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HOMOTOPICALLY TRIVIAL SYMMETRIES OF HAKEN MANIFOLDS ARE TORAL

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WE CONSIDER Haken manifolds M. That is three manifolds which are compact, irreducible, orientable, and sufficiently large.‡ ∂M may or may not be empty. This class of 3-manifolds shares many properties with closed surfaces ($\neq S^2$ or RP^2). On an algebraic level, both are examples of $K(\pi, 1)$'s. More geometrically the hierarchy structure of a Haken manifold [9] is analogous to a sequence of cuts turning a surface into a disk. With the aid of minimal surfaces we are able to find a sufficiently natural hierarchy to establish a new similarity between Haken manifolds and surfaces. Our main theorem is:

THEOREM. *Let* $0 \to G \xrightarrow{\phi} \mathrm{Diff}^+(M)$ *be an effective orientation preserving action of a finite group* G *on a Haken manifold* M. *Exactly one of the following two alternatives hold.*

(1) *There is a torus* T, $T = S^1$ *or* $S^1 \times S^1$ *or* $S^1 \times S^1 \times S^1$, *and an effective action* Φ *of* T *on* M *in which* ϕ *imbeds.*

$$\begin{tikzcd} 0 \arrow[dr] & 0 \arrow[d] & \\ & G \arrow[d] \arrow[r, "\phi"] & \mathrm{Diff}^+(M) \\ 0 \arrow[r] & T \arrow[ur, "\Phi"'] & \end{tikzcd}$$

or

(2) *The induced map* $\bar{\phi}$: $G \to Out\ (\pi_1 M)$ *to the outer automorphisms of* π_1 *is nontrivial.*

We recall that if M is replaced by a compact surface $\neq S^2$ and $T = S^1$ or $S^1 \times S^1$ then the theorem is still true and is a classical result. In fact the result for surfaces may be proved by taking an averaged metric and analyzing the action of G on simple closed geodesics which minimize length in their free homotopy classes. We use a similar approach, looking at how G moves imbedded least area surfaces inside M.

We warn the reader that ϕ will usually be omitted from our notation; if $\gamma \in G$, $\phi\gamma$: $M \to M$ will simply be written γ; $M \to M$.

Our starting point will be the relative imbedding theorem proved in §7 of [1].

LEMMA 1. *Let* h: $(\Sigma, \partial) \to (M, \partial)$ *be a proper imbedding of an oriented incompressible surface in a Haken manifold* M. *Suppose that* ∂M *has non-negative mean curvature*

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‡Our definition differs from Jaco's [10] definition in that we exclude the 3-disk.

with respect to an outward pointing normal. Then there is a smooth immersion g: $(\Sigma, \partial)\to(M, \partial)$, $g(\partial\Sigma) = h(\partial\Sigma)$ *with area* $(g) \le$ *area* (g') *for all smooth maps* g' *homotopic (rel* ∂*) to* h. *Furthermore* g *is an imbedding except in the case* $\partial\Sigma$ *is empty. If* $\partial\Sigma = \emptyset$ *then* g *is either an imbedding or is a double cover onto an embedded non-orientable surface* $\Sigma/2$. ■

We call a map least area (l.a.) if it is a smooth immersion with area smaller or equal to any smooth (or piecewise smooth) map in its homotopy class. In the bounded case we compare area only with smooth maps which are homotopic rel boundary.

Lemma 1 imbeds l.a. surfaces; we need a disjointness result. The tool here is a lemma of Waldhausen's, essentially Proposition 5.4 [9]. Surfaces are to be presumed oriented unless otherwise stated.

LEMMA 2. *Suppose* Σ_1 *and* Σ_2 *are properly imbedded incompressible surfaces in a Haken manifold. Suppose* Σ_1 *and* Σ_2 *are in general position, homotopic (as maps of pairs), and* $\partial\Sigma_1 \cap \partial\Sigma_2 = \emptyset$. *Then for some compact surface* F_1 *there is an imbedded submanifold* $X \subset M$, $X \overset{\text{homeo}}{} F_1 \times [0, 1]/\sigma \times r \sim \sigma \times s$ *for* $\sigma \in \partial F_1$ *and* $0 \le r, s \le 1$, *with* $\partial_0 X \subset \Sigma_1$ *and* $\partial_1 X \subset \Sigma_2$ $(\partial_0 X = F_1 \times 0$ *and* $\partial_1 X_1 = F_1 \times 1$; $\partial X = \partial_0 X \partial F_1 \cup \partial_1 X)$. ■

X will be called a "product region."

The importance of Lemma 2 in the present context is that a product region $X \subset M$ invites an exchange. Replace Σ_1 by $(\Sigma_1 - \partial_0 X) \cup \partial_1 X$ and replace Σ_2 by $(\Sigma_2 - \partial_1 X) \cup \partial_0 X$. Clearly this exchange preserves homotopy classes and cannot fail to reduce the area of one of the surfaces Σ_1 or Σ_2 (or at least preserve the areas of both and create a corner which can be rounded to reduce area). This line of reasoning quickly leads to a disjointness lemma for l.a. surfaces. One technical point is the least area surfaces may not be in general position. This problem is handled by making an *a priori* estimate of the area ϵ to be saved by any potential exchange. Perturb one of the maps into general position w.r.t. the other being careful to increase its area less than ϵ. For details of this and the exchange argument we refer the reader to [1, 4, 5]. We will be content to state the consequent disjointness lemmas.

LEMMA 3. *Let* g_1, g_2: $(\Sigma, \partial)\to(M, \partial)$ *be proper incompressible l.a. imbeddings into a Haken manifold (with* ∂M *having nonnegative mean curvature). We assume that either* $\partial\Sigma = \emptyset$ *or* $g_1|\partial\Sigma$ *and* $g_2|\partial\Sigma$ *are disjoint imbeddings. If* g_1 *and* g_2 *are homotopic (as maps of pairs) then either they are disjoint or have equal images.* ■

To better understand the role of ∂M, consider Propositions 1 and 2.

PROPOSITION 1. *Let* G *be a finite group* $(G \ne \{e\})$ *and* ϕ *an effective action of* G *on a Haken manifold* M *with* $\tilde{\phi}$ *equal to zero. Each boundary component* S *of* M *is a torus and unless* $M = S^1 \times D^2$ *each boundary component is incompressible.*

PROPOSITION 2. *Let* M, G, *and* ϕ *be as above. If* $c \subset \partial M$ *is an oriented simple closed curve then for all* $\gamma \in G$, c *is freely homotopic to* $\gamma(c)$ *in* ∂M.

Proof. Let T be the boundary component containing c. Since $\tilde{\phi} = 0$ the action of G on $H_*(M; Z)$ is trivial, it follows that B preserves each boundary component so for all $\gamma \in G$, (c) T. If γ: $T \to T$ is homotopic to id_T we are finished. If not γ: $T\to T$ is equivalent to one of 4 periodic linear maps $\left(\begin{vmatrix} -0 & 0 \\ 0 & -1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}, \begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix}, \text{or } \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix}\right)$. Of these only the first has an invariant one-dimensional subspace. Thus, in the last three cases $H_1(T;\, Z) \xrightarrow{\text{inc}_*} H_1(M;\, Z)$ must be an injection, contradicting the triviality of $\tilde{\phi}$. Finally suppose γ acts on T by $\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}$. $\text{inc}_* H_1(T;\, Q)$ has rank 1 or 2 and is acted on nontrivially by $\tilde{\phi}$, again a contradiction.

Proof of Proposition 1. Let S be a boundary component of M. ϕ preserves S. Let $K \subset H_1(S;\, Z)$ be the kernel induced by $S \subset M$. Clearly ϕ leaves K invariant; we claim ϕ actually fixes K. To see this lift K to $L \subset H_2(M, S)$, thus $\partial | L\colon L \to K$ is an isomorphism. Let $\bar{L}$ be the image of L in $H_2(M, \partial)$. The commutative diagram below shows $\bar{L}$ is isomorphic to L.

$$\begin{tikzcd}
L \arrow[r, "\cong"] \arrow[dr] \arrow[dd] & K \arrow[dr, hook] & 0 \arrow[d] \\
& H_2(M, S) \arrow[r, "\partial"] \arrow[d] & H_1(S) \arrow[d] \\
\bar{L} \arrow[r, hook] & H_2(M, \partial) \arrow[r, "\partial"] & H_1(\partial)
\end{tikzcd} \qquad \text{Coef.} = Z$$

ϕ acts trivially on $H_3(M, \partial;\, Z)$ and $H^1(M;\, Z)$ and every $x \in H_2(M, \partial;\, Z)$ is of the form $[M, \partial] \cap \alpha$ for some $\alpha \in H^1(M;\, z)$, thus the formula $g_*([M, \partial] \cap \alpha) = (g_*[M, \partial] \cap g^{*-1}\alpha) = [M, \partial] \cap \alpha$ shows ϕ acts trivially on $H_2(M, \partial, Z)$. Thus $\bar{L}$ and K are fixed.

But ϕ also fixes $H_1(S;\, Z)/K$. The addition formula for traces applied to the action of ϕ on $0 \to K \to H_1(S;\, Z) \to H_1(S;\, Z)/K \to 0$ yields: for all $g \in G$ trace $\phi(g)_*$: $H_1(S;\, Z) \leftarrow =$ trace$_g = b_1 =$ the first Betti number of S. Consequently the Lefschetz number of $\phi(g)$: $S \to S$ is the Euler characteristic of S. But computing the Lefschetz number by adding local indexes, we see that an orientation preserving surface automorphism of finite period must have nonnegative Lefschetz number. It follows that $S =$ the 2-sphere or torus. Irreducibility of M rules out $S =$ 2-sphere.

Finally if some boundary component of M is compressible, Dehn's lemma and another application of irreducibility shows that $M = S^1 \times D^2$. ■

LEMMA 3′. *Let (M, ∂) be a Haken manifold with ϕ as above. Suppose (M, ∂) has a G-invariant metric. Let $(F, \partial) \subset (M, \partial)$ be a proper incompressible* 1.*a. imbedding with ∂F consisting of closed geodesics each of minimum length in its free homotopy class in* ∂M. For all $\gamma \in G$, $\gamma F = F$ *or* $\gamma F \cap F = \phi$.

Proof. $\gamma F \cap F = \phi$ follows from Lemma 3 if $\gamma(\partial F) \cap \partial F = \phi$. Conversely, we will argue that $\gamma(\partial F) = \partial F$ implies $\gamma F = F$.

Suppose $\gamma(\partial F) = \partial F$ and $\gamma F \neq F$. γF and F must have disjoint interiors or there will be an area reducing exchange [1]. Thus the inward normals ν and ν_γ to ∂F inside F and γF satisfy an inequality for all p belonging to any boundary component c of F: angle $\nu_\gamma(p) \leq$ angle $(\nu(p)$, where angle refers to the dihedral angle made with the boundary with an appropriate orientation. Since γ is an isometry $\int_c \nu_\gamma(p) = \int_{\gamma^{-1}c} \nu(p) =$

$\int_c \nu(p)$ showing that equality holds above for all p. But a minimal surface is determined by its boundary and inward normal over an open subset of the boundary. So $F = \gamma F$.

To finish the proof we must show $\gamma(\partial F)$ is equal or disjoint from ∂F. By Proposition 2 $\gamma(c)$ is freely homotopic in ∂M to c. Since the boundary consists of minimum length curves, c_i, an exchange argument shows that for each i, $\gamma c_i = c_i$ or γc_i $c_i = \emptyset$. If one alternative holds for c_i and the other for c_j a contradiction is obtained from the preceding integral argument. ■

Note. Unless otherwise constructed, we will always assume that the boundary of a properly imbedded surface is a closed loop minimizing length in its free homotopy class in ∂ (3-manifold).

Aside. If M is closed, $Z = H^3(M; Z) = H^3(\pi_1(M); Z)$. Since $\tilde{\phi}$ is trivial the induced action on group cohomology is trivial. Thus ϕ automatically acts through orientation preserving maps. We point out that this assumption is not superfluous when $\partial M \neq \phi$. Consider $D^2 \times S^1 = \{(z, s)|z, s)|z, s \in \mathbb{C}, |z| \leq 1, |s| = 1\}$ with Z_2 acting by $(z, s) \to (\bar{z}, s)$.

We begin the proof of the theorem.

We note (1) implies that (2) is false; the action of a connected group must be trivial on outer automorphisms of π_1. Thus to prove the theorem, we assume for the remainder of the proof that (2) is false, i.e. that $G \xrightarrow{\phi} (\pi_1 M)$ is the zero map. We proceed to construct the torus action Φ.

We suppose that M has been given a G-invariant Riemannian metric with ∂M having non-negative mean curvature. As M is Haken, we can find an oriented incompressible surface $H: \Sigma \to M$. Using Lemma 1 we find a l.a. map homotopic to h. Let $F \subset M$ denote the image of this l.a. map. Either $F \subset M$ is homotopic to h or is a nonorientable surface covered by a map homotopic to h. In either case let f: $\Sigma \to F \subset M$ be the l.a. map. Let $G(F)$ denote the orbit of F under G.

PROPOSITION 3. *In the case that F is nonorientable, that is $F = \Sigma/Z_2$, we have $G(F) = F$.*

Proof. Since Lemma 3 does not apply as stated to nonorientable surfaces we pass to the cover $\tilde{M}$ determined by $h_{\#}\pi_1(\Sigma, *) = h_{\#}\pi_1(\Sigma) \subset \pi_1(M)$. It follows from Corollary VII, 12 [J] that $\tilde{M}$ is homeomorphic to $\Sigma \times R$ union limit points.† Clearly f lifts to an imbedding $\tilde{f}$: $\Sigma \to \tilde{M}$. Since G acts trivially on $\text{Out}(\pi_1 M)$, for each $\gamma \in G$, there is an arc α in M from $f(*)$ to $\gamma \circ f(*)$ such that α conjugates, in the fundamental groupoid, $f_{\#}(\pi_1(\Sigma, *))$ to $(\gamma \circ f)_{\#}(\pi_1(\Sigma, *))$. Such α is not necessarily unique up to homotopy (rel. endpoints). However, a choice of such an α determines a lifting $\widetilde{\gamma \circ f}$: $\Sigma \to \tilde{M}$ of $\gamma \circ f$. Since G is finite we may find a real number n and lifts $\widetilde{\gamma \circ f}$ for all $\gamma \in G$ such that image $\widetilde{\gamma \circ f} \subset \Sigma \times [-N, N] \subset \Sigma \times R \subset \tilde{M}$. Since covering projections are local isometries, all the lifts have equal area and all are freely homotopic. Lemma 3 implies that any two lifts $\widetilde{\gamma_1 \circ f}$ and $\widetilde{\gamma_2 \circ f}$ are either equal or disjoint.

Let $\bar{M}$ be the cover associated to $\pi_1(F) \subset \pi_1 M$ and p: $M \to \bar{M}$ the 2-fold covering projection. By the method above, the inclusions $\gamma F \subset M$ lifts to $\bar{M}$. For each $\gamma \in Gp|\widetilde{\gamma \circ f}(\Sigma)$ is two to one onto its image; $\widetilde{\gamma \circ f}(\Sigma)$ is invariant under the covering involution. But $\widetilde{\gamma \circ f}(\Sigma)$ separates M and the covering involution interchanges the

†Shortly, we will see that in this case $M \cong \Sigma \times R$.

components of $\widetilde{M - \gamma \circ f(\Sigma)}$. It follows that all $\widetilde{\gamma_i \circ f(\Sigma)}$ have the same image. Now projecting, we see that $G(F) = F$. ■

PROPOSITION 4. *If F is orientable, then $G(F) = (F = F_0) \| \cdots \| F_{n-1}$ a disjoint union of n parallel copies of F where $n / |G|$.*

Proof. Our hypothesis $G \to \mathrm{Out}(\pi_1 M)$ is zero implies that there is a free homotopy of the 1-skeleton of F_i taking $SK^1(F_i)$ $1-1$ onto $SK^1(F_j)$ for all i and j. Since M is a $K(\pi_1(M), 1)$ we may extend over the top cell. Thus F_i and F_j are freely homotopic for all $1 \leq i, j \leq n$. Disjointness follows from Lemma 2.

Any two surfaces F_i and F_j, $i \neq j$ form the boundary of a submanifold $X_{i,j} \subset M$ which is homeomorphic to $F \times [0, 1]$. In other words, F_i and F_j are *parallel*. To see this consider the covering space $\tilde{M}^{\pi_1(F)}$. The inclusions $F_i \subset M$ and $F_j \subset M$ lift into $F \times R \subset M^{\pi_1(F)}$ and are disjoint incompressible surfaces there. It follows from §3[9] that any lifts of F_i and F_j are parallel in $\tilde{M}^{\pi_1(F)}$. Fix the lift $\tilde{F}_i$ of F_i and choose the lift $\tilde{F}_j$ of F_j (there is more than one choice) so that no other lift of F_j lies between $\tilde{F}_i$ and $\tilde{F}_j$. With F_i and F_j so chosen the product region will map $1-1$ under the covering projection. Call the image $X_{i,j}$. The order of the stabilizer of F is the integer $|G|/n$. ■

We call an element $\gamma \in G$ *passive* (more precisely passive on an incompressible surface F with respect to the action ϕ) if $\gamma F = F$ and there is a homotopy of pairs h_t: $(M, F) \to (M, F)$ with $h_0 = \phi(\gamma)$ and $h_1 = id_{(M,F)}$. Fixing F (and of course ϕ) we denote the subgroup of passive elements of $P \subset G$. We will call $\gamma \in G$ *active* if it is not passive.

Note. The homotopy h_t which carries the transformation of a passive element $\phi(\gamma)$ to the identity can itself be altered by a homotopy (rel. $(M, F) \times \{0, 1\}$) so as to respect the complement $M - F$. We call such a homotopy, h_t: $(M; F, M - F) \to (M; F, M - F)$ "*prepared*". The alteration takes a hierarchy $\{F_i\}$ for M beginning with $F = F_0$ and inductively moves $h_t | F_s$ off F. The details are given in the proof of Lemma 7.2[9].

PROPOSITION 5. *In the case that F is nonorientable and $F = \Sigma / Z_2$, we have $P = G$.*

Proof. Let $\gamma \in G$. Then by Proposition 3, there is a homotopy H from id_F to $\gamma | F$ where H runs through M. Let $\bar{F}$ be the boundary of a regular neighborhood of F. Then $\bar{F}$ is an orientable incompressible surface in M. By Proposition 6.5 [2] H may be homotoped relative to its boundary to make $H^{-1}(\bar{F})$ a closed incompressible surface $J \subset F \times I$. But since the normal bundle of J is trivial, the classification of incompressible surfaces in (surface $\times I$)[9] forces $J = \phi$. Thus the modified H lies in interior (reg. neib. F) and may be compressed into F. Since the inclusion induces an injection $\pi_1(M - F) \to \pi_1(M)$, G induces the identity on $\pi_1(M - F)$. It follows by a standard argument that H extends to $M \times I$. The proposition follows. ■

A *torus action* will refer to an effective action of $\{e\}$, S^1, $S^1 \times S^1$, or $S^1 \times S^1 \times S^1$.

LEMMA 6. *Let $F \subset M$ be an oriented incompressible* 1.*a. surface in a Haken manifold with a finite group G acting by ϕ and with $\tilde{\phi}$: $G \to \mathrm{Out}(\pi_1 M)$ equal to zero. Suppose that some element of G is active on F. Then there is an orientation preserving periodic map σ: $F \to F$ such that M is diffeomorphic to the mapping torus $M = F \times [0, 1]/(x, 0) \sim (\sigma(x), 1)$. Furthermore (M, G, ϕ) is constrained by the following* *classification*:

Case 1. *$F \neq$ torus, annulus or disk.*
$P = \{e\}$, G is cyclic, ϕ imbeds in a circle action.

Case 2. *F = torus but $\sigma \neq id_F$.*
Equivalently, M is a torus bundle over S^1 different from T^3. $P = \{e\}$, G is cyclic, ϕ imbeds in a circle action.

Case 3. *F = torus, annulus, or disk, $\sigma = id_F$* (i.e. $M = T^3$ or $T^2 \times I$, or $S^1 \times D^2$). P = Stabilizer $(F) = St(F)$ and $\phi|P$ imbeds in a torus action α. G is a central cyclic extension of P with quotient map $G \xrightarrow{\pi} G/P$.

(a) If π is split by s: $G/P \to G$ let $s(\text{generator}) = y$, y is an active element generating a direct summand C of G. Then $\phi|C$ imbeds in a circle action which commutes with α. Thus G imbeds in the torus action $\alpha \oplus \beta$: $T_\alpha \oplus T_\beta \to \text{Diff}^+(M)$.

(*b*) If π is not split then there is a $\gamma \in G$ and a circle action β commuting with α and containing $\phi|\langle\gamma\rangle$. $\alpha \oplus \beta$ has a finite kernel K and ϕ imbeds in $\alpha \oplus \beta/K$: $T_\alpha \oplus T_{\beta/K} \to \text{Diff}^+(M)$.

Note. Lemma 6 completes the proof of the theorem in the case that some oriented incompressible surface $\Sigma \subset M$ is homotopic to a least area *imbedding* $F \subset F$ for which some $\gamma \in G$ is active.

Proof of Lemma 6. Consider first the case G is cyclic of order m, $G = \{e, \gamma, \ldots, \gamma^{m-1}\}$. γ: $M \to M$ is homotopic to id_M. Let H_t be a homotopy of γ to id_M and let $\tilde{H}_t$ be the lift to $\tilde{M} = M^{\pi_1 F} \times I$. We have a commutative diagram:

$$\begin{array}{ccc} \tilde{M} \times I & \xrightarrow{\tilde{H}} & \tilde{M} \\ \downarrow{\scriptstyle p \times id} & & \downarrow{\scriptstyle p} \\ M & \xrightarrow{H} & M \end{array}$$

Let $\tilde{\gamma}$ be the restrictions of $\tilde{H}$ to the end covering γ. Let $\tilde{F}$ be a fixed lift of F and let B be the product region $\tilde{B} \subset \tilde{M}$ bounded by $\tilde{F} \| \tilde{\gamma}\tilde{F}$.

Let $k-1$ be the number of components of $P^{-1}(G(F)) \subset$ interior B. Establish coordinates $B \cong F \times [0, k]$ on B so that $P(F \times \text{integer}) \subset (G(F))$. Translation by powers of $\tilde{\gamma}$ establishes coordinates on $\tilde{M} \simeq F \times R$. For some integer r dividing km $F \times [0, r] \subset \tilde{M}$ is a fundamental domain for p:$\tilde{M} \to M$. That is $F \times [0, r]/(x, 0) \sim (g(x), r) \cong M \cong F \times R/(x, t) \sim (g(x), t+r)$ for some isotopy class $[g]$ of self maps of F.

We claim $g^{km/r}$ is homotopic to the identity. To see this take the product structure on $F \times [0, r]$ and extend using the covering translations of P:$\tilde{M} \to M$ to a product structure ϵ on $F \times [0, km]$. This can differ from the existing structure δ on $F \times [0, km]$ only by an isotopy. Using P to identify $F = 0$ and $F \times mk$ with $F \subset M$, the structures ϵ and δ each give rise to an automorphism ($\bar{\epsilon}$ and $\bar{\delta}$ resp.) of F. $\bar{\epsilon} = g^{km/r}$ and $\bar{\delta} = \gamma^m = 0$. Since $\bar{\epsilon}$ is isotopic to $\bar{\delta}$ the claim follows.

By Nielsen's theorem we now choose σ from the isotopy class $[g]$ so that $\sigma^{km/r} = 1$. Using σ as a monodromy map it is now easy to build a Seifert fibered model for (M, G) and construct an equivalent diffeomorphism.

Set $\bar{M} = F \times [0, r]/\sigma$. Let S^1 be the circle of circumference km (i.e. $R/, mZ$); S^1 has a natural action on $\bar{M}$ by translation in the second coordinate. We denote by $\bar{G}$ the cyclic subgroup of S^1 of order m (i.e. the integral multiples of k in S^1). Our claim is that the pairs (M, G) and $(\bar{M}, \bar{G})$ are equivariantly diffeomorphic. The diffeomorphism extends the identifications: $\gamma^l(x) \mapsto (x, lk)(x \in F \subset M)$ of $G(F) \subset M$ with the equivalence class $[F \times \text{integers}] \subset \bar{M}$. A necessary verification is: For all integers q if $l_1 + l_2 = l_3 + qm$ then:

$$\gamma^{l_1+l_2}(x) \mapsto (x, (l_1 + l_2)k) = (x, l_3k + qmk) = (\sigma^{(-km/r)q}(x), l_3^k) = (x, l_3^k) \leftarrow\!\shortmid \gamma^{l_3}(x).$$

Let $C_i \subset M$ be a product region bounded by $F \amalg \gamma^i F$. Among these choose C with minimum volume (equivalently such that interior $C \cap G(F) = \emptyset$), call $\partial C = F \| \gamma' F$. It follows from our mapping cylinder description of M that for some integer $l\sigma^l/F$: $F \to F$ and γ'/F: $F \to \gamma' F$ will be homotopic as maps into C (this integer will later be defined as a "length of homotopy"). Now establish a product structure on C, $C \overset{\theta}{\cong} F \times [0, 1] \subset \bar{M}$ with product lines connecting $\sigma^l(x)$ to $\gamma'(x)$. The desired equivariant diffeomorphism is:

$$\begin{array}{ccccccc}
\gamma^i(C) & \xrightarrow{\gamma^{-i}} & C & \xrightarrow{\theta} & F \times [0, 1] & \xrightarrow{\quad\text{translation}\quad} & F \times [ik, ik + 1] \\
\cap & & & & & & \cap \\
M & & & & & & \bar{M}.
\end{array}$$

Case 1. As remarked in the introduction, for surfaces ($\neq S^2$) homotopically trivial finite actions imbed in torus actions. Since $F \neq$ torus, annulus, or disk, F has no circle action and hence no homotopically trivial finite action. Thus $P = \{e\}$. Thus it is only necessary to show that G is cyclic. Let $\gamma \in G - \{e\}$. We know that $\gamma|F$ is homotopic to id_F if the homotopy is permitted to travel through M. Proposition 6.5 [2] allows us to homotope H (rel. $h_0 \amalg h_1$) so that H is transverse to $G(F) \subset M$ and $H^{-1}G(F) = F \times 0 \amalg F \times 1 \amalg F^1, \ldots, \amalg F^{i'}$ are disjoint parallel copies of F, horizontal in the product structure. (For the last part of this statement use Lemma 3.1 [9]: incompressible surfaces in surface $\times I$ are boundary parallel.) Each $F^1, \ldots, F^{i'}$, $F \times 0$, and $F \times 1$ has an associated sign $\pm$. A further homotopy of H allows us to assume that sign $(F \times 0) = -$ sign $(F \times 1)$. Call a homotopy H with the above properties "good". Define the "length homomorphism" l: $G \to Z/qrZ'$ by $l(\gamma) =$ the number of components of $H^{-1}(G(F))$ counted according to sign where H is a good homotopy from $\gamma|F$ to id_F. As before r is the number of components of $G(F)$; q is the order of σ. There are, of course, homotopies from $e|F = id_F$ to id_F with length equal to any integral multiple of qr (the length of H being naturally defined as an integer); this accounts for the indeterminacy $= qrZ$.

Only a passive element $\gamma \in G$ can have $l(\gamma) = 0$. This is because a homotopy H from $\gamma|F$ to id_F can be cut and pasted to eliminate an adjacent pair of surfaces carrying opposite signs. If $l(\gamma) = 0$ this leads to a compression of H into F, H': $F \times I \to F$. Since $\gamma|M - F \simeq id_{M-F}$, a standard argument extends H' over $M \times I$. Hence γ is passive. Consequently l is an injection and G is cyclic.

Case 2. A periodic map of the torus $\neq$ identity is fixed point free if and only if it is homotopic to the identity. Since $\sigma \neq id_F$, the fixed point set of σ, Fix σ, is nonempty. If $\gamma \in P$ then the mapping cylinder description of M yields $\sigma\phi(\gamma) = \phi(\gamma)\sigma$ as maps $F \to F$. By the note preceding Lemma 6, σ will also commute with a good homotopy to

id_F, $\phi_t(\gamma)$. Thus Fix σ must be invariant under $\phi_t(\gamma)$. Fix σ is discrete so $\phi(\gamma) = \phi_0(\gamma)$ must fix each $x \in \text{Fix}\,\sigma$.

By continuity in t, Fix $\sigma \subset$ Fix $\phi_t(\gamma)$ for all values of t $0 \le t \le 1$. In particular $\phi(\gamma)$: $F \to F$ must have a fixed point. Since $\phi(\gamma)$ is periodic and homotopic to id_F we have $\gamma = e$ and $p = \{e\}$.

The rest of Case 2 follows from Case 1.

Case 3. Let $\gamma \in St(F)$. Choose a good homotopy H of γ to id_F. Since the monodromy σ is homotopic to id_F, H may be cut and pasted to compress H into F. Ultimately this process results in a compression of H into F, showing that $\gamma \in P$.

To understand the action of P on M we need:

LEMMA 7. *Let $F = T^2$ or $S^1 \times I$. Let $(N;\ \partial^+N,\ \partial^-N)$ be a 3-manifold diffeomorphic to $(F \times I{:}\ F \times 0,\ F \times 1)$. Suppose ϵ: $P \to \text{Diff}^+(N)$ is an effective action of a finite group and that the action is homotopically trivial (i.e. $\bar{\epsilon}$: $P \to \text{Out}(\pi_1 F)$ is zero). Then there are (possible periodic) coordinates on ∂^+N so that the action on ∂^+N is by translation and a product structure $N \xrightarrow{\text{diffeomorphism}} \partial^+N \times I$ so that the action on N is simply a product of the action on ∂^+N with I. Furthermore this product structure may be arranged so that the map $\partial^+N \overset{\pi}{\to} \partial^-N$ obtained by following product lines is any prescribed ϕ-equivariant diffeomorphism (in the homotopy class determined by $F \times 0 \to F \times 1$ $(x \times 0 \to x \times 1)$).*

Note. The verification of the Smith conjecture and recent work of Meeks[3] suggests that all finite (orientation preserving) group actions on orientable surface $\times\, I$ may be product actions.

The proof of Lemma 7 is deferred until Lemma 6 is complete.

By the classical result the action of P on F imbeds in a torus action $\alpha^=$ on F. This action is by translations. Let the conclusion of Lemma 7 determine a product structure on M^- ($= M$ cut along F) which is equivariant with respect to the action of P on M^- and in which σ is the map π identifying ends of product lines. This product structure can be used to extend $\alpha^=$ to a torus action α^- on M^-. Identifying by σ we have the action α on M.

Assume $P \neq G$, we must show G is a central cyclic extension of P. Let $F = F_0$ and let F_1 by characterized among $F_1, \dots, F_{r-1}$ by volume $X_{0,1} \le \text{vol}\, X_{0,k}$, $2 \le k \le n-1$. Let $\gamma \in G$ satisfy $\gamma F_0 = F_1$. $\{\gamma^k F_0,\ 0 \le k \le r-1\} = \{F_0, \dots, F_{n-1}\}$ for otherwise some F_j would divide some translate $\gamma^k X_{0,1}$ into two pieces Y and Z with $\partial Y = \gamma^k F_0 \amalg F_j$ and $\partial Z = F_j \| \gamma^{k+1} F_0$. Now volume $\gamma^{-k}(Y) <$ volume $X_{0,1}$ so F_0 and $\gamma^{-k}(F_j)$ bounds a smaller volume than F_0 and F_1, a contradiction.

The action of P preserves the various volumes {Volume $X_{0,k}$} so P leaves each $\gamma^k F_0 = F_k$ invariant. It follows that P is normal in G and that G/P is isomorphic to Z/rZ. Using Lemma 7 find a ϕ/P-equivariant product structure on each $X_{k,k+1}$ so that $\pi(x) = \gamma(x)$ for $x \in F_k$. By construction of α the product structures are also α-equivariant. Let $p \in P$ $x \in F_0$, $p^{-1}\gamma^{-1}p\gamma(x) = p^{-1}\pi^{-1}p\pi(x) = p^{-1}p\pi^{-1}\pi(x) = x$. So F_0, hence M is fixed by the above commutator. Thus $\gamma \in Z(P)$, the centralizer of P. The circle action β is constructed by following the parameterized product lines in $X_{k,k+1}$. β commutes with α since these product lines are α-equivariant. Thus we may construct the product action $T_\alpha \oplus T_\beta \xrightarrow{\alpha \oplus \beta} \text{Diff}^+(M)$.

Case 3(a) occurs if and only if $\gamma^r = e$. If $\gamma^r \neq e$ we must divide $T_\alpha \oplus T_\beta$ by the skew-diagonal subgroup $\langle(\gamma^r, -\gamma^r)\rangle$ to obtain an effective action Φ: $T_\alpha \oplus T_\beta / \langle \gamma^r, -\gamma^r \rangle \xrightarrow{\alpha\oplus\beta} \mathrm{Diff}^+(M)$ in which ϕ imbeds. (The embedding $G \to T_\alpha \oplus T_\beta/\langle \gamma^r, -\gamma^r\rangle$ may be defined unambiguously by writing $g \in G$ as $g = p \cdot \gamma^k$, $p \in P$ and sending $g \to (\mathrm{inclusion}_\alpha p, \mathrm{inclusion}_\beta \gamma^k)$.)

This completes the proof of Lemma 6.

ADDENDUM TO LEMMA 6. *Retain the hypothesis of Lemma 6 and assume $M \neq \emptyset$. Also assume ϕ restricted to ∂M is already embedded in a torus action. Then the embedding into Φ which we construct may be chosen to extend the embedding of actions on ∂M.*

Proof of Addendum. The proof differs from the absolute case only when $M = S^1 \times (S^1 \times I)$ and $\sigma \simeq id$: $S^1 \times I \to S^1 \times I$. Here we need a version of Lemma 7, Lemma 7′, which gives a relative product structure for finite actions on ($F \times I$, equivariant product structure on $\partial F \times I$). Instead of isotoping σ to id_F isotope it (rel. $(\partial F) \times I$) to some rotation (in the G-averaged metric) on F. σ will commute with the torus (= circle) action $\alpha^=$ on F so an extension to α is obtained. Otherwise the argument is the same. ∎

Proof of Lemma 7 for $F = T^2$. Let N be a given G-invariant metric in which mean curvature $\partial N \geq 0$ (see [5]). Find two simple closed curves r^+, $s^+ \subset \partial^+ N$ each an absolute minimum for length in its free homotopy class and meeting in one point. Any homotopic $r^{+\prime}$ also of minimum length must be equal (set wise) to r^+ or disjoint, similarly for s^+. Since the action is homotopically trivial the free homotopy classes of r^+ and s^+ are preserved. Thus $\gamma(r^+) = r^+$ (setwise), or $\gamma(r^+) \cap r^+ = \phi$, similarly for s^+. The orbit $P(r^+ \cup s^+)$ is a lattice of circles. Let D be a rectangular domain in F, interior $D \cap P(r^+ \cup s^+) = \emptyset$ and $\partial F = $ arcs of $P(r^+ \cup s^+)$. D is a fundamental domain for the action of P on $\partial^+ N$. The required P-equivariant diffeomorphism with an action by translation is obtained by smoothly identifying D with the fundamental domain of translation group $Z/kZ \oplus Z/lZ$ when k is the number of lines of $P(s^+ \cup r^+)$ meeting s^+ and l is the number of such lines meeting r^+.

Similarly consider r^-, $s^- \subset \partial^- N$ in the same free homotopy class (in N) as r and s (resp.). Let r, s be immersed annuli (for existence, see [SY]) having minimum area with respect to the constraints $\partial s = s^+ \cup s^-$, $\partial r = r^+ \cup r^-$. By [1] r and s are imbedded. By an exchange argument, given explicitly in [1], $r \cap s = $ one imbedded arc. (The argument cited above applies to tori, Klein bottles, and annuli in Haken manifolds. In this case the exchange argument can be done by hand; one must show that any additional curves of intersection give rise to exchanges of subsurfaces which reduce the area of either r or s.) Similarly $P(r)$ and $P(s)$ are disjoint unions of annuli. $P(r \cup s)$ is topologically just $P(r^+ \cup s^+) \times I$. A fundamental domain for the action $\in$ consists of the closure of a connected component W of $N - P(r \cup s)$. This domain has the structure of $D \times I$. Any such product structure which satisfies the appropriate condition (if for some $\gamma \in P$ and $x \in \partial \bar{W}$, $\gamma(x) \in \partial \bar{W}$ then x and $\gamma(x)$ must have the same second coordinate under $\bar{W} \xrightarrow{\text{diffeo}} D \times I$) determines a P-equivariant homeomorphism to a product action on $F \times I$. With a little care, corners at $P(r \cup s)$ can be avoided and the P-equivariant map is a diffeomorphism.

Suppose f: $\partial^- N \to \partial^- N$ is a P-equivariant map and $f \simeq id_{\partial^- N}$ then f covers g in the commutative diagram:

$$\begin{array}{ccc} \partial^- N & \xrightarrow{f} & \partial^- N \\ {\scriptstyle |P|\text{-fold cover}}\downarrow & & \downarrow{\scriptstyle |P|\text{-fold cover}} \\ \partial^- N/P & \xrightarrow{g} & \partial^- N/P. \end{array}$$

The diagram on the fundamental group level shows that $g_\#$ carries a subgroup of index $|P|$ isomorphically to a subgroup of index $|P|$. It follows that $g_\#$ is an isomorphism and that g is isotopic to the identity. The resulting P-equivariant isotopy of f to $id_{\partial^- N}$ can be used to change the product structure in a collar of $\partial^- N$ to achieve the final requirement of Lemma 7. ■

We state and prove the stronger relative version of Lemma 7 for $F = S^1 \times I$.

LEMMA 7'. *Let $F = S^1 \times I$ and N and ϵ be as in Lemma 7. Further suppose that $\partial^0 N = \partial N - (\partial^+ N \cup \partial^- N)$ has a product structure, λ: $\partial^0 N \xrightarrow{\text{diffeomorphism}} \partial F \times I$, which is equivariant with respect to ϵ. Then there is an interval valued and a periodic coordinate on $\partial^+ N$ so that ϵ on $\partial^+ N$ acts by translation of the periodic coordinate. Also there is a p-equivariant product structure $N \xrightarrow{\text{diffeomorphism}} \partial^+ N \times I$ (ϵ acts on $N = \partial^+ N \times I$ as a product) extending the structure on given $\partial^0 N$. The map $\partial^+ N \xrightarrow{\pi} \partial^- N$ can be chosen to be any P-equivariant diffeomorphism extending $\partial(\partial^+ N) \xrightarrow{\pi_\lambda} \partial(\partial^- N)$.*

Proof of Lemma 7'. Give N a P-averaged metric in which $\partial^0 N$, $\partial^+ N$, and $\partial^- N$ are flat and $\partial^0 N$ meets $\partial^+ N \cup \partial^- N$ is a right dihedral angle. Let l_1 and l_2 be any two product structure lines in different components of $\partial^0 N$. Let $r^+(r^-)$ be a minimum length arc in $\partial^+(\partial^- N)$ joining the upper (lower) endpoints of l_1 and l_2. Since P acts freely on $\partial(\partial^+ N)\gamma(r^+)$ and r^+ meet only in their interiors, for $e \neq \gamma \in P$. Since γ has finite order an intersection argument shows that the total sign of such interior intersections is zero. The usual exchange argument shows that r^+ is disjoint from (r^-). Similarly $r^- \cap \gamma(r^-) = \emptyset$. As in the proof of Lemma 7 the closure of a component of $\partial^+ N - P(r^+)$ is a fundamental domain D and can be used to establish coordinates on $\partial^+ N$.

Set $c = l_1 \cup l_2 \cup r^+ \cup r^-$. The metric assumption on the boundary enable us to find an imbedded solution d[4] to the Plateau problem determined by c. By[5] $\gamma d \cap d = \phi$. The closure of a connected component $\bar{E}$ of $N - G(d)$ can be given the structure $\bar{E} \xrightarrow{\text{diffeomorphism}} D \times I$. In addition to the constraints occurring in the proof of Lemma 7 we must require that this structure extend the P-equivariant structure induced on $\bar{E}\partial^0 N$. Under the action of P this structure on $\bar{E}$ generates the desired P-equivariant structure on N. The last assertion is proved as in Lemma 7. ■

We complete the proof of our theorem by applying Lemma 6 and its addendum to a hierarchy for M. Recall the minimal imbedded surface $F(F = \Sigma$ or $\Sigma/Z_2)$. If there is some $\gamma \in G$ which is active on F we are finished by Proposition 5 and the note following Lemma 6. Assume not, $G(F) = F$ and $P = G$. Set M_1 = the compact "M cut along F." 6 acts by ϕ_1 on M_1 and the action $\partial\phi_1$ of G on ∂M_1 imbeds in a torus action $\partial_1\Phi$. $\partial_1\Phi$ is equivariant w.r.t. the regluing of ∂M_1 to form M. Fix this imbedding.

LEMMA 8. *The action ϕ_1 of G on M_1 is homotopically trivial. That is $\tilde{\phi}_1$:$\rightarrow$ $\mathrm{Out}(\pi_1 M)$ is zero.*

Proof. Since $P = G$, to each $\gamma \in G$ there is a prepared homotopy h_t as constructed in the note preceding Lemma 6. These homotopies can be restricted to $M - F \simeq M_1$. Hence $\phi_1(\gamma)$: $M_1 \to M_1$ is homotopic to the identity. The lemma follows. ∎

We apply Lemma 8 and Propositions 1 and 2 to M_1.

Let T be a torus in ∂M_1. For homological reasons there is a simple closed curve $a \subset T$ which is part (or all) of the boundary of a surface $F^1 \subset M_1$ with $\partial F^1 \cap T = a$. We may arrange that F is incompressible and boundary incompressible. Geometrically we may assume w.l.o.g. that ∂F^1 consist of geodesic loops which are minimal in their free homotopy (in ∂M_1) classes and that F^1 has least area (by Lemma 1) among all smooth surfaces in its homotopy class (∂F^1 fixed by the homotopy).

We have two cases: (1) there is some $\gamma \in G$ which is active on F^1, (2) $P_1 = G$. In case (1) the proof is completed by using the addendum to Lemma 6. Extend the imbedding of $\partial_1\phi$ in $\partial_1\Phi$ to an imbedding of ϕ_1 in a torus action Φ_1 on M_1. Reglue to construct $\phi \subset \Phi$ acting on M.

Case 2 can only occur if $F^1 = S^1 \times I$ or D^2. $F^1 = D^2$ is only possible if $M_1 = S^1 \times D^2$; we call this the final case. If $F^1 = D^2 \times I$ and G is passive on F^1 form M_2 by cutting along F^1. The two cases again present themselves but we have reduced the number of boundary components by 1. Eventually we must either arrive at a surface $F^i \subset M_i$ for which there is an active element or come to the final case. If there is a $\gamma \in G$ which is active on F^i then as we found for $i = 1$, the action of G on M_i can be imbedded in a torus action (Lemma 6 and addendum) in such a way that reglueing along the hierarchy to form M preserves the torus action.

The final case is $F^i = D^2$, $M_i = S^1 \times D^2$, $M_{i+1} = D^3$. The action ϕ_{i+1} of G on M_{i+1} preserves the decomposition of ∂D^3 as $\partial D^3 = D^2_+ \cup S^1 \times I \cup D^2_-$ so ϕ_{i+1} must be a cyclic action. It is a standard cyclic action on ∂D^3. By the solution of Smith conjecture [3] ϕ_{i+1} is standard on $(D^3, \partial D^3)$. Thus Φ_{i+1} will be a standard circle action on D^3. As before reglue to obtain Φ. ∎

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