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A REMARK ON THE EXISTENCE OF SPHERE WITH PRESCRIBED MEAN CURVATURE*

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In 1976, the author (see the open problem section in [3]) asked the following question: if ρ is a function defined on R^3, what is the condition on ρ so that there is a sphere in R^3 whose mean curvature is given by ρ. There were important works by Treibergs-Wei [2] and others on this question. Recently in a lecture that I gave in the Chinese University of Hong Kong, I realized that my work with R. Schoen on the existence of black hole can be used to settle the prescribed mean curvature question for a large class of ρ.

Recall that in [1], we are given on a three dimensional manifold M, two tensors (g_{ij}, p_{ij}) where $g_{ij} > 0$ is a Riemannian metric.

Let

$$\begin{aligned} \mu =& \frac{1}{2}\left[R - \sum p^{ij}p_{ij} + \left(\sum p_i^i\right)^2\right] \\ J^i =& \sum_j D_j\left[p^{ij} - \left(\sum p_k^k\right)g^{ij}\right] \end{aligned}$$

where R is the scalar curvature of M.

For a domain Ω in M, we define

$$Rad(\Omega) = \sup\Big\{r : \text{for any simple closed curve } \Gamma \text{ in } \Omega \text{ which is homotopically trivial, } \Gamma \text{ does not bound a disk in a tubular neighborhood } N_r(\Gamma) \text{ of radius } r\Big\}$$

Then the major theorem in [1] says

THEOREM 1. *Let Ω be a bounded region on which $\mu - |J| \geq \Lambda > 0$ holds. Assume that $\Omega \subset \Omega_1$ where the mean curvature of $\partial\Omega_1$ (with respect to the outer normal) is strictly greater than the absolute value of the trace of $p_{ij}|\partial\Omega_1$. If $Rad(\Omega) \geq \sqrt{\frac{3}{2}}\frac{\pi}{\sqrt{\Lambda}}$, then Ω_1 contains a sphere Σ such that the mean curvature of $\Sigma = \pm\,\mathrm{tr}_\Sigma(p_{ij})$. Moreover, any such Σ lying within Ω has diameter at most $\frac{2\pi}{\sqrt{3}\sqrt{\Lambda}}$ and any such Σ intersecting $\partial\Omega$ has the property that $\Omega \cap \Sigma$ has everywhere within distance $\frac{2\pi}{\sqrt{3}\sqrt{\Lambda}}$ of $\partial\Omega$.*

In this note, we apply this theorem to the prescribed mean curvature problem:

Let $p_{ij} = \frac{\rho}{2}g_{ij}$. Then

$$\begin{aligned} 2\mu =& R + \frac{3}{2}\rho^2 \\ J^i =& (D_j\rho)g^{ij} \end{aligned}$$

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and $\mu - |J| = \frac{1}{2}R + \frac{3}{4}\rho^2 - |D\rho|$.

Hence we have the following theorem

THEOREM 2. *Let Ω_1 be a compact domain in M so that the mean curvature of $\partial\Omega_1$ (with respect to its outer normal) is greater than $|\rho|$. Let $\Omega \subset \Omega_1$ be a subdomain on which*

$$\frac{1}{2}R + \frac{3}{4}\rho^2 - |D\rho| \geq \Lambda > 0$$

where R is the scalar curvature of M.

If $Rad(\Omega) \geq \sqrt{\frac{3}{2}\frac{\pi}{\Lambda}}$, then Ω_1 contains a sphere Σ intersecting Ω so that the mean curvature of $\Sigma = \pm\rho$. Moreover, any such Σ lying within Ω has diameter at most $\frac{2\pi}{\sqrt{3}\sqrt{\Lambda}}$ and any such Σ intersecting $\partial\Omega$ has the property that $\Omega \cap \Sigma$ lies everywhere within distance $\frac{2\pi}{\sqrt{3}\sqrt{\Lambda}}$ of $\partial\Omega$.

In [4], we also observed the following: Let $\lambda_1 \geq \lambda_2 \geq \lambda_3$ be eigenvalue of the symmetric tensor p_{ij}. Then if for some nonnegative function f and $\Omega \subset \Omega_1$ so that

$$\int_{\partial\Omega} f + \int_{\Omega} |\nabla f| < \int_{\Omega} f \min(\lambda_2 + \lambda_3, \lambda_2 + 2\lambda_3)$$

then there is a closed surface Σ such that the mean curvature of $\Sigma = \pm\,\mathrm{tr}_{\Sigma}(p_{ij})$.

THEOREM 3. *Let Ω_1 be a compact domain in M so that the mean curvature of $\partial\Omega_1$, (with respect to outer normal) is greater than $|\rho|$. Suppose for some domain $\Omega \subset \Omega_1$, and for some nonegative function f,*

$$\int_{\partial\Omega} f + \int_{\Omega} |\nabla f| < \int_{\substack{\Omega \\ \rho>0}} f\rho + \frac{3}{2}\int_{\substack{\Omega \\ \rho<0}} f\rho$$

Then there is a closed surface Σ whose mean curvature $= \pm\rho$.

REMARK It is an interesting question to determine the genus of Σ.

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