CHAPTER VIII

The Equivariant Loop Theorem for Three-Dimensional Manifolds and a Review of the Existence Theorems for Minimal Surfaces

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The details of this chapter appeared in Meeks and Yau [4, 5]. The equivariant loop theorem that is needed in settling the Smith conjecture can be described as follows.

Let G be finite group acting smoothly on a compact three-dimensional manifold M. Let $S = \amalg_j S_j$ be a union of components of ∂M such that $g(S) = S$ for all $g \in G$. For all j, we have the inclusion $i_j\colon S_j \to M$. Let $K_j \subset \pi_1(S_j)$ be the kernel of $(i_j)_*\colon \pi_1(S_j) \to \pi_1(M)$. The equivariant version of the loop theorem says that there are a finite number of properly embedded disks $D_1, \ldots, D_k$ in M which satisfy the following properties:

(1) K_j is the normal subgroup of $\pi_1(S_j)$ generated by the boundary circles ∂D_i, which lie in S_j.

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(2) For any $g \in G$ and $1 \leq i, i' \leq k$, either $D_i \cap g(D_{i'}) = \emptyset$ or $D_i = g(D_{i'})$.

The major point here is that our solution to the loop theorem respects the action of the group G in a suitable manner. The classical proof of Papakyriakopoulos, Whitehead and Shapiro, and Stallings does not seem to be readily generalizable to cover this case.

Our proof can be sketched in the following manner. We put a metric on M so that the group G acts isometrically and so that ∂M is convex with respect to the outward normal. Then with respect to this metric, we demonstrate the existence of an immersed disk D_1 in M whose boundary ∂D_1 represents a nontrivial element in $\pi_1(S)$ and whose area is minimal among all such disks. Partially using the classical topological methods developed in Papakyriakopoulos [8], Shapiro and Whitehead [11], and Stallings, [12] and partially using the properties of minimal surfaces, we show that D_1 is embedded.

If the smallest normal subgroup of $\pi_1(S)$ containing $[\partial D_1]$ is not equal to $\pi(S)$, then we minimize the area of all disks whose boundary curve is an element in $\pi_1(S)$, which does not belong to this group. In this way, we construct another embedded minimal disk D_2. Continuing this process, we obtain embedded disks $D_1, D_2, \ldots$. This process has to stop because there is only a finite number of pairwise disjoint Jordan curves that are not isotopic to each other.

Having constructed $D_1, \ldots, D_k$, we can prove (2) by using the minimality of the disks. The point is that when two minimal disks intersect nontrivially along a Jordan arc or a closed Jordan curve they must intersect transversally except at finite number of points. One can then prove that this is in contradiction to the minimality of the area by cutting the disks and deforming along their intersection curve. In this way we prove that two minimal disks with the properties described above are either equal or disjoint. Since g is an isometry, it is clear that if D is a minimal disk, then $g(D)$ has similar properties. Property (2) follows easily from this remark.

Up to now, the argument sketched above was described in detail in Meeks and Yau [4]. For the rest of this chapter, we review some of the existence theorems for minimal surfaces that may be useful to the study of the topology of three-dimensional manifolds. For that reason, we generalize some of these classical theorems to a somewhat more general category.

1. Morrey's Solution for the Plateau Problem in a General Riemannian Manifold

Let M be a complete m-dimensional manifold that is homogeneously regular in the following sense of Morrey [7]: There exists a constant $C > 0$ such that for each point $x \in M$, there exists a bi-Lipschitz homeomorphism of a neighborhood of x onto the unit ball in $\mathbf{R}^m$ with Lipschitz constants less than C. By Nash's isometric embedding theorem, we can assume that M is a properly embedded submanifold of a higher-dimensional euclidean space $\mathbf{R}^n$. Let Σ be a compact Riemann surface with boundary $\partial\Sigma$. Let $f: \Sigma \to M$ be a smooth map and $\mathscr{F}$ be the family of maps $g: \Sigma \to M$ so that the energy of g is

$$E(g) = \frac{1}{2}\int_\Sigma |\nabla g|^2 < \infty \tag{1.1}$$

and $g|\partial\Sigma = f|\partial\Sigma$.

THEOREM 1 (Morrey). *There exists a map $f_0 \in \mathscr{F}$ such that $E(f_0) = \inf\{E(g)|g \in \mathscr{F}\}$, and any such f_0 is smooth.*

Proof. Let g_i be a sequence in $\mathscr{F}$ so that $\lim_{i\to\infty} E(g_i) = \inf\{E(g)|g \in \mathscr{F}\}$. Then, since $g_i|\partial\Sigma = f$ for each i and $E(g_i)$ has an upper bound independent of i, a subsequence of g_i converges weakly in the Hilbert space of vector valued mappings g from Σ into R^n with $g|\partial\Sigma = f$ and $\|g\|^2 = \int_\Sigma |g|^2 + \int_\Sigma |\nabla g|^2 < \infty$. We may assume the subsequence is $\{g_i\}$ itself and the weak limit is g_0. It is easy to check that $g_0 \in \mathscr{F}$. As $E(g_0) \le \lim_{i\to\infty} E(g_i)$, $E(g_0) = \inf\{E(g)|g \in \mathscr{F}\}$. It remains to check that g_0 is smooth.

Let x be a point in the interior of Σ. Let $B_x(r)$ be disks of radius r around x. Then we assert that for some constant $a > 0$,

$$\int_{B_x(r)} |\nabla g_0|^2 \le ar \int_{\partial B_x(r)} |\nabla g_0|^2 \tag{1.2}$$

for r smaller than some positive constant independent of a.

If the length of $g_0(\partial B_x(r))$ is greater than $1/c$, where c is the Lipschitz constant that appears in the definition of homogeneous regularity of M, then

$$2\pi r \int_{\partial B_x(r)} |\nabla g_0|^2 \ge \left(\int_{\partial B_x(r)} |\nabla g_0|\right)^2 \ge \left(\frac{1}{c}\right)^2 \tag{1.3}$$

and (1.2) follows by choosing $a = 2\pi c^2 \int_\Sigma |\nabla g_0|^2$.

If the length of $g_0(\partial B_x(r))$ is smaller than $1/c$, we can assume that the image of $\partial B_x(r)$ under g_0 lies in the unit ball in the coordinate system that appeared in the definition of homogeneous regularity. Then we define a map from $B_x(r)$ into this unit ball by requiring each component of the map to be harmonic (with respect to the coordinate system) and its restriction on $\partial B_x(r)$ to be given by $g_0|\partial B_x(r)$. Call this map h. We can define a new map $\tilde{g}_0 \in \mathscr{F}$ by requiring that $\tilde{g}_0 = g_0$ on $\Sigma \setminus B_x(r)$ and $\tilde{g}_0 = h$ on $B_x(r)$.

Since g_0 minimizes the energy in $\mathscr{F}$, it is clear that

$$\int_{B_x(r)} |\nabla g_0|^2 \le \int_{B_x(r)} |\nabla h|^2. \tag{1.4}$$

Because $g_0 | \partial B_x(r) = h | \partial B_x(r)$, it suffices to prove (1.2) by demonstrating that

$$\int_{B(x,r)} |\nabla h|^2 \leq \text{ar} \int_{\partial B(x,r)} |\nabla_{(1/r)(\partial/\partial\theta)} h|^2. \tag{1.5}$$

However, this last inequality follows easily by expanding each coordinate of h in Fourier series.

Therefore, inequality (1.2) is proved and we can rewrite it as

$$\frac{d}{dr} \log\left(\int_{B_x(r)} |\nabla g_0|^2 \right) \geq \frac{1}{a} \frac{d}{dr} \log r. \tag{1.6}$$

By integrating, we find that

$$\int_{B_x(r)} |\nabla g_0|^2 \leq r^{1/a} \left(R^{-1/a} \int_{B_x(R)} |\nabla g_0|^2 \right), \tag{1.7}$$

where $0 \leq r < R$ and $B_x(R)$ is a fixed ball around x.

As $\int_{B_x(R)} |\nabla g_0|^2$ is bounded, (1.7) measures how $\int_{B_x(r)} |\nabla g_0|^2$ decays when $r \to 0$. A calculus lemma of Morrey [7] then shows that g_0 is Hölder continuous at x with constants that depend only on a and $R^{-1/a} \int_{B_x(R)} |\nabla g_0|^2$.

Therefore g_0 is continuous in the interior of Σ. By using the fact that $g_0 | \partial\Sigma$ is Lipschitz, one can use an argument similar to that given before to prove that g_0 is Hölder continuous near $\partial\Sigma$.

The Hölder continuity of g_0 guarantees that the image of a suitable disk $B_x(r)$ of any point $x \in \Sigma$ lies in a coordinate neighborhood of M. Using the coordinate system and the fact that g_0 minimizes energy in $\mathscr{F}$, g_0 must satisfy the variational equation

$$\frac{\partial}{\partial x}\left(\sum_i h_{ij}(g_0) \frac{\partial g_0^i}{\partial x} \right) + \frac{\partial}{\partial y}\left(\sum_i h_{ij}(g_0) \frac{\partial g_0^i}{\partial y} \right) = 0, \tag{1.8}$$

where $\sum_{i,j} h_{ij}\, dx^i\, dx^j$ is the metric tensor of M.

Since we are dealing with vector-valued functions, a difference-quotient argument (see Morrey [7]) can be used to prove the higher differentiability of g_0. This completes the proof of Theorem 1. ∎

In order to state the second theorem, one needs to introduce some terminology. Let Σ and Σ' be two not necessarily connected surfaces. Let $f: \partial\Sigma \to M$ and $f': \partial\Sigma' \to M$ be smooth diffeomorphisms, each mapping onto the same disjoint union of Jordan curves in M. Then we say that $(\Sigma, f) > (\Sigma', f')$ if Σ' can be obtained from Σ by surgery on a collection of disjoint, simple, closed curves in the interior of Σ (i.e., cut along the curve and sew back disks in neighborhoods defined as "homogeneously regular manifolds.") and f' is the same as f up to reparametrization of $\partial\Sigma$. (If Σ is oriented, we can require this reparametrization to preserve the orientation.)

For each pair (Σ, f), let $\mathscr{F}(\Sigma, f)$ be the family of all maps $g: \Sigma \to M$ so that for some conformal structure on Σ, $\int_\Sigma |\nabla g|^2 < \infty$ and $g|\partial\Sigma = f$ up to reparametrization. Let $A(\Sigma, f)$ be the infimum of all $\frac{1}{2}\int_\Sigma |\nabla g|^2$, where ∇ is taken with respect to some conformal structure over Σ and $g|\partial\Sigma$ is equal to f up to reparametrization. Using the existence of isothermal coordinates on a surface, one verifies that $A(\Sigma, f)$ is simply the infimum of the area of all possible maps $g: \Sigma \to M$ so that $g|\partial\Sigma$ is equal to f up to reparametrization.

THEOREM 2. *Suppose for each $(\Sigma', f') < (\Sigma, f)$, that $A(\Sigma', f') > A(\Sigma, f)$. Then there exists a conformal structure over Σ and a smooth conformal map $g_0: \Sigma \to M$ whose area is equal to $A(\Sigma, f)$.*

Proof. First, we fix a conformal structure on Σ and we minimize the energy $E(g)$ over all maps $g: \Sigma \to M$, so that $g|\partial\Sigma = f$ up to a reparametrization. Let g_i be a minimizing sequence. By Theorem 1, we can assume that each g_i is harmonic, and for the latter purpose we may assume that the choice of the conformal structure gives rise to $\lim_{i\to\infty} E(g_i) < A(\Sigma', f')$ whenever $(\Sigma', f') < (\Sigma, f)$.

Each g_i satisfies the equation for harmonicity. Hence the (standard) arguments in Theorem 1 of Meeks and Yau [5] show that we may assume g_i converges smoothly on compact subsets of the interior of Σ_0 to a smooth harmonic map g_0'.

We are now going to prove that g_0' is continuous in a neighborhood of $\partial\Sigma$ and is equal to $f|\partial\Sigma$ up to reparametrization. There are two cases to be discussed. If Σ is not the disk, we proceed as follows.

Let x be an arbitrary point on a component σ of $\partial\Sigma$. Then, by using the argument of Lebesgue (see the proof of Theorem 1 in [5]), we can find a number $0 < r^2 < r_i < r$ so that the length of $g_i(\partial B_x(r) \cap \Sigma)$ is not greater than $\sqrt{2\pi E(g_i)}(\log r)^{-1/2}$. Since $E(g_i)$ is uniformly bounded, the last number is arbitrarily small when r is small enough. When r is small, the arc $B_x(r) \cap \sigma$ is either mapped to an arc of $f(\sigma)$ with length small, compared with r, or mapped to the complement of such an arc on $f(\sigma)$. If the former case occurs for all $x \in \sigma$, then g_i is equicontinuous on σ and the replacement arguments in Theorem 1 show that g_0' is Hölder continuous near $\partial\Sigma$. If the latter case occurs, both $g_j(\partial B_x(r) \cap \Sigma)$ and $g_j(\sigma)\backslash g_j(B_x(r) \cap \sigma)$ have small length and hence bound a disk with small area. We can form a new surface Σ' by putting two new disks with $g_j(B_x(r) \cap \Sigma)$ and $g_j(\Sigma)\backslash g_j(B_x(r) \cap \Sigma)$ together, respectively. In this way, we form a new pair (Σ', f') with an area close to the area of g_j. This gives a contradiction to $A(\Sigma, f) < A(\Sigma'; f')$.

In case Σ is a disk, one has to fix the points 1, -1, and $\sqrt{-1}$ on the unit circle and then fix the points $g_j(1)$, $g_j(-1)$, and $g_j(\sqrt{-1})$ in $f(\sigma)$. One can make this assumption because any three distinct points on the unit circle can be mapped to the other three by a conformal automorphism of Σ and the total energy is invariant under conformal parametrization. With this assumption, the maps g_j always map small arcs to small arcs and hence are equicontinuous. Therefore, g_0' is Hölder continuous in a neighborhood of $\partial\Sigma$.

Now we change the conformal structures over Σ. The treatment here is the same as the one in Schoen and Yau [10]. We only treat the case when Σ is not the disk or the annulus. By doubling Σ and putting the Poincaré metric on the doubled Riemann surface, we can assume that Σ admits a Poincaré metric whose boundary consists of geodesics. The space of conformal structures over Σ can be identified with the space of these metrics. For each fixed conformal structure ω, which satisfies the assumption mentioned in the beginning of the proof, we can choose a map g_ω by the procedure mentioned above and the energy of this map defines a lower semicontinuous function over the space of conformal structures. However, the last space is noncompact. Hence, in order to prove that the lower semicontinuous function has a minimum, we demonstrate that it is proper. (Strictly speaking, we have to study on the Teichmüller space instead of the moduli space. The method in [10] can be used to overcome this problem.) This follows because a sequence of conformal structure tends to infinity iff for each of these conformal structures there exists an embedded closed geodesic or a geodesic arc joining the boundaries whose length with respect to the Poincaré metric tends to zero. If the length of the image of these curves under g_ω is bounded away from zero, then the arguments in [10] demonstrate that the energy of g_ω tends to infinity. Otherwise, the length of the image of the geodesics tends to zero (these geodesics must be closed geodesics because the curves in $f(\partial\Sigma)$ are fixed in M). Hence, eventually the image of the geodesics bound a disk with small area and the arguments used above show that we will violate the condition $A(\Sigma, f) < A(\Sigma', f')$.

In conclusion, we have found a conformal structure on Σ and a map $g_0 : \Sigma \to M$ such that $E(g_0) = A(\Sigma, f)$ and $g_0 | \partial\Sigma = f$ up to reparametrization. Furthermore, the arguments also showed that g_0 is smooth in the interior of Σ and Hölder continuous in a neighborhood of $\partial\Sigma$. The arguments of Hildebrandt [2] then show that g_0 is in fact smooth in a neighborhood of $\partial\Sigma$. This finishes the proof of Theorem 2. ■

Remark. In Hildebrandt [2] a proof of the theorem of Lewy and Morrey was also given. The proof states that if M is real analytic and if the image curves $f(\partial\Sigma)$ are real analytic, then the minimal surface constructed in Theorem 2 must be real analytic. The proof consists of estimating the derivatives of g_0 carefully and proving the convergence of the Taylor series of g_0.

2. The Existence Theorem for Manifolds with Boundary

In this section we extend Theorem 2 to the case in which M is allowed to have boundary ∂M. We say M is homogeneously regular if M is a subdomain of another homogeneously regular manifold N that has no boundary.

THEOREM 3. *Let M be a three-dimensional, homogeneously regular manifold whose boundary ∂M has nonnegative mean curvature with respect to the outward normal. Let Σ be a compact surface with boundary and $f: \Sigma \to M$ be a smooth map so that $f: \partial\Sigma \to M$ is an embedding. Suppose that $A(\Sigma, f) < A(\Sigma', f')$ for all $(\Sigma', f') < (\Sigma, f)$.* (See the definitions in Section 1.) *Then there exists a conformal structure over Σ and a conformal map $g: \Sigma \to M$ so that $g \mid \partial\Sigma$ is equal to f up to reparametrization of $\partial\Sigma$ and the area of g is not greater than the area of any map with the same property.*

Proof. First we notice that in case M is compact and ∂M has nonnegative mean curvature with respect to the outward normal, then Theorem 3 is valid. Furthermore, the same theorem remains valid if M is the intersection of a finite number of compact domains of the above form. This was carried out in Meeks and Yau [6].

In general, let Ω_i be an increasing sequence of compact, smooth domains in M such that $M = \bigcup_{i=1}^{\infty} \Omega_i$, $\partial M = \bigcup_{i=1}^{\infty} (\partial M \cap \Omega_i)$, and $\bigcup_{x \in \Omega_i} B_x(1) \subset \Omega_{i+1}$ for all i. Then for each i, we can change the metric in $\Omega_i \backslash \Omega_{i-1}$ so that $\partial\Omega_i$ has nonnegative mean curvature with respect to the outward normal. By the remark in the last paragraph, we can then find a conformal structure ω_i over Σ and a conformal map $g_i: \Sigma_i \to \Omega_i$ that minimizes area with respect to the changed metric on Ω_i.

We claim that for i large enough, we may assume that $\omega_i = \omega_{i+1} = \cdots$ and $g_i = g_{i+1} = \cdots$. This claim clearly implies the theorem.

For each $j \leq i - 2$ and $x \in g_i(\Sigma) \cap \Omega_j$, let $B_x(1)$ be the ball with center x and radius 1 in N. Then, as $B_x(1) \subset \Omega_{i-1}$ and the metric on Ω_{i-1} is unchanged, there is a bi-Lipschitz diffeomorphism φ of $B_x(1)$ to R^n whose Lipschitz constant is not greater than a constant C. (This comes from the definition of homogeneous regularity of N.)

We are going to bound the area of $g_i(\Sigma) \cap B_x(1)$ from below by a positive constant that is independent of x and i. Hence, we may assume that $g_i(\Sigma) \cap B_x(1)$ minimizes area in $B_x(1)$ instead of minimizing area in $B_x(1) \cap \Omega_i$. For almost every $0 < r \leq 1$, we may assume that $B_x(r) \cap g_i(\Sigma)$ is a disjoint union of Jorden curves $\sigma_1(r), \ldots, \sigma_k(r)$. Let $D_i(r)$ be the minimal disk in $\mathbf{R}^n$ whose boundary is given by $\varphi(\sigma_i(r))$. Then the area of $\varphi^{-1}(D_i(r))$ is not greater than $C^2 A_e(D_i(r))$, where $A_e(D_i(r))$ is the euclidean area of $D_i(r)$. Since $g_i(\Sigma) \cap B_x(1)$ minimizes area in $B_x(1)$, the area of $g_0(\Sigma) \cap B_x(r)$ is not greater than the sum of the areas of $\varphi^{-1}(D_i(r))$, and hence not greater than

$$C^2 \sum_{i=1}^{k} A_e(D_i(r)).$$

By the isoperimetric inequality for minimal surfaces in $\mathbf{R}^n$, $(1/4\pi)A_e(D_i(r))$ is not greater then the square of the euclidean length of $\varphi(\sigma_i(r))$. Hence the area $A(g_i(\Sigma) \cap B_x(r))$ is not greater than $4\pi C^4 l^2(g_i(\Sigma) \cap \partial B_x(r))$, where $l(g_i(\Sigma) \cap \partial B_x(r))$ is the length of $g_i(\Sigma) \cap \partial B_x(r)$.

Since $|\nabla r| \leq 1$ on $g_i(\Sigma)$ when r is restricted to $g_i(\Sigma)$, the coarea formula [1] shows that

$$4\pi C^4 \left(\frac{d}{dr}(A(g_i(\Sigma) \cap B_x(r))\right)^2 \geq A(g_i(\Sigma) \cap B_x(r)). \tag{2.1}$$

By integrating this inequality, we obtain

$$A(g_i(\Sigma) \cap B_x(r)) \geq r^2/\sqrt{\pi}\, C^2 \tag{2.2}$$

for $0 < r \leq 1$.

In particular, $A(g_i(\Sigma) \cap B_x(1)) \geq (\sqrt{\pi}\, C^2)^{-1}$. Without loss of generality, we may assume $\partial(g_i(\Sigma)) \subset \Omega_1$. If $g_i(\Sigma) \cap (\Omega_j \backslash \Omega_{j-1}) \neq \varnothing$, then we can find points $x_1, \ldots, x_{[(j-1)/2]}$ with $x_k \in g_i(\Sigma) \cap (\Omega_{2k-1} \backslash \Omega_{2k-2})$ (Here $[(j-1)/2]$ is the largest integer less than $(j-1)/2$). The balls $B_{x_1}(1), B_{x_2}(1), \ldots$ are disjoint and hence the area of $g_i(\Sigma)$ is not less than $\Sigma_k A(g_i(\Sigma) \cap B_{x_k}(1)) \geq ((j-1)/2)(\sqrt{\pi}\, C^2)^{-1}$. This implies $j \leq 2 + 2\sqrt{\pi}\, C^2 A(\Sigma, f)$ and that $g_j(\Sigma)$ lies in a fixed compact set of M. This finishes the proof of Theorem 3. ■

3. Existence of Closed Minimal Surfaces

In this section we shall record the existence of incompressible, closed, minimal surfaces in a three-dimensional homogeneous regular manifold M (possibly with boundary). If M is compact, without boundary, this was proved independently by Sacks and Uhlenbeck [9] and Schoen and Yau [10].

THEOREM 4. *Let Σ be a compact surface without boundary. Let $f: \Sigma \to M$ be a smooth map, so that $f_*: \pi_1(\Sigma) \to \pi_1(M)$ is injective and f is not homotopic to the sum of two spheres and a map which can be deformed off of every compact subset of M. If ∂M has nonnegative mean curvature with respect to the outword normal, then there exists a conformal structure on Σ and a conformal map $g: \Sigma \to M$, so that $g_*|\pi_1(\Sigma) = f_*|\pi_1(\Sigma)$ and the area of g is not greater then the area of any map from Σ to M that is homotopic to g.*

Proof. First, we treat the following special case. Assume that M has no boundary and that for any point $x \in M$ there is a contractible neighborhood Ω_x of x with $d(x, \partial\Omega_x) \to \infty$ when x tends to infinity. (In this case, we do not need the assumption that f cannot be homotopic to infinity.)

We minimize, for each conformal structure over Σ, the energies of maps whose action on $\pi_1(\Sigma)$ is the same as f. Thus let ω be any conformal structure over Σ. Then we can find a sequence of smooth maps $g_1, g_2, \ldots$ whose action on $\pi_1(\Sigma)$ is the same as f and $\lim_{i\to\infty} E(g_i)$ is the infinmum of the energies of such maps. By following the same procedure as in [10], we can produce a harmonic map g_ω with minimal energy in our class if we can prove that for some $y \in M$, there exists $\varepsilon > 0$ and $L > 0$ such that the set $\{x \in \Sigma \mid d(g_i(x),y) \le L\}$ has measure greater than ε. (This is true because after isometric embedding of M into R^n and applying the Poincaré inequality for vector-valued functions over Σ, we can estimate the L^2-norm of g_i in terms of ε, L, and the energy of g_i. Then we can take a weak convergent subsequence of g_i and proceed as in [10].)

If the last statement were wrong, then by passing to a sequence of g_i, we may assume that for some $\varepsilon_i \to 0$ and $L_i \to \infty$ the measure of

$$\{x \in \Sigma \mid d(g_i(x),y) \le L_i\}$$

is less than ε_i. Assume $\Sigma \neq \mathbf{R}P^2$ and fix an annulus region $[0, a] \times S^1$ in Σ so that $f(\{0\} \times S^1)$ is homotopically nontrivial in M. By Fubini's theorem, we can find $[0, a] \times \{\tau\}$ in $[0, a] \times S^1$ so that, except for a set of measure δ_i in $[0, a] \times \{\tau\}$ with $\delta_i \to 0$, $d(g_i(t, \tau), y) > L_i$. By assumption, there exists contractible domain $\Omega_{(t,\tau)}$ containing $g_i(t, \tau)$ so that $d(g_i(t, \tau), \partial\Omega_{(t,\tau)})$ tends to infinity uniformly as $d(g_i(t, \tau), y) > L_i$ and $i \to \infty$. Since $g_i(\{t\} \times S^1)$ is homotopically nontrivial, its length $L(g_i(\{t\} \times S^1))$ tends to infinity uniformly also. Therefore $\int_0^a L(g_i(\{t\} \times S^1))\, dt \to \infty$. Because the energy of g_i over $[0, a] \times S^1$ is dominated from below by $\{\int_0^a L(g_i(\{t\} \times S^1))\, dt\}^2$ up to a constant independent of i, $E(g_i) \to \infty$. This is a contradiction.

If Σ is $\mathbf{R}P^2$, one can proceed as follows. Let $U(\Sigma)$ be the unit tangent bundle of Σ and let $\mathscr{C}$ be the two-dimensional surface that parametrizes the set of all closed geodesics in Σ. Since the measure of $\{x \in \Sigma \mid d(g_i(x), y) \le L_i\}$ is less than ε_i, the measure of the closed geodesics that pass through

$$\{x \in \Sigma \mid d(g_i(x), y) \le L_i\}$$

is small compared with ε_i. We can consider $|\nabla g_i|^2$ as a function over $U(\Sigma)$, which fibers over $\mathscr{C}$. Hence, $E(g_i)$ can be obtained by integrating $|\nabla g_i|^2$ over the closed geodesics first and then over $\mathscr{C}$. As above, if there is a point x on the closed geodesic so that $d(g_i(x), y) > L_i$, then the integral is dominated from below by L_i^2. This also gives a contradiction and we have proved the existence of g_ω.

As in [10], we have to change the conformal structures ω and minimize $E(g_\omega)$ to achieve a map with minimal area. This can be done in exactly the same way as in [10]. This proves the theorem in the special case described above.

By using the method of [4] and [6], we see that Theorem 4 also holds if M is compact with nonnegative mean curvature with respect to the outward normal.

For the general case, we proceed as in Theorem 3. We construct an increasing sequence of compact domains Ω_i in M so that $M = \bigcup_{i=1}^{\infty} \Omega_i$ and $\bigcup_{x \in \Omega_i} B_x(1) \subset \Omega_{i+1}$. For each Ω_i, we construct a metric so that $\partial\Omega_i$ has nonnegative mean curvature with respect to the outward normal and the metric coincide with the original one on Ω_{i-1}. We may also assume that $f(\Sigma) \subset \Omega_1$.

We can then minimize area in Ω_i and obtain $g_i : \Sigma \to \Omega_i$, which is homotopic to f up to the connected sum of two spheres. By the topological assumption on f, we may find a point $x_i \in \Sigma$ so that $\lim_{i \to \infty} g_i(x_i)$ exists. This fact and the arguments provided in the proof of Theorem 3 then imply that $g_i(\Sigma)$ stays in a fixed compact set of M. This finishes the proof of Theorem 4. ■

COROLLARY. *Let Ω be a domain in $\mathbf{R}^3$ that has nonnegative mean curvature with respect to the outward normal. Then there exists no compact, incompressible surface with nontrivial fundamental group in Ω.*

Proof. Let Σ be the compact, incompressible surface with nontrivial fundamental group in Ω. Then we can choose a two-dimensional sphere S (with positive mean curvature) in $\mathbf{R}^3$ that encloses Σ. The domain $\Omega \cap S$ becomes a compact manifold with nonnegative mean curvature with respect to the outward normal. Hence we can minimize the area of Σ within $\Omega \cap S$ and obtain a compact minimal surface in $\mathbf{R}^3$. (The previous theorem applies, by smoothing, even if $\Omega \cap S$ has corners [6].) Since $\mathbf{R}^3$ has no compact minimal surface, this is a contradiction. ■

By using some topological arguments, one can then derive the following.

COROLLARY. *Let Σ be a properly embedded minimal cylinder in $\mathbf{R}^3$. Then Σ is isotopic to the catenoid.*

4. Existence of the Free Boundary Value Problem for Minimal Surfaces

By using the arguments of the above sections and [5], we can generalize Theorem 1 of [5] in the following way.

Let M be a three-dimensional, homogeneously regular manifold whose boundary ∂M has nonnegative mean curvature with respect to the outward normal. Let Σ be a compact surface with boundary and let $\mathscr{F}_\Sigma$ be the family of smooth maps $f:\Sigma \to M$ so that $f(\partial\Sigma) \subset \partial M$, so that f is not homotopic rel $\partial\Sigma$ to a map whose image is in ∂M and so that there is a fixed compact set $K \subset M$ that meets the image of every map $g:\Sigma \to M$ homotopic, as a map of pairs, to f. Let A be the infimum of the area of the maps in $\mathscr{F}_\Sigma$. We say that $\Sigma' < \Sigma$ if Σ' can be obtained by surgery along a simple closed curve or a Jordan arc of Σ which disconnects Σ.

THEOREM 5. *Let M be a three-dimensional, homogeneously regular manifold whose boundary has nonnegative mean curvature with respect to the outward normal. Let Σ be a compact surface with $A_\Sigma < A_{\Sigma'}$, for any $\Sigma' < \Sigma$. Then we can find a conformal structure over Σ and a smooth conformal map $f \in \mathscr{F}_\Sigma$ so that the area of f is equal to A_Σ.*

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