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Yukawa Couplings on Quintic Threefolds

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Abstract

We compute the particle spectrum and some of the Yukawa couplings for a family of heterotic compactifications on quintic threefolds X involving bundles that are deformations of $TX \oplus \mathcal{O}_X$. These are then related to the compactifications with torsion found recently by Li and Yau. We compute the spectrum and the Yukawa couplings for generic bundles on generic quintics, as well as for certain stable non-generic bundles on the special Dwork quintics. In all our computations we keep the dependence on the vector bundle moduli explicit. We also show that on any smooth quintic there exists a deformation of the bundle $TX \oplus \mathcal{O}_X$ whose Kodaira-Spencer class obeys the Li-Yau non-degeneracy conditions and admits a non-vanishing triple pairing.

1 Introduction

In their proposed superstring compactification [1], Candelas, Horowitz, Strominger and Witten took the product of a maximal symmetric four dimensional space-time with a six dimensional Calabi-Yau manifold X as the ten dimensional space-time. In addition, they identified the Yang-Mills connection with the $SU(3)$ connection of the Calabi-Yau metric and set the dilaton to be a constant. In conformal field theory language these compactifications are referred to as $(2,2)$ models since they admit a $(2,2)$ world sheet supersymmetry. Using explicit formulas for the four dimensional super and Kaehler potentials [23, 3], models with three chiral families have been studied in some depth [8, 9, 12, 11]. In these models, the breaking of the E_6 gauge group to a GUT group or to the Standard Model gauge group was done at the field theoretic level.

A proposal of Witten [4] was to use bundles with $SU(4)$ or $SU(5)$ structure group in order for the GUT group to arise as the gauge group at the string level. Mathematically, this approach relies on the Donaldson-Uhlenbeck-Yau theorem [5, 6] about the existence of Hermite-Yang-Mills connections on stable bundles. In conformal field theory language these models are referred to as $(0,2)$ models. The most widely used technique to find such bundles was Monad construction [7, 12].

During the second string revolution, Horava-Witten [13] proposed a string compactification which relaxed the Green-Schwarz Anomaly cancellation condition by allowing $M5$ branes. Using the newly gained freedom and a recent mathematical technique called Spectral cover construction [14, 15], several GUT models with three families were found [16]. Using Spectral cover construction in conjunction with constructing stable bundles as non-trival extensions, a variety of heterotic Standard Model vacua was found [17, 18, 19]. In particular, these constructions involve building stable bundles on Calabi-Yau threefolds with non-trivial fundamental groups.

In [20], Strominger analyzed a more general heterotic superstring background by allowing non-zero torsion and a scalar "warp factor" $D(x)$ in the metric. More specifically, he considered a ten dimensional space-time that is the product $M \times X$ of a maximally symmetric four dimensional space-time M and an internal space X such that the metric g_{MN} on $M \times X$ takes the form

$$g_{MN}(m,x) = e^{2D(x)} \begin{pmatrix} g_{\mu\nu}(m) & 0 \\ 0 & g_{ij}(x) \end{pmatrix}, \quad m \in M, \qquad x \in X.$$

Strominger showed that in order to achieve space-time supersymmetry the internal six manifold X must be a complex manifold with a non-vanishing holomorphic three form Ω and the dilaton field ϕ must be identified with $D(x)$. In addition, the gauge connection on the heterotic vector bundle E over X has to be hermitian Yang-Mills with respect to the hermitian form $\omega = \sqrt{-1}g_{i\bar{j}}dz^i d\bar{z}^j$. In summary he proposed to solve the system

$$
\begin{aligned}
F \wedge \omega^2 &= 0 && (1.1)\\
F^{2,0} = F^{0,2} &= 0 && (1.2)\\
\partial\bar{\partial}\omega &= \sqrt{-1}trF \wedge F - \sqrt{-1}trR \wedge R && (1.3)\\
d^*\omega &= \sqrt{-1}(\partial - \bar{\partial}) \ln \|\Omega\|. && (1.4)
\end{aligned}
$$

Here R denotes the curvature tensor of the hermitian metric ω, F the curvature of the vector bundle E, tr is the trace of the endomorphism bundle of either E or TX, and the norm $||\cdot||$ and dualization d^* in the last equation are taken with respect to ω.

Given such solution, Strominger shows that the Kalb-Ramond field H, the dilaton ϕ and the physical metric g^0_{ij} are given by

$$
H = \frac{\sqrt{-1}}{2}(\bar{\partial} - \partial)\omega, \quad \phi = \frac{1}{8}\ln\|\Omega\| + \phi_0, \quad g^0_{ij} = e^{2\phi_0}\|\Omega\|^{\frac{1}{4}}g_{ij},
$$

where ϕ_0 is an arbitrary constant.

In a recent paper [21] Li and Yau have given the first irreducible non-singular solution of this system of equations for $SU(4)$ and $SU(5)$ principal bundles. In more concrete terms, they consider a smooth Calabi-Yau threefold (X, ω_0) and the vector bundle

$$
(E, D_0'') = \mathcal{O}_X^{\oplus r} \oplus TX, \tag{1.5}
$$

where D_0'' denotes the standard holomorphic structure on $\mathcal{O}_X^{\oplus r} \oplus TX$. In addition, they consider h_1 to be the constant metric on $\mathcal{O}_X^{\oplus r}$ and h_2 the metric on the Calabi-Yau threefold X induced by the Kaehler form ω_0, so that $\det(h_1 \oplus h_2)$ is the constant metric on $\wedge^{r+3}E = \mathcal{O}_X$. The pair $h_1 \oplus h_2$ and ω_0 is a reducible solution of Strominger's system with vanishing H. Li and Yau show that under certain algebraic conditions, small perturbations (E, D_s'') give irreducible solutions of Strominger's system.

More precisely, they consider a small perturbation (E, D_s'') and its first order approximation which are described by its Kodaira-Spencer class in

$$
H^1_{\bar{\partial}}(X, E^* \otimes E). \tag{1.6}
$$

This vector space consists of two by two matrices whose off diagonal terms correspond to a column vector $(\alpha_1, \ldots, \alpha_r)^t$ where $\alpha_i \in H^1(X, TX^*)$ and a row vector $(\beta_1, \ldots, \beta_r)$ with

$\beta_i \in H^1(X, TX)$. Under the assumption that for the chosen deformation (E, D''_s), $\{\alpha_i\}$ and $\{\beta_i\}$ are separately linearly independent, Li and Yau [21] show that there exists a family of pairs of hermitian metrics and hermitian forms (h_s, ω_s) solving Strominger's system for the holomorphic bundle (E, D''_s). In the limit as s goes to zero, this solution converges to a metric h_0 whose metric connection is a hermitian Yang-Mills connection of E over (X, ω_0).

In this paper we consider deformations $\mathcal{E} = (E, D''_s)$ of the rank four vector bundle

$$E = \mathcal{O}_X \oplus TX \tag{1.7}$$

on smooth quintic threefolds X. We compute the cohomology groups for $\mathcal{E}$ and $\wedge^2\mathcal{E}$ and the triple pairing

$$H^1(X, \mathcal{E}) \times H^1(X, \mathcal{E}) \times H^1(X, \wedge^2\mathcal{E}) \to \mathbb{C}. \tag{1.8}$$

We will keep the dependence on the bundle moduli in our computations explicit which gives, in particular, an explicit expression for the triple pairing 1.8[a].

In section 2 we give a short review of the family $\tilde{\mathcal{E}}$ of deformations of $\mathcal{O}_X \oplus TX$ whose generic member $\mathcal{E}$ obeys the condition on its Kodaira-Spencer class required by Li and Yau. Note that this condition is also sufficient to guarantee the stability of $\mathcal{E}$ with respect to the Kaehler class ω_0.

In section 3 we compute $H^*(X, \mathcal{E})$ and $H^*(X, \wedge^2\mathcal{E})$ for a generic $\mathcal{E}$ on a generic quintic X. We find as the only non-vanishing group $h^1(X, \mathcal{E}) = 100$.

In section 4 we begin our analysis for non-generic compactifications. We fix X to be the Dwork quintic threefold and consider a specific $\mathcal{E}$. We find non-vanishing $H^*(X, \mathcal{E})$ and $H^*(X, \wedge^2\mathcal{E})$. Since $\mathcal{E}$ is not generic, we include an algebraic proof of stability. In section 5 we compute the triple pairing 1.8 for the chosen bundle and give an explicit parameterization of its moduli dependence.

In section 6 we generalize these results. We show that for any smooth quintic there exists a deformation $\mathcal{E}$ whose Kodaira-Spencer class obeys the above stated conditions and admits non-vanishing triple pairing.

To place the results in physical context, let us recall the general strategy of finding vacua in heterotic string theory with spacetime supersymmetry. Instead of solving the full fledged string equations of motion, which include all massive modes, one finds a supersymmetric configuration in the field theory approximation. In particular, one solves for a bosonic configuration in which all fermions can be consistently set to zero. Hence the supersymmetry

[a] R.R thanks B. Ovrut for pointing out the importance of such parameterization.

variations for the gravitino ψ_M, the dilatino λ, and the gluino χ

$$\delta\psi_M = \nabla_M\epsilon + \frac{1}{48}e^{2\phi}\big(\gamma_M H - 12H_M\big)\epsilon \tag{1.9}$$

$$\delta\lambda = \nabla\phi\epsilon + \frac{1}{24}e^{2\phi}H\epsilon \tag{1.10}$$

$$\delta\chi = e^{\phi}F_{ij}\Gamma^{ij}\epsilon \tag{1.11}$$

have to vanish. Here ϵ denotes the Majorana-Weyl spinor which generates the supersymmetry transformation in the field theory limit. These equations have to be supplemented by the anomaly cancellation condition

$$dH = \mathrm{tr}\, R\wedge R - \mathrm{tr}\, F\wedge F. \tag{1.12}$$

Having found a solution to these equation, one has found, in particular, a supersymmetric solution of the string theory equations of motion to lowest order in the dimensionless parameter $\frac{\alpha'}{R^2}$ where R^2 denotes the radius of the compact space X. Using non-renomalization theorems for the effective four dimensional superpotential one can argue that such solution can be completed to a solution to any finite power of $\frac{\alpha'}{R^2}$ in a perturbative expansion. Non-perturbative corrections have to be considered separately. One important feature of the full solution is that it will not modify existing zero modes of the four dimensional theory, that is, it does not modify the four dimensional particle spectrum.

The simplest way to find a solution of the equations above is to set $H = d\phi = 0$. An exact solution to the equations is given by the choice of a Calabi-Yau threefold X with a Kaehler metric. The gauge connection is determined by the spin-connection of X. These compatifications are the previously mentioned $(2,2)$ models, and their massless particle spectrum, which is charged under the low energy gauge group, is determined by the cohomology of X.

A generalization of these solutions is to consider the equations as lowest order approximations in $\frac{\alpha'}{R^2}$ and solve them order by order. In particular, one can start again by setting $H = d\phi = 0$, and solve the supersymmetry variations by choosing a Calabi-Yau threefold X with Kaehler metric, but choose a general gauge connection on a vector bundle E which solves the hermitian Yang-Mills equations. The anomaly cancellation condition implies that H vanishes to order $\frac{1}{R}$, but will be generically non-zero to order $\frac{1}{R^3}$. Witten argued that this correction is of string theoretic nature and that these solutions can also be consistently computed to any finite order of $\frac{\alpha'}{R^2}$. These compactifications are referred to as $(0,2)$ models and their charged massless particle spectrum is determined by the cohomology of E and $\wedge^2 E$. Note that these models have already non-vanishing torsion H.

The 1-parameter family (h_s, ω_s) of solutions of Strominger's equations found by Li and Yau determines, in particular, a 1-parameter family (E, D''_s) of complex structures on the underlying vector bundle. It is plausible, but not obvious to us, that their 1-parameter family (h_s, ω_s) can be obtained via Witten's procedure, i.e. that these are the α' corrections of the $(0,2)$ models (E, D''_s) determined by h_s, for a suitable choice of $\alpha' = \alpha'(s)$. If this is indeed so, it would follow from the non-renormalization theorems that the particle spectrum of the Li-Yau solutions with torsion is identical with that of the $(0,2)$ models which we analyze in this paper. Some progress towards determining the corrected particle content of models with torsion has been made recently in [22].

2 Irreducible $SU(4)$ bundles

In this section we review the explicit construction [21] of the family $\tilde{\mathcal{E}}$ of deformations of

$$\mathcal{O}_X \oplus TX, \tag{2.1}$$

such that for a generic member $\mathcal{E}$ the off-diagonal terms in the Kodaira-Spencer class do not vanish.

To begin with, consider the combination of the normal bundle sequence with the Euler sequence.

$$\begin{tikzcd} & \mathcal{O}_X \arrow[d] & \\ & \mathcal{O}_X(1)^{\oplus 5} \arrow[d] & \\ TX \arrow[r] & T\mathbb{P}_4|_X \arrow[r, "\phi"] & \mathcal{O}_X(5). \end{tikzcd} \tag{2.2}$$

This defines a non-trivial canonical extension F

$$\begin{tikzcd} \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[d] & \\ F \arrow[r] \arrow[d] & \mathcal{O}_X(1)^{\oplus 5} \arrow[r] \arrow[d] & \mathcal{O}_X(5) \arrow[d] \\ TX \arrow[r] & T\mathbb{P}_4|_X \arrow[r] & \mathcal{O}_X(5), \end{tikzcd} \tag{2.3}$$

which corresponds (up to rescaling) to the unique element β in $H^1(X, TX^*)$. Using β, it is straighforward to construct a family $\tilde{\mathcal{F}}$ describing a deformation of the $\mathcal{O}_X \oplus TX$ such that its Kodaira-Spencer class has the form $\left(\begin{array}{c|c} 0 & 0 \\ \hline \beta & 0 \end{array}\right)$. Consider the two projections

$$\pi_1 : X \times A^1 \to X, \quad \pi_2 : X \times A^1 \to A^1, \tag{2.4}$$

and let t be the standard coordinate function on A^1. The class

$$t \cdot \beta \in Ext^1_{X\times A^1}(\pi_1^* TX, \mathcal{O}_{X\times A^1}) = H^0(A^1, \mathcal{O}_{A^1}) \otimes Ext^1_X(TX, \mathcal{O}_X) \tag{2.5}$$

defines a locally free extension $\tilde{\mathcal{F}}$ on $X \times A^1$. Its restriction $\mathcal{F}_t$ to $X \times t$ is isomorphic to F for non-vanishing t and isomorphic to $\mathcal{O}_X \oplus TX$ for $t = 0$.

We will now construct a deformation of $\mathcal{O}_X \oplus TX$ such that its Kodaira-Spencer class is of the form $\left(\begin{array}{c|c} 0 & \alpha \\ \hline 0 & 0 \end{array}\right)$ with $\alpha \neq 0$. Pick a section $u \in H^0(X, \mathcal{O}_X(5))$ and define a vector bundle $\tilde{\mathcal{F}}'$ on $X \times A^1$ as the kernel of the map $\Phi = (tu, \pi_1^*\phi)$, that is

$$\tilde{\mathcal{F}}' \longrightarrow \mathcal{O}_{X\times A^1} \oplus \pi_1^* T\mathbb{P}_4|_X \xrightarrow{\Phi} \pi_1^*\mathcal{O}_X(5). \tag{2.6}$$

Note that this sequence fits naturally into the diagram

$$\begin{tikzcd}
\pi_1^* TX \arrow[r] \arrow[d] & \pi_1^* TX \arrow[d] & \\
\tilde{\mathcal{F}}' \arrow[r] \arrow[d] & \mathcal{O}_{X\times A^1} \oplus \pi_1^* T\mathbb{P}_4|_X \arrow[r, "\Phi"] \arrow[d, "\mathbb{I}\oplus\pi_1^*\phi"] & \pi_1^*\mathcal{O}_X(5) \arrow[d] \\
\mathcal{O}_{X\times A^1} \arrow[r] & \mathcal{O}_{X\times A^1} \oplus \pi_1^*\mathcal{O}_X(5) \arrow[r] & \pi_1^*\mathcal{O}_X(5)
\end{tikzcd} \tag{2.7}$$

Denote the restriction of $\tilde{\mathcal{F}}'$ to $X \times t$ by $\mathcal{F}_t'$. Clearly, $\mathcal{F}_0' = \mathcal{O}_X \oplus TX$ and the Kodaira-Spencer class is for a generic t and u is $\left(\begin{array}{c|c} 0 & \alpha \\ \hline 0 & 0 \end{array}\right)$ [21].

Finally, we will construct a family of holomorphic bundles which contains $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$ as subfamilies, that is it has to contain

$$\mathcal{F}_t \to \mathcal{O}_X(1)^{\oplus 5} \to \mathcal{O}_X(5), \quad \mathcal{F}_t' \to \mathcal{O}_X \oplus T\mathbb{P}_4|_X, \to \mathcal{O}_X(5) \tag{2.8}$$

with

$$\mathcal{F}_0 = \mathcal{F}_0' = \mathcal{O}_X \oplus TX. \tag{2.9}$$

The required family will be given by a universal bundle over the total space of a vector bundle. The base A^1 of the vector bundle will parameterize the extension

$$\eta : \mathcal{O}_X \to \mathcal{O}_X(1)^{\oplus 5} \to T\mathbb{P}_4|_X, \tag{2.10}$$

and the fiber will correspond to the vector space $Hom(\mathcal{O}_X(1)^{\oplus 5}, \mathcal{O}_X(5))$. It follows from the discussion above that η allows the construction of an extension $\mathcal{W}$ as

$$\mathcal{O}_{X\times A^1} \to \mathcal{W} \to \pi_1^* T\mathbb{P}_4|_X \tag{2.11}$$

on $X \times A^1$ such that

$$\mathcal{W}|_{X\times 0} = \mathcal{O}_X \oplus T\mathbb{P}_4|_X, \quad \mathcal{W}|_{X\times t} = (\mathcal{O}_X(1))^{\oplus 5},\ t \neq 0. \tag{2.12}$$

One can show that $\pi_{2*}(\mathcal{W}^* \otimes \pi_1^*\mathcal{O}_X(5))$ is a vector bundle on A^1 of rank 350. The total space W of this vector bundle will be the parameter space of our family. To find the universal bundle, consider more generally, any vector bundle $\mathcal{M}$ over $\pi : X \times A \to A$, and denote the total space of $\pi_*\mathcal{M}$ by M. The fiber of $\pi_*\mathcal{M}$ at $a \in A$ is $H^0(X, \mathcal{M}|_{X\times a})$. Therefore we find that $p^*\mathcal{M}$ for

$$\begin{tikzcd} p^*\mathcal{M} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\ X \times M \arrow[r, "p"] & X \times A \end{tikzcd} \tag{2.13}$$

has a canonical global section which maps (x, w) to $w(x)$. Returning to our original family, we find that $p^*(\mathcal{W}^* \otimes \pi_1^*\mathcal{O}_X(5))$ has a canonical global section, hence, we find over $X \times W$ a canonical homomorphism with kernel $\tilde{\mathcal{E}}$

$$\tilde{\mathcal{E}} \to p^*\mathcal{W} \to p^*\pi_1^*\mathcal{O}_X(5). \tag{2.14}$$

It is now straightforward to see that both families $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$ are contained in this family, hence its generic member will correspond to a Kodaira-Spencer class $\left(\begin{array}{c|c} 0 & \alpha \\ \hline \beta & * \end{array}\right)$. The generic restriction of $\tilde{\mathcal{E}}$ to $X \times w$ by fits into the exact sequence

$$\mathcal{E} \to \mathcal{O}_X(1)^{\oplus 5} \to \mathcal{O}_X(5). \tag{2.15}$$

We will use this definition of $\mathcal{E}$ to compute its cohomology.

3 Cohomology for generic $\mathcal{E}$

In this section we study the cohomology of a generic element $\mathcal{E}$ in the previously described family of $SU(4)$ bundles on a generic quintic X. It follows from our discussion in the introduction that such an $\mathcal{E}$ will be stable. We give an explicit parameterization of its cohomology groups in terms of the bundle moduli. Various algebraic-geometric techniques are introduced as needed.

3.1 $H^*(X, \mathcal{E})$

Recall from section 2 that the $SU(4)$ bundle is deformation of $TX \oplus \mathcal{O}_X$, given by the kernel of the map $\omega : \mathcal{O}_X^{\oplus 5}(1) \to \mathcal{O}_X(5)$. Note that w is given by five global sections $s_i,\ i = 1, \ldots, 5$ in $H^0(X, \mathcal{O}_X(4))$. Using the definitions

$$P = \mathcal{O}_X^{\oplus 5}(1), \quad V = \mathcal{O}_X(5) \tag{3.1}$$

we find $\mathcal{E}$ fits in the exact sequence

$$\mathcal{E} \longrightarrow P \xrightarrow{w} V. \tag{3.2}$$

Using upper-semi continuity the dimension of the various cohomology groups of a generic $\mathcal{E}$ must be smaller or equal than the dimensions of the cohomology groups of $TX \oplus \mathcal{O}_X$. For convenience we recall

$$h^0(X, TX) = 0,\ h^1(X, TX) = 101,\ h^2(X, TX) = 1,\ h^3(X, TX) = 0. \tag{3.3}$$

and

$$h^0(X, \mathcal{O}_X) = 1,\ h^1(X, \mathcal{O}_X) = 0,\ h^2(X, \mathcal{O}_X) = 0,\ h^3(X, \mathcal{O}_X) = 1. \tag{3.4}$$

Observe that

$$H^i(X, \mathcal{O}_X(n)) = 0, \quad i > 0, n \geq 0 \quad \text{or} \quad i < 3, n \leq 0. \tag{3.5}$$

This is clear for $n = 0$, and for non-zero n, it follows simply from Kodaira vanishing theorem and Serre duality. In particular, we find

$$H^i(X, P) = 0, \quad H^i(X, V) = 0, \quad i > 0. \tag{3.6}$$

Hence the long exact sequence in cohomology related to (3.2) reduces to

$$\begin{tikzcd} 0 \arrow[r] & H^0(X,\mathcal{E}) \arrow[r] & H^0(X,P) \arrow[r] & H^0(X,V) \arrow[dll, rounded corners] \\ & H^1(X,\mathcal{E}) \arrow[r] & 0 \end{tikzcd} \tag{3.7}$$

The only non-vanishing cohomology group of $\mathcal{E}$ is $H^1(X, \mathcal{E})$, given as a quotient of

$$H^0(X, P) \xrightarrow{w} H^0(X, V) \longrightarrow H^1(X, \mathcal{E}) \longrightarrow 0, \tag{3.8}$$

where the map w was given in the defining equation above and represents the explicit dependence on the vector bundle moduli. To parameterize $H^0(X, P)$ and $H^0(X, V)$ we will use the exact sequence

$$\mathcal{O}_{\mathbb{P}^4}(p-5) \xrightarrow{X} \mathcal{O}_{\mathbb{P}^4}(p) \longrightarrow \mathcal{O}_X(p). \tag{3.9}$$

on $\mathbb{P}^4$ given by the multiplication with the defining equation X of the quintic. Using its induced long exact sequence on cohomology we find

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}) \xrightarrow{X} H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5)) \longrightarrow H^0(X, V) \longrightarrow 0. \tag{3.10}$$

and

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 5}) = H^0(X, P). \tag{3.11}$$

Also, it follows from this argument that the map w is a restriction from a map $\omega_{\mathbb{P}^4}$ on $\mathbb{P}^4$. We are left to describe $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(m))$. More generally, consider the polynomial ring

$$\mathbb{C}[x_0, \ldots, x_l], \tag{3.12}$$

generated by the coordinates of $\mathbb{P}^l$. The global sections of $\mathcal{O}_{\mathbb{P}^l}(m)$ are simply the homogeneous elements of this ring of degree m. In particular,

$$h^0(\mathbb{P}^l, \mathcal{O}_{\mathbb{P}^l}(m)) = \begin{pmatrix} l+m \\ m \end{pmatrix}. \tag{3.13}$$

Therefore we find

$$h^0(X, P) = 25, \quad h^0(X, V) = 125, \tag{3.14}$$

hence,

$$h^1(X, \mathcal{E}) = 100. \tag{3.15}$$

Its explicit parameterization is given as a quotient of the vector space of all homogenous polynomials in the coordinates of $\mathbb{P}^4$ of degree five (modulo the unique relation imposed by the quintic) modulo five sets of all homogenous linear polynomials in the coordinates of $\mathbb{P}^4$ with the map explicitly given by w. Comparing the dimensions of the cohomology groups of $\mathcal{E}$ with 3.3 and 3.4 gives an answer consistent with semicontinuity and preservation of Euler characteristic.

3.2 $H^*(X, \wedge^2\mathcal{E})$

To compute the cohomology of higher antisymmetric powers of $\mathcal{E}$ we observe that the map (3.2) induces the exact sequence

$$\wedge^2\mathcal{E} \longrightarrow \wedge^2 P \longrightarrow \mathcal{E} \otimes V. \tag{3.16}$$

In order to study the cohomology of $\wedge^2\mathcal{E}$, let us first consider the cohomology of $\wedge^2 P$. Since $\wedge^2 P \approx \mathcal{O}_X(2)^{\oplus 10}$ it follows that

$$h^0(X, \wedge^2 P) = 150, \quad h^i(X, \wedge^2 P) = 0, \quad i > 0. \tag{3.17}$$

Hence the long exact sequence with respect to (3.16) implies

$$\begin{tikzcd} 0 \arrow[r] & H^0(X,\wedge^2\mathcal{E}) \arrow[r] & H^0(X,\wedge^2 P) \arrow[r] & H^0(X,\mathcal{E}\otimes V) \arrow[dll, rounded corners] \\ & H^1(X,\wedge^2\mathcal{E}) \arrow[r] & 0 \arrow[r] & H^1(X,\mathcal{E}\otimes V) \arrow[dll, rounded corners] \\ & H^2(X,\wedge^2\mathcal{E}) \arrow[r] & 0, & \end{tikzcd} \tag{3.18}$$

Since $\wedge^2\mathcal{E}$ is stable, $H^0(X,\wedge^2\mathcal{E})$ vanishes and we are left to study the cohomology of $\mathcal{E}\otimes V$. We consider

$$\mathcal{E}\otimes V \longrightarrow P\otimes V \xrightarrow{w} V\otimes V. \tag{3.19}$$

and its corresponding long exact sequence in cohomology

$$\begin{tikzcd} 0 \arrow[r] & H^0(X,\mathcal{E}\otimes V) \arrow[r] & H^0(X,P\otimes V) \arrow[r, "w"] & H^0(X,V\otimes V) \arrow[dll, rounded corners] \\ & H^1(X,\mathcal{E}\otimes V) \arrow[r] & 0. & \end{tikzcd} \tag{3.20}$$

The map $w: H^0(X,P\otimes V)\to H^0(X,V\otimes V)$ maps a 1025 dimensional vector space to a 875 dimensional vector space. We will show in the next subsection that for a generic quintic X and a generic w, this map is surjective. This implies, that all cohomology of $\wedge^2\mathcal{E}$ vanishes.

$$H^i(X,\wedge^2\mathcal{E}) = 0,\ \forall i. \tag{3.21}$$

3.2.1 $w: H^0(X,P\otimes V)\to H^0(X,V\otimes V)$

In this subsection we will study the map $w: H^0(X,P\otimes V)\to H^0(X,V\otimes V)$. We will see that, for generic X and generic w, it is surjective. By the openness of this condition, we have to find one specific X with a specific $\mathcal{E}$ such that w is surjective. To begin with, note that

$$P\otimes V = \mathcal{O}_X(6)^{\oplus 5}, \quad V\otimes V = \mathcal{O}_X(10) \tag{3.22}$$

We will study w via pull-back to $\mathbb{P}^4$. Using (3.9) we find the resolutions of $H^0(X,P\otimes V)$ and $H^0(X,V\otimes V)$ in the commuting diagram

$$\begin{tikzcd} H^0(X,P\otimes V) \arrow[r, "w"] & H^0(X,V\otimes V) \\ H^0(\mathbb{P}^4,\mathcal{O}_{\mathbb{P}^4}(6)^{\oplus 5}) \arrow[u] \arrow[r, "w_{\mathbb{P}^4}"] & H^0(\mathbb{P}^4,\mathcal{O}_{\mathbb{P}^4}(10)) \arrow[u] \\ H^0(\mathbb{P}^4,\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 5}) \arrow[u, "X"] \arrow[r] & H^0(\mathbb{P}^4,\mathcal{O}_{\mathbb{P}^4}(5)), \arrow[u, "X"'] \end{tikzcd} \tag{3.23}$$

where all vertical lines are short exact. To study $w_{\mathbb{P}^4}$ consider, more generally, the homogeneous coordinate ring

$$S := \mathbb{C}[x_0, \ldots, x_l] \tag{3.24}$$

of $\mathbb{P}^l$. Consider a homogenous ideal $J \subset S$ generated by a regular sequence $f_1, \ldots, f_n$ of n homogeneous polynomials [29]. For example, the derivatives of a smooth hypersurface form a regular sequence and its corresponding ideal is called Jacobian ideal of the hypersurface. If the ideal J is generated by a regular sequence, we have the Koszul complex for $R = S/I$ [29], that is

$$S \otimes \wedge^n E \to \ldots \to S \otimes \wedge^2 E \to S \otimes E \to S \to R, \tag{3.25}$$

where all maps are exact. E is simply an n dimensional vector space with basis $\{e_i\}$ and the map from $S \otimes E \to S$ is given by $e_i \to f_i$. Note that the Koszul resolution preserves the grading of S.

Returning to the case of the quintic in $\mathbb{P}^4$, let as assume that the five global sections s_i defining w are derivatives of the Fermat quintic. We find

$$\begin{tikzcd}
& H^0(X, P \otimes V) \arrow[r, "w"] & H^0(X, V \otimes V) \\
H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) \otimes \wedge^2 E \arrow[r] & H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(6)) \otimes E \arrow[u] \arrow[r, "w_{\mathbb{P}^4}"] & H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(10)) \arrow[u, "r"] \\
& H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)) \otimes E \arrow[u] \arrow[r] & H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5)). \arrow[u, "X"]
\end{tikzcd} \tag{3.26}$$

In terms of dimensions, this diagram reads as

$$\begin{tikzcd}
& H^0(X, P \otimes V) \arrow[r, "w"] & H^0(X, P \otimes V) \\
150 \arrow[r] & 1050 \arrow[u] \arrow[r, "w_{\mathbb{P}4}"] & 1001 \arrow[u, "r"] \\
& 25 \arrow[u] \arrow[r] & 126 \arrow[u, "X"]
\end{tikzcd} \tag{3.27}$$

In order to prove surjectivity of w we are left to find an X such that the map

$$w_{\mathbb{P}^4} \oplus X : H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(6)) \otimes E \oplus H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5)) \to H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(10)) \tag{3.28}$$

is surjective. The ideal spanned by $w_{\mathbb{P}^4}$ in $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(10))$ is of the form $\sum_i (x_i)^4$. Hence we are missing all monomials in $\{x_i\}_i$ of degree ten whose highest coefficient is smaller than four. This is a 101 dimensional vector space and a convenient parameterization (up to permutation) is given by

$$M = \sum_{i=1}^{5} M_i = (33310) \oplus (33220) \oplus (33211) \oplus (32221) \oplus (22222) \tag{3.29}$$

where $i_0 ... i_5$ stands for $\prod_{j=0}^{5} x_j^{i_j}$ and $(i_0 ... i_5)$ stands for all permutations. Similarly we can describe all monomials in $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5))$ which are not in the the ideal generated by w. They are simply all monomials in $\{x_i\}_i$ of degree five whose highest coefficient is smaller than four. Using the same convention as above, a convenient parameterization is given by

$$N = \sum_{i=1}^{5} N_i = (32000) \oplus (31100) \oplus (22100) \oplus (21110) \oplus (11111) \tag{3.30}$$

Both N and M are 101 dimensional vector spaces and we have to find a $X : N \to M$ which is an isomorphism. Lets chose $X = \prod x_i$. The induced map splits in the direct sum

$$X = \oplus_i X_i : \sum_i N_i \to \sum_i M_i \tag{3.31}$$

where X_1 and X_2 are the zero map and $X_i,\ i = 3, 4, 5$ are isomorphisms. Being invertible on $N_i,\ i = 3, 4, 5$ is an open condition, hence for a sufficiently small deformation of X we keep this property. Explicitly we deform X by

$$X = \prod x_i + \sum_{i \leqslant j\ k} \epsilon_{ij,k} v_{ij,k} \tag{3.32}$$

where $v_{ij,k}$ stands for the monomial $x_i x_j x_k^3$ and $\epsilon_{ij,k}$ are sufficiently small parameters. The induced map is again a direct sum $X = \oplus_i X_i$. We will show that is induced map on N_1 and N_2 is also an isomorphism. It is not to difficult to show that the 30×30 matrix X_2 splits into 10 blocks of 3×3 matrizes. For example for the three monomials $\{31100, 01103, 01130\} \in N_2$ we find as the only non-vanishing components of X_2

$$\left(\begin{array}{c|c|c|c} & 02203 & 32230 & 32203 \\ \hline 31100 & 0 & \epsilon_{23,4} & \epsilon_{23,5} \\ \hline 01103 & \epsilon_{23,4} & 0 & 0 \\ \hline 01130 & \epsilon_{23,5} & \epsilon_{23,1} & 0 \end{array} \right). \tag{3.33}$$

Note that the determinant of this 3×3 matrix is $\epsilon_{23,4}\epsilon_{23,5}\epsilon_{23,1}$. It follows from this discussion that X_2 will be an isomorphism iff $\epsilon_{ij,k} \neq 0,\ \forall i, j, k$. Unfortunatly, the 20×20 matrix X_1 does not seem to have such a clear structure. Nevertheless, a quick check in Maple ensures that X_1 is an isomorphism as well.

4 A non-generic E on the Dwork quintic

In the previous section we have shown that $H^*(X, \wedge^2\mathcal{E})$ vanishes for generic $\mathcal{E}$ on generic quintics X. In this section we show that this is only a generic result. We give a concrete example of $\mathcal{E}$ on the Dwork quintic threefold, such that

$$h^1(X, \wedge^2\mathcal{E}) = h^2(X, \wedge^2\mathcal{E}) = 50. \tag{4.1}$$

Since this $\mathcal{E}$ is not generic, we will include a proof of its stability. We will see later that the Yukawa coupling for this vector bundle also does not vanish.

Our bundle is given by

$$\mathcal{E} \longrightarrow P \xrightarrow{w} V, \tag{4.2}$$

where w is given by the Jacobian ideal of the Fermat quintic. We will consider this bundle on the Dwork quintic, which is given by the polynomial

$$\sum x_i^5 + \prod x_i = 0. \tag{4.3}$$

To begin with let us prove the stability of $\mathcal{E}$. It follows from the appendix that it is sufficient to show that the maps

$$H^0(X, P) \to H^0(X, V) \tag{4.4}$$

and

$$H^0(X, \wedge^2 P) \to H^0(X, P \otimes V) \tag{4.5}$$

are injective. To check these conditions we will use techniques developed in section 3.2.1. In particular, to show (4.4), we use the resolution

$$\begin{tikzcd}
H^0(X,P) \arrow[r, "w"] & H^0(X,V) \\
H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)) \otimes E \arrow[u] \arrow[r, "w_{\mathbb{P}^4}"] & H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5)) \arrow[u, "r"'] \\
& H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}). \arrow[u, "X"']
\end{tikzcd} \tag{4.6}$$

where $w_{\mathbb{P}^4}$ is given by the five partial derivatives of the Fermat quintic and X by the Dwork (4.3). To prove injectivity of w we have to show that X is not in the ideal generated by $w_{\mathbb{P}^4}$. But what is in the Jacobian ideal of the Fermat quintic? Certainly the Fermat quintic itself. Borrowing a result from [25] shows that a quintic will be in this ideal iff it is isomorphic to the Fermat quintic. Since the Dwork quintic is not isomorphic to the Fermat, the image of

$w_{\mathbb{P}^4}$ does not intersect the image of X, hence w is injective. To show that condition (4.5) is obeyed we will show that the map

$$H^0(X, P \otimes P) \to H^0(X, P \otimes V) \tag{4.7}$$

is injective. To guarantee this we simply need to show the injectivity of the map

$$H^0(X, P(1)) \to H^0(X, V(1)). \tag{4.8}$$

Again, pulling it back to $\mathbb{P}_4$ we find

$$\begin{tikzcd}
H^0(X, P(1)) \arrow[r, "w"] & H^0(X, V(1)) \\
H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) \otimes E \arrow[u] \arrow[r, "w_{\mathbb{P}^4}"] & H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(6)) \arrow[u, "r"] \\
 & H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)). \arrow[u, "X"]
\end{tikzcd} \tag{4.9}$$

where $w_{\mathbb{P}^4}$ and X are as above. Recall from section 3.2.1 that the image of $w_{\mathbb{P}^4}$ misses all monomials of degree six whose highest power is smaller then 4. The image of X in that quotient is given by degree six polynomials whose highest power is two. Hence these images don't intersect and w is injective.

What about the cohomology of $\mathcal{E}$? It is easy to see that all arguments of section 3.1 apply. We recall the results for convenience. All cohomology of $\mathcal{E}$ vanishes except $H^1(X, \mathcal{E})$ which can be explicitly parameterized as the quotient of

$$H^0(X, P) \xrightarrow{w} H^0(X, V) \longrightarrow H^1(X, \mathcal{E}). \tag{4.10}$$

Let us consider the cohomology of $\wedge^2\mathcal{E}$. Tracing through (3.18) and (3.20) we find that

$$H^1(X, \wedge^2\mathcal{E}) = \frac{ker(H^0(X, P \otimes V) \to H^0(X, V \otimes V))}{H^0(X, \wedge^2 P)} \tag{4.11}$$

and

$$H^2(X, \wedge^2\mathcal{E}) = \frac{H^0(X, V \otimes V)}{H^0(X, P \otimes V)}. \tag{4.12}$$

To determine the dimensionality we consider the diagram (3.26) and the remarks which followed. We find

$$h^1(X, \wedge^2\mathcal{E}) = h^2(X, \wedge^2\mathcal{E}) = 50. \tag{4.13}$$

5 Yukawa couplings

In this section we make some comments about the Yukawa couplings corresponding to the triple product

$$H^1(X, \mathcal{E}) \otimes H^1(X, \mathcal{E}) \otimes H^1(X, \wedge^2 \mathcal{E}) \to H^3(X, \wedge^4 \mathcal{E}) \cong \mathbb{C}. \tag{5.1}$$

In particular we give their explicit dependence on the vector bundle moduli, that is, their dependence on

$$w \in H^0(X, \mathcal{O}_X(4)^{\oplus 5}). \tag{5.2}$$

First note that the triple product (5.1) can be rewritten as the pairing:

$$H^1(X, \mathcal{E}) \otimes H^1(X, \mathcal{E}) \to H^2(X, \wedge^2 \mathcal{E}). \tag{5.3}$$

By (3.8) and (4.12), this becomes a pairing:

$$\frac{H^0(X, V)}{H^0(X, P)} \otimes \frac{H^0(X, V)}{H^0(X, P)} \to \frac{H^0(X, V \otimes V)}{H^0(X, V \otimes P)}. \tag{5.4}$$

By tracing through a few commuting diagrams, we identify this pairing as the natural multiplication. It follows from (3.8) and (4.12) that the inclusions of the denominators into the numerators are given by the map w defining $\mathcal{E}$. In particular, the dependence of the Yukawa couplings on the moduli space of $\mathcal{E}$ is made manifest.

A well known special case occurs for the specific choice of w which makes $\mathcal{E}$ an extension of the tangent bundle by the trivial bundle. Recall from (2.3) that this $\mathcal{E}$ fits into the commuting diagram

$$\begin{tikzcd} \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[d] & \\ \mathcal{E} \arrow[r] \arrow[d] & \mathcal{O}_X(1)^{\oplus 5} \arrow[r] \arrow[d, "\nabla_i"] & \mathcal{O}_X(5) \arrow[d] \\ TX \arrow[r] & T\mathbb{P}_4|_X \arrow[r, "\nabla_i Q"] & \mathcal{O}_X(5). \end{tikzcd} \tag{5.5}$$

In particular, it follows from this that w is given by the partials of the quintic X. It follows from the long exact sequence in cohomology corresponding to the first vertical short exact sequence that

$$H^1(X, \mathcal{E}) = H^1(X, TX), \quad H^2(X, \wedge^2 \mathcal{E}) = H^2(X, \wedge^2 TX) \tag{5.6}$$

Hence for this particular $\mathcal{E}$ the pairing is given by

$$H^1(X, TX) \otimes H^1(X, TX) \to H^2(X, \wedge^2 TX). \tag{5.7}$$

But now we are in the situation studied by Carlson and Griffiths, [26, 27]. Their result states that the pairing pairTX is given by the polynomial multiplication of the rings

$$\frac{H^0(X,\mathcal{O}_X(5))}{H^0(X,\mathcal{O}_X(1)^{\oplus 5})}\otimes\frac{H^0(X,\mathcal{O}_X(5))}{H^0(X,\mathcal{O}_X(1)^{\oplus 5}}\to\frac{H^0(X,\mathcal{O}_X(10))}{H^0(X,\mathcal{O}_X(6)^{\oplus 5})}, \tag{5.8}$$

where the maps $H^0(X,\mathcal{O}_X(1)^{\oplus 5})\to H^0(X,\mathcal{O}_X(5))$ and $H^0(X,\mathcal{O}_X(6)^{\oplus 5})\to H^0(X,\mathcal{O}_X(10))$ are given by partials of X.

6 Non-vanishing Yukawa couplings for all quintics X

We now generalize the result of the previous section: we show the existence of a stable bundle $\mathcal{E}$ with non-vanishing $H^2(X,\mathcal{E})$ for every smooth quintic. In particular, this implies the existence of heterotic vacua with non-vanishing Yukawa coupling for every quintic.

To do so we use the analytic result by Li and Yau [21] stated in the introduction. Recall that the sufficient condition for the existence of a hermitian Yang-Mills connection and a solution for Strominger's equation on $\mathcal{E}$ was that the off diagonal terms in $Ext^1(TX\oplus\mathcal{O}_X,TX\oplus\mathcal{O}_X)$ do not vanish. A sufficient condition for the non-vanishing of the anti-diagonal is that neither $\mathcal{E}$ nor $\mathcal{E}^*$ admits global sections. The vanishing of $h^0(X,\mathcal{E}^*)=h^3(X,\mathcal{E})$ follows from its definition (3.2) and the vanishing theorem (3.6). The condition for the vanishing of global sections of $\mathcal{E}$ was analyzed in 4.4. In particular, in diagram (4.6) we choose the map X to be the defining equation of our quintic. For the map $w_{\mathbb{P}_4}$ we use the partials of a different quintic X' which is a small deformation of X, that is

$$X'=X+\epsilon \tag{6.1}$$

If we choose a smooth quintic X' which is not isomorphic to X, it follows from the arguments presented in section 4 that $\mathcal{E}$ has no global sections. Note that the set of all quintics X' is a 126 dimensional space, and teh condition is violated on a 25 dimensional subset. Note that it follow from this discussion that our example in section 4 can be made to fit in this framework.

To show the non-vanishing of $H^2(X,\wedge^2\mathcal{E})$ we recall its description in (4.12). To compute it we have to trace through diagram (3.26). We have to show that

$$im\ \left(w_{\mathbb{P}_4}:H^0(\mathbb{P}_4,\mathcal{O}_{\mathbb{P}_4}(5))\right)\to H^0(\mathbb{P}_4,\mathcal{O}_{\mathbb{P}_4}(10))\bigcup X\cdot H^0(\mathbb{P}_4,\mathcal{O}_{\mathbb{P}^4}(5)) \tag{6.2}$$

does not span the 1001 dimensional vector space $H^0(\mathbb{P}_4,\mathcal{O}_{\mathbb{P}^4}(10))$ for some X'. Note that the image of $w_{\mathbb{P}^4}$ is 900 dimensional and the image of the fixed space $X\cdot H^0(\mathbb{P}_4,\mathcal{O}_{\mathbb{P}^4}(5))$ is 126 dimensional. Therefore we have to study the sublocus in the Grassmannian $G(900, 1001)$ of subspaces which do not intersect the 126 dimensional vector space transversally. We find its codimension to be 26. Among all quintics X', which depend on 126 parameters, each component of those not transversal to the 126 dimensional space is at least 100 dimensional. (Since X is contained in this locus, it is not empty). Therefore we can find X' such that $H^0(X, \mathcal{E}) = 0$ and $H^1(X, \wedge^2\mathcal{E}) \neq 0$.

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Appendix

In this section we will consider the stability of $\mathcal{E}$ on the quintic X. To begin with, we will show that the conditions

$$0 = H^0(X, \mathcal{E}) = H^0(X, \wedge^2\mathcal{E}) = H^0(X, \wedge^3\mathcal{E}) \tag{6.3}$$

are sufficient to guarantee stability of $\mathcal{E}$. More generally, to define Mumford-Takemoto stability on a torsion free coherent sheaf W on a complex Kaehler manifold of dimension n one introduces its slope with respect to a Kaehler class H

$$\mu_H(W) = \frac{c_1(W) \cdot H^{n-1}}{\mathrm{rk}(W)}. \tag{6.4}$$

A sheaf is stable if all torsion free coherent sub-sheaves $\mathcal{F}$ whose rank is smaller the the rank of W obey

$$\mu_H(\mathcal{F}) < \mu_H(W). \tag{6.5}$$

In general the condition of stability will depend on the chosen Kaehler class H. However, if we restrict X to be a smooth quintic in $\mathbb{P}_4$, then the Lefschetz Hyperplane theorem ensures that $Pic(X) = \mathbb{Z}$ and $h^{1,1}(X) = \mathbb{C}$. Therefore, (up to re-scaling), H is uniquely determined and, henceforth, we will suppress it.

Assume there exists a torsion free destabilizing sub-sheaf $\mathcal{F}$ of rank r of $\mathcal{E}$, that is

$$\mathcal{F} \subset \mathcal{E} \tag{6.6}$$

with $\mu(\mathcal{F}) \geq 0$. Therefore there is the non-zero map

$$\wedge^r \mathcal{F} \to \wedge^r \mathcal{E} \tag{6.7}$$

and a non-zero map

$$(\wedge^r \mathcal{F})^{**} \to \wedge^r \mathcal{E}^{**} = \wedge^r \mathcal{E} \tag{6.8}$$

Note that for torsion free sheaves the first Chern class of $\mathcal{F}$ can be defined [24] by

$$c_1(\mathcal{F}) = c_1((\wedge^r \mathcal{F})^{**}). \tag{6.9}$$

Since $(\wedge^r \mathcal{F})^{**}$ is a reflexive torsion free sheaf of rank one, it is a line bundle. Therefore we find a destabilizing line bundle

$$(\wedge^r \mathcal{F})^{**} \subset \wedge^r \mathcal{E}. \tag{6.10}$$

It follows from this discussion that in order to prove stability of $\mathcal{E}$, we simply have to show that no destabilizing line bundle of $\wedge^i \mathcal{E},\ i = 1, 2, 3$ exists. Recall from above that every line bundle on X is of the form $\mathcal{O}_X(n)$ with the property that $h^0(X, \mathcal{O}_X(n)) > 0$. That is, every destabilizing line bundle has at least one global section. Assume that we have shown that $H^0(X, \wedge^i \mathcal{E}) = 0,\ i = 1, 2, 3$. Assume that there is a destabilizing line bundle

$$\mathcal{O}_X(n) \subset \wedge^r \mathcal{E} \tag{6.11}$$

for some $0 < r < 4$. Then we have the inclusion

$$H^0(X, \mathcal{O}_X(n)) \subset H^0(X, \mathcal{E}), \tag{6.12}$$

a contradiction. Hence, (6.3) is sufficient to guarantee the stability of $\mathcal{E}$.

To show that $H^0(X, \wedge^i \mathcal{E}) = 0,\ i = 1, 2, 3$ we recall the defining sequence (3.2). For $i = 1$ we must prove that

$$H^0(X, P) \to H^0(X, V) \tag{6.13}$$

is injective. It follows from (3.16) that in the case of $i = 2$ we must show that

$$H^0(X, \wedge^2 P) \to H^0(X, \mathcal{E} \otimes V) \tag{6.14}$$

is injective. If we combine this map with the injective map

$$H^0(X, \mathcal{E} \otimes V) \to H^0(X, P \otimes V) \tag{6.15}$$

we are left to show that the map

$$H^0(X, \wedge^2 P) \to H^0(X, P \otimes V) \tag{6.16}$$

is injective. Note that this map is the restriction of the map

$$1 \otimes w : H^0(X, P \otimes P) \to H^0(X, P \otimes V). \tag{6.17}$$

To show that $H^0(X, \wedge^3 \mathcal{E}) = 0$, recall that

$$H^0(X, \wedge^3 \mathcal{E}) = H^0(X, \mathcal{E}^*) = H^3(X, \mathcal{E})^*. \tag{6.18}$$

The vanishing of $H^3(X, \mathcal{E})$ follows from the long exact sequence in cohomology associated to (3.2) and the vanishing of the non-zero cohomology group (3.6) of P and V .

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