

Argyres-Douglas matter and $\mathcal{N} = 2$ dualities

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ABSTRACT: We study S duality of four dimensional $\mathcal{N} = 2$ Argyres-Douglas (AD) theory engineered from 6d A_{N-1} (2,0) theory. We find a (p, q) sequence of SCFTs, here (p, q) is co-prime and class \mathcal{S} theory defined on sphere corresponds to class (0,1) theory. We represent these theories by a sphere with marked points, and S duality is interpreted as different pants decompositions of the same punctured sphere. The weakly coupled gauge theory description involves gauging AD matter which is represented by three punctured sphere.

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1 Introduction

S duality of four dimensional supersymmetric field theories has been extensively explored in the past several decades. The classical example is $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [1], where S duality exchanges the gauge group G and Langlands dual gauge group G^L . Similar S duality has been found for $\mathcal{N} = 2$ $SU(2)$ gauge theory coupled with four fundamental hypermultiplets [2]. For these examples, theories in different duality frames admit Lagrangian descriptions. Argyres-Seiberg [3] generalized S duality to $\mathcal{N} = 2$ $SU(3)$ with six fundamental hypermultiplets, and the new feature is that one of the dual theory involves a strongly coupled matter system. Gaiotto [4] found a remarkable generalization of Argyres-Seiberg duality by using 6d $(2, 0)$ construction. These so-called class \mathcal{S} theories tremendously improve the space of theories whose S duality behavior is known.

One feature of class \mathcal{S} theory is that the scaling dimension of Coulomb branch operator is integral. There are another class of $\mathcal{N} = 2$ models called Argyres-Douglas (AD) theories whose Coulomb branch operators have fractional scaling dimensions [5, 6]. These models seem to be much more general than the theory with integral scaling dimensions, and we would like to understand its S duality property. Two very interesting examples have been studied in [7], and several infinite class of self-dual models have been studied in [8]. Full S duality property of certain class of AD theories admitting 3d mirror has been found in [9] using the decomposition of their 3d mirrors.

The purpose of this paper is to find the S duality property of all Argyres-Douglas theories engineered using 6d A_{N-1} type $(2, 0)$ theories [15]. We first classify AD theories with exact marginal deformations; and for AD theories without exact marginal deformations, we call them AD matter. We found that it is natural to organize the theory space by a (p, q) label with $q > 0, p \geq -q + 1$, here (p, q) is co-prime and class \mathcal{S} theory defined on sphere corresponds to class $(0, 1)$ theory.

S duality of class \mathcal{S} theory is elegantly solved by representing the SCFT by a Riemann surface with marked points [4], and different S duality frames correspond to different degeneration limits of punctured Riemann surface into three punctured sphere representing matter systems. Motivated by class \mathcal{S} theory and the S duality of AD theories found in [9], we successfully represent our class (p, q) SCFT by a sphere with marked points: each punctured sphere has a label (p, q) , and each marked point is labeled by a Young Tableaux $[a_1, a_2, \dots, a_i]$ with arbitrary size (notice that for class S theory the size of Young Tableaux is the same for all the marked points), see left hand side of figure. 1 for an example. Now we have following S duality picture of our class (p, q) theory:

- The AD matter is represented by a sphere with three marked points.
- The number of exact marginal deformations of a SCFT is identified with the number of complex structure deformations of the punctured sphere.
- Different duality frames are represented by different degeneration limits of the same punctured sphere into three punctured sphere. We can read off the field theory description of each duality frame from the degeneration limit.

See figure. 1 for an example. We have made several checks such as the match of the Coulomb branch spectrum, central charges, and the vanishing of β function, etc. There are several new features of S duality of our general class (p, q) theories: a): Generically (except class $(p, 1)$ and $(1, q)$ theory), there are three types of marked points (which we label them black, red and blue), and the AD matter is represented by a sphere with one each of each color; This restricts the possible pants decompositions; b): Higher genus version is only possible for class \mathcal{S} theory.

This paper is organized as follows: section II reviews basic facts of exact marginal deformations of 4d $\mathcal{N} = 2$ SCFT; Section III describes the construction of $\mathcal{N} = 2$ SCFT

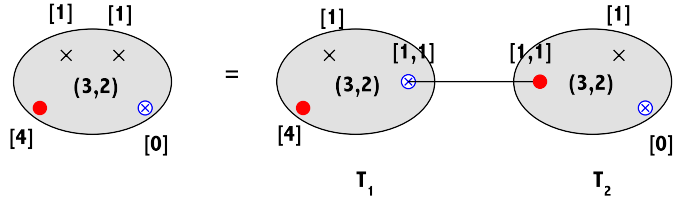


Figure 1. (A_3, A_5) theory is represented by a sphere with four marked points, here we need to use three types of marked points besides the Young Tableaux data. This theory belongs to class $(3, 2)$ theory, and its weakly coupled gauge theory description can be found from the degeneration limit of the punctured sphere. There are two duality frames as we can only exchange two black marked points.

from 6d A_{N-1} theory; Section IV describes the S duality of theories with 3d mirror, which is also called class $(p, 1)$ theory; Section V describes S duality of class (p, q) theory; Section VI discusses SCFT formed by conformally gauging AD matter, and we argue that higher genus version of general (p, q) class theory is not possible. Finally a conclusion is given in section VII.

2 Exact marginal deformations of 4d $\mathcal{N} = 2$ SCFT

The representation theory of four dimensional $\mathcal{N} = 2$ superconformal algebra (SCA) has been described in [10]. $\mathcal{N} = 2$ SCA has an important $SU(2)_R \times U(1)_R$ R symmetry, and the half BPS operators have been classified in [10]. There are two kinds of important half-BPS operators: $\mathcal{E}_{r,(0,0)}$ and \hat{B}_R , here r is the $U(1)_R$ charge and R is the integer specifying the $SU(2)_R$ representations. We are mostly interested in the operators $\mathcal{E}_{r,(0,0)}$ and its scaling dimension is given by the following formula

$$\Delta[\mathcal{E}_{r,(0,0)}] = r. \quad (2.1)$$

If $r = 2$, we have the following marginal deformation

$$\delta S = \tau \int d^4x \tilde{Q}^4 \mathcal{E}_{2,(0,0)} + c.c. \quad (2.2)$$

Such deformations are actually exact marginal, and it is proven in [11] that this type of deformations are the only exact marginal deformations for a $\mathcal{N} = 2$ SCFT.

Given a $\mathcal{N} = 2$ SCFT, we might want to answer the following questions about exact marginal deformations:

- Counting the number of exact marginal deformations.
- Determine the Zamolodchikov metric on the conformal manifold ¹.
- Determine the weakly coupled gauge theory descriptions at singularity of the conformal manifold.

¹More precisely, the space of exact marginal deformation $\mathcal{N} = 2$ SCFT is often not a manifold, but a moduli stack.

The first question is easiest and can often be read from the Seiberg-Witten (SW) curve, while the second and third questions are significantly harder. The major task of this paper is to understand the first and the third question for $\mathcal{N} = 2$ AD SCFT engineered from $6d$ $A_{N_1}(2,0)$ theory.

3 $\mathcal{N} = 2$ SCFT from M5 branes

A large class of four dimensional $\mathcal{N} = 2$ SCFTs can be engineered by putting $6d(2,0)$ theory of type J on Riemann surface with various type of regular and irregular punctures. A SCFT can be defined using following configurations:

- A genus g Riemann surface with arbitrary number of regular singularities.
- A sphere with one irregular singularity.
- A sphere with one irregular singularity and one regular singularity.

The classification of SCFT is then reduced to the classification of punctures. The regular singularity has been classified in [4, 12, 13], and is closed related to the classification of nilpotent orbit of Lie algebra J [14]. The irregular singularity has been classified in [15, 16], and is related to the classification of positive grading of Lie algebra J [17].

3.1 Classification of punctures

3.1.1 Irregular puncture

Let's start with six dimensional $A_{N-1}(2,0)$ theory and compactify it on a Riemann surface Σ to get a 4d $\mathcal{N} = 2$ SCFT. Hitchin's equation is defined on Σ and the classification of irregular singularity is reduced to the classification of higher order singular boundary condition of Hitchin's equation. Hitchin's equation involves a pair of fields (A_μ, Φ) on Σ , and Φ is called Higgs field. For irregular singularity, the Higgs field has the following behavior near the singularity [15, 16] (see also [18–20] for A_1 case):

$$\Phi = \frac{T_k}{z^{2+k/b}} + \sum_{-b \leq k' < k} \frac{T_{k'}}{z^{2+k'/b}}; \quad (3.1)$$

Here z is the local coordinate near the singularity, and T_k is a regular semi-simple element of Lie algebra A_{N-1} ; (b, k) is coprime, i.e. $(k, b) = 1$. We also ignore the regular terms which can freely fluctuate. The coefficients T_i are semi-simple and can also be simultaneously diagonalized. In going around the puncture for one loop ($z \rightarrow ze^{2\pi i}$), the Higgs field changes as follows

$$\Phi' = \frac{\exp^{-\frac{2\pi ik}{b}} T_k}{z^{2+k/b}} + \sum_{-b \leq k' < k} \frac{\exp^{-\frac{2\pi ik'}{b}} T_{k'}}{z^{2+k'/b}}. \quad (3.2)$$

The above solution is consistent if one can find an inner automorphism σ of Lie algebra A_{N-1} such that

$$\sigma T_{k'} \sigma^{-1} = \exp^{\frac{2\pi ik'}{b}} T_{k'}, \quad (3.3)$$

So $\sigma\Phi'\sigma^{-1} = \Phi$ up to the difference of regular terms. The above inner automorphism exists if the Lie algebra has the following grading [17]:

$$g = \bigoplus_{i \in \mathbb{Z}/b} g^i. \quad (3.4)$$

Let's take $k' = nb + l$, with $0 \leq l < b$, then $T_{k'} \in g^l$.

All such gradings are classified in [21], which is related to the classification of **cyclic** element of the corresponding Lie algebra. Here we give a brief review. Let's start with a nilpotent element e , then by the Morozov-Jacobson theorem, the element e can be included in an sl_2 -triple (e, h, f) , so that $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. Then the eigenspace decomposition of g with respect to $\text{ad } h$ action is a \mathbb{Z} -grading of g :

$$g = \bigoplus_{j=-d}^d g_j. \quad (3.5)$$

where $g_{\pm d} \neq 0$. The positive integer d is called depth of the \mathbb{Z} -grading. An element of g of the form $e + F$, where F is a non-zero element of g_d , is called a **cyclic** element, associated to e . A cyclic element is called **semi-simple** if the generic element of $e + F$ is semi-simple, and it is called **regular** semi-simple if $e + F$ is regular semi-simple.

For each nilpotent orbit e , one can associate a weighted Dynkin diagram, and the weights take values in 0, 1 and 2. A nilpotent element is called **even** if all the weights are **even**. For each nilpotent orbit, we can associate a positive integer

$$m = \sum a_i s_i + 2. \quad (3.6)$$

Let ϵ be a primitive m th root of 1. For a nilpotent orbit, we can define an inner automorphism σ_e of g by letting

$$\sigma_e(e_{\alpha_i}) = \epsilon^{s_i} e_{\alpha_i}, \quad \sigma_e(e_{-\alpha_i}) = \epsilon^{-s_i} e_{-\alpha_i}, \quad i = 1, \dots, r. \quad (3.7)$$

Where $e_{\pm\alpha_i}$ are Lie algebra elements attached to roots $\pm\alpha_i$. This action is extended to the whole Lie algebra using the Cartan-Weyl basis. The automorphism σ_e defines a \mathbb{Z}/m grading on Lie algebra

$$g = \bigoplus_{j \in \mathbb{Z}/m} g^j; \quad (3.8)$$

Here $g^0 = g_0$ is a reductive subalgebra of g . If e is even, then the lowest non-zero part is $g^2 = g_2 + g_{-d}$, and all the odd part in the above decomposition is missing. So the order of automorphism is actually $m' = \frac{m}{2}$.

Let's focus on $J = A_{N-1}$, and the Lie algebra is identified with the $N \times N$ traceless matrices. A nilpotent orbit is labeled by a Young Tableaux $[n_1, \dots, n_r]$ with $\sum n_i = N$, and the corresponding sl_2 triple (e, h, f) has the following standard form

$$e = \begin{pmatrix} J_{n_1} & 0 & 0 & 0 \\ 0 & J_{n_2} & 0 & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & J_{n_r} \end{pmatrix}, \quad h = \begin{pmatrix} D_{n_1} & 0 & 0 & 0 \\ 0 & D_{n_2} & 0 & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & D_{n_r} \end{pmatrix}. \quad (3.9)$$

	Young Tableaux	order m'
Semi-simple	$[n_1, \dots, n_1, 1, \dots, 1]$	n_1
Regular semi-simple	$[n_1, \dots, n_1]$	n_1
	$[n_1, \dots, n_1, 1]$	n_1

Table 1. The corresponding nilpotent orbit of semi-simple and regular semi-simple cyclic element of Lie algebra A_{N-1} .

Here J_{n_i} is the Jordan matrix with size n_i , and D_{n_i} is the $n_i \times n_i$ dimensional diagonal matrix with diagonal entries $(n_i - 1, n_i - 3, \dots, -(n_i - 3), -(n_i - 1)$. The corresponding weighted Dynkin diagram is found as follows: rearrange the eigenvalues of h such that they are monotonically decreasing $a_1 \geq a_2 \geq \dots \geq a_{N-1} \geq a_N$, and the weighted Dynkin diagram is

$$\overset{a_1 - a_2}{\bullet} - \overset{a_2 - a_3}{\bullet} \dots - \overset{a_N - a_{N-1}}{\bullet} \quad (3.10)$$

The semi-simple and regular semi-simple cyclic elements are classified in [21], and the result is listed in table. 1.

Example 1: Consider Lie algebra sl_4 and its nilpotent orbit corresponding to partition [4]. The corresponding sl_2 triple is

$$e = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}. \quad (3.11)$$

So the weighted Dynkin diagram is $\overset{2}{\bullet} - \overset{2}{\bullet} - \overset{2}{\bullet}$. After some computations, we find the following decomposition of lie algebra

$$g_0 = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 9 \\ 0 & 0 & 0 & a_{44} \end{pmatrix}, g_2 = \begin{pmatrix} 0 & a_{12} & 0 & 0 \\ 0 & 0 & a_{23} & 0 \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}, g_4 = \begin{pmatrix} 0 & 0 & a_{13} & 0 \\ 0 & 0 & 0 & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, g_6 = \begin{pmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$g_{-2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_{21} & 0 & 0 & 0 \\ 0 & a_{32} & 0 & 0 \\ 0 & 0 & a_{43} & 0 \end{pmatrix}, g_{-4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{31} & 0 & 0 & 0 \\ 0 & a_{42} & 0 & 0 \end{pmatrix}, g_{-6} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{41} & 0 & 0 & 0 \end{pmatrix}. \quad (3.12)$$

We have following grading on Lie algebra $sl(4)$:

$$g^0 = g_0, \quad g^2 = g_2 + g_{-6}, \quad g^4 = g_4 + g_{-4}, \quad g^6 = g_6 + g_{-2}; \quad (3.13)$$

and the order of actual automorphism is four, so we have a refined grading

$$sl(4) = \bigoplus_{Z/4} g^l. \quad (3.14)$$

Example 2: Consider SL_4 Lie algebra and the nilpotent element corresponding to the partition $[2, 2]$; and the standard triple is

$$e = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (3.15)$$

Using the matrix h , we find that the weighted Dynkin diagram is $\overset{0}{\bullet} - \overset{2}{\bullet} - \overset{0}{\bullet}$. Using h , we have the following decomposition of Lie algebra:

$$g_0 = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & a_{22} & 0 & a_{24} \\ a_{31} & 0 & a_{33} & 0 \\ 0 & a_{42} & 0 & a_{44} \end{pmatrix}, g_2 = \begin{pmatrix} 0 & a_{12} & 0 & a_{14} \\ 0 & 0 & 0 & 0 \\ 0 & a_{32} & 0 & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}, g_{-2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_{21} & 0 & a_{23} & 0 \\ 0 & 0 & 0 & 0 \\ a_{41} & 0 & a_{43} & 0 \end{pmatrix}. \quad (3.16)$$

The Lie algebra has the following grading

$$g^0 = g_0, \quad g^2 = g_2 + g_{-2}. \quad (3.17)$$

and the order of actual automorphism is two:

$$sl_4 = \bigoplus_{\mathbb{Z}/2} g^l. \quad (3.18)$$

Classifications: Now let's explain how to define our irregular singularity using the grading of Lie algebra. Start with $J = A_{N-1}$, and given a Young Tableaux $[n_1, \dots, n_1]$ or $[n_1, \dots, n_1, 1]$ which define a regular semi-simple cyclic element, the Higgs field takes the form:

$$\Phi = \frac{T_k}{z^{2+\frac{k}{n_1}}} + \sum_{-n_1 \leq k' < k} \frac{T_{k'}}{z^{2+\frac{k'}{n_1}}}; \quad (3.19)$$

Here $T_{k'} \in g^l$ with $k' = nn_1 + l$ and $0 \leq l < n_1$. They are all taken to be diagonal matrices, and the eigenvalues take the following form

$$\begin{aligned} T_i &= \text{diag}(a_1 A_i, \dots, a_2 A_i, \dots, a_{\frac{N}{n_1}} A_i) & Y &= [n_1, \dots, n_1], \\ T_i &= \text{diag}(a_0, a_1 A_i, \dots, a_2 A_i, \dots, a_{\frac{N}{n_1}} A_i) & Y &= [n_1, \dots, n_1, 1]. \end{aligned} \quad (3.20)$$

Here A_i are fixed diagonal matrix with size n_1 , and traceless condition would force some coefficients a_i to be zero. The detailed classification of SCFT is given as follows:

1. Let's take the trivial grading corresponding to partition $[1, \dots, 1]$, then the corresponding irregular singularity is

$$\Phi = \frac{T_n}{z^n} + \dots + \frac{T_{n-1}}{z^{n-1}} + \dots + \frac{T_1}{z}. \quad (3.21)$$

Here T_i s are diagonal matrix with distinct eigenvalues.

2. Let's take the grading corresponding to partition $[N]$, the Higgs field takes the following form

$$\Phi = \frac{T}{z^{2+k/N}} + \dots \quad (3.22)$$

Here k and N are coprime. More generally, we can choose the grading corresponding to the partition $[n_1, \dots, n_1]$ with n_1 a divisor of N , and the Higgs field takes the form

$$\Phi = \frac{T}{z^{2+k/n_1}} + \dots \quad (3.23)$$

One can also engineer this class of theory using type IIB string theory on a 3-fold singularity of the following form [15, 22]

$$x_1^2 + x_2^2 + x_3^N + z^{\frac{N}{n_1}k} = 0. \quad (3.24)$$

3. Let's take the grading corresponding to partition $[N-1, 1]$, the Higgs field takes the form

$$\Phi = \frac{T}{z^{2+\frac{k}{N-1}}} + \dots \quad (3.25)$$

here k and $N-1$ is coprime. More generally, one can take the grading corresponding to partition $[n_1, \dots, n_1, 1]$ with n_1 a divisor of $N-1$, and the Higgs field takes the following form

$$\Phi = \frac{T}{z^{2+\frac{k}{n_1}}} + \dots \quad (3.26)$$

One can also engineer this class of theory using type IIB string theory on a 3-fold singularity of the following form [15]:

$$x_1^2 + x_2^2 + x_3^N + x_3 z^{\frac{N-1}{n_1}k} = 0. \quad (3.27)$$

Remark: It is interesting to explore whether one could get interesting SCFT using general semi-simple grading listed in table. 1.

Degenerations: Let's first consider the irregular singularity corresponding to the partition $[1, \dots, 1]$, see (3.21). Here T_i are taken to be diagonal matrices. The degeneration is defined by a sequence of Levi sub-algebra $L_1 \subset L_2 \subset \dots \subset L_n$ so that L_i commutes with the set $(T_n, T_{n-1}, \dots, T_{n-i})$ [23]. Let's use a sequence of Young Tableaux Y_i to denote the degeneracy of the eigenvalue of T_n , then the constraints coming from the inclusion relation of the Levi sub-algebra is that the Young Tableaux Y_i is derived by decomposing the columns of Y_{i+1} .

One can generalize the above consideration to the situation corresponding to partition $[n_1, \dots, n_1]$ or $[n_1, \dots, n_1, 1]$. For the partition $[n_1, \dots, n_1]$, the irregular singularity is

$$\Phi = \frac{T_k}{z^{2+\frac{k}{n_1}}} + \sum_{-n_1 \leq k' < k} \frac{T_{k'}}{z^{2+\frac{k'}{n_1}}}; \quad (3.28)$$

Each diagonal matrix T_i takes the form in (3.20), and one can use a Young Tableaux with size $\frac{N}{n_1}$ to denote the eigenvalue degeneracy of the coefficient a_i . Now the degeneracy can be defined by choosing a sequence of $k + n_1 + 1$ Young Tableaux with size $\frac{N}{n_1}$:

$$Y_k \subset Y_{k-1} \dots \subset Y_{-n_1}. \quad (3.29)$$

Similarly, one have a sequence of Young Tableaux to describe the degeneration of irregular singularity defined using partition $[n_1, \dots, n_1, 1]$. Note however not all the irregular singularity defined from degeneration define a SCFT, and those which actually define SCFTs will be classified later.

3.1.2 Regular puncture

One still get AD SCFT if we add an extra regular singularity besides an irregular singularity. The regular singularity for A_{N-1} Hitchin system has been classified in [4], and the classification coincides with the classification of nilpotent orbit which is labeled by a Young Tableaux $[n_s^{h_s}, \dots, n_1^{h_1}]$ with $n_s > \dots > n_1$, whose flavor symmetry is

$$G_Y = [\prod_{i=1}^s U(h_i)]/U(1). \quad (3.30)$$

The favor central charge depends on the detailed form of the other irregular singularity and will be computed later.

3.2 Newton Polygon and SW curve

One can represent the irregular singularity (3.1) of A_{N-1} theory by a Newton polygon, see figure. 2. The slop of the boundary line represents the leading order of pole (minus two) of the irregular singularity. A regular singularity adds a further boundary line which connects point $(-N, 0)$ and point $(0, N)$. The SW curve is identified with the spectral curve of the Hitchin system

$$\det(x - \Phi) = 0. \quad (3.31)$$

The SW curve of Coulomb branch can be easily found from Newton polygon, i.e. we associate to a point with coordinate (m, n) a monomial $x^m z^n$, and the SW curve is simply (Consider theory defined using partition $[n_1, \dots, n_1]$)

$$x^N + z^k + \sum_{(m,n) \in S} u_{m,n} x^m z^n = 0, \quad (3.32)$$

where the coefficients $u_{m,n}$ label the parameters (couplings, Coulomb branch operators and masses) of the AD theory. Notice that we only count square points within the Newton polygon. One can find the scaling dimensions of these parameters by demanding each term in SW curve to have the same scaling dimension and that the SW differential $\lambda = xdz$ has the scaling dimension one. Consider theory defined using partition $[n_1, \dots, n_1]$, the SW curve of the AD point is

$$x^k + z^N = 0. \quad (3.33)$$

Then we have the following two equations

$$k[x] = N[z], \quad [x] + [z] = 1. \quad (3.34)$$

and it is easy to find $[x] = \frac{N}{N+k}$, $[z] = \frac{k}{N+k}$, then it is straightforward to find the full spectrum of the theory.

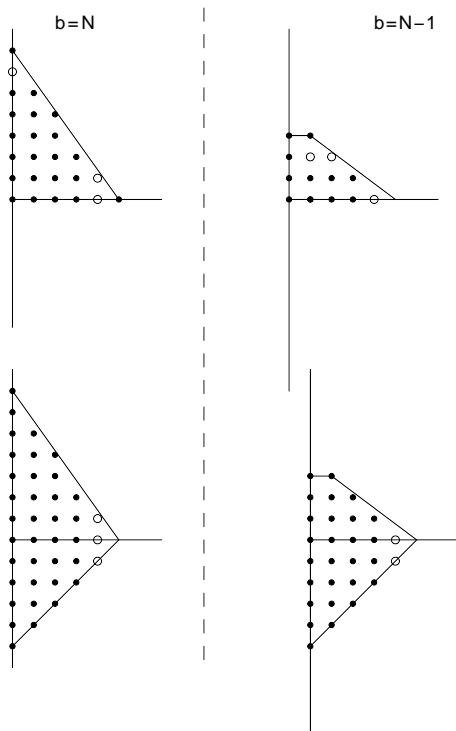


Figure 2. Newton polygon of SCFT defined by 6d $A_{N-1}(2,0)$ theory on a sphere with an irregular singularity and a regular singularity. The SW curve can be found from the monomials associated with black bullets within Newton polygon.

3.3 Argyres-Douglas matter

One can read off the number of exact marginal deformations and the mass parameters from the SW curve. On the other hand, one can read those numbers from the data in defining irregular singularity: the number of parameters in leading order matrix T_k gives the maximal possible exact marginal deformations, and the number of mass parameters are identified with the parameters in diagonal matrix T_0 . For the AD theories using the regular semi-simple cyclic element, those numbers are listed in table. 2.

Argyres-Douglas matters are defined as those SCFTs satisfying the following two questions: a) there are no exact marginal deformations. b) there is non-abelian flavor symmetry; A detailed classification will be given in following sections.

Young Tableaux	exact marginal parameters	mass parameters
$[n_1, \dots, n_1]$	$\frac{N}{n_1} - 1$	$\frac{N}{n_1} - 1$
$[n_1, \dots, n_1, 1]$	$\frac{N}{n_1} - 1$	$\frac{N}{n_1} - 1$

Table 2. Number of maximal possible exact marginal deformations and mass parameters for the AD theories defined by the corresponding cyclic element of A_{N-1} Lie algebra.

4 Theory of class $(p, 1)$

Let's consider 4d $\mathcal{N} = 2$ SCFTs defined by compactifying 6d $(2, 0)$ A_{N-1} theory on a sphere with following irregular singularity:

$$\Phi = \frac{T_n}{z^n} + \dots + \frac{T_1}{z}. \quad (4.1)$$

The eigenvalue degeneracies of matrices T_i are encoded by a sequence of Young Tableaux

$$Y_n \subset Y_{n-1} \subset \dots \subset Y_1. \quad (4.2)$$

This class of theories are defined using the partition $[1, \dots, 1]$, see the notation in last section. We also consider theories defined using above irregular singularity plus an extra regular singularity, which is denoted by a Young Tableaux Y_0 . We call these models as class $(p, 1)$ theories with $p = n - 2$, and the reasoning will be clear in next section. A theory might be denoted by $(A_{N-1}; Y_n, \dots, Y_1; Y_0)$; Not all of them describe a SCFT and we will classify those configurations which actually define SCFTs.

A special feature of these theories is that they admit 3d mirror with Lagrangian descriptions. The 3d mirror is quite useful in understanding many properties of original 4d theory. In particular, it was shown in [9] that one can interpret S duality of 4d theory as different decompositions of 3d mirror. In [9], we outline the main idea of finding S duality for this class of theories. Here we will introduce further combinatorial tools and provide a systematical study of S duality of above SCFTs.

4.1 Coulomb branch spectrum

Let's first review how to find Coulomb branch spectrum for this class of theories [15]. The SW curve is identified with the spectral curve of the corresponding Hitchins system:

$$\det(x - \Phi) = 0 \rightarrow x^N + \sum_{i=2}^N \phi_i(z) x^{N-i} = 0. \quad (4.3)$$

Here z denotes the coordinate on Riemann surface Σ on which we compactify 6d $(2, 0)$ theory. At the SCFT point, the SW curve takes the following form,

$$x^N + z^{(n-2)N} = 0, \quad (4.4)$$

from which we can find out the scaling dimension of x and z :

$$[x] = \frac{n-2}{n-1}, \quad [z] = \frac{1}{n-1}. \quad (4.5)$$

This is determined by requiring the SW differential $\lambda = xdz$ having scaling dimension one, and requiring each term in (4.4) to have the same scaling dimension.

The Coulomb branch spectrum can be computed as follows: Let's consider one of Young Tableaux $Y_j = [n_1, \dots, n_{r_j}]$ in the definition of irregular singularity (4.2), and label the boxes of Y_j as $1, \dots, N$ starting from the bottom-left corner of Y_j and going row by row, see figure. 3; Define a sequence of integral numbers $p_i^{(j)}$:

$$p_i^{(j)} = i - s_i^{(j)}, \quad i = 1, \dots, N \quad (4.6)$$

here $s_i^{(j)}$ is the height of the i th box in Y_j . The independent Coulomb branch operators in $\phi_i(z)$ are found from coefficients of the monomials $z^k x^{N-i}$, $0 \leq k \leq d_i$, and d_i is given by the following formula

$$d_i = \sum_{j=1}^n p_i^{(j)} - 2i. \quad (4.7)$$

Example: Consider an irregular singularity specified by the Young Tableaux $Y_3 = [2, 2, 2], Y_2 = [2, 2, 2], Y_1 = [1, 1, 1, 1, 1, 1]$. Using formula 4.6, we get following set of numbers:

$$p_i^{(3)} = (0, 1, 2, 2, 3, 4), \quad p_i^{(2)} = (0, 1, 2, 2, 3, 4), \quad p_i^{(1)} = (0, 1, 2, 3, 4, 5). \quad (4.8)$$

So the set of numbers d_i are

$$d_i = (-2, -1, 0, -1, 0, 1). \quad (4.9)$$

Negative number means that there is no coulomb branch deformation in the corresponding differential ϕ_i . So the SW curve with independent Coulomb branch operators are

$$x^6 + c_1 x^3 + c_2 x + (z^6 + c_3 z + c_4) = 0. \quad (4.10)$$

The scaling dimensions of these Coulomb branch operators are:

$$[c_1] = \frac{3}{2}, \quad [c_2] = \frac{5}{2}, \quad [c_3] = \frac{5}{2}, \quad [c_4] = 3. \quad (4.11)$$

4	5	6
1	2	3

4	5	6
1	2	3

1	2	3	4	5	6
---	---	---	---	---	---

Y_3

Y_2

Y_1

Figure 3. Young Tableaux with labels. Here $Y_3 = [2, 2, 2], Y_2 = [2, 2, 2], Y_1 = [1, 1, 1, 1, 1, 1]$.

Now let's add an extra regular singularity which is labeled by a Young Tableaux $Y^0 = [n_1, \dots, n_{r_0}]$, and again define a sequence of numbers p_0 using formula 4.6. The maximal order of pole in ϕ_i is just $p_i^{(0)}$. The independent Coulomb branch operators in $\Phi_i(z)$ are found from the coefficients before following monomials

$$z^k x^{N-i}, \quad -p_i^{(0)} \leq k \leq d_i. \quad (4.12)$$

4.2 3d Mirror

Let's compactify a four dimensional $\mathcal{N} = 2$ theory on a circle and flow to IR to get a 3d $\mathcal{N} = 4$ SCFT A . For a 3d $\mathcal{N} = 4$ SCFT A , one can often find a mirror SCFT B [24]. The basic feature of the 3d mirror is that the Coulomb branch of theory A is mapped to Higgs branch of theory B , and vice versa.

The three dimensional mirror theory B of our model is found in [15], and they all admit a Lagrangian description. Consider a theory denoted as $(A_{N-1}; Y_n, \dots, Y_1; Y_0)$, and its 3d mirror is derived step by step as the following:

1. Assume $Y_n = [n_1, n_2, \dots, n_r]$, then assign r quiver nodes with gauge group $U(n_i)$, and draw $n - 2$ quiver arrows connecting any pair of quiver nodes, see figure. 4.
2. If one of column with height n_i of Y_n is further partitioned as $[m_{i1}, m_{i2}, \dots, m_{is}]$ in Young Tableaux Y_{n-1} , we split the quiver node with rank n_i into several quiver nodes with rank m_{ij} , and draw $n - 3$ quiver arrows between those new created quiver nodes, see figure. 4 step 2. One do the similar splitting for each Young Tableaux until Y_2 , and get a quiver with many quiver nodes and nested arrows between them. Notice that the sum of the total rank of all the quiver nodes are N .
3. The special treatment is needed for Y_1 : if one of the column of Y_2 has height l and is further partitioned as $[l_1, l_2, \dots, l_t]$ in Y_1 , we do not split the quiver node with rank l , instead we attach a quiver tail as follows: define $h_i = \sum_t^{j=t-i+1} l_j$, the quiver tail is

$$U(h_1) - U(h_2) - \dots - U(h_{t-1}) - [U(l)]. \quad (4.13)$$

4. If there is an extra regular singularity specified by a Young Tableaux $Y_0 = [p_1, p_2, \dots, p_q]$, define $h_i = \sum_q^{j=q-i+1} p_j$, and assign a quiver tail:

$$U(h_1) - U(h_2) - \dots - U(h_{q-1}) - [U(N)]. \quad (4.14)$$

Then spray the $U(N)$ node as the pattern determined by the Young Tableaux Y_2 of the the irregular singularity. Finally we glue the quiver of irregular singularity and regular singularity by identifying the sprayed nodes of regular singularity tail with the quiver nodes determined by Y_2 .

Two examples are shown in figure. 5. The 3d mirror can be used to learn interesting physical properties of original 4d theory:

- The 3d mirror can be used to find the non-abelian flavor symmetry of the original 4d theory. The Higgs branch of our 4d theory is mapped to the Coulomb branch of the 3d mirror theory B , and the flavor symmetry of 4d theory can be read from the Coulomb branch symmetry of theory B . The method of reading flavor symmetry of Coulomb branch of a quiver gauge theory is developed in [25]. Here we review the basic ingredients. A quiver node is defined as **balanced** if $N_f = 2N_c$. For each balanced ADE chain (which can only be ADE shape), we have a corresponding ADE flavor group on Coulomb branch, and there is a $U(1)$ flavor symmetry for each non-balanced quiver node.

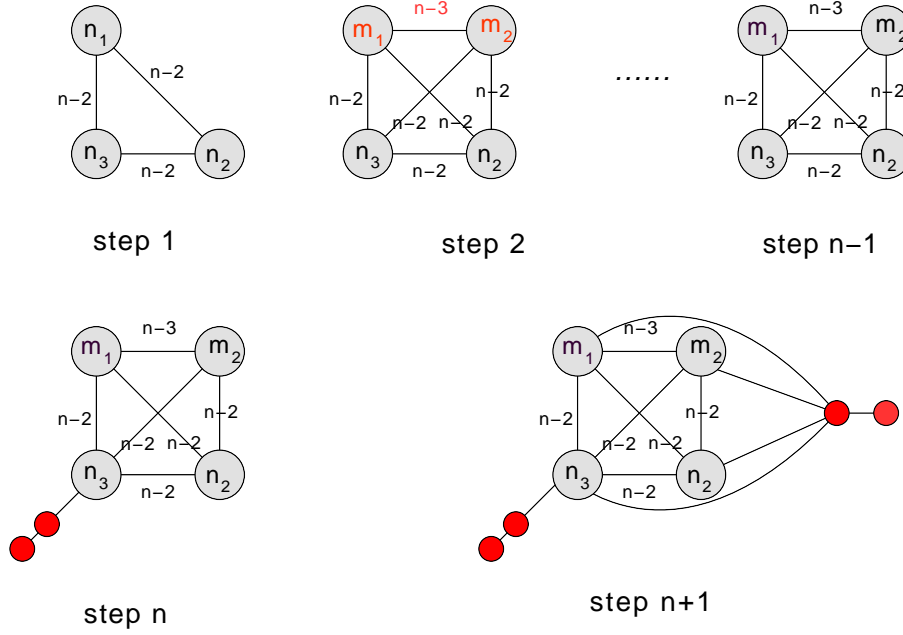


Figure 4. Step 1: If the first Young Tableaux Y_n has partition $[n_1, n_2, n_3]$, first assign a quiver with three nodes and ranks n_i , and then connect $n - 2$ quiver arrows between those nodes. Step 2: If n_1 is further partitioned into $[m_1, m_2]$ in Y_{n-1} , we split the quiver node with rank n_1 into two quiver nodes with rank m_1 and m_2 , and the number of quiver arrows between m_i and n_1, n_2 are still $n - 2$; but the number of arrows between m_1 and m_2 are $n - 3$. Similar procedure is done for other Young Tableaux and we stop at Y_2 . Bottom: If a column with height l in Y_2 is further split into $[l_1, l_2, \dots, l_t]$, one attach a quiver tail to the node with rank l in quiver determined by (Y_n, \dots, Y_2) ; For Y_0 , we attach a quiver tail which is connected to all the quiver nodes determined by Y_2 . The number of quiver arrow is one if there is no label.

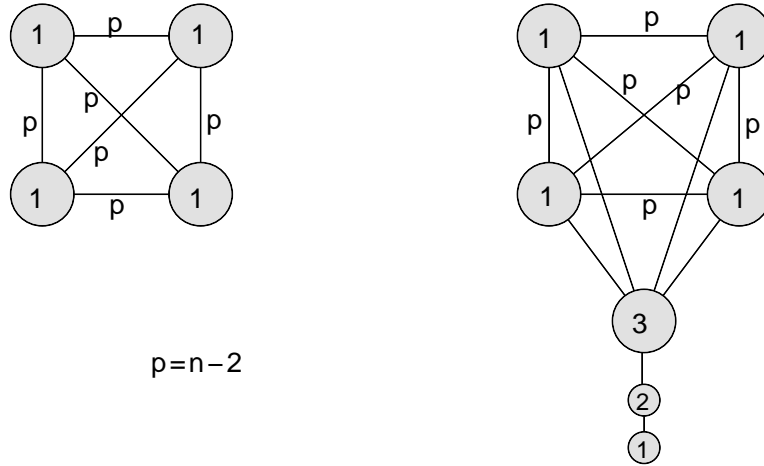


Figure 5. Left: 3d mirror for theory $(A_3; Y_n, \dots, Y_1; Y_0)$ with $Y_n = \dots = Y_1 = [1, 1, 1, 1]$ and $Y_0 = [4]$ (trivial). Right: 3d mirror for theory $(A_3; Y_n, \dots, Y_1; Y_0)$ with $Y_n = \dots = Y_1 = [1, 1, 1, 1]$ and $Y_0 = [1, 1, 1, 1]$. The number of quiver arrow is one if there is no label.

- Sometimes the original 4d theory is not irreducible, i.e. the 4d theory consists an interacting part and a free part. One can use 3d mirror to detect it and actually find the interacting part [26]. Let's define a quiver node to be **bad** if $N_f < 2N_c$, and our original 4d theory is reducible if the mirror quiver has a bad node. To get the interacting part of original 4d theory, we perform the following operation: first replace the rank of a bad node as follows $N'_c = N_f - 2N_c$. After this step one may create other bad nodes, and one need to continue performing the above operation until there is no bad node left. The final quiver describes the mirror of the interacting part of original theory.
- Given a 3d mirror quiver B , one can reverse engineer the M5 configuration of theory A . Often theory A can have more than one M5 brane constructions [15, 27].

4.3 Classification of SCFT

We would like to identify theory $(A_{N-1}; Y_n, \dots, Y_1; Y_0)$ which defines a SCFT. One necessary condition is that the possible number of exact marginal deformations should be larger than the number of dimension two operators. It appears that this is also a sufficient condition for our model.

Let's first assume that $n > 3$ and Y_n has a columns. Then the maximal number of exact marginal deformations are $a - 2$: there are a parameters in defining Y_n , and traceless condition and an overall scaling removes two parameters. Let's now count the number of dimension two operators. The SW curve is

$$x^N + \sum_{i=2}^N \phi_i(z)x^{N-i} = 0. \quad (4.15)$$

The dimension two operator appears as the coefficient of the following monomial in SW curve:

$$z^{b_i}x^{N-i}, \quad b_i = (n-2)i + 2 - 2n. \quad (4.16)$$

Here we need $i \geq 3$. Let's look at differential $\phi_i, 3 \leq i \leq a$, and the maximal order of z in ϕ_i which will give Coulomb branch operator is

$$d_i = n(i-1) - 2i = (n-2)i - n, \quad i \leq a. \quad (4.17)$$

We have used formula (4.7) and the fact that all the Young Tableaux has at least a columns, so each Young Tableaux contributes $i - 1$ to d_i . We have $d_i - b_i = n + 2 > 0$, and this implies that there are at least $a - 2$ dimensional two operators in the Coulomb branch spectrum.

Next let's consider differential ϕ_{a+1} , and the maximal order of z which will give Coulomb branch operator is

$$d_{a+1} = n_1(a+1-2) + n_2(a+1-1) - 2(a+1) = (n-2)(a+1) - 2n + n_2. \quad (4.18)$$

Here n_1 (resp. n_2) is the number of Young Tableaux which has a (resp. more than a) columns, and $n_1 + n_2 = n$. To require $d_{a+1} < b_{a+1}$, we get $n_2 < 2$, so we only have zero or

one Young Tableaux with more than a columns. This implies that the first $n - 1$ Young Tableaux all has a columns, which means that they are all equal (Notice that Y_i is derived by further splitting the columns of Y_{i+1} , so if Y_{i+1} and Y_i have the same columns, they are equal.). The above result is not changed by adding an extra regular singularity. Moreover, once we have $Y_n = \dots = Y_2$, there is no dimension two operator in differential $\phi_i, i > a$. So the number of dimension two Coulomb branch operator matches with the number of exact marginal deformations.

Now let's consider the situation $n \leq 3$, after careful analysis, we found the following two possibilities

- Y_3 and Y_2 both have a columns, and Y_1 is arbitrary, the 3d mirror is shown on the left of figure. 6, and $p = 1$.
- Y_2 and Y_1 both have arbitrary Columns, and Y_0 is arbitrary². The 3d mirror is a star shape quiver, see figure. 6.

For above class of theories, there are only $a - 3$ dimension two operators. The 3d mirror for these SCFTs is shown in figure. 6. The 3d mirror on the left of figure. 6 is actually 3d mirror of class \mathcal{S} theory defined on sphere.

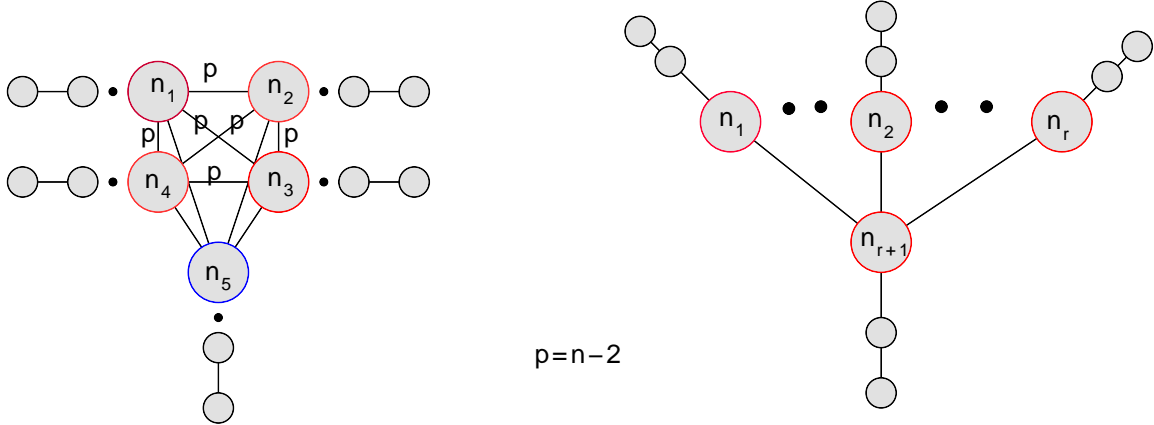


Figure 6. The 3d Mirror of class $(p, 1)$ SCFT. These quiver nodes with explicit ranks form the core of the quiver, and those nodes without rank form a quiver tail. Left: $Y_n = \dots = Y_2 = [n_1, n_2, n_3, n_4]$ with $n \geq 3$, and Y_1, Y_0 are arbitrary. Right: $n = 2$, and (Y_2, Y_1, Y_0) are arbitrary.

Among these SCFTs, the 3d mirror for AD matter is shown in figure. 7. Let's make some comments about these AD matters:

1. Let's choose numbers (n_1, n_2, n_3) in figure. 7 so that the 3d mirror is a good quiver, then the maximal flavor symmetry of AD matter on the left can be $SU(n_1) \times SU(n_2) \times SU(n_3)$ (further enhancement is also possible). The flavor central charge for them is

$$k_{SU(n_1)} = n_1 + \frac{1}{n-1}, \quad k_{SU(n_2)} = n_2 + \frac{1}{n-1}, \quad k_{SU(n_3)} = n_3 + \frac{n-2}{n-1}. \quad (4.19)$$

²Such theories have multiple realizations using different six dimensional theory and irregular singularities, here we choose a realization with maximal number of possible exact marginal deformations.

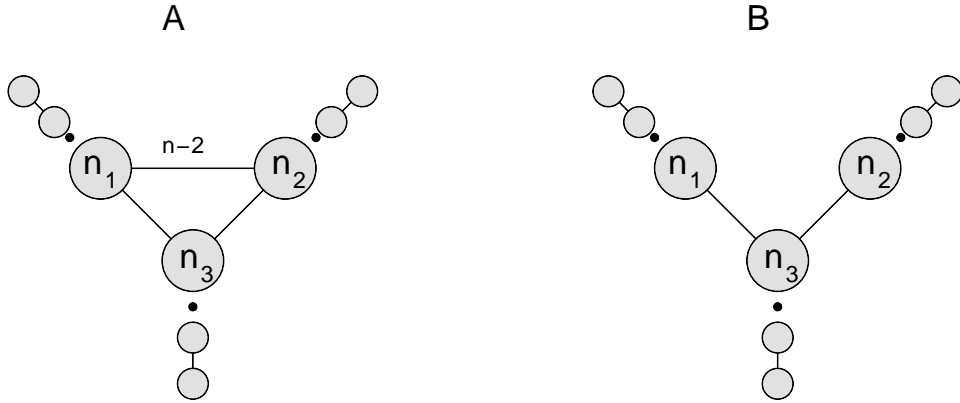


Figure 7. The 3d mirror of AD matter in class $(p, 1)$ theories, here $p = n - 2$.

The flavor symmetry $SU(n_3)$ can be further enhanced to $SU(n_3 + 1)$ with the same flavor central charge.

For matter system on the right of figure. 7, to have a good quiver, we need $n_3 > n_1$ and $n_3 > n_2$. Let's fix n_3 , the maximal flavor symmetry can be $SU(n_3) \times SU(n_3) \times SU(n_3)$. This is actually T_{n_3} theory, and the flavor central charge is

$$k_{SU(n_3)} = n_3. \quad (4.20)$$

We can consider other type of flavor symmetries which is the subgroup of $SU(n_3) \times SU(n_3) \times SU(n_3)$.

2. The AD matter could have enhanced flavor symmetry. The AD matter whose flavor symmetry is enhanced to a single SU group is shown in figure. 8.
3. The Coulomb branch spectrum can be computed using the defining data of the irregular singularity and regular singularity, see formula (4.7). The central charge a and c can be computed using the following formula

$$2a - c = \frac{1}{4} \sum_{i=1}^r (2u_i - 1), \quad a - c = -\frac{d_H}{24}. \quad (4.21)$$

Here r is the dimension of Coulomb branch, and d_H is the Higgs branch dimension which is equal to the dimension of Coulomb branch of 3d mirror.

Remark: An interesting question is the interpretation of other non-conformal theories: they admit four dimensional $\mathcal{N} = 2$ SUSY but not the full superconformal symmetry; some of them can be described by asymptotical free gauge theories, while other cases are more mysterious. It is interesting to further study those theories. To give an example, let's consider theories $(A_2; Y_n, \dots, Y_1; Y_0)$ with $Y_n = \dots = Y_{n_1+1} = [2, 1]$, $Y_{n_1} = \dots = Y_1 = [1, 1, 1]$, here we take $n_1 > 1$. The 3d mirror of this theory is shown in figure. 9.

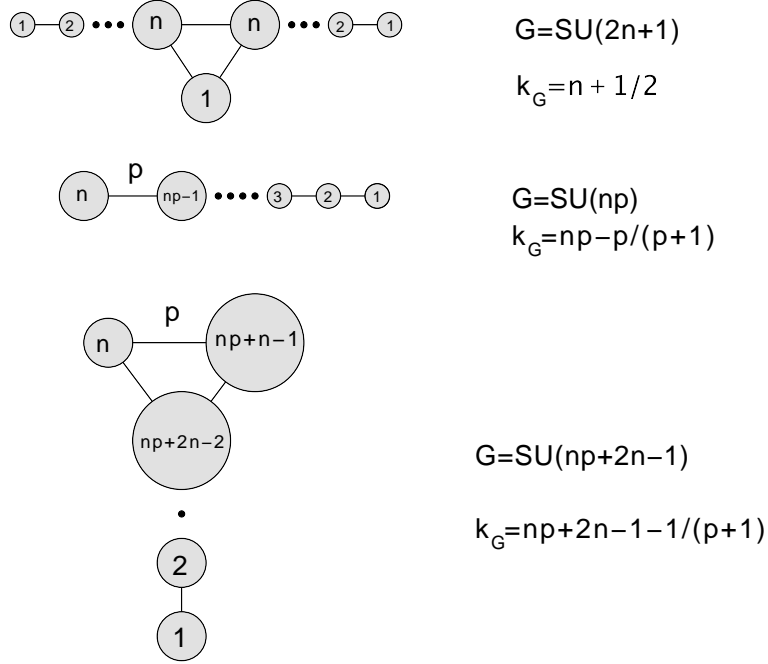


Figure 8. 3d mirror of AD matter whose flavor symmetry is a simple SU group, and we also list the flavor central charge.

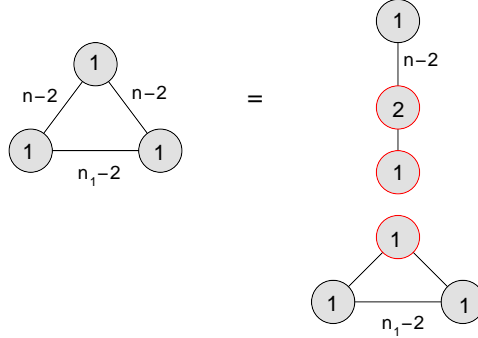


Figure 9. 3d mirror of non-conformal theory, here $n_1 > 1$. The field theory description can be found from decomposing the 3d mirror into two pieces, an the field theory description is $T_1 - SU(2) - T_2$. Here T_1 (resp. T_2) matter contribute $2 + \frac{1}{n-2}$ (resp. $1 + \frac{n_1-1}{n_1-2}$ or 2 for $n_1 = 2$.) to the β function of $SU(2)$ gauge group, so the total contribution of matter to the β function is less than four, and the theory is an asymptotical free theory.

4.4 Duality

Let's summarize the classification of AD matter and SCFT with exact marginal deformations found in previous section. Let's start with 6d $A_{N-1}(2,0)$ theory on a sphere with the following irregular singularity:

$$\Phi = \frac{T_n}{z^n} + \dots + \frac{T_1}{z}, \tag{4.22}$$

and the eigenvalue degeneracy of T_i is encoded by a sequence of Young Tableaux

$$Y_n \subset Y_{n-1} \subset \dots \subset Y_1. \quad (4.23)$$

One can add another regular singularity labeled by Y_0 so it is still possible to define a SCFT. Our theory might be denoted as $(A_{N-1}; Y_n, \dots, Y_1; Y_0)$. We have following classification of SCFT:

- For $n \geq 4$, the irregular singularity has the structure $Y_n = Y_{n-1} = \dots = Y_2 = [n_1, n_2, \dots, n_a]$. Y_1 and Y_0 are arbitrary. We label this class of theory as $(\mathbf{p}, \mathbf{1})$ with $p = n - 2 \geq 2$. One of the reason for the labeling is that the Coulomb branch operators have the common denominator $p + 1$.
- For $n \leq 3$, and there are two possibilities:
 1. The irregular singularity has the structure $Y_3 = Y_2 = [n_1, n_2, \dots, n_a]$, Y_1 arbitrary. (Theory defined with an irregular singularity and a regular singularity can be engineered using a single irregular singularity using a different 6d (2,0) theory [15]). We call them theory of class $(\mathbf{1}, \mathbf{1})$.
 2. The irregular singularity has the structure $Y_2 = [n_1, n_2, \dots, n_a]$, and Y_1 and Y_0 are both arbitrary. We call them theory of class $(\mathbf{0}, \mathbf{1})$.

For $n \geq 4$, the number of exact marginal deformations are $a - 2$, and for $n \geq 3$, the number of exact marginal deformations are $a - 3$.

4.4.1 Theory of class (0, 1)

Interestingly, class \mathcal{S} theory defined on sphere is identified as class (0, 1) theory, which is engineered by the following data $(A_{N-1}; Y_2, Y_1; Y_0)$, with $Y_2 = [n_1, \dots, n_a]$. The irregular singularity has the following form

$$\Phi = \frac{T_2}{z^2} + \frac{T_1}{z}. \quad (4.24)$$

There is also an extra regular singularity labeled by Y_0 . This theory has $a - 3$ exact marginal deformations. The 3d mirror of this theory is a star shaped quiver with a central node of rank n_c [28], see also figure. 6.

S duality of class \mathcal{S} theory is best understood by representing SCFT by a sphere with marked points [4]: We represent above SCFT by a sphere with a marked points, and assign a Young Tableaux of size n_c . The number of complex structure deformation of the punctured sphere is $a - 3$ which is equal to the number of exact marginal deformation of SCFT. Different weakly coupled gauge theory descriptions are then identified as different pants decomposition of the same punctured sphere. Moreover the matter contents in the gauge theory description are identified as the theory represented by three punctured spheres. Many aspects of these dualities have been studied in [29–40]. These results generalize S duality results found by Argyres and Seiberg [3, 41, 42].

4.4.2 Theory of class (1, 1)

Let's consider theory $(A_{N-1}; Y_3, Y_2, Y_1)$ with $Y_3 = Y_2 = [n_1, n_2, \dots, n_a]$ and Y_1 arbitrary. The irregular singularity has the following form

$$\Phi = \frac{T_3}{z^3} + \frac{T_2}{z^2} + \frac{T_1}{z^1}. \quad (4.25)$$

This theory has $a - 3$ exact marginal deformations.

Motivated by S duality of class (0, 1) theory (class \mathcal{S} theory) and the result in [9], it might be also possible to represent our theory by a sphere with a marked points: each marked point has a rank n_i , and a Young Tableaux $[m_1^{(i)}, \dots, m_{r_i}^{(i)}]$ such that $\sum_{j=1}^{r_i} m_j^{(i)} = n_i$. See figure. 10 for illustration, notice that in this case the size of Young Tableaux is different for each marked point. One nice consequence of this identification is that the number of exact marginal deformation matches with the complex structure moduli of punctured sphere, and AD matter is represented by a three punctured sphere.

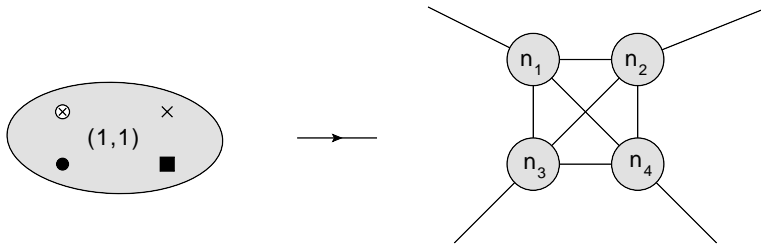


Figure 10. Left: A class (1,1) SCFT is represented by a sphere with marked points, and each marked point has a Young Tableaux with size n_i . Right: The 3d mirror of the theory shown on left, and we omit the quiver tails which are determined by Young Tableaux.

With the above representation of our SCFT, we would like to propose the following method of identifying the duality frames for class (1, 1) theories:

Conjecture 1 *Different duality frames of class (1, 1) theory labeled by the data $(A_{N-1}; Y_3, Y_2, Y_1)$ with $Y_3 = Y_2 = [n_1, n_2, \dots, n_a]$ is identified with different degeneration limit of a sphere with a marked points.*

Example: Let's consider the theory labeled by the data $(A_{N-1}; Y_3, Y_2, Y_1)$ with $Y_3 = Y_2 = Y_1 = [1, \dots, 1]$. This class of theory can also be engineered by Type IIB string theory on a 3-fold singularity $x_1^2 + x_2^2 + x_3^N + x_4^N = 0$ (They are also called (A_{N-1}, A_{N-1}) theory [22]). To study S duality of this theory, we represented it by a sphere with N identical marked points, which is of the type [1]. Different duality frames are identified as different degeneration limits of the same punctured sphere, see figure. 11, 12, 13.

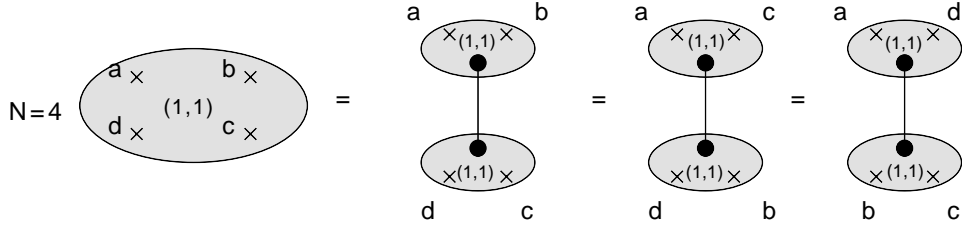


Figure 11. The three duality frames of (A_3, A_3) theory. The cross marked point has Young Tableaux [1].

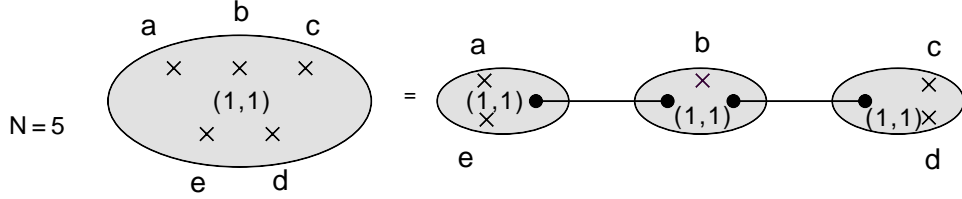


Figure 12. The duality frames of (A_4, A_4) theory. Different labelings for the marked points in the degeneration limit of the punctured sphere give different duality frames. The cross marked point has Young Tableaux [1].

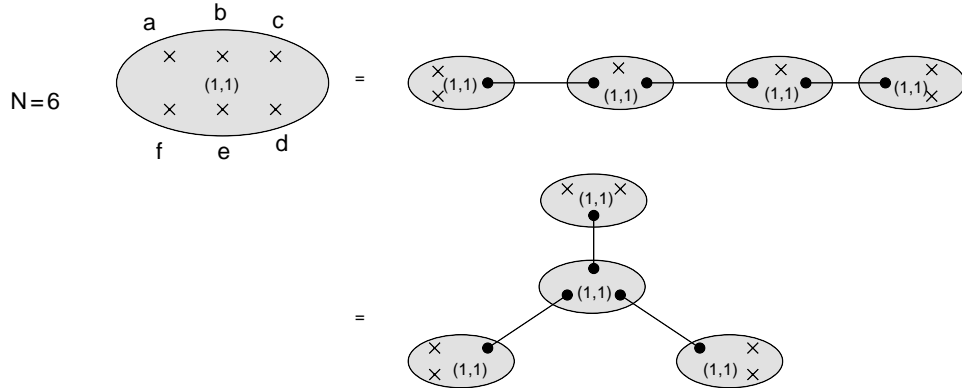


Figure 13. The duality frames of (A_5, A_5) theory. Different labelings for the marked points in the degeneration limit of the punctured sphere give different duality frames. The cross marked point has Young Tableaux [1].

The field theory description for these theories can be read from the decompositions of the punctured sphere:

N=4:

$$\begin{array}{c}
 1 \\
 | \\
 D_2SU(3) \text{ --- } \mathbf{SU(2)} \text{ --- } D_2(SU(3))
 \end{array} \tag{4.26}$$

N=5:

$$D_2SU(3) \text{ --- } \mathbf{SU(2)} \text{ --- } D_2(SU(5)) \text{ --- } \mathbf{SU(2)} \text{ --- } D_2(SU(3)) \tag{4.27}$$

N=6:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & & | & & & & \\
 D_2SU(3) & \text{---} & \mathbf{SU}(2) & \text{---} & D_2(SU(5)) & \text{---} & \mathbf{SU}(3) & \text{---} & D_2(SU(5)) & \text{---} & \mathbf{SU}(2) & \text{---} & D_2(SU(3)) \\
 & & & & & & & & & & & & (4.28) \\
 & & & & D_2SU(3) & & & & & & & & (4.29) \\
 & & & & | & & & & & & & & \\
 & & & & \mathbf{SU}(2) & & & & & & & & \\
 & & & & | & & & & & & & & \\
 D_2SU(3) & \text{---} & \mathbf{SU}(2) & \text{---} & T & \text{---} & \mathbf{SU}(2) & \text{---} & D_2(SU(3))
 \end{array}$$

Let's explain how to find the duality frames for $N = 4$. In the degeneration limit, we get two three punctured spheres: two marked points are of the type $[1]$, while the third one coming from degeneration limit has the type $[1, 1]$. This theory can be engineered using the irregular singularity $(A_3; [1, 1, 2], [1, 1, 2], [1, 1, 1, 1])$. The 3d mirror of this theory is a bad quiver, and use the reduction procedure, we find that the interacting part is engineered by the irregular singularity $(A_2; [1, 1, 1], [1, 1, 1], [1, 1, 1])$. This theory has a $SU(3)$ flavor symmetry, and we gauge a $SU(2)$ subgroup. The extra free hyper on $SU(2)$ gauge group is needed for the conformal invariance. More checks will be given later.

Let's explain more the AD matter used in the field theory description. AD matter $D_2SU(2k + 1)$ is engineered using the following irregular singularity

$$Y_3 = [k, k, 1], Y_2 = [k, k, 1], Y_1 = [1, \dots, 1]. \quad (4.30)$$

and the 3d mirror is shown in figure. 14. This theory has global symmetry $SU(2k + 1)$, and

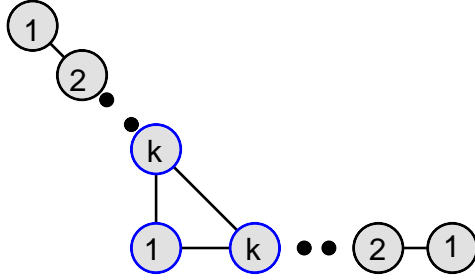


Figure 14. The 3d mirror for $D_2(SU(2k + 1))$ theory which has $SU(2k + 1)$ flavor symmetry.

various data is listed in table. 3. The theory T can be engineered using six dimensional A_5 theory on the following irregular singularity

$$Y_3 = [2, 2, 2], Y_2 = [2, 2, 2], Y_1 = [1, 1, 1, 1, 1, 1]. \quad (4.31)$$

This theory has flavor symmetry $SU(2)^3 \times U(1)^2$. The 3d mirror is shown in figure. 15, and various other data for this SCFT is listed in table. 3.

Let's explain some checks for S dualities we proposed for (A_{N-1}, A_{N-1}) theory:

AD matter	Coulomb branch spectrum	a	c	k_F
$D_2SU(2k+1)$	$\{\frac{2k+1}{2}, \frac{2k-1}{2}, \dots, \frac{3}{2}, \underbrace{1, \dots, 1}_{2k}\}$	$\frac{7}{24}k(k+1)$	$\frac{1}{3}k(k+1)$	$\frac{2k+1}{2}$
T	$\{\frac{3}{2}, \frac{5}{2}, \frac{5}{2}, 3\}$	$\frac{37}{12}$	$\frac{41}{12}$	$k_{SU(2)} = \frac{5}{2}$

Table 3. Various quantities for the AD matter $D_2SU(2k+1)$ and T which appear in S duality of (A_{N-1}, A_{N-1}) theory with $N \leq 6$.

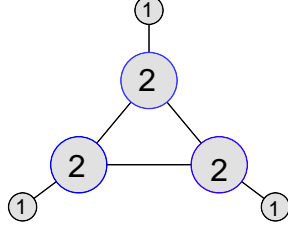


Figure 15. 3d mirror for theory $T = (A_5; [2, 2, 2], [2, 2, 2], [1, 1, 1, 1, 1, 1])$.

- First of all, the sum of β function is zero for all the gauge groups. For $N = 4$, we have $\beta_{D_2(SU(3))} = \frac{3}{2}$ and $\beta_{hyper} = 1$ to the $SU(2)$ gauge group, so the sum of the contribution from various matter contents is 4, which cancels the contribution from the vector multiplet of $SU(2)$ gauge group.
- The central charge a and c of the original theory is computed and listed in table. 3, which can be derived from the formula in (4.21). Let's do the computation for $N = 4$ explicitly. Using gauge theory description, we find

$$\begin{aligned}
 a_{total} &= a_{vec} + a_{hyper} + 2a_{D_2(SU(3))} = \frac{5}{8} + \frac{1}{12} + 2 * \frac{7}{12} = \frac{15}{8}, \\
 c_{total} &= c_{vec} + c_{hyper} + 2c_{D_2(SU(3))} = \frac{1}{2} + \frac{1}{6} + 2 * \frac{2}{3} = 2.
 \end{aligned} \tag{4.32}$$

The answer is the same as the central charge of the original theory listed in table. 4.

- The Coulomb branch spectrum of the original theory is listed in table. 4, which can be either computed from the method reviewed in previous section or the method using singularity theory [43]. We can compute the coulomb branch spectrum using the gauge theory description, and explicitly verify that it agrees with the result in table. 4. For $N = 4$, each $D_2SU(3)$ matter sector contributes a coulomb branch operator with scaling dimension $\frac{3}{2}$ and a mass parameter, and $SU(2)$ gauge group contributes a dimension two operator, finally the free hyper contributes a mass parameter, so the Coulomb branch spectrum is $(1, 1, 1, \frac{3}{2}, \frac{3}{2}, 2)$, which matches with the Coulomb branch spectrum of original theory.

For general N , the matter contents can be found as follows: a tube in the degeneration limit separates the original marked points into two sets with number n_L and n_R respectively; the new marked point in the degeneration limit is of the type $\underbrace{[1, \dots, 1]}_{\min(n_L, n_R)}$. We also

N	Coulomb branch spectrum	a	c
4	$\{1, 1, 1, \frac{3}{2}, \frac{3}{2}, 2\}$	$\frac{15}{8}$	2
5	$\{1, 1, 1, 1, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 2, 2, \frac{5}{2}\}$	$\frac{25}{6}$	$\frac{13}{3}$
6	$\{1, 1, 1, 1, 1, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 2, 2, 2, \frac{5}{2}, \frac{5}{2}, 3\}$	$\frac{185}{24}$	$\frac{95}{12}$

Table 4. Various quantities for (A_{N-1}, A_{N-1}) theories.

need to use the reduction procedure on 3d mirror to get the interacting part. Sometimes, free fundamental matter is also needed to ensure conformal invariance. A duality frame where the interacting matter systems are $D_2SU(2k+1)$ theory is shown in figure. 16 and figure. 17.

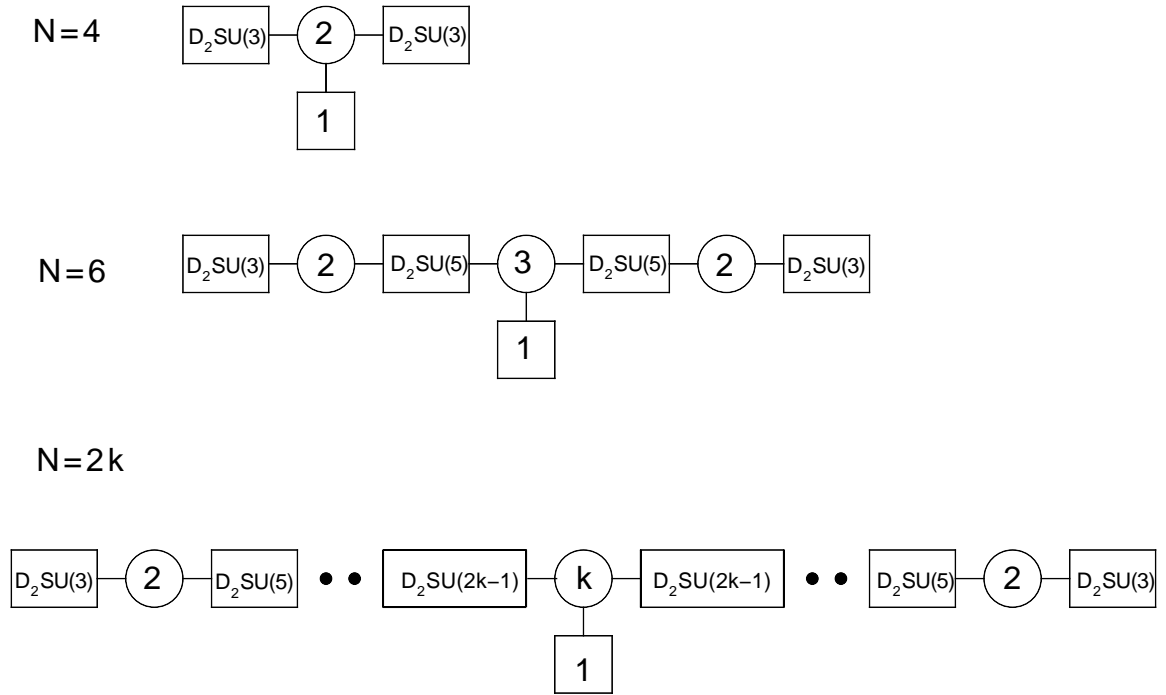


Figure 16. One duality frame for (A_{N-1}, A_{N-1}) theory with even N .

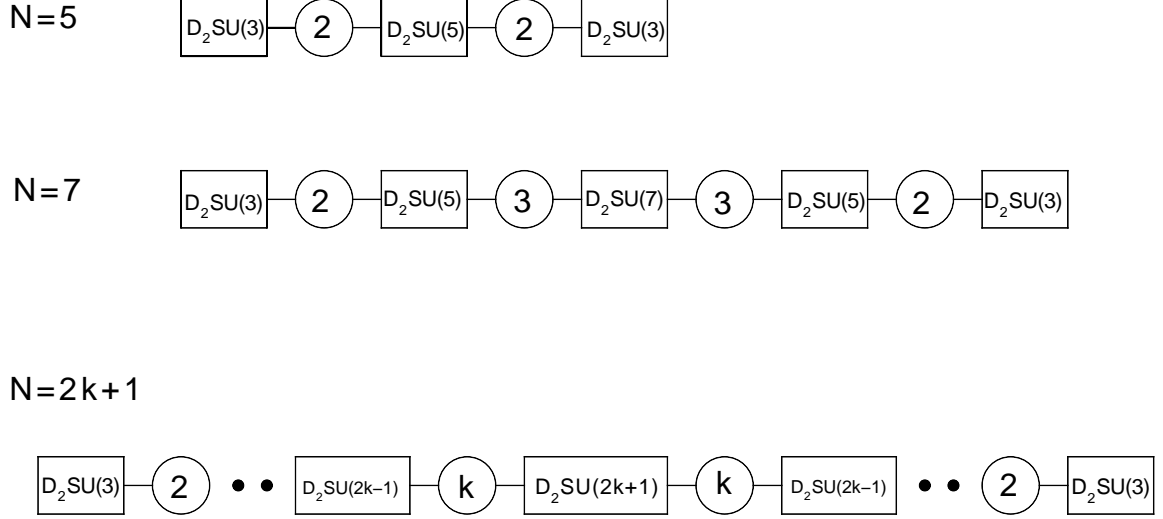


Figure 17. One duality frame for (A_{N-1}, A_{N-1}) theory with odd N .

4.4.3 Theory of class $(p, 1)$

Let's now consider the SCFT $(A_{N-1}; Y_n, \dots, Y_1; Y_0)$ with $Y_n = \dots = Y_2 = [n_1, \dots, n_a]$, $n \geq 4$. The irregular singularity has the following form:

$$\Phi = \frac{T_n}{z^n} + \frac{T_{n-1}}{z^{n-1}} + \dots + \frac{T_1}{z^1}. \quad (4.33)$$

The number of exact marginal deformations is $a - 2$. This is the theory of class $(p, 1)$ with $p = n - 2$.

We now represent our theory by a sphere with $a + 1$ marked points, and we have two kinds of marked points: one represents the data of Y_i , $i \geq 2$ and Y_1 : there is a Young Tableaux attached to each marked point and its total size is n_i (one of the column data of the Young Tableaux Y_2), and we label it as black puncture; We also need an extra Young Tableaux Y_0 with size N , and we need this marked point even if Y_0 is trivial (the Young Tableaux is the type $[N]$). See figure. 18. We label it as a red puncture, and the size of its Young Tableaux is the sum of size of all the black punctures. The AD matter is represented by a three punctured sphere with two black puncture and one red puncture.

Different duality frames are represented by different degeneration limits of the same punctured sphere. In the degeneration limit, one get opposite type of marked points on two degenerated pieces, so that each punctured three sphere in the degeneration limit has only one red puncture and two black punctures.

Example Let's consider SCFT $(A_{N-1}; Y_n, \dots, Y_1; Y_0)$ with $Y_n = \dots = Y_1 = [1, \dots, 1]$ and $Y_0 = [N]$. The same theory can be defined by the singularity $f = z_0^2 + z_1^2 + z_2^N + z_3^{(n-2)N}$. Now we represent the theory by a sphere with $N + 1$ marked points, and the first N marked points are the same and of type $[1]$, while the extra marked point represents the trivial regular puncture with Young Tableaux $[N]$. S duality is interpreted as different degeneration limits of this punctured sphere. To find out the weakly coupled gauge theory description, we need to figure out the new puncture in the degeneration limit. Let's write

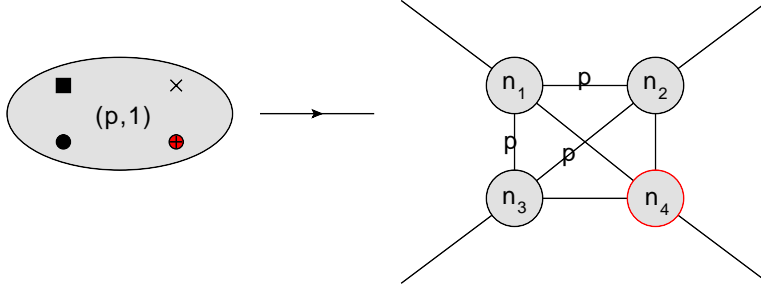


Figure 18. The punctured sphere for class $(p, 1)$ theory. The red marked point representing the regular puncture. The 3d mirror for this theory is shown on the right hand side, and the quiver tail is represented by a straight line whose detailed form is determined by the Young Tableaux of the marked point.

down the field theory for $N = 3$ case (the general case could be worked out following the method used in [32]). The new puncture in the degeneration limit is of the type $[1, 1]$, so the 3d mirror of two matter systems are shown in figure. 20, and the reduced interacting matter system is also shown there from which we can read off the M5 brane configuration. The field theory description is then

$$T_1 - SU(2) - T_2. \quad (4.34)$$

The two matter systems are engineered as follows

$$\begin{aligned} T_1 : & (A_1; Y_n, \dots, Y_1; Y_0), \quad Y_n = \dots = Y_1 = Y_0 = [1, 1] \\ T_2 : & (A_2; Y_n, \dots, Y_1; Y_0), \quad Y_n = \dots = Y_2 = [2, 1], \quad Y_1 = [1, 1, 1], \quad Y_0 = [3]. \end{aligned} \quad (4.35)$$

T_1 has flavor symmetry $SU(2)_1 \times U(1)$ with $k_{SU(2)_1} = 2 + \frac{1}{p+1}$; T_2 also has flavor symmetry $SU(2)_2 \times U(1)$ with $k_{SU(2)_2} = 1 + \frac{p}{p+1}$. So the contribution of matter system T_1 and T_2 to $SU(2)$ gauge group is four, which is needed for the conformal invariance. The interested reader can check that the central charges and Coulomb branch spectrum are the same for the full theory and the field theory description.

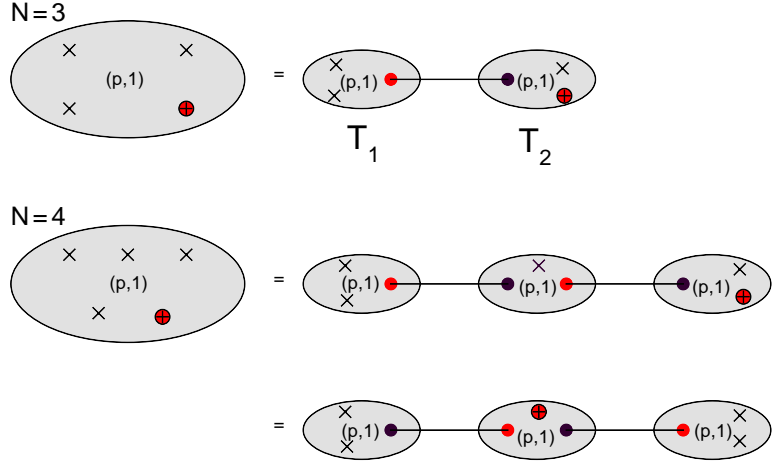


Figure 19. Different duality frames for (A_{N-1}, A_{pN-1}) theory. The black marked point has Young Tableaux [1], the red marked point has Young Tableaux $[N]$. We do not write down the label for each black marked point, and the permutation of these labels will give different duality frames.

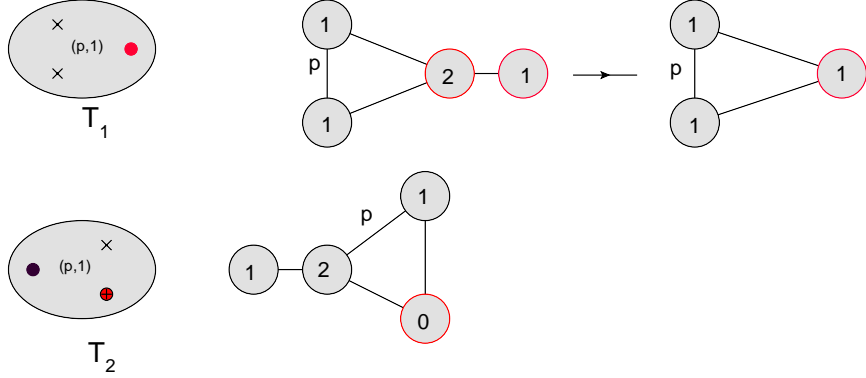


Figure 20. The 3d mirror for two matter systems appearing in the duality frame of (A_2, A_{2p-1}) theory.

5 Theory of class (p, q)

Let's now consider the following class of SCFTs: the Coulomb branch spectrum takes the form $\Delta(u) = \frac{pi}{p+q}$ or $\Delta(u) = \frac{qi}{p+q}$, with $(p, q) = 1$ and i integers. We call them theories of class (p, q) (we also impose the condition $q > 0$). Using irregular and regular punctures reviewed in section II, we have two interesting family of theories in class (p, q) , and we call them class A and class B theory. The corresponding Newton diagram is listed in figure 22, and the number of exact marginal deformations is $r - 1$.

Let's consider class A theory: start with 6d $A_{r-1}(2, 0)$ theory and compactify it on a sphere with following irregular singularity:

$$\Phi = \frac{T_{p+q+1}}{z^{2+\frac{p}{q}}} + \frac{T_{p+q}}{z^{2+\frac{p-1}{q}}} + \dots + \frac{T_1}{z}. \quad (5.1)$$

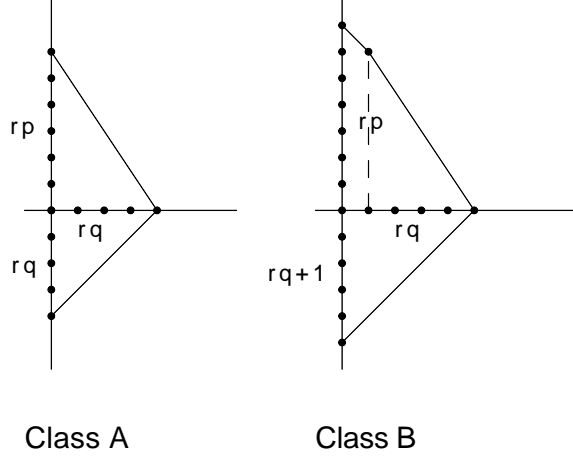


Figure 21. Newton polygon for two family of class (p, q) theories.

The diagonal matrix T_i takes the following form

$$T_i = \text{diag}(a_1 A_i, a_2 A_i, \dots, a_r A_i), \quad a_i \neq a_j. \quad (5.2)$$

Here A_i is a fixed q dimensional diagonal matrix. We can add another regular singularity, sand the theory is still in class (p, q) . In this case p could take negative value with the range $p > -q$. The SW curve at the SCFT point is

$$x^{r q} + z^{r p} = 0 \rightarrow [x] = \frac{p}{p+q}, \quad [z] = \frac{q}{p+q}. \quad (5.3)$$

One can compute the full Coulomb branch spectrum by considering the deformation of above curve, and they can also be read from the under-diagram monomials of the Newton polygon. Similarly, the description of irregular singularity and the Coulomb branch spectrum can be found for type B theory.

We now consider the degeneration of irregular singularity (5.2), and they are classified by a sequence of Young Tableaux

$$Y_{p+q+1} \subset Y_{p+q} \subset \dots \subset Y_1. \quad (5.4)$$

Here Y_i is a Young Tableaux $[n_1, \dots, n_{r_i}]$ with $\sum n_j = r$, and n_i denotes the degeneracy of a_i in formula (5.2). The Coulomb branch spectrum can be found as follows: Define the covering coordinate w using formula $z = w^q$, and $dz = qw^{q-1}dw$, then the irregular singularity has the form (the Higgs field is a one form $\Phi(z)dz$):

$$\begin{aligned} \Phi(w)dw &= \left(\frac{T_{p+q+1}}{w^{q+p+1}} + \frac{T_{p+q}}{w^{q+p}} + \dots + \frac{T_1}{w} \right) dw. \\ Y'_{p+q+1} &\subset Y'_{p+q} \subset \dots \subset Y'_1. \end{aligned} \quad (5.5)$$

and Young tableaux $Y'_i = qY_i = [\underbrace{n_1, \dots, n_1}_q, \dots, \underbrace{n_{r_i}, \dots, n_{r_i}}_q]$. We can now use the formula (4.7) to find the maximal power d_i of the form $\omega^{d_i} x^{r q - i}$ which will give Coulomb branch

operators. We can change back to z coordinate by noticing that ω^{d_i} transforms as a degree i differential on Riemann surface, i.e.

$$\omega^{d_i}(dw)^i = \omega^{d_i}\left(\frac{dz}{w^{q-1}}\right)^i = z^{\frac{d_i-(q-1)i}{q}}(dz)^i. \quad (5.6)$$

The maximal power of $z^{d'_i}x^{rq-i}$ is then

$$d'_i = \left[\frac{d_i - (q-1)i}{q}\right]. \quad (5.7)$$

Here square bracket means taking the integral part of number inside.

Example: Let's consider $A_3(2,0)$ theory on a sphere with following irregular singularity

$$\Phi = \frac{T_6}{z^{2+\frac{3}{2}}} + \frac{T_5}{z^3} + \dots + \frac{T_1}{z}. \quad (5.8)$$

We have $(p,q) = (3,2)$, and the Young Tableaux is $Y_6 = \dots = Y_2 = [2]$, and $Y_1 = [1,1]$. Let's choosing the covering coordinate $x = \omega^2$, and the Higgs field becomes

$$\Phi = \frac{T_6}{w^6} + \dots + \frac{T_1}{w}. \quad (5.9)$$

The Young Tableaux is $Y_6 = \dots = Y_2 = [2,2]$ and $Y_1 = [1,1,1,1]$. We find

$$d_i = (0, 2, 1, 5) \quad (5.10)$$

and using formula (5.7):

$$d'_i = (-1, 0, -1, 0) \quad (5.11)$$

and the SW curve with independent Coulomb branch operators looks like

$$x^4 + [u_1]x + z^6 + [u_2] = 0, \quad (5.12)$$

and the scaling dimensions are $u_2 = \frac{12}{5}$, $u_1 = \frac{6}{5}$. This theory has flavor symmetry $SU(2)$, and we identify it as the rank two H_0 theory [44].

We now classify the configuration which defines a SCFT. Let's consider SW curve of our theory

$$x^{rq} + \sum_{i=2}^{rq} \phi_i x^{rq-i} = 0. \quad (5.13)$$

Let's first consider the case $p > q \geq 2$, then the dimension two operators appear as coefficient of following monomial

$$z^{b(j)}x^{rq-j}, \quad b(j) = -2 + \frac{(j-2)p}{q}, \quad j = 2 + kq, \quad k = 1, \dots, r-1. \quad (5.14)$$

Now let's assume $Y_{p+q+1} = [n_1, \dots, n_a]$ with $\sum_{j=1}^a a_j = r$, then there are at most $a-1$ exact marginal deformations. Let's first analyze the differential ϕ_j , $j = 2 + kq$, $k = 1, \dots, a-1$, and the dimension two operators in those differential are preserved as there is no reduction for these differentials, so there is at least $a-1$ dimension two operators which match with

the number of exact marginal deformations. Let's next consider differential ϕ_{2+aq} , and using formula 5.7, we have

$$d_{2+aq} = \left[\frac{n_1(2+aq-2) + n_2(2+aq-1) - (q+1)(2+aq)}{q} \right] = \left[-2 + ap + \frac{n_2-2}{q} \right] \quad (5.15)$$

here n_1 (resp. n_2) denote the number of Young Tableaux where $aq+2$ is in second row (resp. first row), we used the fact that $n_1 + n_2 = p + q + 1$. To have a SCFT, we need to have $d_{2+aq} < b(2+aq)$ (see formula (5.14)), so $n_2 = 0$ or $n_2 = 1$. The conclusion is that the constraint on the Young Tableaux is again

$$Y_{p+q+1} = \dots = Y_2, \quad Y_1 \text{ arbitrary} \quad (5.16)$$

Similar analysis can be done for the case $-q+1 \leq p < q$, and the result is the same as (5.16).

The AD matter (for $q \geq 2$) (SCFT without exact marginal deformations) is now classified as follows:

- If $p \neq 1$, AD matter is classified by the Young Tableaux $Y_{p+q+1} = \dots = Y_2 = [r]$, and Y_1, Y_0 are arbitrary.
- If $p = 1$ and q arbitrary, we have

$$b(2+kq) = -2+k, \quad k = 1, \dots, r-1. \quad (5.17)$$

If $r = 1$, one get AD matter. If $r = 2$, and there is no regular singularity, then $b(2+q) = -1$, so there is no dimension two operator in Coulomb branch spectrum as the exponent of z variable is non-negative due to the absence of regular singularity; and in this case, the configuration with $Y_{p+q+1} = [1, 1]$ describes AD matter, which is nothing but (A_{q-1}, A_1) theory.

5.1 Duality

To study duality, we would like to represent our theory by a sphere with marked points and identify the exact marginal deformations with the complex structure moduli of the sphere. Let's start with class A theory, and there are a total of $r-1$ exact marginal deformations. Comparing with the class $(p, 1)$ theory, it is tempting to represent the irregular singularity by r marked points which reflect r block of the form of irregular singularity. We can add a further marked point representing the regular singularity (the label could be zero to represent the case where the regular singularity is trivial: $Y_0 = [N]$). Now there are a total of $r+1$ marked points on sphere, and the number of complex structure moduli is $r-2$, which is one less than the number of exact marginal deformation. It appears that we still miss a marked point!

The solution is to look at class B theory, for this class of theory, we have r marked point (we call them black type) from the one compact segment of boundary of Newton polygon, and one marked point (we call them red type) from regular singularity. We need to add another marked point (blue type) with rank one representing the second compact

segment of Newton polygon, see figure. 22. So there is a total of $r + 2$ marked points on sphere, and the number of complex structure moduli matches with the number of exact marginal deformations.

Here comes the representation for class A theory: there is an extra blue type marked point whose rank is zero. Generally, we have the following representation for our SCFT: there are r black marked point, one red marked point, and one blue marked point. We also attach a quiver tail to each black and red marked point, and for the blue marked point, only a $U(1)$ flavor symmetry is found.

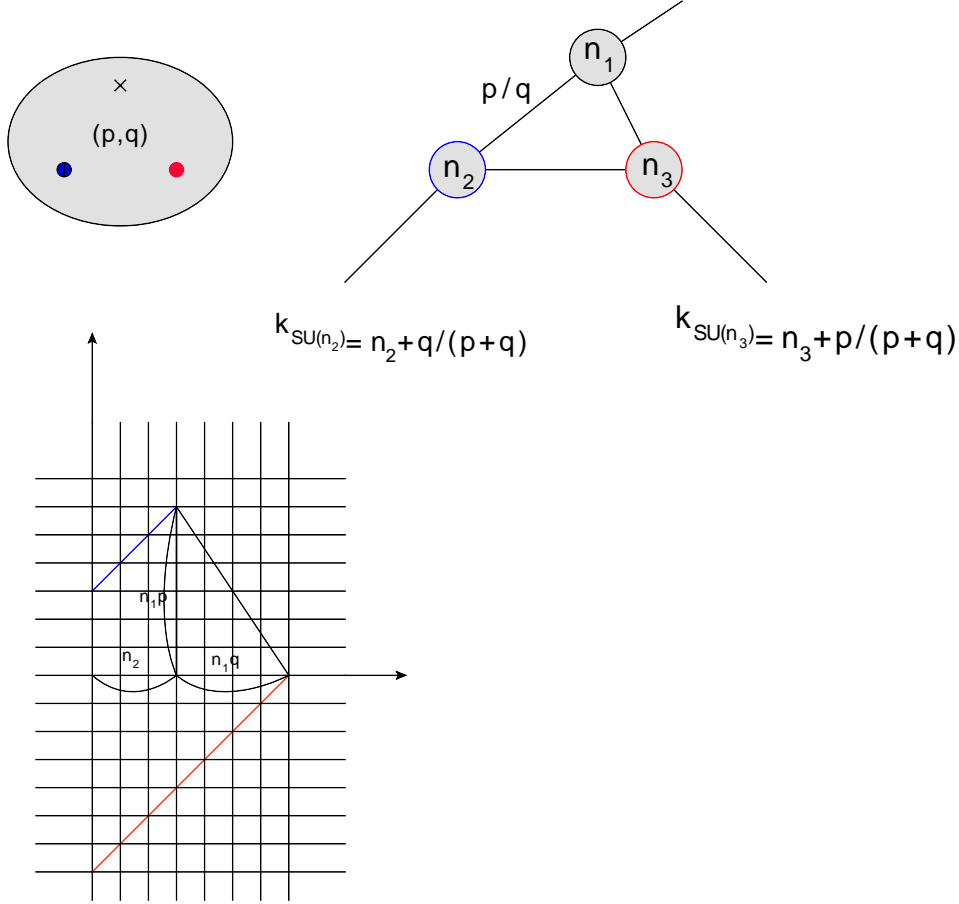


Figure 22. Top Left: The AD matter in class (p, q) theories is represented by a sphere with one black marked point, one red and one blue marked point. Top right: A pseudoquiver representation for the AD matter, and this is not a real quiver since the quiver arrow is fractional. Bottom: Newton polygon for the AD matter and one can read off the Coulomb branch spectrum from the lattice points under the boundary of Newton polygon.

Using the classification of AD matter in last subsection, we see that AD matter is represented by a three sphere with three marked points.

- if $p \neq 1$, AD matter is represented by a sphere with one black, one blue and one blue marked point. Notice that we only have $U(1)$ flavor symmetry for blue marked point.

- if $p = 1$, there are two kinds of AD matter: a): sphere with a black, red, and blue marked point; b): sphere with two black, and one blue marked point.

Our later study suggest that in general we need a theory whose blue marked point has general flavor symmetry, and the Coulomb branch spectrum might be found from a Newton polygon shown in figure. 22. It is not obvious where this theory sits in our M5 brane configurations, and it would be interesting to find some stringy constructions.

The duality is again interpreted as the degeneration limit of the punctured Riemann surface. However, not all the degeneration limit seems possible as there is only one type of AD matter which requires three different marked points (for $p \neq 1$ and $q \neq 1$).

Example Let's take $p = 3$ and $q = 2$, consider the SCFT which is defined by the following irregular singularity

$$\Phi = \frac{T_6}{z^{2+\frac{3}{2}}} + \dots + \frac{T_1}{z}. \quad (5.18)$$

Here T_6 is a $2r \times 2r$ diagonal matrix. This theory can also be engineered by the 3-fold singularity $x_1^2 + x_2^2 + x_3^{2r} + x_4^{3r} = 0$ (they are called (A_{2r-1}, A_{3r-1}) theory.). According to our general rule, it can be represented by a sphere with r black marked point of type $[1]$, and one red marked point of type $[0]$, and one red marked point of type $[2r]$. S duality is interpreted as different degeneration limit of this punctured sphere. Let's take $r = 2$ and $r = 3$ for illustration, see figure. 23. For example, if $r = 2$, the field theory looks like

$$T_1 - SU(2) - T_2. \quad (5.19)$$

Here T_1 and T_2 are shown in figure. 24. The interested reader can check that the Coulomb branch spectrum, central charges of the gauge theory description match with original theory.

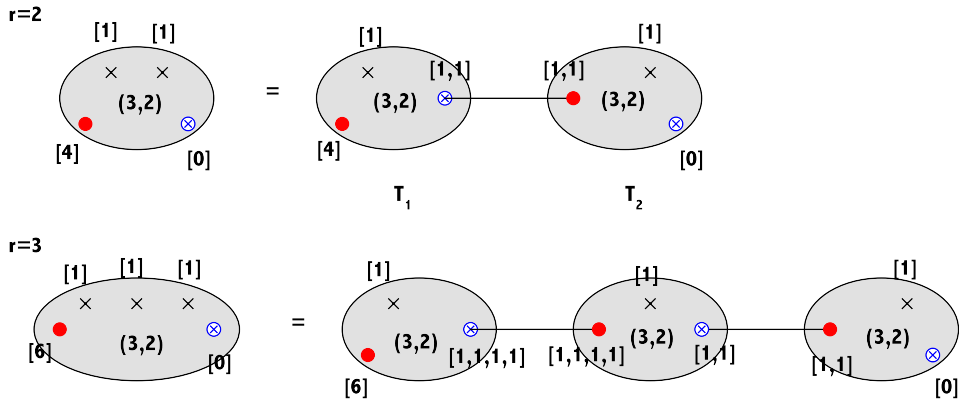


Figure 23. Duality frames for (A_{2r-1}, A_{3r-1}) theory

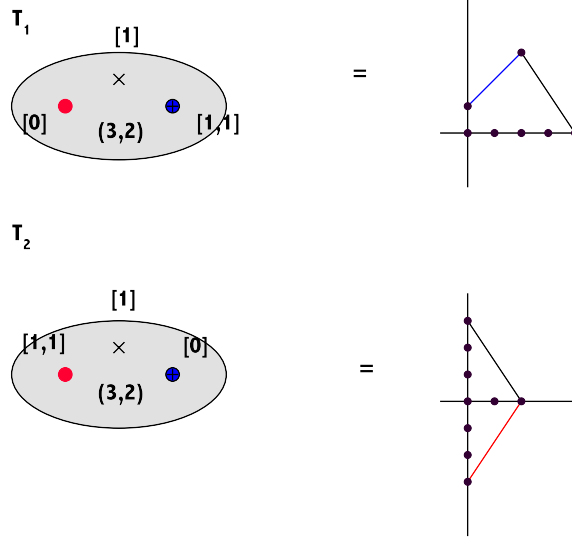


Figure 24. Two AD matter in the field theory limit of the theory engineered using 3-fold singularity $x_1^2 + x_2^2 + x_3^4 + x_4^6 = 0$. The Coulomb branch operators can be found from the monomials under the corresponding Newton polygon.

6 SCFT built from gauging AD matter

We have studied S duality of those SCFTs which admit constructions using 6d $A_{N-1}(2,0)$ theory. More generally, one could just gauge various AD matters to form new SCFTs, and some of them might not have string theory or M5 brane description. The theory space seems pretty large.

In previous sections, all the duality frames of our SCFT is a linear quiver in the sense that a gauge group is coupled to only two AD matters. In general, We could also build D type and E type quiver, namely there would be gauge group coupled with more than two AD matter systems. Some examples are studied in [8, 45]. It would be interesting to perform a full classification of such new theories.

For class \mathcal{S} theory (class $(0,1)$ theory), we can build theories represented by higher genus Riemann surface [4]. The basic field theory interpretation is following: one can gauge two $SU(N)$ flavor symmetries of a single T_N theory to form a SCFT with a $SU(N)$ gauge group which is conformal. One might wonder whether this higher genus version is possible for our general class (p,q) theories. However, it turns out that such higher genus generalization for general class (p,q) theories is not possible. The reason is following: our (p,q) AD matter has flavor symmetry $SU(n_1) \times SU(n_2) \times SU(n_3)$, here $SU(n_3)$ comes from regular singularity, and the flavor central charge is

$$k_{SU(n_2)} = n_2 + \frac{q}{p+q}, \quad k_{SU(n_3)} = n_3 + \frac{p}{p+q}; \quad q > 0 \quad (6.1)$$

Here $SU(n_3)$ flavor symmetry can be enhanced to $SU(n_3 + 1)$ if $n_1 q + n_2 = n_3 + 1$, and $SU(n_2)$ flavor symmetry can be enhanced to $SU(n_2 + 1)$ if $n_3 + n_1 \frac{p}{q} = n_2 + 1$. To get a conformal gauging, we can try following:

1. We take $n_2 = n_3 + 1$ and require the enhancement of $SU(n_3)$ to $SU(n_3 + 1)$, so the flavor symmetry of AD matter has a subgroup $SU(n_3 + 1) \times SU(n_3 + 1)$. We can gauge two $SU(n_3 + 1)$ flavor groups to get a conformal gauging, and we have following two equations:

$$n_1 q + n_2 = n_3 + 1, \quad n_2 = n_3 + 1, \quad q > 0, \quad (6.2)$$

but there is no solution, so we could not form a SCFT through gauging a single AD matter using this method.

2. Instead, we take $n_2 = n_3 - 1$ and require the enhancement of $SU(n_2)$ to $SU(n_2 + 1)$, so we have following two equations:

$$n_3 + n_1 \frac{p}{q} = n_2 + 1, \quad n_2 = n_3 - 1, \quad q \neq 0, \quad (6.3)$$

This is only possible if $p = 0$, and then $q = 1$ by our convention. We get class $(0, 1)$ theory which is just class \mathcal{S} theory.

7 Conclusion

We have found a (p, q) generalization of $\mathcal{N} = 2$ class \mathcal{S} theory, and its S duality behavior is found as follows: these theories can be represented by a sphere with marked points, and S duality is interpreted as different degeneration limits of the same punctured sphere. Unlike class \mathcal{S} theory, the punctured sphere in our case does not arise geometrically. It would be very interesting to find a stringy explanation of our result.

It is surprising that S duality of those more general AD theories can be interpreted in a similar way as class \mathcal{S} theory. We hope that our findings could help us understand better the still mysterious S duality of supersymmetric quantum field theory. More details such as the duality group deserve further study, see also [46, 47].

A natural generalization of our result would be to consider AD theories engineered from D and E type 6d $(2, 0)$ theories [16]. The basic ideas such as representing a theory by a punctured sphere should be similar, and it is interesting to work out the details.

Our initial goal is to understand S duality of theories engineered using Type IIB string theory on a 3-fold singularity [43, 45, 48, 49]. We only solved S duality for a very small subset of these theories: theories engineered using following singularities: $x_1^2 + x_2^2 + x_3^N + z^k = 0$ and $x_1^2 + x_2^2 + x_3^N + x_3 z^k = 0$. It seems that new methods are needed to understand S duality of these general class of theories.

It is interesting to explore the implication of our S duality result on the computation of physical observables such as S^4 partition function, superconformal index, etc.

Acknowledgements

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