Nodal geometry of graphs on surfaces

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Abstract

We prove two mixed versions of the Discrete Nodal Theorem of Davies et. al. [3] for bounded degree graphs, and for three-connected graphs of fixed genus g. Using this we can show that for a three-connected graph satisfying a certain volume-growth condition, the multiplicity of the nth Laplacian eigenvalue is at most $2\left[6(n-1)+15(2g-2)\right]^2$. Our results hold for any Schrödinger operator, not just the Laplacian.

1 Introduction

Let $G(V,E)$ be a finite connected graph. We denote by $x \sim y$ that $(xy) \in E$. The degree of a vertex v will be denoted by $\deg(v)$. The Laplace operator associated to G is a linear operator $\Delta : \mathbb{R}^V \to \mathbb{R}^V$ given by $\Delta f(x) = \sum_{x\sim y} f(x) - f(y)$ for any function $f \in \mathbb{R}^V$. We shall consider the more general class of Schrödinger operators. Let $M = (m_{xy})_{x,y\in V}$ be any symmetric matrix satisfying $m_{xy} < 0$ if $x \sim y$ and $m_{xy} = 0$ otherwise. The diagonal entries m_{xx} can be arbitrary. We denote again by $\Delta : \mathbb{R}^V \to \mathbb{R}^V$ the operator given by $\Delta f(x) = \sum_y m_{xy} f(y)$. Let us denoted the eigenvalues of Δ by $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_{|V|}$, and an eigenfunction corresponding to λ_i by $u^{(i)}$. (By the Perron-Frobenius theorem the multiplicity of λ_1 is 1, since G is connected.)

Let us fix an eigenfunction $u = u^{(n)}$. The vertices where u vanishes are usually referred to as *nodes*. A strong nodal domain for u is a maximal connected induced subgraph $D \le G$ on which u is either strictly positive or strictly negative. Let $D_1, D_2, \ldots, D_t$ be the list of strong nodal domains. Davies et al show in [3] that $t \le n+r-1$ where r is the multiplicity of λ_n. We are interested in an upper bound for t that involves the genus of the graph instead of the multiplicity of λ_n.

Theorem 1. *If the maximum degree is d in G then $t \le d\cdot(n-1)$. If the graph is 3-connected and g denotes its genus then $t \le 6(n-1)+14(2g-2)$.*

Remark 1.1.
- It has been observed in [3] that the star-graph on $N+1$ vertices behaves badly in terms of these type of questions. It has only three different eigenvalues: $\lambda_1 = 0, \lambda_2 = \cdots = \lambda_N = 1$ and $\lambda_{N+1} = N+1$. Furthermore any eigenfunction for λ_2 has exactly N strong nodal domains. This shows that the first statement of Theorem 1 is sharp.

- The double-star $K_{2,N}$ has similar properties: 1 is an eigenvalue of multiplicity N, and any eigenfunction has N strong nodal domains, while the genus is still 0. This shows that 3-connectedness is essential if we want an upper bound depending only on n and g in the second statement.
- One might then think that the triple star $K_{3,N}$ could be a 3-connected counter-example. It is not, however, since its genus becomes suddenly large.

Cheng [1] proved that on a smooth surface of genus g the multiplicity of λ_n is bounded by $(n+2g+1)(n+2g+2)/2$. The idea of his proof is to use the high multiplicity to obtain an eigenfunction which vanishes to a high order. This function will have a lot of sign changes near this zero, and hence it will have many nodal domains. But the number of nodal domains is limited by Courant's original nodal domain theorem. Using our discrete version of the nodal domain theorem we can adapt Cheng's approach for the graph case. However an extra assumption is needed for our graph.

Definition 1.1. A graph G satisfies the quadratic volume-growth condition VG if for any $D \subset V$ such that $|D| \leq |V|/2$ we have $|\partial D| \geq \sqrt{|D|}$. Here ∂D denotes the outer vertex-boundary of D, that is, those vertices of $V \setminus D$ that are adjacent to D.

Theorem 2. *If G is a 3-connected graph that satisfies VG then the multiplicity of λ_n is at most $2\left[6(n-1)+15(2g-2)\right]^2$ where g is the genus of G.*

Remark 1.2. As the volume growth condition is used only at the very last step of the proof, it could be easly replaced by alternative versions, yielding sligtly different bounds in Theorem 2.

2 Nodal geography

Let us fix our graph G. Let λ_n be the n-th eigenvalue of the Laplacian, and let $u = u^{(n)}$ be an eigenfunction for λ_n. We may assume without loss of generality that $\lambda_{n-1} < \lambda_n$, and fix pairwise orthogonal eigenfunctions $u^{(1)}, \ldots, u^{(n-1)}$ corresponding to $\lambda_1, \ldots, \lambda_{n-1}$.

Let $\mathcal{D} = \{D_1, D_2, \ldots, D_t\}$ be the set of strong nodal domains of u. We start by analyzing the relative location of these domains. We say that two domains D_1, D_2 are adjacent if there is an edge $v_1 \sim v_2$ such that $v_1 \in D_1$ and $v_2 \in D_2$. This of course implies that the sign of u on D_1 is different from that on D_2. This defines a graph on the set of domains.

Let us take any connected component of this graph, and take the union of the corresponding domains. We shall call this a (nodal) *region* of u. Each region consist of one or more domains. It is clear from the definition, that any vertex in the boundary of a region is a node. We call a region *small* if it consist of a single strong domain. Otherwise we call it *large*.

We shall group the regions into larger compounds which we call (nodal) *islands* of u. Similarly to regions, we are going to distinguish between *small* islands - meaning they consist of a single strong domain - and *large* islands, which contain more than one strong domain. The construction of islands is done recursively. At the beginning each region is an island on its own (either small or large, depending on the type of the region). In one step we look for a node which is adjacent to exactly two different islands, at least one of which has to be a small island, and unite these two islands into one big island. (The result is then neccessarily a large island.) We repeat this step as long as there are islands to unite. Let $I_1, I_2, \ldots, I_s$ denote the final list of islands. The set of small islands will be denoted by $\mathcal{S}$ and the set of large islands by $\mathcal{L}$. The number of strong domains in an island I shall be denoted by $t(I)$.

Claim 2.1. *Any node adjacent to a small island has to be adjacent to at least 3 different islands.*

Proof. Let us look at a small island I. If v is a node adjacent to I then the function u is non-zero at a neighbor of v. But since $\Delta u(v) = \lambda_n u(v) = 0$, there must be another neighbor of v where u is of the opposite sign. This other vertex cannot be in I since I consist of a single strong domain. Hence it must belong to a different island. Then, by the definition of the islands the node v must be adjacent to at least three islands. □

Let $V_0 \subset V$ denote the set of nodes adjacent to at least one small island. Let us consider now the t-dimensional real Euclidean vector space $\mathbb{R}^{\mathcal{D}}$ with the standard scalar product $(f, g) = \sum_i f(D_i)g(D_i)$, and the s dimensional subspace $W \leq \mathbb{R}^{\mathcal{D}}$ consisting of functions that are constant on the domains of each island. For any node $v \in V_0$ let $\varphi_v \in W$ denote the function defined by

$$\varphi_v(D) = \frac{1}{t(I(D))} \sum_{w \in I(D)} m_{vw} u(w).$$

Here $I(D)$ denotes the island in which the domain D lies. The function φ_v is made so that it is automatically constant on each island. Since $\Delta u(v) = 0$, each φ_v is orthogonal to the constant 1 function.

Lemma 2.1. *The dimension of the subspace $W_0 = \langle \varphi_v : v \in V_0 \rangle \leq W$ is at least*

a) $|\mathcal{S}|/d$ where d denotes the maximum degree of G,

b) $\frac{1}{6}(|\mathcal{S}| - 14(2g-2))$ if the G is 3-connected and g denotes the genus of G.

Proof. Both parts are proved by successively picking nodes $v_1, v_2, \dots \in V_0$ with the property that for every i the node v_i is adjacent to a small island that was not adjacent to any previously picked node. If $I \in \mathcal{S}$ and v is a node adjacent to I then $\varphi_{v_i}(I) \neq 0$. Thus our process guarantees that all the φ_{v_i} are independent.

For the first part the greedy algorithm generates a good sequence $v_1, v_2, \ldots$. In each step we find a small island that is not adjacent to any of the previously selected nodes, and choose any adjacent node as the next v_i. This way the number of small islands we can choose from decreases at most by d, hence the sequence of v_i will be of length at least $|\mathcal{S}|/d$.

For the second part we use a similar greedy algorithm. The idea is that for a fixed genus there is always a vertex of degree at most six, unless the graph is very small. Let us contract each small island to a point by contracting the edges of an arbitrary spanning tree of the island. Denote the resulting set of points by $W = \{w_1, \dots, w_{|\mathcal{S}|}\}$. Let us only keep the subgraph spanned by $V_0 \cup W$ and delete all loop and multiple edges and in general any edge not running between V_0 and W. This way we get a new bipartite graph H that is still embedded in Σ_g. Since G was 3-connected, this means that every small island had to have at least 3 adjacent nodes in V_0. In H this simply means that the degree of each w_i is at least 3. The proof of the following statement will be given below.

Claim 2.2. *If $|W| > 14(2g-2)$ then there is a vertex $v \in V_0$ whose degree is at most 6.*

This is all we need for our greedy algorithm to work: if $|W| \leq 14(2g-2)$ there is nothing to prove. On the other hand if $|W| > 14(2g-2)$ then by the claim there is a vertex $v \in V_0$ with small degree. Let us choose $v_1 = v$ and remove v_1 and all its neighbors from H. This cannot increase the genus of the graph. We repeat the process until the size of W shrinks below $14(2g-2)$. In each step we lose at most 6 vertices from W hence we get at least $\frac{1}{6}(|W| - 14(2g-2)) = \frac{1}{6}(|\mathcal{S}| - 14(2g-2))$ independent φ_v functions, as stated. ☐

Proof of Claim 2.2. Take the minimal genus representation of H. Then every face has to be a disc. Since the graph is bipartite and has no multiple edges, each face is an even cycle of length at least 4. If it is longer, we can cut it into smaller faces of length exactly 4 by drawing some of the diagonals, and keeping the graph bipartite. Finally we can transform the graph in the following way: on each face connect the two vertices belonging to W by a dotted diagonal. The dotted edges form a graph embedded in Σ_g whose vertex set is W and the faces correspond exactly to the vertices of V_0. Denote the new graph by H_1.

Assume every degree in V_0 is at least 7, that is, each face of H_1 has at least 7 sides. Hence for this graph $e \geq 7f/2$, and $e \geq 3v/2$ since each vertex has degree at least 3. Multipying the first bound by 4, the second by 10 and adding them up we get

$$14f + 15v \leq 14e = 14f + 14v + 14(2g-2),$$

that is $|W| = v \leq 14(2g-2)$ and this completes the proof. ☐

Claim 2.3. *Recall that t denoted the total number of strong domains. Let y denote the codimension of W_0 in W. Then we have*

a) $y \leq \frac{d-1}{d} t$ where d denotes the maximum degree of G,

b) $y \leq \frac{5}{6} t + \frac{14}{6}(2g-2)$ if the G is 3-connected and g denotes the genus of G.

Proof. Notice that each large island contains at least two strong domains, hence $t \geq 2|\mathcal{L}| + |\mathcal{S}|$. On the other hand by the lemma in case a) we have $y = |\mathcal{L}| + |\mathcal{S}| - \dim W_0 \leq 2\frac{d-1}{d}|\mathcal{L}| + \frac{d-1}{d}|\mathcal{S}| \leq \frac{d-1}{d}t$, and case b) is entirely analogous. $\square$

Definition 2.1. Let $\psi_1, \ldots, \psi_y$ denote a basis of the orthogonal complement of W_0 in W.

3 Proof of Theorem 1

We use the notation from the previous section. Let $w_i : V \to \mathbb{R}$ be defined by

$$w_i(v) = \begin{cases} u(v) & \text{if } v \in D_i \\ 0 & \text{otherwise.} \end{cases}$$

Let us define $f = \sum c_i w_i$. Suppose we can choose the coefficients such that $(f, f) = 1$ and f is orthogonal to $u^{(1)}, \ldots, u^{(n-1)}$, furthermore the function $c : \mathcal{D} \to \mathbb{R}$ is orthogonal to $\psi_1, \ldots, \psi_y$. We will follow closely the approach of [3] to show that these constraints imply that all the c_i's are equal to zero, which is a contradiction. The proof goes in three steps. First we show that the c_i's are constant in each region, then in each island. Finally using orthogonality to the ψ_i's we get all the c_i's are zero. The first step is explicitly, the second is implicitly contained in [3], but we repeat the arguments here to remain self-contained.

Lemma 3.1. *If $f = \sum c_i w_i$ is orthogonal to $u^{(1)}, \ldots, u^{(n-1)}$ then $\Delta f = \lambda_n f$, and for any two adjacent strong domains D_i, D_j we have $c_i = c_j$.*

Proof. We use Duval and Reiner's [4] formula, which can be verified by straightforward computation. For any self-adjoint operator A:

$$(f, Af) = \sum_{i=1}^{t} c_i^2 (w_i, Au) - \frac{1}{2} \sum_{i,j=1}^{t} (c_i - c_j)^2 (w_i, Aw_j).$$

If we choose $A = \Delta - \lambda_n I$ then since $(f, f) = 1, Au = 0$ and for $i \neq j$ the product $(w_i, Aw_j) = (w_i, \Delta w_j)$ we get

$$(f, \Delta f) - \lambda = -\frac{1}{2} \sum_{i,j=1}^{t} (c_i - c_j)^2 (w_i, \Delta w_j).$$

It is easy to see that $(w_i, \Delta w_j) = 0$ if D_i and D_j are not adjacent. If they are, then w_i and w_j have different signs, hence each non-zero term in $(w_i, \Delta w_j)$ is a product of a positive and two negative numbers.

So we have $(f, \Delta f) \leq \lambda_n$. On the other hand by the well-known min-max principle $(f, \Delta f) \geq \lambda_n (f, f)$ if f is orthogonal to the first $n - 1$ eigenfunctions. Hence in our case $\lambda_n \leq (f, \Delta f) \leq \lambda_n$. This implies by the same min-max principle that $\Delta f = \lambda_n f$. On the other hand it also implies that $(c_i - c_j)^2 (w_i, \Delta w_j) = 0$ for all i, j. If D_i and D_j are adjacent, the argument above shows that in fact $(w_i, \Delta w_j) > 0$, so we must have $c_i = c_j$. This completes the proof. $\square$

Lemma 3.2. *The c_i's are constant in each island.*

Proof. By the previous lemma we see that the c_i's are constant in each region. We prove this lemma recursively as the islands were formed. At the beginning of the process each region is an island, hence the statement is true. The only thing we have to check is whenever two islands are merged into a larger island, the statement remains true. So lets consider a particular step of the process when two islands I, J are merged into one large island. By induction we know that c is constant on I and on J. By the definition of the island forming process, at this time there must be a node v which is adjacent to only these two islands. We know by the previous lemma, that $\Delta f = f$ and $\Delta u = u$. Let us write down what this precisely means for the node v. Let

$$A = \sum_{x \in I} m_{vx} u(x); \qquad B = \sum_{x \in J} m_{vx} u(x).$$

Since $u(v) = f(v) = 0$ we get regardless of the value of λ_n that

$$c_I A + c_J B = (\Delta f)(x) = 0 = (\Delta u)(x) = A + B.$$

Since either I or J had to be a small island at this step of the process, either A or B has to be non-zero. But this implies the other being non-zero as well, and simple computation shows that this implies $c_I = c_J$.

We have showed that in each step when two islands are united, the function c remains constant in each island, hence this holds at the end as well. □

We have shown that if we regard the coefficients c_i as a function $c : \mathcal{D} \to \mathbb{R}$ then actually $c \in W$.

Lemma 3.3. *c is orthogonal to W_0.*

Proof. Let $v \in V_0$ be a node (which is by definition adjacent to at least one small island). Let $J_1, \ldots, J_p$ denote all the islands adjacent to v, and for each j let

$$A(j) = \sum_{x \in I_j} m_{vx} u(x) = \sum_{D \in J_j} \phi_v(D)$$

where D runs over all strong domains in the island J_j. The second equation holds by the definition of φ_v. Let us temporarily denote by $c(J) = c(D)$ the value of c on any domain $D \in J$. We may do this, since c is known to be constant on each island. Now similarly to the previous lemma we have

$$0 = \sum_x m_{vx} f(x) = \sum_{j=1}^{p} c(J_j) A(j) = \sum_{j=1}^{p} \sum_{D \in J_j} c(D) \varphi_v(D) = \sum_{i=1}^{t} c_i \varphi_v(D_i).$$

As this holds for every $v \in V_0$, hence for every φ_v spanning W_0, we have the desired orthogonality. □

Since c is also orthgonal to $\psi_1, \dots, \psi_y$ which is the orthogonal complement of W_0 in W, this means that c is orthogonal to W. This together with $c \in W$ implies that $c = 0$, contradicting our assumption.

Hence the $n+y-1$ orthogonality conditions imply all the c_i's are zero, hence the number of strong domains t is at most $n + y - 1$. Using Claim 2.3 simple computation shows that in case a) we get $t \le d(n-1)$ while $t \le 6(n-1) + 14(2g-2)$ follows in case b). This completes the proof of the theorem.

4 Proof of Theorem 2

Let g denote the genus of G, and let us fix an embedding of G into Σ_g, the closed oriented surface of genus g. Let us fix the n-th eigenvalue of the Laplacian $\lambda = \lambda_n$, and assume that it has multiplicity r. This means there are r linearly independent eigenfunctions $f_1, \dots, f_r$ for λ. Combining these functions we will try to create an eigenfunction which has many strong nodal domains.

First of all pick a set of r vertices $R = \{v_1, \dots, v_r\}$ which exhibit the independence of the functions $f_1, \dots, f_r$. Next choose a connected subgraph $W' \subset V$ of size $|W'| = r/2$. The W' and R sets may overlap.

Claim 4.1. *There is a linear combination $u = \sum a_i f_i$ that vanishes on W' but is non-zero on at least half of R.*

Proof. Those eigenfunctions that vanish on W' constitute an $r/2$ dimensional linear subspace of $\langle f_1, \dots, f_r \rangle$. Suppose that each of these functions vanishes on more than $r/2$ points of R. The set of eigenfunctions that vanish on a fixed vertex set of size $r/2+1$ is an $r/2-1$ dimensional subspace of $\langle f_1, \dots, f_r \rangle$. Hence we could cover an $r/2$ dimensional space with finitely many $r/2-1$ dimensional ones, which is clearly impossible. Hence the desired linear combination exists. $\square$

Let $W \subset V$ denote the connected component of nodes of u that contains W'. Let $Z = \partial(V \setminus W)$ the inner vertex-boundary of W.

Claim 4.2. $|Z| \ge \sqrt{r/2} - 1$.

Proof. Either W or $V \setminus W$ contains at most half of all the vertices. In the second case by the volume-growth property $|\partial(V \setminus W)| \ge \sqrt{|V \setminus W|} \ge \sqrt{r/2}$. In the first case we apply the growth estimate to $W \setminus Z$. Obviously $\partial(W \setminus Z) = Z$, hence $|Z|^2 \ge |W| - |Z| \ge r/2 - |Z|$. From this we get $(|Z|+1)^2 > |Z|^2 + |Z| \ge r/2$ and the claim follows. $\square$

Each vertex in Z is adjacent to a non-node of u, hence it has to be adjacent to at least a positive and a negative vertex.

Let us consider G^*, the dual graph of G on Σ_g. On each face of G^*, let us record the sign of u, whether it is plus, minus or zero.

Let us remove each edge from G^* that has the same sign recorded on its two sides. Any time we find a vertex of degree two, let us replace the two edges with a single edge, thereby removing the vertex. If we find isolated or degree one vertices, let us remove those too. It is clear, that after this process each face of the remaining graph corresponds to a strong domain of u or to a connected group of nodes of u. In particular there is the face corresponding to the nodes in W. By the construction this face now has at least $|Z|$ sides and $|Z|$ vertices. This is because if we trace the boundary of this region from the outside, we encounter at least $|Z|$ sign-changes, one at each vertex of Z.

Next we remove all the faces that correspond to nodes if u. If such a face is a p-gon, then we contract it to a single vertex which will have degree at least p. If the face had more than one boundary component, then we remove the face from Σ, glue a disc to each boundary component, and then contract each of these new faces to single vertices as above. This step might disconnect the surface or decrease its genus, but that will only be to our advantage. If in this process any vertices of degree 2 were created, we remove them as above.

Let us see what remains: each face now corresponds to precisely one strong domain of u. Since adjacent domains have opposite sign, this means that every vertex of the remaining graph has an even degree, which cannot be 2, hence each degree is at least 4. There is one special vertex that has degree at least $|Z|$. (This came from contracting our distinguished face.) The graph is drawn on a disjoint union of surfaces whose total genus is at most g. By connecting the surface-components with small tubes we can get a single surface Σ' of genus at most g in which the graph is embedded. Euler's formula now says that $e \leq 2g - 2 + f + v$ where e is the number of edges, f the number of faces and v the number of vertices. On the other hand $e \geq (|Z| + 4(v-1))/2$ by simple counting. Putting this together we get

$$f \geq 2 - 2g + v + |Z| - 2 \geq |Z| + 1 - 2g.$$

On the other hand from Theorem 1 we know that $f \leq 6(n-1) + 14(2g-2)$. Hence by Claim 4.2 we get $\sqrt{r/2} - 1 \leq |Z| \leq 6(n-1) + 15(2g-2) - 1$, from which $r \leq 2\left[6(n-1) + 15(2g-2)\right]^2$, exactly what we had to prove.

References

[1] S. Y. Cheng, Eigenfunctions and nodal sets, *Comment. Math. Helv.* **51** (1976), 43–55.

[2] R. Courant, D. Hilbert, *Methods of Mathematical Physics, Vol. 1.* Interscience, New York, 1953.

[3] B. E. Davies, G. M. L Gladwell, J. Leydold, P. F. Stadler, Discrete nodal domain theorems, *Lin. Alg. Appl.* **336** (2001) 51–60.

[4] A. M. Duval, V. Reiner, Perron-Frobenius type results and discrete versions of nodal domain theorems, *Lin. Alg. Appl.* **294** (1999) 259–268.