Differential Equations from Mirror Symmetry

Bong H. Lian,[1] and Shing-Tung Yau[2]

Abstract: We discuss a method for deriving differential equations for the prepotential and the mirror map arising from a pair of Calabi-Yau manifold. Examples with one Kähler modulus are given. Here we find that the differential equations we derive almost entirely characterize the prepotential and the mirror map in question. Some of the results here have been announced previously in [3].

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[1] Department of Mathematics, Brandeis University, Waltham, MA 02154.

[2] Department of Mathematics, Harvard University, Cambridge, MA 02138.

1. Introduction

We construct a new type of differential equations which govern the B-model prepotential and the mirror map which arise from a pair of mirror manifolds. By this we mean a pair of Calabi-Yau threefolds whereby the B-model prepotential of one threefold agrees with the A-model prepotential of the other threefold. The existence of those equations are in fact closedly related to the special geometry of the moduli spaces of Calabi-Yau manifolds.

In this paper we will restrict ourselves to the case where the mirror pair X, Y have $h^{1,1}(Y) = 1 = h^{2,1}(X) = 1$. Our construction can be generalized to the multi-moduli cases as well.

2. The mirror map and the prepotential

Let $\pi : \mathcal{X} \to \mathbf{P}^1$ be a smooth algebraic family over $\mathbf{P}^1$. We assume that the fibers $X_z := \pi^{-1}(z)$, except at finitely many points, are Calabi-Yau threefolds with $h^{2,1}(X_z) = 1$, and that the singular locus is a divisor with normal crossing. In a neighborhood of a smooth fiber X_z, we have a map from this into a universal family of Calabi-Yau threefolds, which by the theorem of Bogomolov-Tian-Todorov, is a smooth family over a disk in $H^{2,1}(X_z)$. We assume that this map is locally 1-1. We shall fix a smooth base point $z_0 \in \mathbf{P}^1$. Away from the singularity, there is a natural line bundle over the base whose fiber over z is $H^{3,0}(X_z)$. There is a section Ω whose value $\Omega(z)$ at z is a nowhere vanishing holomorphic 3-form on X_z. It is clear that Ω is unique up to a choice of a holomorphic function. For a given symplectic base $\alpha_1, , \alpha_2, \beta^1, \beta^2$ of $H^3(X_{z_0})$ with $\langle \alpha_i, \beta^j \rangle = \delta_i^j$, we can consider the integrals of $\Omega(z)$ over them near z_0. Put together, they form a 4-vector of (multivalued) holomorphic function which we call a period vector $\vec{\pi}$. It can be shown that (see [2][4]) that the period vector takes the form $(\xi_0, \xi_1, \frac{\partial G}{\partial \xi_0}, \frac{\partial G}{\partial \xi_1})$ where G is a homogeneous function of degree 2 of ξ_0, ξ_1. By a change of variable, $t = \xi_1/\xi_0$, it follows that $F = \xi_0^{-2} G$ is a function of t only, and we can write

$$\vec{\pi} = \xi_0(1, t, F', 2F - tF'). \tag{2.1}$$

It is well-known that the components of the period vectors are flat section of a bundle equipped with a natural flat connection. Concretely, the normalization of Ω can be choosen so that the components of the period vectors are solutions to a fourth order ordinary differential equations on $\mathbf{P}^1$ with regular singularities. It is known as a

Picard-Fuchs equation. In particular it is an ODE with polynomial coefficients.

Under suitable conditions, Mirror Symmetry interpretes the function F as a holomorphic function defined on the tube domain $\mathcal{T} = H^{1,1}(Y, \mathbf{R}) + \sqrt{-1}K(Y)$, where Y is another Calabi-Yau threefold and $K(Y)$ is its Kähler cone. The change of variable $t = \xi_1/\xi_0$ is interpreted as a mapping from $\mathbf{P}^1$ into $\mathcal{T}$. Note that as given, this mapping is only defined near a based point z_0. Inverting the mapping locally, we get a holomorphic function $z(t)$ which we call a *mirror map*. In Physics, F''' is known as a Yukawa coupling.

Note that both $z(t)$ and $F(t)$ are both defined at least as functions from a domain in $\mathbf{C}$, independent of the above Mirror Symmetry interpretation. In the context of mirror symmetry, many physicists have posted the following question (we thank M. Bershadsky and C. Vafa for communicating this question to us): *Is the function F governed by a differential equation?* In this paper, we answer this question in the affirmative. We shall construct a (rather complicated) 7th order polynomial differential equation for the function

$$K := F'''$$

with constant coefficients. More generally, we have

Theorem 2.1. *Let L be a 4th order linear differential operator on $\mathbf{P}^1$ with polynomial coefficients. Suppose $\xi_0, \xi_1, \xi_0 F'(\frac{\xi_1}{\xi_0}), \xi_0(2F(\frac{\xi_1}{\xi_0}) - \frac{\xi_1}{\xi_0}F'(\frac{\xi_1}{\xi_0}))$ are linearly independent solutions to $Lf = 0$ near 0 for some holomorphic function F on a domain. Then $K(t) := F'''(t)$ satisfies a 7th order polynomial differential equation with constant coefficients. Likewise for $z(t)$ as defined above.*

The result is really local in nature and can be formulated entirely in a disk. The details of the construction of the 7th order differential equation for $z(t)$ was done in [3] and shall not be repeated here. We shall first prove the theorem, and then return to examples in the last section. The hypotheses of the theorem will be in force throughout this paper.

The basic idea of the proof is this: ODEs with solutions above can be viewed as the differential algebraic analogue of polynomial equations in two variables. The functions F and z play the roles of the two variables. In the algebraic case, one can use eliminating theory to construct algebraic equations for each variables by "separating" the variables. The strategy is to try to do the same in the differential algebraic case. We achieve this by developing and using some elementary tools in differential graded algebra.

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3. Change of variables

Note that the functions $z(t)$ and $F(t)$ depend only on the ratio of two solutions ξ_1, ξ_0, hence remain unchanged if we perform a change of variables $f(z) = A(z)g(z)$ on the equation $Lf = 0$. We can choose A so that the new equation reads:

$$g''''(z) + a_2(z)g''(z) + a_1(z)g'(z) + a_0(z)g(z) = 0 \qquad (3.1)$$

where the a_i are rational functions.

Lemma 3.1. $a_1(z) = a_2'(z)$.

For the proof of this, see [2]. Under a change of variable $z \mapsto t$, (3.1) becomes

$$\tilde{L}g := g''''(z(t)) + c_2 g''(z(t)) + c_2' g'(z(t)) + c_0 g(z(t)) = 0 \qquad (3.2)$$

where

$$\begin{aligned} c_2 =& a_2 {z'}^2 - \frac{15}{2}(\frac{z''}{z'})^2 + 5\frac{z^{(3)}}{z'} = a_2 {z'}^2 + 5\{z(t), t\} \\ c_0 =& a_0 {z'}^4 + \frac{3}{2}\frac{da_2}{dz}{z'}^2 z'' - \frac{3}{4}a_2 {z''}^2 - \frac{135 {z''}^4}{16 {z'}^4} \\ &+\frac{3}{2}a_2 z' z^{(3)} + \frac{75 {z''}^2 z^{(3)}}{4{z'}^3} - \frac{15 {z^{(3)}}^2}{4{z'}^2} - \frac{15 z'' z^{(4)}}{2{z'}^2} + \frac{3z^{(5)}}{2z'} \end{aligned} \qquad (3.3)$$

The prime here means $\frac{d}{dt}$. Note that c_0, c_2 are differential rational functions of $z(t)$, ie. a rational function in $z, z', z'', \ldots$ Now (3.2)has four linear independent solutions of the form $u(1, t, F', 2F - tF')$ where $u = \xi_0/A$. We may now view (3.2) together with its four special solutions as a system of polynomial ODEs in $z(t), F(t), u(t)$. Our polynomial ODEs for $z(t), F(t)$ with constant coefficients will be obtained by separating the variables.

First we claim that

$$K = const.\ u^{-2} = const.\ \frac{A^2}{\xi_0^2}. \tag{3.4}$$

Substituting the 4 special solutions above in (3.2), we get

$$\begin{aligned} u^{(4)} + c_2 u'' + c_2' u' + c_0 u &= 0 \\ 4u''' + 2c_2 u' + c_2' u &= 0 \\ K(6u'' + c_2 u) + 4K'u' + K''u &= 0 \\ 2Ku' + K'u &= 0. \end{aligned} \tag{3.5}$$

Our claim follows from solving the 4th equation above.

Note that given this K, the 2nd equation in (3.5)can be obtained by differentiating the 3th equation. From now on we want to view the system (3.5)of equations as a system of ODEs in the two dependent variables $K(t), z(t)$ and one independent variable t. The system (3.5)can be written as

$$\begin{aligned} c_2(z;t) &= r_2(K;t) \\ c_0(z;t) &= r_0(K;t) \end{aligned} \tag{3.6}$$

where c_2, c_0 are rational function of $z, z', ..$ defined in (3.3). The r_2, r_0 are rational function of $K, K', ..$ given by

$$\begin{aligned} r_2 &= \frac{2K''}{K} - \frac{5}{2}\left(\frac{K'}{K}\right)^2 \\ r_0 &= \frac{-35\,K'^4}{16\,K^4} + \frac{5\,K'^2\,K''}{K^3} - \frac{5\,K''^2}{4\,K^2} - \frac{2\,K'\,K^{(3)}}{K^2} + \frac{K^{(4)}}{2\,K}. \end{aligned} \tag{3.7}$$

Note that the system (3.6)of equations are derived with no special assumption about the coordinate t other than that it is the ratio of two linearly independent solutions: $t = \xi_1/\xi_0$. Thus the *form* of the system (3.6)must remain the same if we use another ratio $\tilde{t}$. Thus the new equations will read:

$$\begin{aligned} c_2(z(\tilde{t});\tilde{t}) &= r_2(K(\tilde{t});\tilde{t}) \\ c_0(z(\tilde{t});\tilde{t}) &= r_0(K(\tilde{t});\tilde{t}). \end{aligned} \tag{3.8}$$

4. Some differential algebra

If we assign the weight 1 to each t differentiation (ie each prime), then both c_2, r_2 in (3.6)are expressions of weight 2. Similarly both c_0, r_0 are expression of weight 4. This motivates the following considerations. Let $x, x', .., y, y', ..$ be formal variables. We assume that $wt\ x^{(k)} = wt\ y^{(k)} = k$. We may also freely use some other formal variables $\alpha, \alpha', .., \beta, \beta', ..$ which will bear the analogous meaning. A rational function $f(x, x', .., y, y', ..)$ in these variables will be called a differential rational function. We usually denote it as $f[x, y]$. If it depends only on the x, we write $f[x]$. Given two such functions, say $f[x], g[y]$, it makes sense to compose them. Namely we do so in an obvious way $f[g[x]]$ according to the usual rule of differentiation. Let $X[x], Y[x], A[y], B[y]$ be differential rational functions, homogeneuos with positive weights $|X| = |A|$, $|Y| = |B|$. We would like to study the system of ODEs:

$$\begin{aligned} X[x] &= A[y] \\ Y[x] &= B[y]. \end{aligned} \tag{4.1}$$

We write

$$I(A[y], B[y]) := \{P[a, b] | P[A[y], B[y]] \equiv 0\}. \tag{4.2}$$

This is obviously a differential ideal in the ring of differential polynomials in two variables α, β. If α, β are assign the weight $|A|, |B|$ respectively, then $I(A[y], B[y])$ is in fact a graded ideal. Note that if $x(t), y(t)$ are solutions to the system (4.1), then every element $P[a, b] \in I(A[y], B[y])$ gives a differential equation

$$P[X[x(t)], Y[x(t)]] = 0 \tag{4.3}$$

for the function $x(t)$. This equation can of course be trivial.

We shall construct a nontrivial differential equation for the function $K(t)$ by first studying the differential graded ideal $I(c_2, c_0)$ where c_2, c_0 are given in (3.3). We will prove that there exists an element of minimal weight in this ideal. In this sense, this will be the simplest polynomial differential equation for $K(t)$.

Let's consider the effect under the formal substitution law $t \mapsto \tilde{t} = \frac{at+b}{ct+d}$, $x \mapsto \tilde{x}$, $y \mapsto \tilde{y}$ on a homogeneous differential rational function of the form $P[X[x], Y[x]]$ of weight l. It is called covariant if it transforms like $P[X[x], Y[x]] \mapsto \gamma^{2l} P[X[\tilde{x}], Y[\tilde{x}]]$. Suppose the differential rational functions $X[x], Y[x]$ transform as follows:

$$
\begin{aligned}
X[z] &\mapsto \sum_{j=0}^{n} \gamma^{2n-j} c^j Q_j[X[\tilde{z}], Y[\tilde{z}]] \\
Y[z] &\mapsto \sum_{j=0}^{m} \gamma^{2m-j} c^j R_j[X[\tilde{z}], Y[\tilde{z}]]
\end{aligned}
\tag{4.4}
$$

where $\gamma = ct + d$, n, m are the respective weights of X, Y, and the Q_j, R_j are some fixed differential polynomials of respective weights $n - j, m - j$. We also assume the same transformation law for the differential polynomials $A[y], B[y]$, but only with $x, \tilde{x}$ replaced by $y, \tilde{y}$ respectively. Note that the leading term must be given by $Q_0[X, Y] = X$ and $R_0[X, Y] = Y$. The assumption above simply says that *the differential polynomial* $\mathbf{C}(t)$*-algebra generated by* X, Y *is closed under the transformations from* $SL(2, \mathbf{C})$.

Lemma 4.1. *Suppose that* $P[\alpha, \beta]$ *is an element in* $I(A[y], B[y])$ *of minimal weight. Then* $P[X[x], Y[x]]$ *is covariant.*

Proof: It is enough to show that $P[\alpha, \beta] \mapsto \gamma^{2l} P[\alpha, \beta]$ under formal substitution:

$$
\begin{aligned}
\alpha &\mapsto \sum_{j=0}^{n} \gamma^{2n-j} c^j Q_j[\alpha, \beta] \\
\beta &\mapsto \sum_{j=0}^{m} \gamma^{2m-j} c^j R_j[\alpha, \beta].
\end{aligned}
\tag{4.5}
$$

Obvioiusly under (4.5), we have

$$
P[\alpha, \beta] \mapsto \sum_{j=0}^{l} \gamma^{2l-j} c^j S_j[\alpha, \beta] \tag{4.6}
$$

for some differential polynomials S_j of weights $l - j$. We must show that $S_j = 0$ for all $j > 0$. By assumption A, B have similar transformation law (4.5). More precisely we have

$$
P[A[y], B[y]] \mapsto \sum_{j=0}^{l} \gamma^{2l-j} c^j S_j[A[\tilde{y}], B[\tilde{y}]] \tag{4.7}
$$

Since $P[\alpha, \beta]$ is an element of the ideal $I(A[y], B[y])$, it follows that $P[A[y], B[y]]$ is identically zero. Hence the right hand side of (4.7)must also be identically zero. Thus we have $S_j[A[\tilde{y}], B[\tilde{y}]] = 0$ for all j. Since $\tilde{y}$ is just a formal variable, the same holds true with $\tilde{y}$ replaced by y. This means that the $S_j[\alpha, \beta]$ are in the ideal $I(A[y], B[y])$. But S_j has weight $l - j$ and so by the minimality of $P[\alpha, \beta]$, we have $S_j[\alpha, \beta] = 0$ for all $j > 0$. Thus (4.6)now says exactly that P is covariant of weight l. This completes the proof. □

4.1. SL(2) action on solution set

We now return to our original system of equations (3.6), and diagress slightly to consider its general solutions. Given a complex parameter t, let $\tilde{t} = \frac{at+b}{ct+d}$ where $ad - bc = 1$. Viewing $t \to \tilde{t}$ as a change of coordinates, we have the formal relations:

$$\begin{aligned} c_2(z(\tilde{t}); \tilde{t}) &= \gamma^4 c_2(z(\tilde{t}); t) \\ c_0(z(\tilde{t}); \tilde{t}) &= \gamma^8 c_0(z(\tilde{t}); t) + 3\gamma^7 c c_2'(z(\tilde{t}); t) + 6\gamma^6 c^2 c_2(z(\tilde{t}); t). \end{aligned} \tag{4.8}$$

Proposition 4.2. *Let $z(t), K(t)$ be solutions to (3.6). Then so are*

$$\begin{aligned} \tilde{z}(t) &:= z(\frac{at+b}{ct+d}) \\ \tilde{K}(t) &:= C\gamma^{-4} K(\frac{at+b}{ct+d}) \end{aligned} \tag{4.9}$$

for any constant C.

Proof: It follows immediately from (3.6)that the equations are invariant under the scaling of K by a constant. So without loss of generality, we can set $C = 1$. By direct computation, we find that

$$\begin{aligned} r_2(K(\tilde{t}); \tilde{t}) &= \gamma^4 r_2(\gamma^{-4} K(\tilde{t}); t) \\ r_0(K(\tilde{t}); \tilde{t}) &= \gamma^8 r_0(\gamma^{-4} K(\tilde{t}); t) + 3\gamma^7 c r_2'(\gamma^{-4} K(\tilde{t}); t) + 6\gamma^6 c^2 r_2(\gamma^{-4} K(\tilde{t}); t). \end{aligned} \tag{4.10}$$

Applying (3.6), (4.8) and (4.10), we get

$$\begin{aligned} c_2(z(\tilde{t}); t) &= \gamma^{-4} c_2(z(\tilde{t}); \tilde{t}) = \gamma^{-4} r_2(K(\tilde{t}); \tilde{t}) = r_2(\gamma^{-4} K(\tilde{t}); t) \\ c_0(z(\tilde{t}); t) &= \gamma^{-8} c_0(z(\tilde{t}); \tilde{t}) - 3\gamma^{-1} c c_2'(z(\tilde{t}); t) - 6\gamma^{-2} c^2 c_2(z(\tilde{t}); t) \\ &= r_0(\gamma^{-4} K(\tilde{t}); t) \quad \square \end{aligned} \tag{4.11}$$

It is interesting to see how the new solution given is *constructed.* Fix a solution $(z(t), K(t))$ to the system (3.6). And consider the same system with z, K replaced by $\tilde{z}, \tilde{K}$. Set $\tilde{z}(t) = z(\frac{at+b}{ct+d})$ where $z(t)$ is assumed to be a given solution. We would like to solve for $\tilde{K}(t)$ in terms of the given $K(t)$. We shall see that the general solution is precisely $\tilde{K}(t) = C\gamma^{-4}K(\frac{at+b}{ct+d})$. We must solve

$$\begin{aligned} r_2(K(\tilde{t}); \tilde{t}) =& \gamma^4 r_2(\tilde{K}(t); t) \\ r_0(K(\tilde{t}); \tilde{t}) =& \gamma^8 r_0(\tilde{K}(t); t) + 3\gamma^7 c r_2'(\tilde{K}(t); t) + 6\gamma^6 c^2 r_2(\tilde{K}(t); t). \end{aligned} \tag{4.12}$$

Set $y(t) = Log\ K(t)$. Then we get

$$\begin{aligned} r_2(K(t); t) =& 2y'' - \frac{y'^2}{2} \\ r_0(K(t); t) =& \frac{y^{(4)}}{2} + \frac{y''^2}{4} - \frac{y''y'^2}{2} + \frac{y'^4}{16}. \end{aligned} \tag{4.13}$$

Thus the first equation in (4.12)becomes

$$\frac{d^2y(\tilde{t})}{d\tilde{t}^2} - \frac{1}{4}(\frac{dy(\tilde{t})}{d\tilde{t}})^2 = \gamma^4(\frac{d^2\tilde{y}(t)}{dt^2} - \frac{1}{4}(\frac{d\tilde{y}(t)}{dt})^2). \tag{4.14}$$

Since $y(t)$ is given, this is a Ricatti equation in the variable $\frac{d\tilde{y}(t)}{dt}$. It's easy to verify that its general solution is

$$\tilde{y}(t) := y(\frac{at+b}{ct+d}) - 4Log\ \gamma. \tag{4.15}$$

By the standard classical method, we find that the general solution to (4.14) is

$$y_{gen}(t) = \tilde{y}(t) - \int \frac{4C exp(\frac{1}{2}\tilde{y}(t))}{1 + C\int exp(\frac{1}{2}\tilde{y}(t))} \tag{4.16}$$

where C is an arbitrary constant. Writing it in terms of $K(t)$, we get

$$\tilde{K}(t) = \frac{\gamma^{-4}K(\frac{at+b}{ct+d})}{\left(B + C\int \gamma^{-2}K(\frac{at+b}{ct+d})^{1/2}\right)^4}. \tag{4.17}$$

Now this satisfy the second equation in (4.12) iff $C = 0$.

5. ODEs for the Yukawa Coupling – Existence

Our problem now is to construct an element $P[\alpha,\beta]$ of $I(c_2,c_0)$, which will give a nontrivial differential equation for $K(t)$. The last section indicates that we should look for a *covariant* element $P[\alpha,\beta]$ such that

$$P[c_2,c_0]\equiv 0 \tag{5.1}$$

identically. We shall do this in three steps. Let α,β be formal variable of weight 2,4 respectively. We shall call a differential polynomial $P[\alpha,\beta]$ a *simple covariant* of weight k if

$$P[c_2,c_0]=p(z){z'}^k \tag{5.2}$$

for some rational function $p(z)$. It is obvious that $P[\alpha,\beta]$ is automatically a covariant. First we find two simple covariants $P_1[\alpha,\beta], Q_1[\alpha,\beta]$ of lowest weights. Our original system (3.6) then becomes

$$\begin{aligned} P_1[r_2[K],r_0[K]] &=p(z){z'}^{|P_1|}\\ Q_1[r_2[K],r_0[K]] &=q(z){z'}^{|Q_1|} \end{aligned} \tag{5.3}$$

for some rational functions $p(z),q(z)$. We then eliminate z from this pair of equations. Finally we prove that the resulting equation governing K is nontrivial.

Proposition 5.1. *Let*

$$\begin{aligned} \rho[\alpha,\beta] &=100\beta-9\alpha^2-30\alpha''\\ \chi[\alpha,\beta] &=-32\rho[\alpha,\beta]^2\alpha-45(\frac{d}{dt}\rho[\alpha,\beta])^2+40\rho[\alpha,\beta]\frac{d^2}{dt^2}\rho[\alpha,\beta]. \end{aligned} \tag{5.4}$$

They are the unique simple covariants of weights 4,10 respectively. Moreover any other simple covariant not proportional to either of them has higher weight.

Note that these two simple covariants are in fact universal: they are independent of the original 4th order differential operator. Thus they are independent of the initial data (a_2,a_0).

Proof: We prove this by solving exactly the condition (5.2) in the differential graded algebra generated by α,β up to weight 10. Since α,β have weights $2,4$, the only graded pieces we have to look are of weights between $2,10$. It is easy to work out a basis in terms of $\alpha, \alpha', .., \beta, \beta'$ for each graded pieces. Solving the above condition yields the desired result. □

It is interesting to note that the equations (5.4)defines an isomorphism

$$\iota : (\alpha, \beta) \mapsto (\rho[\alpha, \beta], \chi[\alpha, \beta]) \tag{5.5}$$

between the two fields of differential rational functions: one is generated by ρ, χ, the other by α, β.

We also note that

$$\begin{aligned} \rho[c_2, c_0] &= \rho[a_2(z), a_0(z)] z'^4 \\ \chi[c_2, c_0] &= \chi[a_2(z), a_0(z)] z'^{10}. \end{aligned} \tag{5.6}$$

Here we have slightly abused the notation: $\rho[a_2(z), a_0(z)]$ really means $100a_0(z) - 9a_2(z)^2 - 30a_2''(z)$, and similarly for $\chi[a_2(z), a_0(z)]$. Thus both are just rational functions in the variable z.

If $\rho[a_2(z), a_0(z)] = 0$ (or $\chi[a_2(z), a_0(z)] = 0$), then it's easy to show that $\rho[r_2[K], r_0[K]] = 0$ (or $\chi[r_2[K], r_0[K]] = 0$) is a nontrivial ODE for K. Thus from now on, we assume that neither is zero. Set

$$\begin{aligned} p(z) &= \rho[a_2(z), a_0(z)] \\ q(z) &= \chi[a_2(z), a_0(z)] \end{aligned} \tag{5.7}$$

Then we have

$$5\chi[c_2, c_0]\rho[c_2, c_0]' - 2\chi[c_2, c_0]'\rho[c_2, c_0] = (5q\frac{dp}{dz} - 2p\frac{dq}{dz})z'^{15}. \tag{5.8}$$

It follows that

$$\begin{aligned} \frac{\rho[c_2, c_0]^5}{\chi[c_2, c_0]^2} &= \frac{p^5}{q^2} \\ \frac{(5\chi[c_2, c_0]\rho[c_2, c_0]' - 2\chi[c_2, c_0]'\rho[c_2, c_0])^2}{\chi[c_2, c_0]^3} &= \frac{(5q\frac{dp}{dz} - 2p\frac{dq}{dz})^2}{q^3}. \end{aligned} \tag{5.9}$$

Lemma 5.2. *For any $f(z), g(z) \in \mathbf{C}(z)$, there is a nonzero polynomial $Q(a, b)$ such that $Q(f(z), g(z)) \equiv 0$ identically. If we write $f = f_1/f_2$, $g = g_1/g_2$, then such a Q may be choosen so* *that $deg_a\ Q$ and $deg_b\ Q$ are both bounded from above by $N_{max} := max(max(deg\ f_1, deg\ f_2) + max(deg\ g_1, deg\ g_2) - 2, 1)$.*

Proof: We consider the system of homogeneous linear equations resulting from the coefficients of the powers of z in

$$\sum_{i,j=0}^{N} a_{ij} f(z)^i g(z)^j = 0. \tag{5.10}$$

It is easy to verify that for $N = N_{max}$, the linear system has more variables than equations. □

In particular if we now take f, g to be the two right hand sides of (5.9), then it follows that we have a nonzero polynomial Q with the above vanishing property. Fix such a Q and define a new differential polynomial in the variables ξ, η of weights 4,10 respectively:

$$R[\xi, \eta] := \eta^{5N} Q(\frac{\xi^5}{\eta^2}, \frac{(5\eta\xi' - 2\eta'\xi)^2}{\eta^3}). \tag{5.11}$$

It is obvious that $R[\xi, \eta]$ is nonzero. Also by construction we have $R[\rho[c_2, c_0], \chi[c_2, c_0]] \equiv 0$ identically.

Proposition 5.3. $R[\rho[r_2[K], r_0[K]], \chi[c_2[K], c_0[K]]] = 0$ *is a non-*

trivial ODE for K.

Proof: By direct computation, we find

$$\rho[r_2[K], r_0[K]] = 175\, K'^4 - 280\, K \frac{K'^2\, K'' + 49\, K^2\, K''^2 + 70\, K^2\, K'\, K^{(3)} - 10\, K^3\, K^{(4)}}{K^4}$$

$$\begin{aligned}
\chi[r_2[K], r_0[K]] =4\, \Big(&1225000\, K'^{10} - 4900000\, K\, K'^8\, K'' \\
&+ 6737500\, K^2\, K'^6\, K''^2 r - 3626000\, K^3\, K'^4\, K''^3 \\
&+ 689675\, K^4\, K'^2\, K''^4 - 360836\, K^5\, K''^5 \\
&+ 1225000\, K^2\, K'^7\, K^{(3)} - 2940000\, K^3\, K'^5\, K''\, K^{(3)} \\
&+ 1690500\, K^4\, K'^3\, K''^2\, K^{(3)} \\
&+ 695800\, K^5\, K'\, K''^3\, K^{(3)} \\
&+ 367500\, K^4\, K'^4\, {K^{(3)}}^2 - 705600\, K^5\, K'^2\, K''\, {K^{(3)}}^2 \\
&- 235200\, K^6\, K''^2\, {K^{(3)}}^2 + 117600\, K^6\, K'\, {K^{(3)}}^3 \\
&- 175000\, K^3\, K'^6\, K^{(4)} + 420000\, K^4\, K'^4\, K''\, K^{(4)} \\
&- 562800\, K^5\, K'^2\, K''^2\, K^{(4)} + 203000\, K^6\, K''^3\, K^{(4)} \\
&+ 42000\, K^5\, K'^3\, K^{(3)}\, K^{(4)} + 142800\, K^6\, K'\, K''\, K^{(3)}\, K^{(4)} \\
&- 16800\, K^7\, {K^{(3)}}^2\, K^{(4)} - 12000\, K^6\, K'^2\, {K^{(4)}}^2 \\
&- 26400\, K^7\, K''\, {K^{(4)}}^2 + 94500\, K^5\, K'^3\, K''\, K^{(5)} \\
&- 103950\, K^6\, K'\, K''^2\, K^{(5)} - 31500\, K^6\, K'^2\, K^{(3)}\, K^{(5)} \\
&+ 37800\, K^7\, K''\, K^{(3)}\, K^{(5)} + 9000\, K^7\, K'\, K^{(4)}\, K^{(5)} \\
&- 1125\, K^8\, {K^{(5)}}^2 - 17500\, K^5\, K'^4\, K^{(6)} \\
&+ 28000\, K^6\, K'^2\, K''\, K^{(6)} - 4900\, K^7\, K''^2\, K^{(6)} \\
&- 7000\, K^7\, K'\, K^{(3)}\, K^{(6)} + 1000\, K^8\, K^{(4)}\, K^{(6)}\Big) \,/K^{10}
\end{aligned} \tag{5.12}$$

Observe that $R[\xi, \eta]$ has either a nontrivial leading power in η', or else it has a leading power in η. In the first case, we see

$R[\rho[r_2[K], r_0[K]], \chi[c_2[K], c_0[K]]]$ has a nontrivial leading term containing the 7th derivative $K^{(7)}$. In the second case we have a nontrivial leading term containing $K^{(6)}$. In either case, the resulting ODE is nontrivial. □

This completes the proof of our main theorem.

6. Remarks and examples

The proof above indicates also that we can put a bound on the minimal weight of our ODE for K. Since Q obviously has weight zero, the differential polynomial R above has weight $5N$. Thus the minimal weight is no more than $5N_{max}$.

We now return to the discussion of the Picard-Fuchs equation of a family of Calabi-Yau 3-folds. In the case of the mirror quintic, we have the famous Picard-Fuchs equation

$$\left(\Theta^4 - 5z(5\Theta+4)(5\Theta+3)(5\Theta+2)(5\Theta+1)\right) f(z)) = 0 \qquad (6.1)$$

where $\Theta := z\frac{d}{dz}$. In this case one finds that $N_{max} = 1400$. This turns out to be far from optimum. Using the computer, we find an ODE for K of weight 360.

In practice, we can simplify the computation for an ODE significantly as follows. If we regard $5\eta\xi' - 2\eta'\xi$ as another variable, then the construction in the last section guarantees that there have a polynomial in $\xi, \eta, \delta := 5\eta\xi' - 2\eta'\xi$ which gives a nontrivial ODE for K.

Thus it is enough to look in the polynomial ring in ξ, η, δ where the weights of the variables are respectively $4, 10, 15$. Our construction guarantees that any zero polynomial quasi-homogeneous polynomial $P(\xi, \eta, \delta)$ satisfying

$$P(\rho[c_2, c_0], \chi[c_2, c_0], 5\rho[c_2, c_0]'\chi[c_2, c_0] - 2\rho[c_2, c_0]\chi[c_2, c_0]') \equiv 0 \quad (*) \quad (6.2)$$

identically gives a a nontrivial ODE for K. We want to find one with the minimum weight.

Lemma 6.1. *A necessary and sufficient condition for a nonzero polynomial $P(\xi, \eta, \delta)$ satisfying (*) to be a minimum of weight n is that every polynomial $T(\xi, \eta, \delta)$ satisfying (*) of weight $n-1, n-2, n-3,$ or $n-4$ is zero.*

Proof: The necessity of the condition is clear. To show sufficiency, note that if $T(\xi, \eta, \delta)$ is a polynomial of weight lower that $n-4$ and satisfies (*), then we can multiply it by a suitable power – say ξ^k – so that $\xi^k T(\xi, \eta, \delta)$ has weight $n-1, n-2, n-3$, or $n-4$ because ξ has weight 4. Now it must be zero, implying that T itself is zero. □

When we try to find a polynomial $P(\xi, \eta, \delta)$ satisfying (*) and of minimal weight, say n, the lemma saves us from checking the the lower weight polynomials – all we have to check is that there is none in weights $n-4$ to $n-1$. For example, in the case of the mirror quintic. We verify by computer that there are no nonzero polynomial $T(\xi, \eta, \delta)$ satisfying (*) and has weight between 359 and 356. Moreover there are exactly 127 monomials of weight 360 in the three variables. Up to multiple, exactly one linear combination of them satisfies (*).

Here we shall give the simplest example known to us: here we consider the family of Calabi-Yau 3-folds X mirror to the complete intersection of 4 quadrics in $\mathbf{P}^7$. In this case, the minimal polynomial satisfying (*) has weight 180:

$$\begin{aligned}
&P(\delta, \chi, \rho) = \\
&3783403212890625\,\chi^{18} + 52967644980468750\,\chi^{15}\,\delta^2 \\
&+ 292835408677734375\,\chi^{12}\,\delta^4 + 833559395864062500\,\chi^9\,\delta^6 \\
&+ 1301823644717109375\,\chi^6\,\delta^8 + 1064406315612768750\,\chi^3\,\delta^{10} \\
&+ 357449882108765625\,\delta^{12} + 909717589887890625 0\,\chi^{16}\,\rho^5 \\
&+ 7554368090695078125 0\,\chi^{13}\,\delta^2\,\rho^5 \\
&- 551687817628209375 00\,\chi^{10}\,\delta^4\,\rho^5 \\
&- 12359332799277384375 00\,\chi^7\,\delta^6\,\rho^5 \\
&- 26283288293882470687 50\,\chi^4\,\delta^8\,\rho^5 \\
&- 16694424211736222437 50\,\chi\,\delta^{10}\,\rho^5 \\
&- 3163959222224629737093 75\,\chi^{14}\,\rho^{10} \\
&- 10413036935813864040750 00\,\chi^{11}\,\delta^2\,\rho^{10} \\
&+ 13970612413905450453112 50\,\chi^8\,\delta^4\,\rho^{10} \\
&- 978071752628929206000\,\chi^5\,\delta^6\,\rho^{10} \\
&+ 30889615158829455201731 25\,\chi^2\,\delta^8\,\rho^{10} \\
&+ 72637526392158281350412 2500\,\chi^{12}\,\rho^{15} \\
&- 24019675673062579829188 92000\,\chi^9\,\delta^2\,\rho^{15} \\
&+ 29319060393675698423999 77800\,\chi^6\,\delta^4\,\rho^{15} \\
&- 25920077297305483107297 52000\,\chi^3\,\delta^6\,\rho^{15} \\
&+ 264772114318563252921325 00\,\delta^8\,\rho^{15} \\
&+ 731773527868504699561324929024\,\chi^{10}\,\rho^{20} \\
&- 1384453886791545382987331665920\,\chi^7\,\delta^2\,\rho^{20}
\end{aligned}$$

$$
\begin{aligned}
&+ 1003786188392583028918031769600\,\chi^4\,\delta^4\,\rho^{20} \\
&- 92650299984331138894225408000\,\chi\,\delta^6\,\rho^{20} \\
&+ 26437995071637403548055756603392 0\,\chi^8\,\rho^{25} \\
&- 32365388499667835941553990270976 0\,\chi^5\,\delta^2\,\rho^{25} \\
&+ 10512210115205768202081722695680 0\,\chi^2\,\delta^4\,\rho^{25} \\
&+ 488537001674142496400389234386534 40\,\chi^6\,\rho^{30} \\
&- 331534237606649896835138314672537 60\,\chi^3\,\delta^2\,\rho^{30} \\
&+ 52812067925398832115636932444160 0\,\delta^4\,\rho^{30} \\
&+ 49655388960105132238220106179962470 40\,\chi^4\,\rho^{35} \\
&- 12389340807480736990290861242921779 20\,\chi\,\delta^2\,\rho^{35} \\
&+ 26502177116226635590076194581676818432 0\,\chi^2\,\rho^{40} \\
&+ 5822406825670998196401392296588763725824\,\rho^{45}
\end{aligned}
$$

(6.3)

Note that since δ, χ, ρ are of weights 15,10,4 respectively, P is a quasi-homogeneous polynomial of weight 180. Each of the 37 terms in this polynomial corresponds to a partition of 180 by 15,10,4.

In the second example, we consider the complete intersection of two cubic in $\mathbf{P}^5$. In this case, we find that the ODE for K is given by a polynomial of weight 330. There are 108 partitions of 330 by 15,10,4. Thus this polynomial has a maximum length of 108.

We have verified that in all cases above, our differential equation determines the Yukawa coupling $K(t)$ up to the first two Fourier coefficients.

References

[1] P. Candelas, X. de la Ossa, P. Green, and L. Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, Nucl. Phys. B359 (1991) 21-74.

[2] S. Ferrara, C. Kounnas, D. Lüst, and F. Swirner, *Automorphic Functions and Special Kähler Geometry*, In *Essays on Mirror Manifolds*, Ed. S.T. Yau, International Press 1992.

[3] A. Klemm, B.H. Lian, S.S. Roan, and S.T. Yau, *A Note on ODEs from Mirror Symmetry*, Conference Proceeding in honor of I. Gel'fand (1995).

[4] A. Strominger, *Commun. Math. Phys. 133 (1990) 163.*