Hyperbolic Harmonic Mapping for Surface Registration

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Abstract—Automatic computation of surface correspondence via harmonic map is an active research field in computer vision, computer graphics and computational geometry. It may help document and understand physical and biological phenomena and also has broad applications in biometrics, medical imaging and motion capture. Although numerous studies have been devoted to harmonic map research, limited progress has been made to compute a diffeomorphic harmonic map on general topology surfaces with landmark constraints. This work conquer this problem by changing the Riemannian metric on the target surface to a hyperbolic metric, so that the harmonic mapping is guaranteed to be a diffeomorphism under landmark constraints. The computational algorithms are based on the Ricci flow method and the method is general and robust. We apply our algorithm to study constrained surface registration problem which applied to both medical and computer vision applications. Experimental results demonstrate that, by changing the Riemannian metric, the registrations are always diffeomorphic, and achieve relative high performance when evaluated with some popular surface registration evaluation standards.

1 Introduction

Harmonic mapping has been commonly applied for brain cortical surface registration. Physically, a harmonic mapping minimizes the "stretching energy", and produces smooth registration. The harmonic mappings between two hemsiphere cortical surfaces, which were modeled as genus zero closed surfaces, are guaranteed to be diffeomorphic, and angle-preserving [1]. Furthermore, all such kind of harmonic mappings differ by the Möbius transformation group. Numerically, finding a harmonic mapping is equivalent to solve an elliptic partial differential equation, which is stable in the computation and robust to the input noises.

Unfortunately, harmonic mappings with constraints may not be diffeomorphic any more, and produces invalid registrations with flips. In order to overcome this shortcoming, in this work we propose a novel brain registration method, which is based on hyperbolic harmonic mapping. Conventional registration methods map the template brain surface to the sphere or planar domain [1], [2], then compute harmonic mappings from the source brain to the sphere or planar domain. When the target domains are with complicated topologies, or the landmarks, the harmonic mappings may not be diffeomrophic. In contrast, in our work, we slice the brain surfaces along the landmarks, and assign a unique hyperbolic metric on the template brain, such that all the boundaries become geodesics, harmonic mappings are established and guaranteed to be diffeomorphic.

2 Theoretic Background

Hyperbolic Harmonic Map: Suppose S is an oriented surface with a Riemannian metric $\mathbf{g}$. One can choose a special local coordinates (x, y), the so-called *isothermal parameters*, such that $\mathbf{g} = \sigma(x,y)(dx^2+dy^2) = \sigma(z)dzd\bar{z}$, where the complex parameter $z = x + iy$, $dz = dx + idy$. An atlas consisting of isothermal parameter charts is called an *conformal structure*.

Definition 2.1 (Harmonic Map): The *harmonic energy* of the mapping is defined as $E(f) = \int_S \rho(z)(|w_z|^2 + |w_{\bar{z}}|^2)dxdy$. If f is a critical point of the harmonic energy, then f is called a *harmonic map*.

The necessary condition for f to be a harmonic map is the Euler-Lagrange equation $w_{z\bar{z}} + \frac{\rho_w}{\rho} w_z w_{\bar{z}} \equiv 0$. The theory on the existence, uniqueness and regularity of harmonic maps have been thoroughly discussed in [3]. The following theorem lays down the theoretic foundation of our proposed method.

Theorem 2.2: **[3]** Suppose $f : (S_1, \mathbf{g}_1) \to (S_2, \mathbf{g}_2)$ is a degree one harmonic map, furthermore the Riemann metric on S_2 induces negative Gauss curvature, then for each homotopy class, the harmonic map is unique and diffeomorphic.

Ricci Flow: Ricci flow deforms the Riemannian metric proportional to the curvature, such that the curvature evolves according to a heat diffusion process and eventually becomes constant everywhere.

Definition 2.3 (Ricci Flow): Hamilton's surface Ricci flow is defined as $\frac{dg_{ij}}{dt} = -2Kg_{ij}$.

Theorem 2.4 (Hamilton): Let $(S, \mathbf{g})$ be compact. If $\chi(S) < 0$, then the solution to Ricci Flow equation exists for all $t > 0$ and converges to a metric of constant curvature.

Fundamental Group and Fuchs Group: Let S be a surface, all the homotopy classes of loops form the fundamental group (homotopy group), denoted as $\pi_1(S)$. A surface $\tilde{S}$ with a projection map $p : \tilde{S} \to S$ is called the *universal covering space* of S. The Deck transformation $\phi : \tilde{S} \to \tilde{S}$ satisfies $\phi \circ p = p$ and form a group $Deck(\tilde{S})$. Let γ be a loop on the hyperbolic surface, then its homotopy class $[\gamma]$ corresponds to a unique Möbius transformation ϕ_γ. As the Gauss curvature of S is negative, in each homotopy class $[\gamma]$, there is a unique geodesic loop given by the axis of ϕ_γ.

Hyperbolic Pants Decomposition: Given a topological surface S, it can be decomposed to pairs of pants. Each pair of pants is a genus zero surface with three boundaries. If the surface is with a hyperbolic metric, then each homotopy class has a unique geodesic loop. Suppose a pair of hyperbolic pants with three boundaries $\{\gamma_i, \gamma_j, \gamma_k\}$, which are geodesics. Let $\{\tau_i, \tau_j, \tau_k\}$ be the shortest geodesic paths connecting each pair of boundaries. The shortest paths divide the surface to two identical hyperbolic hexagons with right inner angles. When the hyperbolic hexagon with right inner angles is isometrically embedded on the Klein disk model, it is identical to a convex Euclidean hexagon.

3 Algorithms

We explain our registration algorithm pipeline as illustrated in Alg. 1 and Fig. 1:

Algorithm 1 Brain Surface Registration Algorithm Pipeline.

1. Slice the cortical surface along the landmark curves.
2. Compute the hyperbolic metric using Ricci flow.
3. Hyperbolic pants decomposition, isometrically embed them to Klein model.
4. Compute harmonic maps using Euclidean metrics between corresponding pairs of pants, with consistent curvature based boundary matching constraints computed by the DWT algorithm.
5. Use nonlinear heat diffusion to improve the mapping to a global harmonic map on Poincare disk model.

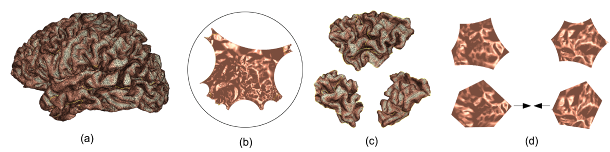

Fig. 1. Algorithm Pipeline (suppose we have 2 brain surface M and N as input): (a). One of the input brain models M, with landmarks being cut open as boundaries. (b). Hyperbolic embedding of the M on the Poincaré disk. (c). Decompose M into multiple pants by cuting the landmarks into boundaries, and each pant is further decomposed to 2 hyperbolic hexagons. (d). Hyperbolic hexagons on Poincaré disk become convex hexagons under the Klein model, then a one-to-one map between the correspondent parts of M and N can be obtained. Then we can apply our hyperbolic heat diffusion algorithm to get a global harmonic diffeomorphism.

References

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