariety in the target space(-time) of strings. Such a subtlety, among other things, was recognized seriously only by a smaller group of string theorists, e.g. [G-S] and [H-S-T]. For most of the stringy literatures, the simpler cycle-picture are more dominating (in the region of the related Wilson's theory-space where "branes are really branes"). In the hind sight, there might be a reason for this: Recall that an open string interacts with D-branes via its end-points. In most disscussions/literatures, these end-points are only taken to be simple points (i.e. reduced points in the algebro-geometric language) and hence, despite the fact that D-brane warpping can be a more complicated notion than usually thought of, open strings do not see anything beyond the cycle picture with a gauge bundle supported thereon. Should one remember that an end-point is attached to the open string and there are jets (in the sense of differential topology or, in the open-string world-sheet picture, in the sense of real algebraic geometry) at the end-point, then one may expect to draw out some open-string-parameterization-invariant details of such hidden "thickened structure" (e.g. non-reducedness of subschemes, embedded points, torsion-subsheaves within a torsion sheaf, ..., etc.). (However, except in the elementary discussion of momentum conservation of open strings, in which 1-jet is involved, we are not aware of any other use of jets at the end-point of open string in string theory.)

On the other hand, since a D-brane (again in the "brane is really a brane" region) is now taken as an extended dynamical object in its own right and hence has its own definition and deformation-obstruction theory, while it must contain contents induced from open strings, it is completely legitimate that it could also have contents without contradictions with open strings and yet open strings cannot see. In the current example and in Polchinski's

Finally, the map that sends φ_m to the diagonal of J_m gives rise to a continuous map π_{Chow} : $Map\,((Space\,M_n(\mathbb{C});\mathbb{C}^n), \mathbb{A}^1) \to S^n(\mathbb{A}^1)$. It has Φ_{Chow} as a section.

Associated quiver.

Given a finite-dimensional $\mathbb{C}$-algebra R, one can associate a *quiver*[36] Γ_R to R as follows:

(1) Let $\{e_1, \cdots, e_k\}$ be a complete set of primitive orthogonal idempotents in R. Then associate to each e_i a vertex, denoted also by e_i.

(2) Let $J(R)$ be the radical of R. Then, associate $dim_{\mathbb{C}}\, e_i(J(R)/J(A)^2)e_j$-many arrows from e_i to e_j.

Applying this to φ_m, representing a point in $Map\,((Space\,M_n(\mathbb{C});\mathbb{C}^n)\,,\,\mathbb{A}^1)$, by associating a graph to the Artinian $\mathbb{C}$-algebra $\mathbb{C}[y]/Ker\,\varphi_m \simeq \langle \mathbf{1}, m\rangle$, following the rules above, we obtain a quiver Γ_{φ_m} that captures part of the geometry of the D0-brane on $\mathbb{A}^1$ associated to φ_m:

· a *vertex* e_i for the connected component Z_i of $Spec\,\mathbb{C}[y]/Ker\,\varphi_m = Im\,\hat{\varphi}_m = \amalg_{i=1}^k Z_i$ of the D0-brane on $\mathbb{A}^1$;

· an *arrow* with both ends attached to e_i if Z_i has the embedded dimension 1 (i.e. if Z_i is a non-reduced point on $\mathbb{A}^1$); there are no other arrows for any pair (e_i, e_j), $1 \le i, j \le k$.

The Chan-Paton module discussed in an earlier theme is realized now as a representation of Γ_{φ_m}: (without loss of generality, we take m to be the Jordan form $J = J_m$ and adopt earlier notations)

· assign the $\mathcal{O}_{Z_i}$-module $(\hat{\varphi}_{m*}\mathbb{C}^n)|_{Z_i} = \hat{\varphi}_{m*}V_i$ to vertex e_i for $i = 1, \ldots, k$;

· if there is an arrow on e_i, then assign to that arrow the nilpotent endomorphism on $\hat{\varphi}_m V_i$ associated to the multiplication by $(y - \lambda_i)$ (i.e. the push-forward of the endomorphism $A_i - \lambda_i \mathbf{1}_{n_i}$ on V_i).

The quiver Γ_{φ_m} together with this representation now encodes the full geometry of the connected components of the D0-brane on $\mathbb{A}^1$ except their exact locations $y = \lambda_1, \cdots, \lambda_k$.

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picture, open strings do not "see" the (more complicated) structure sheaf $\mathcal{O}_{Z_p}$ when it is non-reduced but, rather, only see the (simpler) cycle $\Sigma_{i=1}^k n_i p_i$ with the sub-Chan-Paton space $\mathbb{C}^{n_i}$ attached to each p_i. Furthermore, what open strings do not see is nevertheless transformable to what open strings do see via an isotopic decay. This is actually a general feature. In our setting, we take both as different yet allowable existences of D-branes on the target space(-time) from deformations of D-branes in the sense of deformations of morphisms from an Azumaya-type noncommutative space to the open-string target space(-time). This explains also the term in Definition 4.1.4, cf. footnote 30.

[36] A few definitions/remarks for readers' reference are put here to make precise of the discussion while avoiding distractions. A '*quiver*' is an oriented graph Γ introduced in, e.g., the work of Gabriel in early 1970s to study representations of algebras. A *representation of a quiver* Γ over $\mathbb{C}$ is an assignment to each vertex $v_i \in \Gamma$ a $\mathbb{C}$-vector space V_i and to each arrow (i.e. oriented edge) $\in \Gamma$ from v_i to v_j a $\mathbb{C}$-linear homomorphism $\varphi_{ij} : V_i \to V_j$. Such representations have now become also a standard tool for string theorists to encode the field contents in a supersymmetric gauge field theory coupled with matters. Such field theories occur particularly on (the world-volume of) D-branes. Due to the rigidity of supersymmetric field theory, a quiver representation pretty much fixes the combinatorial type of the field theory under investigation.

There are *different* quivers that can be associated to a finite-dimensional $\mathbb{C}$-algebra R, regarded as a (left) R-module from the algebra multiplication. The one we choose here encodes the embedded dimension (i.e. the dimension of the tangent space when re-phrased in geometry) of of the Artinian $\mathbb{C}$-algebra in our problem. See, e.g., [A-R-S], [G-R], and [Jat] for more discussions.

Higgsing/un-Higgsing of D-branes via deformations of morphisms.[37]

The important open-string-induced Higgsing (i.e. gauge symmetry-breaking)/un-Higgsing (i.e. gauge symmetry enhancement) behavior on D-branes can be reproduced in the current content as follows. As any associative $\mathbb{C}$-algebra R gives rise to a Lie algebra $(R,[\cdot,\cdot])$ over $\mathbb{C}$ by taking the Lie bracket to be $[m_1,m_2]=m_1m_2-m_2m_1$, we can equivalently make the discussion directly for associative algebras in our problem.

Since on $\mathit{Space}\,M_n(\mathbb{C})$, $M_n(\mathbb{C})$ acts on the Chan-Paton space $\mathbb{C}^n$ as the endomorphism algebra $\mathit{End}(\mathbb{C}^n)$ of the Chan-Paton space, this is the counterpart of (the Lie algebra of) the gauge symmetry on a D-brane in physicists' picture. Given a $[\varphi_m:\mathbb{C}[y]\to M_n(\mathbb{C})]\in \mathit{Map}((\mathit{Space}\,M_n(\mathbb{C});\mathbb{C}^n),\mathbb{A}^1)$, the Chan-Paton space $\mathbb{C}^n$ on $\mathit{Space}\,M_n(\mathbb{C})$ is turned into the Chan-Paton module on $\mathit{Im}\,\hat{\varphi}_m$ by taking $\mathbb{C}^n$ now as a (left) $\langle\mathbf{1},m\rangle$-module, as discussed earlier. To distinguish them, we will denote the latter by ${}_{\langle\mathbf{1},m\rangle}\mathbb{C}^n$. Let

$$\mathit{Centralizer}\,\langle\mathbf{1},m\rangle\;:=\;\{m''\in M_n(\mathbb{C})\,:\,m''m'=m'm''\;\text{ for all }m'\in\langle\mathbf{1},m\rangle\}$$

be the centralizer of $\langle\mathbf{1},m\rangle$ in $M_n(\mathbb{C})$. Then,

Lemma 4.1.8 [centralizer vs. pushed-forward endomorphism]. *A $\mathbb{C}$-vector-space endomorphism $m''\in M_n(\mathbb{C})$ of $\mathbb{C}^n$ can be pushed forward to a $\langle\mathbf{1},m\rangle$-module endomorphism on ${}_{\langle\mathbf{1},m\rangle}\mathbb{C}^n$ if and only if $m''\in \mathit{Centralizer}\,\langle\mathbf{1},m\rangle\subset M_n(\mathbb{C})$.*

This gives a correspondence:

$$\mathit{Centralizer}\,\langle\mathbf{1},m\rangle\subset M_n(\mathbb{C})\qquad\Longleftrightarrow\qquad \textit{gauge symmetry on the D0-brane } \mathit{Im}\,\hat{\varphi}_m \textit{ on } \mathbb{A}^1.$$

Recall further from earlier discussions the connected-component-decomposition $\mathit{Im}\,\hat{\varphi}_m=:Z=\amalg_{i=1}^k Z_i$ and the $\langle\mathbf{1},m\rangle$-module direct-sum decomposition ${}_{\langle\mathbf{1},m\rangle}\mathbb{C}^n=\sum_{i=1}^k V_i$ with $\hat{\varphi}_{m*}V_i$ supported on Z_i. Then, there is a natural direct-product decomposition as $\mathbb{C}$-algebras:

$$\mathit{Centralizer}\,\langle\mathbf{1},m\rangle\;=\;\prod_{i=1}^k \mathit{Centralizer}\,\langle\mathbf{1},m\rangle_{(i)}\;\subset\;\prod_{i=1}^k \mathit{End}(V_i)\;\simeq\;\prod_{i=1}^k M_{n_i}(\mathbb{C})\,.$$

Up to conjugation, we may assume that $m=J_m=J$ a Jordan form, then $\mathit{Centralizer}\,\langle\mathbf{1},m\rangle_{(i)}\subset M_{n_i}(\mathbb{C})$ consists of $n_i\times n_i$-matrices is of the form B_i given in Remark 4.1.6. Thus, each Z_i can be regarded as a D0-brane on $\mathbb{A}^1$ in its own right, associated to $[\varphi_{B_i}]\in \mathit{Map}((\mathit{Space}\,M_{n_i}(\mathbb{C});\mathbb{C}^{n_i}),\mathbb{A}^1)$, with the Chan-Paton module $\hat{\varphi}_{B_i*}\mathbb{C}^{n_i}$ and the gauge symmetry associated to the endomorphism subalgebra $\mathit{Centralizer}\,\langle\mathbf{1}_{n_i},B_i\rangle$ in $M_{n_i}(\mathbb{C})$. When φ_m varies, this gives rise to Higgsing/un-Higgsing of gauge symmetry of D0-branes on $\mathbb{A}^1$.

In particular, if we restrict φ_m to vary in $\Phi_{\mathit{Chow}}(S^n(\mathbb{A}^1))\subset \mathit{Map}((\mathit{Space}\,M_n(\mathbb{C});\mathbb{C}^n),\mathbb{A}^1)$, then the Higgsing/un-Higgsing pattern of n D0-branes on $\mathbb{A}^1$ is as follows:

(1) For φ_m in the stratum associated to the type $(n=d_1+\cdots+d_k)$:

- [*D-branes on* $\mathbb{A}^1$]
  $Z=\amalg_{i=1}^k Z_i\simeq\amalg_{i=1}^k \mathit{Spec}\,\mathbb{C}$, (i.e. k-collection of stacked D0-branes on $\mathbb{A}^1$);
- [*the Chan-Paton space*]
  $\mathbb{C}^{n_i}$ supported at the D0-brane Z_i on $\mathbb{A}^1$ for $i=1,\ldots k$;

[37] Readers who already know the stringy side of Polchinski's D-branes are suggested to compare it with the mathematical picture described in this theme. The Higgsing/un-Higgsing phenomenon described in this theme following Definition 2.2.3 is a general feature.

- [*gauge symmetry*]
  a factor $M_{n_i}(\mathbb{C}) \simeq End(\mathbb{C}^{n_i})$ on Z_i for $i = 1, \cdots, k$; the total gauge symmetry of the k-many D0-brane system is the Lie algebra associated to the product $\prod_{i=1}^k M_{n_i}(\mathbb{C})$.

(2) As a consequence of Item (1) above, when we vary $[\varphi_m] \in \Phi_{Chow}(S^n(\mathbb{A}^1))$ so that, for example,

- [*Higgsing*]
  Z_1 splits to j-many separated D0-brane collections $Z'_1, \cdots, Z'_j$ on $\mathbb{A}^1$, governed by the partition $n_1 = n'_1 + \cdots + n'_j$. Then the Chan-Paton space $\mathbb{C}^{n_1}$ splits as well and turns into a Chan-Paton-space $\mathbb{C}^{n'_i}$ at Z'_i for $i = 1, \ldots, j$. The gauge symmetry associated to $M_{n_1}(\mathbb{C})$ is now broken to the one associated to the sub-endomorphism-algebra $\prod_{i=1}^j M_{n'_i}(\mathbb{C})$ with the factor $M_{n'_i}(\mathbb{C})$ assigned to $(Z'_i, \mathbb{C}^{n'}_i)$ for $i = 1, \ldots, j$.

- [*un-Higgsing*]
  $Z_1, \cdots, Z_j$ collide/merge to a new Z'_j. Then there is now a D0-brane collection at Z'_j with Chan-Paton space $\mathbb{C}^{n_1+\cdots+n_j}$. The original gauge symmetry for the collection $\{(Z_1, \mathbb{C}^{n_1}) \cdots, (Z_j, \mathbb{C}^{n_j})\}$, which is the one associated to $M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_j}(\mathbb{C})$, is now enhanced to the gauge symmetry associated to $M_{n_1+\cdots+n_j}(\mathbb{C})$, acting on $(Z'_j, \mathbb{C}^{n_1+\cdots+n_j})$.

Except that we have to use algebraic groups – in particular the $GL_\bullet(\mathbb{C})$-series in the current content – in the pure algebro-geometric setting, this is exactly the pattern of the oriented-open-string-induced Higgsing/un-Higgsing of unitary gauge symmetry of D-branes that Polchinski concluded in [Pol3: Sec. 3.3 and Sec. 3.4][38]. In summary:

Proposition 4.1.9 [Higgsing/un-Higgsing of D0-branes on $\mathbb{A}^1$].[39] *The pattern of open-string-induced Higgsing/un-Higgsing behavior of n D0-branes on $\mathbb{A}^1$ can be reproduced in the current content via deformations of morphisms $[\varphi_m : \mathbb{C}[y] \to M_n(\mathbb{C})]$ in $\Phi_{Chow}(S^n(\mathbb{A}^1)) \subset Map((Space\, M_n(\mathbb{C}); \mathbb{C}^n), \mathbb{A}^1)$.*

Comparison with the spectral cover construction and the Hitchin system.

Fix a complex line bundle $\pi_L : L \to pt$ over a point pt. We will identify pt with the zero-section of L whenever needed. Let λ be the tautological section of $\pi_L^* L$ over L.

Definition 4.1.10 [semi-simple pair].[40] A pair (E, ϕ), where $\pi_E : E \to pt$ is a rank-n complex vector bundle over pt and $\phi : E \to E \otimes L$ a complex-vector-bundle-homomorphism over pt is called *semi-simple* if ϕ is semi-simple (i.e. diagonalizable) with respect to a (hence any) trivialization $E \simeq \mathbb{C}^n$ and $L \simeq \mathbb{C}$.

[38] E-print version: `hep-th/9611050`: Sec. 2.3 and Sec. 2.4.

[39] For non-string-theorists: On the physics side, the Higgsing of gauge symmetry on D-branes in the sense of Polchinski is originated from the induced stretching of open strings whose end-points are attached to D-branes that are originally stacked and then are deformed and separated. Such stretching turns part of the massless spectrum of open strings that contribute to the gauge fields on the D-branes into massive spectrum and hence reduces the gauge fields on the D-branes. The fact that this crucial open-string-induced behavior of D-branes can be reproduced by following Definition 2.2.3 alone *without* resorting to open strings is what convince us that it makes sense to take Definition 2.2.3 as the prototype intrinsic mathematical definition for Polchinski's D-branes. Unfamiliar readers are encouraged to study [P-S] and [Pol4] to get a feeling.

[40] The adjoint action of $GL_n(\mathbb{C})$ on $M_n(\mathbb{C})$ does not have stable points in the sense of Mumford in [M-F-K]. With Polchinski's D-branes in mind, we choose semi-simple pairs for the role of stable pairs in [Hi].

Associated to a semi-simple pair (E, ϕ), with the ϕ of type $(n = n_1 + \cdots + n_k)$, are the following objects:

(1) the reduced zero-locus $Z_\phi = \amalg_{i=1}^k Z_{\phi;i}$ of the section $\det(\pi_L^*\phi - \mathbf{1}\otimes\lambda)$ of $\det(\pi_L^* E)\otimes(\pi_L^* L)^{\otimes n}$;

(2) a direct-sum decomposition $E = \sum_{i=1}^k V_i$ of bundles over pt so that $\hat{V}_i := (\pi_L^* V_i)|_{Z_{\phi,i}} = (Ker(\pi_L^*\phi - \mathbf{1}\otimes\lambda))|_{Z_{\phi;i}}$ for $i = 1, \ldots, k$;

(3) $\prod_{i=1}^k End(V_i) \subset End(E) \simeq M_n(\mathbb{C})$ acting on E leaving each V_i invariant for $i = 1, \ldots, k$.

This is the 0-dimensional spectral cover construction in the sense of [Hi]; see also [B-N-R], [Don1], and [Ox]. The Hitchin system in this content takes the form of the isomorphism $S^n\mathbb{C} \xrightarrow{\sim} \mathbb{C}^n$ that sends $[\lambda_1, \cdots, \lambda_n]$ to the monic polynomial $\prod_{i=1}^n(\lambda - \lambda_i)$ of degree n in λ.

Now identify L with $\mathbb{A}^1$ by $y \mapsto \lambda$ and E with the Chan-Paton space of n D0-branes stacked at the origin $y = 0$. Then ϕ corresponds to a D0-brane configuration supported at Z_ϕ, with the Chan-Paton space $\hat{V}_i$ and endomorphism algebra $End(\hat{V}_i) = End(V_i)$ at $Z_{\phi,i}$. One may regard Z_ϕ as a deformation of the stacked D0-branes at $y = 0$ (which corresponds to $\phi = 0$). *This reproduces also the Higgsing/un-Higgsing behavior of Polchinski's D-branes.* Note that D0-branes on $\mathbb{A}^1$ described through this construction correponds to the locus $Im\,\Phi_{Chow}$ in $Map((Space\, M_n(\mathbb{C}); \mathbb{C}^n), \mathbb{A}^1)$.

This spectral cover picture of D-branes is particularly fascinating when one recalls the Seiberg-Witten integrable system and the associated gauge-symmetry-breaking pattern revealed there; cf. [S-W1] and [Don2], [D-W], [Le].[41]

For the rest of this section, we will focus mainly on the moduli problem.

4.2 D0-branes on the complex projective line $\mathbb{P}^1$.

Let Y be the projective line over $\mathbb{C}$:

$$Y \;=\; \mathbb{P}^1 \;=\; U_0 \cup_{U_0 \cap U_\infty} U_\infty \;=\; Spec\,\mathbb{C}[y_0] \cup_{Spec\,\mathbb{C}[y_0\,,\,1/y_o] \simeq Spec\,\mathbb{C}[1/y_\infty\,,\,y_\infty]} Spec\,\mathbb{C}[y_\infty]\,,$$

where $Spec\,\mathbb{C}[y_0\,,\,1/y_o] \xrightarrow{\sim} Spec\,\mathbb{C}[1/y_\infty\,,\,y_\infty]$ is given by $y_\infty \mapsto 1/y_0$. Having discussed the details of D0-branes on $\mathbb{A}^1$ in Sec. 4.1, we focus now on the issue of gluings for D0-branes on $\mathbb{P}^1$.

Recall the Grassmannian-like manifold $Gr^{(2)}(n; d, n-d)$; the idempotents $\mathbf{1}_d$, $d = 0, \ldots, n$, in $M_n(\mathbb{C})$; and the notation '$m_1 \sim m_2$' for similar matrices in $M_n(\mathbb{C})$ from Sec. 3.2. Then, the ring-set representation variety

$$\begin{aligned} Rep^{\text{ring-set}}(\mathbb{C}[y], M_n(\mathbb{C})) \;&=\; \{(e, m) \in M_n(\mathbb{C}) \times M_n(\mathbb{C}) \,:\, e^2 = e\,,\; em = me = m\} \\ &\subset\; \mathbb{A}^{n^2} \times \mathbb{A}^{n^2} \end{aligned}$$

has $(n+1)$-many connected components, given by

$$Rep^{\text{ring-set}}(\mathbb{C}[y], M_n(\mathbb{C}))_{(d)} \;:=\; \{(e, m) \in Rep^{\text{ring-set}}(\mathbb{C}[y], M_n(\mathbb{C})) \,:\, e \sim \mathbf{1}_d\}\,,$$

[41] However, this setting has two drawbacks one should be aware of: (1) it obscures the important noncommutative nature of D-branes for it treats D-branes (of B-type) only as coherent torsion sheaves with a gauge symmetry, which we know now is not a complete picture, (see also [Di-M] for subtleties in the case of D-brane bound-state systems), and (2) while this construction is immediately generalizable to D-branes of complex codimension-1 in a complex target space, the further extension to describe higher-*co*dimensional D-branes becomes cumbersome. These indicate that the spectral cover setting might be just accidental for the cases it is applicable and is overall not most natural for D-branes. Cf. [Liu1].

$d = 0, \ldots, n$. (Here we identify the pair (e, m) with the ring-set-homomorphism

$$\varphi_{(e,m)} : \mathbb{C}[y] \to M_n(\mathbb{C}) \qquad \text{with } 1 \mapsto e \text{ and } y \mapsto m.)$$

$\mathit{Rep}^{\text{ring-set}}(\mathbb{C}[y], M_n(\mathbb{C}))_{(d)}$ is a $GL_n(\mathbb{C})$-manifold that goes with a natural $GL_n(\mathbb{C})$-equivariant bundle map $\mathit{Rep}^{\text{ring-set}}(\mathbb{C}[y], M_n(\mathbb{C}))_{(d)} \to Gr^{(2)}(n; d, n-d)$ with fiber $\simeq M_d(\mathbb{C})$. In particular, $\dim_{\mathbb{C}} \mathit{Rep}^{\text{ring-set}}(\mathbb{C}[y], M_n(\mathbb{C}))_{(d)} = d^2 + 2d(n-d) = n^2 - (n-d)^2$, which increases strictly when d goes from 0 to n. The space $\mathit{Mor}^{\text{ring-set}}(\mathbb{C}[y], M_n(\mathbb{C}))$ of ring-set-homomorphisms from $\mathbb{C}[y]$ to $M_n(\mathbb{C})$ can be thought of as the $GL_n(\mathbb{C})$-space $\amalg_{d=0}^n \mathit{Rep}^{\text{ring-set}}(\mathbb{C}[y], M_n(\mathbb{C}))_{(d)}$, but with the topology $\mathcal{T}$ in Definition 3.2.6. It has the following properties:

- $\mathit{Rep}^{\text{ring-set}}(\mathbb{C}[y], M_n(\mathbb{C}))_{(n)} = \mathit{Rep}(\mathbb{C}[y], M_n(\mathbb{C}))$ is an open dense subset of $\mathit{Mor}^{\text{ring-set}}(\mathbb{C}[y], M_n(\mathbb{C}))$.
- A neighborhood of (e, m) with $e \sim \mathbf{1}_d$ consists of all $(e', m') \in \mathit{Rep}^{\text{ring-set}}(\mathbb{C}[y], M_n(\mathbb{C}))$ such that
  - $e' \sim \mathbf{1}_{d'}$ for some $d' \geq d$;
  - there is an idempotent e'' in $Z(\langle e', m' \rangle)$ with the properties:
    - $e'' \sim \mathbf{1}_d$ and is in a neighborhood of e,
    - $e''m'$ is in a neighborhood of m in $M_n(\mathbb{C})$,
    - besides the characteristic value 0 of multiplicity $d + (n - d')$, the matrix

      $$(e' - e'')m' \;=\; (e' - e'')m'(e' - e'') \in M_n(\mathbb{C})$$

      has all the remaining $(d' - d)$-many characteristic values in a neighborhood of ∞ in $\mathbb{C} \cup \{\infty\}$.

The space $\mathit{Mor}(\mathit{Space}\, M_n(\mathbb{C}), \mathbb{P}^1)$ of morphisms from $\mathit{Space}\, M_n(\mathbb{C})$ to $\mathbb{P}^1$ is given by the locus in $\mathit{Mor}^{\text{ring-set}}(\mathbb{C}[y_0], M_n(\mathbb{C})) \times \mathit{Mor}^{\text{ring-set}}(\mathbb{C}[y_\infty], M_n(\mathbb{C}))$ described by the following conditions:

$$(\varphi_{(e_{(0)}, m_{(0)})}, \varphi_{(e_{(\infty)}, m_{(\infty)})}) \in \mathit{Mor}^{\text{ring-set}}(\mathbb{C}[y_0], M_n(\mathbb{C})) \times \mathit{Mor}^{\text{ring-set}}(\mathbb{C}[y_\infty], M_n(\mathbb{C})),$$

(1) $e_{(0)}e_{(\infty)} = e_{(\infty)}e_{(0)}$, $\mathbf{1} = e_{(0)} + e_{(\infty)} - e_{(0)}e_{(\infty)}$;

(2) $e_{(0)}m_{(\infty)} = m_{(\infty)}e_{(0)}$, $e_{(\infty)}m_{(0)} = m_{(0)}e_{(\infty)}$;

(3) $e_{(\infty)}\langle e_{(0)}, m_{(0)} \rangle = e_{(0)}\langle e_{(\infty)}, m_{(\infty)} \rangle$ in $M_n(\mathbb{C})$, (note that under Condition (2), $e_{(\infty)}\langle e_{(0)}, m_{(0)} \rangle = \langle e_{(\infty)}e_{(0)}, e_{(\infty)}m_{(0)} \rangle$ and $e_{(0)}\langle e_{(\infty)}, m_{(\infty)} \rangle = \langle e_{(0)}e_{(\infty)}, e_{(0)}m_{(\infty)} \rangle$);

(4) $e_{(\infty)}m_{(0)}$ is invertible in $\langle e_{(\infty)}e_{(0)}, e_{(\infty)}m_{(0)} \rangle$, $e_{(0)}m_{(\infty)}$ is invertible in $\langle e_{(0)}e_{(\infty)}, e_{(0)}m_{(\infty)} \rangle$;

(5) The identity in Condition (3) takes $e_{(\infty)}m_{(0)}$ to the inverse of $e_{(0)}m_{(\infty)}$ and $e_{(0)}m_{(\infty)}$ to the inverse of $e_{(\infty)}m_{(0)}$.

Note that Conditions (1) and (2) says that

$$\mathbf{1} \in \langle e_{(0)}, e_{(\infty)} \rangle \subset Z(\langle e_{(0)}, e_{(\infty)}, m_{(0)}, m_{(\infty)} \rangle) \subset M_n(\mathbb{C})\,.$$

Conditions (3), (4), and (5) are the descendability to localizations and the gluability of pairs of ring-set-morphisms in $\mathit{Mor}^{\text{ring-set}}(\mathbb{C}[y_0], M_n(\mathbb{C})) \times \mathit{Mor}^{\text{ring-set}}(\mathbb{C}[y_\infty], M_n(\mathbb{C}))$. $GL_n(\mathbb{C})$ acts diagonally on $\mathit{Mor}^{\text{ring-set}}(\mathbb{C}[y_0], M_n(\mathbb{C})) \times \mathit{Mor}^{\text{ring-set}}(\mathbb{C}[y_\infty], M_n(\mathbb{C}))$, leaving Conditions (1) - (5) invariant.

Lemma 4.2.1 [closed condition]. *Assuming Conditions* (1) *and* (2), *then Conditions* (3), (4), *and* (5) *together are equivalent to*

$(3')$ $e_{(0)}e_{(\infty)}m_{(0)}m_{(\infty)} = e_{(0)}e_{(\infty)}$.

In particular, the system $\{(1),(2),(3),(4),(5)\}$ *realizes* $Mor\,(Space\,M_n(\mathbb{C}),\mathbb{P}^1)$ *as a* $GL_n(\mathbb{C})$*-invariant closed subset in* $Mor^{\text{ring-set}}(\mathbb{C}[y_0],M_n(\mathbb{C}))\times Mor^{\text{ring-set}}(\mathbb{C}[y_\infty],M_n(\mathbb{C}))$.

$Map\,((Space\,M_n(\mathbb{C});\mathbb{C}^n),\mathbb{P}^1) = Mor\,(Space\,M_n(\mathbb{C}),\mathbb{P}^1)/\sim$ is now given by the orbit-space of the $GL_n(\mathbb{C})$-action on the above locus in $Mor^{\text{ring-set}}(\mathbb{C}[y_0],M_n(\mathbb{C}))\times Mor^{\text{ring-set}}(\mathbb{C}[y_\infty],M_n(\mathbb{C}))$.

For $\mathcal{R} = (\varphi_{(e_{(0)},m_{(0)})},\varphi_{(e_{(\infty)},m_{(\infty)})}) \in Map\,((Space\,M_n(\mathbb{C});\mathbb{C}^n),\mathbb{P}^1)$, the Chan-Paton module on each local chart U, where $U=U_0$ or U_∞, is given by the (e,m)-module $e\cdot\mathbb{C}$ but now regarded as a $\mathbb{C}[y]$-module ${}_{\mathbb{C}[y]}(e\cdot\mathbb{C})$ via $\varphi_{(e,m)}$. We will denote this $\mathcal{O}_U$-module on U by $\hat{\varphi}_{(e,m)*}(e\cdot\mathbb{C}^n)$. It is supported on the image scheme $Im\,\hat{\varphi}$ on U associated to the ideal $Ker\,\varphi_{(e,m)}$ in $\mathbb{C}[y]$. Here, $(e,m)=(e_{(0)},m_{(0)})$ or $(e_{(\infty)},m_{(\infty)})$ respectively and $\mathbb{C}[y]=\mathbb{C}[y_0]$ or $\mathbb{C}[y_\infty]$ respectively. Except that $e\cdot\mathbb{C}^n$ now replaces $\mathbb{C}^n$, all the local details of $\hat{\varphi}_{(e,m)*}(e\cdot\mathbb{C}^n)$ are the same those in the case $Y=\mathbb{A}^1$. The total length of $\hat{\varphi}_{(e,m)*}(e\cdot\mathbb{C}^n)$ is $dim_{\mathbb{C}}(e\cdot\mathbb{C}^n)$, $(=d$ for $e\sim\mathbf{1}_d)$. The pair $\{Im\,\hat{\varphi}_{(e_{(0)},m_{(0)})}\,,\,Im\,\hat{\varphi}_{(e_{(\infty)},m_{(\infty)})}\}$ of local image schemes glue to a 0-dimensional subscheme, denoted $Im\,\hat{\varphi}_{\mathcal{R}}$ or $\hat{\varphi}_{\mathcal{R}}(Space\,M_n(\mathbb{C}))$, of length $\le n$ on $\mathbb{P}^1$. Idempotency of $e_\bullet$ and Conditions (1) and (2) imply that $\{\hat{\varphi}_{(e_{(0)},m_{(0)})*}(e_{(0)}\cdot\mathbb{C}^n)\,,\,\hat{\varphi}_{(e_{(\infty)},m_{(\infty)})*}(e_{(\infty)}\cdot\mathbb{C}^n)\}$ glues to a (torsion) $\mathcal{O}_{\mathbb{P}^1}$-module on $\mathbb{P}^1$. This is the push-forward $\hat{\phi}_{\mathcal{R}*}\mathbb{C}^n$ of $\mathbb{C}^n$ on $Space\,M_n(\mathbb{C})$ to $\mathbb{P}^1$ under $\hat{\phi}_{\mathcal{R}}$; cf. footnote 32. It is the Chan-Paton module of the D0-branes $\hat{\varphi}(Space\,M_n(\mathbb{C}))$ on $\mathbb{P}^1$ in the current setting. Note that the total length of $\hat{\phi}_{\mathcal{R}*}\mathbb{C}^n$ on $\mathbb{P}^1$ remains n. The Higgsing/un-Higgsing behavior of Chan-Paton modules of D0-branes on any target Y is a local issue and hence, for $Y=\mathbb{P}^1$, is the same as that for $Y=\mathbb{A}^1$ in Sec. 4.1.

The local discussions in Sec. 4.1 can be glued to global statements. In particular,

Proposition 4.2.2 [D0-branes on $\mathbb{P}^1$]. *There is an embedding* $\Phi_{Hilb}: Hilb^{\,n}_{\mathbb{P}^1} =: (\mathbb{P}^1)^{[n]} \to Map\,((Space\,M_n(\mathbb{C});\mathbb{C}^n),\mathbb{P}^1)$*, whose image is characterized by* $\varphi_{\mathcal{R}}$ *such that* $Im\,\hat{\varphi}_{\mathcal{R}}$ *is a subscheme of length* n *on* $\mathbb{P}^1$*. There is an embedding* $\Phi_{Chow}: S^n(\mathbb{P}^1)\to Map\,((Space\,M_n(\mathbb{C});\mathbb{C}^n),\mathbb{P}^1)$*, whose image is characterized by* $\varphi_{\mathcal{R}}$ *such that* $Im\,\hat{\varphi}_{\mathcal{R}}$ *is a reduced subscheme (of length* $\le n$*) on* $\mathbb{P}^1$*. There is a map* $Map\,((Space\,M_n(\mathbb{C});\mathbb{C}^n),\mathbb{P}^1)\to S^n(\mathbb{P}^1)$ *that has* Φ_{Chow} *as a section. The pattern of open-string-induced Higgsing/un-Higgsing behavior of* n *D0-branes on* $\mathbb{P}^1$ *can be reproduced in the current content via deformations of morphisms* $[\varphi_{\mathcal{R}}]$ *in* $\Phi_{Chow}(S^n(\mathbb{P}^1))\subset Map\,((Space\,M_n(\mathbb{C});\mathbb{C}^n),\mathbb{P}^1)$.

Remark 4.2.3 [*strict morphism*]. A strict morphism from $Space\,M_n(\mathbb{C})$ to $\mathbb{P}^1$ is given by a strict morphism (cf. Definition 1.2.11 and Definition 1.1.1) from $[(\{\mathbb{C}[y_0],\mathbb{C}[y_\infty]\}\rightrightarrows\{\mathbb{C}[y,1/y]\})]$ to $[\{M_n(\mathbb{C})\}]$. Since $Z(M_n(\mathbb{C}))=\mathbb{C}$, such a morphism factors as

$$[(\{\mathbb{C}[y_0],\mathbb{C}[y_\infty]\}\rightrightarrows\{\mathbb{C}[y,1/y]\})]\longrightarrow[\{\mathbb{C}\}]\longrightarrow[\{M_n(\mathbb{C})\}]$$

and, hence, corresponds to a morphism $Spec\,\mathbb{C}\to\mathbb{P}^1$. The corresponding D0-brane on $\mathbb{P}^1$ is supported at a reduced $\mathbb{C}$-point on $\mathbb{P}^1$ with the Chan-Paton module $\mathbb{C}^n$, i.e. n-many coincident D0-branes on $\mathbb{P}^1$ in the picture of Polchinski. The moduli space of such morphisms (i.e. coincident D0-branes) is $\mathbb{P}^1$. Thus, we see that the inclusion of general morphisms (cf. Definition 1.2.14 and Definition 1.1.1) in the definition of $Mor\,(Space\,M_n(\mathbb{C}),\mathbb{P}^1)$ and, hence, in the definition of $Map\,((Space\,M_n(\mathbb{C});\mathbb{C}^n),\mathbb{P}^1)$ is also required if one wants to incorporate the Higgsing/un-Higgsing behavior of, in this case, D0-branes on $\mathbb{P}^1$. Similar phenomenon occurs for other projective target spaces as well. This is another incident of the mysterious harmony between stringy requirement and mathematical naturality for a string-theory-related mathematical object.

4.3 D0-branes on the complex affine plane $\mathbb{A}^2$.

For a commutative Y of dimension ≥ 2, an additional ingredient than those in Sec. 4.1 and Sec. 4.2 is commuting schemes/varieties[42]. We discuss in this subsection the case $Y = \mathbb{A}^2$, for which the commuting variety that occurs is known slightly better.

Let $Y = \mathbb{A}^2 = \mathit{Spec}\,\mathbb{C}[y_1, y_2]$ be the affine plane over $\mathbb{C}$. Then $\mathit{Mor}(\mathit{Space}\,M_n(\mathbb{C}), Y) = \mathit{Mor}(\mathbb{C}[y_1, y_2], M_n(\mathbb{C}))$ is the variety[43] what parameterizes the elements in the set

$$C_2M_n(\mathbb{C}) \;:=\; \{(m_1, m_2) \in M_n(\mathbb{C}) \times M_n(\mathbb{C}) \,:\, m_1m_2 = m_2m_1\}$$

of pairs of commuting matrices in $M_n(\mathbb{C})$. This variety is identical with $\mathbb{C}$-algebra representation variety $\mathit{Rep}\,(\mathbb{C}[y_1, y_2], M_n(\mathbb{C}))$ with a point represented by $(m_1, m_2) \in M_n(\mathbb{C}) \times M_n(\mathbb{C})$ corresponding to the $\mathbb{C}$-algebra-homomorphism

$$\varphi_{(m_1,m_2)} : \mathbb{C}[y_1, y_2] \to M_n(\mathbb{C})\,, \text{ generated by } 1 \mapsto \mathbf{1},\ y_1 \mapsto m_1, \text{ and } y_2 \mapsto m_2\,.$$

Proposition 4.3.1 [irreducibility]. ([Ge], [Bas], and [Vac2].) *$\mathit{Rep}\,(\mathbb{C}[y_1, y_2], M_n(\mathbb{C}))$ is an irreducible variety of dimension n^2+n in $\mathbb{A}^{n^2} \times \mathbb{A}^{n^2}$. The $GL_n(\mathbb{C})$-action on $\mathit{Rep}\,(\mathbb{C}[y_1, y_2], M_n(\mathbb{C}))$ has stabilizer subgroups of minimal dimension n. A generic $GL_n(\mathbb{C})$-orbit thus has dimension $n^2 - n$, that achieves the maximum orbit-dimension and the subset that consists of $\varphi_{(m_1,m_2)}$, where (m_1, m_2) is a diagonalizable commuting pair with both m_1 and m_2 having distinct characteristic values, is a smooth open dense subset in $\mathit{Rep}\,(\mathbb{C}[y_1, y_2], M_n(\mathbb{C}))$.*

It follows that

$$\mathit{Map}\,((\mathit{Space}\,M_n(\mathbb{C}); \mathbb{C}^n), \mathbb{A}^2) \;\simeq\; \mathit{Map}\,(\mathit{Space}\,M_n(\mathbb{C}), \mathbb{A}^2) \;=\; \mathit{Rep}\,(\mathbb{C}[y_1, y_2], M_n(\mathbb{C}))/\!\sim\,,$$

the orbit-space of the $GL_n(\mathbb{C})$-action with the quotient topology, is a connected non-Hausdorff topological space that contains a connected smooth open dense Hausdorff subset of dimension $2n$, namely the subset of $S^n(\mathbb{A}^2)$ that consists of $[(\lambda_1, \mu_1), \cdots, (\lambda_n, \mu_n)]$ such that λ_i, $i = 1, \ldots, n$, are all distinct from each other and so are μ_i, $i = 1, \ldots, n$. Here $S^n(\mathbb{A}^2) := (\mathbb{A}^2)^n/\mathit{Sym}_n$ is the n-th symmetric product of $\mathbb{A}^2$.

The complete set of dominance relations of the $GL_n(\mathbb{C})$-orbits in $\mathit{Rep}\,(\mathbb{C}[y_1, y_2], M_n(\mathbb{C}))$, which generalizes [Ge], are not known. However, there are two distinguished Hausdorff subspace in $\mathit{Map}\,((\mathit{Space}\,M_n(\mathbb{C}); \mathbb{C}^n), \mathbb{A}^2)$ that can be understood through the work of Nakajima [Na] and of Vaccarino [Vac2]:

(1) the naturally embedded image of the Hilbert scheme $(\mathbb{A}^2)^{[n]} := \mathit{Hilb}^{\,n}_{\mathbb{A}^2}$ (with the reduced scheme structure) of 0-dimensional subschemes of length n on $\mathbb{A}^2$;

(2) the naturally embedded image of the Chow variety $\mathit{Chow}^{(n)}_{0,\mathbb{A}^2} = S^n(\mathbb{A}^2)$ of 0-cycles of order n on $\mathbb{A}^2$.

---

[42] For pure algebraic geometers: Moduli problems in commutative algebraic geometry tends to boil down to Hilbert schemes, which in projective cases are realized as a locus in an appropriate Grassmannian variety. In that sense, commuting schemes/varieties play the same fundamental role as Grassmannian varieties do for the moduli problem of morphisms from an Azumaya-type noncommutative space to a commutative variety. We hope this gives further motivation to study commuting schemes/varieties. See, e.g., [Bas], [Ge], [Ri], [S-T], [Vac1], [Vac2].

[43] Throughout, we only consider the reduced scheme structure on a commuting scheme or a representation scheme that occurs in the problem.

We now explain the details.

Proposition 4.3.2 [regular representation]. *Let R be a commutative Artinian algebra over $\mathbb{C}$ of dimension n. Then, the regular representation*[44] *of R realizes R as a maximal commutative subalgebra R' of $M_n(\mathbb{C})$. Furthermore, as an R'-module, ${}_{R'}\mathbb{C}^n \simeq R'$.*

Proof. This is an immediate corollary of [S-T: Sec.2.7, Theorem 11]. When R is generated by two commuting elements and the identity, as is in our case, there are two other independent proofs: (1) The first part of the proof of [Na: Sec. 1.2, Theorem 1.9] can be adapted directly to give another more analytic proof of the statement, cf. proof of Proposition 4.3.3 below. (2) This is a corollary of [Ge], which says that the maximum dimension of a commutative subalgebra in $M_n(\mathbb{C})$ generated by two commuting matrices and the identity is n.

$\square$

Note that, in the above statement, different choices of $R \simeq \mathbb{C}^n$ as $\mathbb{C}$-vector spaces give rise to R''s in the same adjoint $GL_n(\mathbb{C})$-orbit. It follows that there is an embedding of sets

$$\Phi_{\mathit{Hilb}} : (\mathbb{A}^2)^{[n]} \longrightarrow \mathit{Map}\left((\mathit{Space}\, M_n(\mathbb{C}); \mathbb{C}^n), \mathbb{A}^2\right), \quad \mathbb{C}[y_1, y_2]/I \longmapsto \varphi_{(m_1,m_2)}\,.$$

Here, I is an ideal of $\mathbb{C}[y_1, y_2]$ so that $\mathit{dim}_{\mathbb{C}}(\mathbb{C}[y_1, y_2]/I) = n$; it gives then the subalgebra $(\mathbb{C}[y_1, y_2]/I)' \subset M_n(\mathbb{C})$ as in Proposition 4.3.2, unique up conjugation; the corresponding matrix m_i for y_i, $i = 1, 2.$ under the built-in $\mathbb{C}$-algebra-isomorphism $\mathbb{C}[y_1, y_2]/I \stackrel{\sim}{\rightarrow} (\mathbb{C}[y_1, y_2]/I)'$ determines then $\varphi_{(m_1,m_2)}$.

Proposition 4.3.3 [stable subset]. (Cf. [Na: Theorem 1.9].) *Let $\mathit{Rep}\,(\mathbb{C}[y_1, y_2], M_n(\mathbb{C}))^{st}$ be the subset of $\mathit{Rep}\,(\mathbb{C}[y_1, y_2], M_n(\mathbb{C}))$ that consists of $\varphi_{(m_1,m_2)}$ such that ${}_{\langle \mathbf{1}, m_1, m_2\rangle}\mathbb{C}^n \simeq \langle \mathbf{1}, m_1, m_2\rangle$ as $\langle \mathbf{1}, m_1, m_2\rangle$-modules. Then $\mathit{Rep}\,(\mathbb{C}[y_1, y_2], M_n(\mathbb{C}))^{st}$ is smooth and $GL_n(\mathbb{C})$-invariant with stabilizers all of the same dimension n.*

Proof. This is actually [Na: Theorem 1.9] in disguise. Note that the stability condition in the defining condition of the set $\tilde{H}$ in ibidem is precisely the condition "${}_{\langle \mathbf{1}, m_1, m_2\rangle}\mathbb{C}^n \simeq \langle \mathbf{1}, m_1, m_2\rangle$ as $\langle \mathbf{1}, m_1, m_2\rangle$-modules" in the statement here. Having said so, let us give a sketch of the proof in terms of the current setting.

Using the trace map $M_n(\mathbb{C}) \to \mathbb{C}$ as a complex bilinear inner product on the $\mathbb{C}$-vector space $M_n(\mathbb{C})$, one can show that the (analytic quadric) commutator map (on analytic spaces)

$$c \;:\; M_n(\mathbb{C}) \times M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})\,, \quad (m_1, m_2) \longmapsto [m_1, m_2] := m_1 m_2 - m_2 m_1$$

has cokernel $\mathit{coker}\, dc_{(m_1,m_2)}$ at (m_1, m_2) being $\{\xi \in M_n(\mathbb{C}) : [\xi, m_1] = [\xi, m_2] = 0\}$, i.e. the centralizer $\mathit{Centralizer}\,\langle \mathbf{1}, m_1, m_2\rangle$ of the subalgebra $\langle \mathbf{1}, m_1, m_2\rangle$ in the algebra $M_n(\mathbb{C})$. Note that for $(m_1, m_2) \in C_2 M_n(\mathbb{C})$, $\langle \mathbf{1}, m_1, m_2\rangle \subset \mathit{Centralizer}\,\langle \mathbf{1}, m_1, m_2\rangle$.

If, furthermore, $\varphi_{(m_1,m_2)} \in \mathit{Rep}\,(\mathbb{C}[y_1, y_2], M_n(\mathbb{C}))^{st}$, then ${}_{\langle \mathbf{1}, m_1, m_2\rangle}\mathbb{C}^n = \langle \mathbf{1}, m_1, m_2\rangle \cdot v_0$ for some $v_0 \in \mathbb{C}^n$. The $\mathbb{C}$-linear map $\mathit{Centralizer}\,\langle \mathbf{1}, m_1, m_2\rangle \to \mathbb{C}^n$, defined by $\xi \mapsto \xi \cdot v_0$, is then invertible and hence a $\mathbb{C}$-vector-space-isomorphism. It follows that $\langle \mathbf{1}, m_1, m_2\rangle = \mathit{Centralizer}\,\langle \mathbf{1}, m_1, m_2\rangle$ and $\dim_{\mathbb{C}} \mathit{coker}\, dc_{(m_1,m_2)} = n$ for $\varphi_{(m_1,m_2)} \in \mathit{Rep}\,(\mathbb{C}[y_1, y_2], M_n(\mathbb{C}))^{st}$. This shows that $\mathit{Rep}\,(\mathbb{C}[y_1, y_2], M_n(\mathbb{C}))^{st}$ is smooth.

Finally, note that $\mathit{Stab}\,(\varphi_{(m_1,m_2)}) = GL_n(\mathbb{C}) \cap \mathit{Centralizer}\,\langle \mathbf{1}, m_1, m_2\rangle$, which has the same dimension as $\mathit{Centralizer}\,\langle \mathbf{1}, m_1, m_2\rangle$. The proposition follows.

$\square$

[44] Recall that a *regular representation* of an algebra R is the representation of R on R itself by, in our convention, left multiplications; i.e. R as a (left) R-module.

Since the closure of $\overline{O}$ of a G-orbit O of an action of a reductive algebraic group G on an affine variety V (both over $\mathbb{C}$) is a union of O with G-orbits of strictly smaller dimension, one has:

Corollary 4.3.4 [good quotient]. *All the $GL_n(\mathbb{C})$-orbits are closed in $Rep\,(\mathbb{C}[y_1,y_2],M_n(\mathbb{C}))^{st}$ and the map $Rep\,(\mathbb{C}[y_1,y_2],M_n(\mathbb{C}))^{st} \to Im\,\Phi_{Hilb}$ to the orbit-space is a good quotient.*

This realizes the map $\Phi_{Hilb} : (\mathbb{A}^2)^{[n]} \to Map\,((Space\,M_n(\mathbb{C});\mathbb{C}^n),\mathbb{A}^2)$ as an embedding of the (reduced) Hilbert scheme as a variety/analytic space.

Let $\mathbb{C}^n$ parameterizes the diagonal matrices in $M_n(\mathbb{C})$. Then, the embedding

$$\begin{array}{ccc} \mathbb{C}^n \times \mathbb{C}^n = (\mathbb{C}^2)^n & \hookrightarrow & M_n(\mathbb{C}) \times M_n(\mathbb{C}) \\ ((\lambda_1,\mu_1),\,\cdots\,,(\lambda_n,\mu_n)) & \mapsto & (Diag\,(\lambda_1,\,\cdots\,,\lambda_n)\,,\,Diag\,(\mu_1,\,\cdots\,,\mu_n)) \end{array}$$

descends to an embedding

$$\begin{array}{cccc} \Phi_{Chow} & : & S^n(\mathbb{A}^2) & \longrightarrow & Map\,((Space\,M_n(\mathbb{C});\mathbb{C}^n),\mathbb{A}^2) \\ & & [(\lambda_1,\mu_1),\,\cdots\,,(\lambda_n,\mu_n)] & \longmapsto & \varphi_{(Diag\,(\lambda_1,\cdots,\lambda_n)\,,\,Diag\,(\mu_1,\cdots,\mu_n))} \end{array}$$

of the Chow variety.

$S^n(\mathbb{A}^2)$ is the categorical quotient of $Rep\,(\mathbb{C}[y_1,y_2],M_n(\mathbb{C}))$ under the adjoint $GL_n(\mathbb{C})$-action. The affine morphism $Rep\,(\mathbb{C}[y_1,y_2],M_n(\mathbb{C})) \to S^n(\mathbb{A}^2)$ induced by the $GL_n(\mathbb{C})$-invariant function ring on $Rep\,(\mathbb{C}[y_1,y_2],M_n(\mathbb{C}))$ descends to a morphism $Im\,\Phi_{Hilb} \to Im\,\Phi_{Chow}$ of varieties that realizes $(\mathbb{A}^2)^{[n]}$ as a desingularization of $S^n(\mathbb{A}^2)$. $Im\,\Phi_{Chow}$ consists of all the closed points in $Map\,((Space\,M_n(\mathbb{C});\mathbb{C}^n),\mathbb{A}^2)$ and the closure of any point in $Map\,((Space\,M_n(\mathbb{C});\mathbb{C}^n),\mathbb{A}^2)$ contains a unique point in $Im\,\Phi_{Chow}$. Cf. [Na], [Pro], [Ri], and [Vac2].

Note that, for $(m_1,m_2) \in C_2M_n(\mathbb{C})$, as m_1 and m_2 commute, they can be simultaneously triangularized. If they have a simultaneous triangularization with the diagonal entries $(\lambda_1,\,\cdots\,,\lambda_n)$ and $(\mu_1,\,\cdots\,,\mu_n)$ respectively, let $I_{\{(\lambda_1,\mu_1),\,\cdots,(\lambda_n,\mu_n)\}} := (y_1-\lambda_1, y_2-\mu_1)\cap\cdots\cap(y_1-\lambda_n, y_2-\mu_n)$ be the ideal in $\mathbb{C}[y_1,y_2]$ for the set of closed points $\{(\lambda_1,\mu_1),\,\cdots\,,(\lambda_n,\mu_n)\}$ (as points on the analytic space $\mathbb{C}^2$ with repeated points dropped). Then,

$$I^n_{\{(\lambda_1,\mu_1),\,\cdots,(\lambda_n,\mu_n)\}} \subset Ker\,\varphi_{(m_1,m_2)} \subset I_{\{(\lambda_1,\mu_1),\,\cdots,(\lambda_n,\mu_n)\}}\,.$$

In particular, the image scheme $Im\,\hat{\varphi}_{(m_1,m_2)} \simeq Spec\,(\mathbb{C}[y_1,y_2]/Ker\,\varphi_{(m_1,m_2)})$ on $\mathbb{A}^2$ has the reduced scheme structure exactly the set $\{(\lambda_1,\mu_1),\,\cdots\,,(\lambda_n,\mu_n)\}$ above.

For $\varphi_{(m_1,m_2)} \in Im\,\Phi_{Hilb}$, the Chan-Paton module $\hat{\varphi}_{(m_1,m_2)*}\mathbb{C}^n$, as a $\mathcal{O}_{Im\,\hat{\varphi}_{(m_1,m_2)}}$-module, is isomorphic to the structure sheaf $\mathcal{O}_{Im\,\hat{\varphi}_{(m_1,m_2)}}$ while for $\varphi_{(m_1,m_2)} \in Im\,\Phi_{Chow}$, the Chan-Paton module $\hat{\varphi}_{(m_1,m_2)*}\mathbb{C}^n$, as a $\mathcal{O}_{Im\,\hat{\varphi}_{(m_1,m_2)}}$-module, is isomorphic to $\oplus_{i=1}^n\mathcal{O}_{(\lambda_i,\mu_i)}$. Here, (λ_i,μ_i), $i=1,\,\ldots\,,n$, are the image point from earlier notations with repeated (λ_i,μ_i) kept to contribute to the direct sum. Behavior of Higgsing/un-Higgsing follows similar pattern as in Sec. 4.1.

4.4 D0-branes on a complex quasi-projective variety.

A picture of D0-branes on a (commutative) complex quasi-projective variety that follows from a combination and an immediate generalization of Sec. 4.1 - Sec. 4.3 is given in this subsection. A comparison with gas of D0-branes in [Vafa1] of Vafa is given in the end.

D0-branes on $\mathbb{P}^r$.

Let Y be the projective space over $\mathbb{C}$:

$$Y \;=\; \mathbb{P}^r \;=\; \mathit{Proj}\,\mathbb{C}[y_0, y_1, \cdots, y_r] \;=\; \cup_{i=0}^r U_i \;=\; \cup_{i=0}^r \mathit{Spec}\,\mathbb{C}[\tfrac{y_0}{y_i}, \cdots, \tfrac{y_r}{y_i}]\,.$$

Here $y_\bullet/y_i$ are treated as formal variables with $y_i/y_i =$ the identity 1 of the ring $\mathbb{C}[\frac{y_0}{y_i}, \cdots, \frac{y_r}{y_i}]$; the gluings $U_i \supset U_{ij} := U_i \cap U_j \xleftarrow{\sim} U_{ji} := U_j \cap U_i \subset U_j$ of local affine charts are given by

$$\begin{array}{ccccccc}
\mathbb{C}[\frac{y_0}{y_i}, \cdots, \frac{y_r}{y_i}] & \hookrightarrow & \dfrac{\mathbb{C}[\frac{y_0}{y_i}, \cdots, \frac{y_r}{y_i}, \frac{y_i}{y_j}]}{\left(\frac{y_j}{y_i}\cdot\frac{y_i}{y_j}-1\right)} & \stackrel{\sim}{\longrightarrow} & \dfrac{\mathbb{C}[\frac{y_0}{y_j}, \cdots, \frac{y_r}{y_j}, \frac{y_j}{y_i}]}{\left(\frac{y_i}{y_j}\cdot\frac{y_j}{y_i}-1\right)} & \hookleftarrow & \mathbb{C}[\frac{y_0}{y_j}, \cdots, \frac{y_n}{y_j}] \\
 & & \frac{y_\bullet}{y_i} & \longmapsto & \frac{y_\bullet}{y_j}\cdot\frac{y_j}{y_i} & & \\
 & & \frac{y_i}{y_j} & \longmapsto & \frac{y_i}{y_j} & & .
\end{array}$$

Let

$$C_{r+1}M_n(\mathbb{C}) \;:=\; \{(m_0, \cdots, m_r) \in M_n(\mathbb{C})^{r+1} \;:\; m_i m_j = m_j m_i\,, i, j = 0, \ldots, r\}\,.$$

The ring-set representation variety

$$\begin{aligned}
&\mathit{Rep}^{\text{ring-set}}(\mathbb{C}[\tfrac{y_0}{y_i}, \cdots, \tfrac{y_n}{y_i}], M_n(\mathbb{C})) \\
&\quad = \{(m_{(i),0}, \cdots, m_{(i),r}) \in C_{r+1}M_n(\mathbb{C}) \;:\; m_{(i),i}m_{(i),i'} = m_{(i),i'}m_{(i),i} = m_{(i),i'},\, i' = 0, \ldots, r\} \\
&\quad \subset \prod_{r+1} \mathbb{A}^{n^2} \;=\; \mathbb{A}^{n^2(r+1)}\,,
\end{aligned}$$

(in particular, $e_{(i)} := m_{(i),i}$ is an idempotent), is a disjoint union of

$$\begin{aligned}
&\mathit{Rep}^{\text{ring-set}}(\mathbb{C}[\tfrac{y_0}{y_i}, \cdots, \tfrac{y_n}{y_i}], M_n(\mathbb{C}))_{(d)} \\
&\quad := \;\{(m_{(i),\bullet})_\bullet \in \mathit{Rep}^{\text{ring-set}}(\mathbb{C}[\tfrac{y_0}{y_i}, \cdots, \tfrac{y_n}{y_i}], M_n(\mathbb{C})) \;:\; m_{(i),i} \sim \mathbf{1}_d\}\,, \quad d \;=\; 0, \ldots, n\,.
\end{aligned}$$

Here, again, we identify the ring-set-homomorphism $\varphi_{(m_{(i),0}, \cdots, m_{(i),r})} : \mathbb{C}[\frac{y_0}{y_i}, \cdots, \frac{y_r}{y_i}] \to M_n(\mathbb{C})$ that sends $y_\bullet/y_i$ to $m_{(i),\bullet}$ with $(m_{(i),0}, \cdots, m_{(i),r}) \in C_{r+1}M_n(\mathbb{C})$.

Similar to the case $Y = \mathbb{P}^1$ in Sec. 4.2, the space $\mathit{Mor}\,(\mathit{Space}\,M_n(\mathbb{C}), \mathbb{P}^r)$ of morphisms from $\mathit{Space}\,M_n(\mathbb{C})$ to $\mathbb{P}^r$ is given by the locus on $\prod_{i=0}^r \mathit{Mor}^{\text{ring-set}}(\mathbb{C}[\frac{y_0}{y_i}, \cdots, \frac{y_r}{y_i}], M_n(\mathbb{C}))$ described by the following conditions:[45]

$$(\varphi_{(m_{(0),\bullet})_\bullet}, \cdots, \varphi_{(m_{(r),\bullet})_\bullet}) \in \textstyle\prod_{i=0}^r \mathit{Mor}^{\text{ring-set}}(\mathbb{C}[\frac{y_0}{y_i}, \cdots, \frac{y_r}{y_i}], M_n(\mathbb{C})),$$

(1) $m_{(i),i}m_{(j),j} = m_{(j),j}m_{(i),i}\,,$
$\mathbf{1} = \sum_i m_{(i),i} - \sum_{i_1<i_2} m_{(i_1),i_1}m_{(i_2),i_2} + \cdots$
$\qquad + (-1)^{k-1}\sum_{i_1<\cdots<i_k} m_{(i_1),i_1}\cdots m_{(i_k),i_k} + \cdots + (-1)^r m_{(0),0}\cdots m_{(r),r};$

(2) $m_{(i),i}m_{(j),\bullet} = m_{(j),\bullet}m_{(i),i},\; i, j, \bullet = 0, \ldots, r;$

(3) $(m_{(j),j}m_{(i),j})\,(m_{(j),j}m_{(j),i}) \;=\; m_{(i),i}m_{(j),j}\,,\; i, j = 0, \ldots, r;$ cf. Lemma 4.2.1;

(4) $m_{(j),j}m_{(i),\bullet} \;=\; m_{(j),\bullet}\cdot(m_{(j),j}m_{(i),j})\;\; i, j, \bullet = 0, \ldots, r;$ cf. the gluing $U_{ij} \xleftarrow{\sim} U_{ji}$.

[45]For readers who are familiar with toric geometry: Such system of conditions can be formally associated to the fan (or polytope in the projective case) of a toric variety.

$GL_n(\mathbb{C})$ acts diagonally on $\prod_{i=0}^{r} Mor^{\text{ring-set}}(\mathbb{C}[\frac{y_0}{y_i}, \cdots, \frac{y_r}{y_i}], M_n(\mathbb{C}))$, via the post-composition with the adjoint $GL_n(\mathbb{C})$-action on $M_n(\mathbb{C})$, and the above system of conditions describes a $GL_n(\mathbb{C})$-invariant closed subset therein. The space of D0-branes on $\mathbb{P}^r$ is given by $Map\,((Space\, M_n(\mathbb{C}); \mathbb{C}^n), \mathbb{P}^r) = Mor\,(Space\, M_n(\mathbb{C}), \mathbb{P}^r)/\sim$, described by the orbit-space of the $GL_n(\mathbb{C})$-action on the above subset in $\prod_{i=0}^{r} Mor^{\text{ring-set}}(\mathbb{C}[\frac{y_0}{y_i}, \cdots, \frac{y_r}{y_i}], M_n(\mathbb{C}))$.

The Chan-Paton modules of D0-branes on $\mathbb{P}^r$ and their Higgsing/un-Higgsing behavior follow the reasoning that combines the cases $Y = \mathbb{P}^1$ and $Y = \mathbb{A}^2$. Together with the simultaneous triangularizability of any family of commuting matrices and the map that takes a tuple of triangularized matrices to the tuple of the respective diagonal, one has: (cf. Proposition 4.2.2)

Proposition 4.4.1 [D0-branes on $\mathbb{P}^r$]. *There is an embedding $\Phi_{Hilb} : Hilb_{\mathbb{P}^r}^{n} =: (\mathbb{P}^r)^{[n]} \to Map\,((Space\, M_n(\mathbb{C}); \mathbb{C}^n), \mathbb{P}^r)$. $\varphi_{\mathcal{R}} \in \Phi_{Hilb}((\mathbb{P}^r)^{[n]})$ has the property that $Im\,\hat{\varphi}_{\mathcal{R}}$ is a subscheme of length n on $\mathbb{P}^r$. There is an embedding $\Phi_{Chow} : S^n(\mathbb{P}^r) \to Map\,((Space\, M_n(\mathbb{C}); \mathbb{C}^n), \mathbb{P}^r)$, whose image is characterized by $\varphi_{\mathcal{R}}$ associated to a system of commuting diagonalizable matrices. (In particular, $Im\,\hat{\varphi}_{\mathcal{R}}$ is a reduced subscheme of length $\leq n$ on $\mathbb{P}^r$.) There is a map $Map\,((Space\, M_n(\mathbb{C}); \mathbb{C}^n), \mathbb{P}^r) \to S^n(\mathbb{P}^r)$ that has Φ_{Chow} as a section. The pattern of open-string-induced Higgsing/un-Higgsing behavior of n D0-branes on $\mathbb{P}^r$ can be reproduced in the current content via deformations of morphisms $[\varphi_{\mathcal{R}}]$ in $\Phi_{Chow}(S^n(\mathbb{P}^r)) \subset Map\,((Space\, M_n(\mathbb{C}); \mathbb{C}^n), \mathbb{P}^r)$.*

D0-branes on a quasi-projective variety.

Let Y be a quasi-projective variety and suppose that Y is embedded in $\mathbb{P}^r$ as $Y_1 - Y_2$, where both Y_1 and Y_2 are closed subschemes of $\mathbb{P}^r$. Let $I_1 = \langle f_{11}, \cdots, f_{1l_1}\rangle$ (resp. $I_2 = \langle f_{21}, \cdots, f_{2l_2}\rangle$) be the homogeneous ideal in $\mathbb{C}[y_0, \cdots, y_r]$ associated to Y_1 (resp. Y_2) in $\mathbb{P}^r$. Recall the local affine charts $\cup_{i=0}^{r} U_i$ of $\mathbb{P}^r$. Consider the (in general only quasi-affine) open cover $\cup_{i=0}^{r}((Y_1 - Y_2) \cap U_i)$ of Y. Then, the pair (I_1, I_2) gives rise to a pair

$$\Big(I_{1,(i)} = (f_{11,(i)}, \cdots, f_{1l_1,(i)})\,,\, I_{2,(i)} = (f_{21,(i)}, \cdots, f_{2l_2,(i)}) \Big)$$

of ideals in $\mathbb{C}[\frac{y_0}{y_i}, \cdots, \frac{y_r}{y_i}]$ via the dehomogenization of (I_1, I_2) on the affine chart U_i of $\mathbb{P}^r$ for $i = 0, \ldots, r$. The space $Mor\,(Space\, M_n(\mathbb{C}), Y)$ of morphisms from $Space\, M_n(\mathbb{C})$ to Y is given by further restricting the locus $Mor\,(Space\, M_n(\mathbb{C}), \mathbb{P}^r)$ in $\prod_{i=0}^{r} Mor^{\text{ring-set}}(\mathbb{C}[\frac{y_0}{y_i}, \cdots, \frac{y_r}{y_i}], M_n(\mathbb{C}))$, described by Conditions (1) - (4) in the previous theme, to the following system of incidence relation from I_1 and exclusion relations from I_2:

(5) [(*closed*) *incidence conditions from* I_1]:

$$f_{1\bullet,(i)}(m_{(i),0}, \cdots, m_{(i),r}) = 0 \in M_n(\mathbb{C})\,,\ \bullet = 1, \ldots, l_1\,,\ i = 0, \ldots, r\,;$$

(6) [(*open*) *exclusion conditions from* I_2]:

$$m_{(i),i} \in \Big\langle f_{2\bullet,(i)}(m_{(i),0}, \cdots, m_{(i),r}) \Big\rangle_{\bullet=1}^{l_2} \subset M_n(\mathbb{C})\,,\ i = 0, \ldots, r\,.$$

The diagonal $GL_n(\mathbb{C})$-action on $\prod_{i=0}^{r} Mor^{\text{ring-set}}(\mathbb{C}[\frac{y_0}{y_i}, \cdots, \frac{y_r}{y_i}], M_n(\mathbb{C}))$ leaves the locally-closed subset that satisfies Conditions (1) - (6) invariant. The space of D0-branes on Y is given then by $Map\,((Space\, M_n(\mathbb{C}); \mathbb{C}^n), Y) = Mor\,(Space\, M_n(\mathbb{C}), Y)/\sim$, described by the orbit-space of the $GL_n(\mathbb{C})$-action on the above locally-closed subset in $Mor\,(Space\, M_n(\mathbb{C}), \mathbb{P}^r)$.

Remark 4.4.2 [Independence of embedding]. The open cover $\cup_{i=0}^{r}((Y_1 - Y_2) \cap U_i)$ of Y can be refined to an affine open cover of Y, which realizes Y as a gluing system of rings. Different embeddings of Y in projective spaces realizes Y as different gluing systems of rings that have a common refinement. It follows then from Sec. 1.2 that $Map\,((Space\,M_n(\mathbb{C});\mathbb{C}^n),Y)$ thus constructed is independent of the embedding of Y in a projective space.

Proposition 4.4.1 implies then:

Theorem 4.4.3 [D0-branes on quasi-projective variety]. *Let Y be a quasi-projective variety over $\mathbb{C}$. (1) There is an embedding $\Phi_{Hilb}: Hilb_Y^n =: Y^{[n]} \to Map\,((Space\,M_n(\mathbb{C});\mathbb{C}^n),Y)$. $\varphi_{\mathcal{R}} \in \Phi_{Hilb}(Y^{[n]})$ has the property that $Im\,\hat{\varphi}_{\mathcal{R}}$ is a subscheme of length n on Y. (2) There is an embedding $\Phi_{Chow}: S^nY \to Map\,((Space\,M_n(\mathbb{C});\mathbb{C}^n),Y)$, whose image is characterized by $\varphi_{\mathcal{R}}$ associated to a system of commuting diagonalizable matrices. (In particular, $Im\,\hat{\varphi}_{\mathcal{R}}$ is a reduced subscheme of length $\leq n$ on Y.) (3) There is a map $Map\,((Space\,M_n(\mathbb{C});\mathbb{C}^n),Y) \to S^nY$ that has Φ_{Chow} as a section. (4) The pattern of open-string-induced Higgsing/un-Higgsing behavior of n D0-branes on Y can be reproduced in the current content via deformations of morphisms $[\varphi_{\mathcal{R}}]$ in $\Phi_{Chow}(S^nY) \subset Map\,((Space\,M_n(\mathbb{C});\mathbb{C}^n),Y)$.*

Remark 4.4.4 [*toric variety*]. The discussions for D0-branes on $\mathbb{P}^r$ (resp. a quasi-projective variety) generalize immediately to D0-branes on a toric variety (resp. a subscheme of a toric variety).

D0-branes, gauged matrix models, and quantum moduli spaces.

When Y is a closed subvariety of a toric variety/$\mathbb{C}$, the space $Map\,((Space\,M_n(\mathbb{C});\mathbb{C}^n),Y)$ are described by a system of noncommutative-polynomial-like algebraic equations that give only closed conditions. In this case, $Map\,((Space\,M_n(\mathbb{C});\mathbb{C}^n),Y)$ is realizable as the classical moduli space of vacua (also known as vacuum manifold/variety) of a gauged matrix model. The construction is similar to that of [Wi1] but adjusted to $d = 0 + 1$ matrix models. See also the discussions in [D-G-M], [Do-M], and [G-L-R] for related situations and [L-Y5] for further discussions. The real issue, particularly from the mathematical/geometric aspect, is whether there is or needs to be also a good/mathematical notion of *quantum moduli space* in this case to incorporate more physics into the current mathematical setting. In the next theme, we will see an example from string theory in which $Map\,((Space\,M_n(\mathbb{C});\mathbb{C}^n),Y)$ already contains both a classical and a quantum moduli space of D0-branes on Y in the sense of [Vafa1].

A comparison with the moduli problem of gas of D0-branes in [Vafa1] **of Vafa.**

In [Vafa1], Vafa studied, among other things, the physics of finitely many D0-branes and D4-branes. In particular, for a gas of n-many identical D0-branes on one D4-brane supported on a complex torus $\mathbb{T}^4$, except the additional $U(1)$-factor in the whole gauge group that comes from the simple D4-brane, the Higgsing/un-Higgsing behavior of such D0-D4 systems is the same as that for n-many D0-branes alone and the classical moduli/configuration space of the n-many D0-branes on the $\mathbb{T}^4$ is given by $S^n(\mathbb{T}^4)$, which is a singular complex space. This moduli space is subject to a quantum correction to a quantum moduli space $\widetilde{S^n(\mathbb{T}^4)}$, dictated by the requirement that the cohomology $H^*(\widetilde{S^n(\mathbb{T}^4)},\mathbb{C})$ should be the orbifold cohomology (e.g. [V-W1] and [V-W2]) of $S^n(\mathbb{T}^4)$ from string theory. It is also anticipated that $\widetilde{S^n(\mathbb{T}^4)}$ should be a hyperkähler resolution of $S^n(\mathbb{T}^4)$. See also related discussions in [B-V-S1], [B-V-S2], and [Vafa2].

The related orbifold cohomology was later constructed mathematically by Chen and Ruan in [C-R1] and [C-R2]. In [Ru: Conjecture 6.3], Ruan conjectured in particular that, for Y a smooth projective surface over $\mathbb{C}$ such that $Y^{[n]}$ has a hyperkähler structure, the orbifold cohomology ring $H^*_{\mathrm{orb}}(S^nY,\mathbb{C})$ of S^nY is isomorphic to the (ordinary) cohomology ring $H^*(X^{[n]};\mathbb{C})$ of $X^{[n]}$. For the case Y is a smooth projective surface/$\mathbb{C}$ with trivial canonical line bundle, this was proved by Uribe [Ur: Theorem 3.2.3] together with previous result of Lehn and Sorger in [L-S]. Thus, for Y a smooth projective Calabi-Yau surface, the $\widetilde{S^nY}$ anticipated in [Vafa1] is $Y^{[n]}$.

In our current setting, a gas of n-many D0-branes on a D4-brane[46], supported on a smooth projective surface Y, is regarded as the image of a morphism from $(\mathit{Space}\, M_n(\mathbb{C});\mathbb{C}^n)$ to Y. The moduli space $\mathit{Map}\,((\mathit{Space}\, M_n(\mathbb{C});\mathbb{C}^n),Y)$ of such morphisms contains both $Y^{[n]}\simeq \mathit{Im}\,\Phi_{\mathit{Hilb}}$ and $S^nY\simeq \mathit{Im}\,\Phi_{\mathit{Chow}}$, and the restriction of $\pi_{\mathit{Hilb}}:\mathit{Map}\,((\mathit{Space}\, M_n(\mathbb{C});\mathbb{C}^n),Y)\to S^nY$ to $\mathit{Im}\,\Phi_{\mathit{Hilb}}$ realizes the resolution $Y^{[n]}\to S^nY$. In the special case that Y is in addition Calabi-Yau, we see that $\mathit{Map}\,((\mathit{Space}\, M_n(\mathbb{C});\mathbb{C}^n),Y)$ contains *both* the classical and the quantum moduli space of D0-brane configurations on Y in [Vafa1].

4.5 A remark on D-branes and universal moduli space.

In the previous subsections, we see an interesting feature of the moduli space of D0-branes on a (commutative) quasi-projective variety: namely, it incorporates both the Hilbert scheme and the Chow variety. We also see in the end of Sec. 4.4 that in a special occasion this is interpreted as containing both the classical and the quantum moduli space of D0-branes in physics.

While the encompassing of both the classical and the quantum moduli space of a D-brane system on a string target space in general is an issue that will be subject to how we formulate the intrinsic definition of D-brane bound system, the unifying feature of the moduli space of D-branes on a target space (in the sense of Definition 2.2.3 and its extension/generalization to systems that contains NS-branes as well) for different moduli spaces (e.g. Hilbert schemes and Chow varieties in the above example) in commutative geometry should be an anticipated feature when the mathematical definition/formulation of D-branes is "correct". Indeed, since 1995 new stringy dualities have made predictions that relate invariants of different mathematical origins, e.g. from the stable maps, the stable/torsion sheaves, and subschemes respectively (when put in the setting of algebraic geometry). These stringy dualities involve D-branes at work. It is thus natural to anticipate that all these standard moduli spaces that appear in the mathematical definition of these invariants should live in different, possibly partially-overlapped regions/corners of the moduli space of D-branes (or in general D-branes coupled with NS-branes) on a target space. This anticipation is particularly compelling from the viewpoint of Wilson's theory-space underlying these stringy dualities; cf. [Liu2] and [L-Y1: appendix A.1].

[46] A complete treatment of this involves an intrinsic mathematical construction/definition of a bound system of D-branes. Here, we only consider the pure D0-brane sector/factor in such a system.

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