October 2014

yymm.nnnn [hep-th]
D(11.2): fermionic D-brane

D-branes and Azumaya/matrix noncommutative differential geometry, II: Azumaya/matrix supermanifolds and differentiable maps therefrom – with a view toward dynamical fermionic D-branes in string theory

Chien-Hao Liu and Shing-Tung Yau

Abstract

In this Part II of D(11), we introduce new objects: super-C^k-schemes and Azumaya super-C^k-manifolds with a fundamental module (or synonymously matrix super-C^k-manifolds with a fundamental module), and extend the study in D(11.1) ([L-Y3], arXiv:1406.0929 [math.DG]) to define the notion of 'differentiable maps from an Azumaya/matrix supermanifold with a fundamental module to a real manifold or supermanifold'. This allows us to introduce the notion of 'fermionic D-branes' in two different styles, one parallels Ramond-Neveu-Schwarz fermionic string and the other Green-Schwarz fermionic string. A more detailed discussion on the Higgs mechanism on dynamical D-branes in our setting, taking maps from the D-brane world-volume to the space-time in question and/or sections of the Chan-Paton bundle on the D-brane world-volume as Higgs fields, is also given for the first time in the D-project. Finally note that mathematically string theory begins with the notion of a differentiable map from a string world-sheet (a 2-manifold) to a target space-time (a real manifold). In comparison to this, D(11.1) and the current D(11.2) together bring us to the same starting point for studying D-branes in string theory as dynamical objects.

Key words: D-brane, fermionic D-brane, sheaf of spinor fields; super-C^k-ring; supermanifold, super-C^k-scheme; Azumaya supermanifold, matrix supermanifold; C^k-map; Higgs mechanism, generation of mass.

MSC number 2010: 81T30, 58A40,14A22; 58A50, 16S50, 51K10, 46L87, 81T60, 81T75, 81V22.

Acknowledgements. We thank Cumrun Vafa for lectures and discussions that influence our understanding of string theory. C.-H.L. thanks in addition Harald Dorn for illuminations on nonabelian Dirac-Born-Infeld action for coincident D-branes; Gregory Moore, Cumrun Vafa for illuminations on Higgs fields on D-branes; Dennis Westra for thesis that influences his understanding of super-algebraic geometry in line with Grothendieck's Algebraic Geometry; Murad Alim, Gaëtan Borot, Daniel Freed, Siu-Cheong Lau, Si Li, Baosen Wu for discussions on issues beyond; Alison Miller, Freed, Vafa for topic and basic courses, fall 2014; Gimnazija Kranj Symphony Orchestra for work of Nikolay Rimsky-Korsakov that accompanies the typing of the notes; Ling-Miao Chou for discussions on electrodynamics, comments on illustrations, and moral support. D(11.1) and D(11.2) together bring this D-project to another phase; for that, special thanks also to Si Li, Ruifang Song for the bi-weekly Saturday D-brane Working Seminar, spring 2008, that gave him another best time at Harvard and tremendous momentum to the project. The project is supported by NSF grants DMS-9803347 and DMS-0074329.

Chien-Hao Liu dedicates this note
to his another advisor Prof. Orlando Alvarez
during his Berkeley and Miami years,
who gave him the first lecture on D-branes
and brought him to the amazing world of stringy dualities;
and to Prof. Rafael Nepomechie,
a pioneer on higher-dimensional extended objects beyond strings,
who gave him the first course on supersymmetry.

0. Introduction and outline

As a preparation to study D-branes in string theory as dynamical objects, in [L-Y3] (D(11.1)) we developed the notion of 'differentiable maps from an Azumaya/matrix manifold with a fundamental module to a real manifold' along the line of Algebraic Geometry of Grothendieck and synthetic/C^k-algebraic differential geometry of Dubuc, Joyce, Kock, Moerdijk, and Reyes; and gave examples to illustrate how deformations of differentiable maps in our setting capture various behaviors of D-branes.

In this continuation of [L-Y3] (D(11.1)), we extend the study to the notion of 'differentiable maps from an Azumaya/matrix supermanifold with a fundamental module to a real supermanifold'. This allows us to introduce the notion of 'fermionic D-branes' in two different styles, one parallels Ramond-Neveu-Schwarz fermionic string and the other Green-Schwarz fermionic string. [L-Y1] (D(1)), [L-L-S-Y] (D(2)), [L-Y3] (D(11.1)) and the current note (D(11.2)) together bring

- the study of D-branes in string theory as dynamical objects in the context/realm/language of algebraic geometry or differential/symplectic/calibrated geometry, without supersymmetry or with supersymmetry

all in the equal footing. This brings us to the door of a new world on dynamical D-branes, whose mathematical and stringy-theoretical details have yet to be understood.

The organization of the current note is as follows. In Sec. 1, we brings out the notion of differential maps from an Azumaya/matrix brane with fermions in a most primitive setting based on [L-Y3] (D(11.1)). In Sec. 2 – Sec. 4, we first pave our way toward uniting the new fermionic degrees of freedom into the Azumaya/matrix geometry involved, as is done in the study of supersymmetric quantum field theory to the ordinary geometry, and then define the notion of 'differentiable maps from an Azumaya/matrix supermanifold with a fundamental module to a real manifold' in Sec. 4.2 and further extend it to the notion of 'differentiable maps from an Azumaya/matrix supermanifold with a fundamental module to a real supermanifold' in Sec. 4.3. To give string-theory-oriented readers a taste of how such notions are put to work for D-branes, in Sec. 5.1, we introduce the two notions of fermionic D-branes, one following the style of Ramond-Neveu-Schwarz fermionic string and the other the style of Green-Schwarz fermionic string; and in Sec. 5.2 we give a more precise discussion on the Higgs mechanism on dynamical D-branes in our setting for the first time in this D-project. Seven years have passed since the first note [L-Y1] (D(1)) in this project in progress. In Sec. 6, we reflect where we are in this journey on D-branes, with a view toward the future.

Convention. Standard notations, terminology, operations, facts in (1) superring theory toward superalgebraic geometry; (2) supersymmetry, supersymmetric quantum field theory; (3) Higgs mechanism, gauge symmetry breaking; grand unification theory can be found respectively in (1) [Wes]; (2) [Arg1], [Arg2], [Arg3], [Arg4], [D-E-F-J-K-M-M-W], [Freed], [Freund], [G-G-R-S], [St], [Wei], [W-B]; (3) [I-Z], [P-S], [Ry]; [B-H], [Mo], [Ros]. There are several inequivalent notions of (4) 'supermanifold'; all intend to (and each does) capture (some part of) the geometry behind supersymmetry in physics. The setting in (4) [Man], [S-W] is particularly in line with Grothendieck's Algebraic Geometry and hence relevant to us.

- '*field*' in the sense of quantum *field* theory (e.g. *fermionic field*) vs. '*field*' as an algebraic structure in ring theory (e.g. the *field* $\mathbb{R}$ *of real numbers*).

- For clarity, the *real line* as a real 1-dimensional manifold is denoted by $\mathbb{R}^1$, while the *field of real numbers* is denoted by $\mathbb{R}$. Similarly, the *complex line* as a complex 1-dimensional manifold is denoted by $\mathbb{C}^1$, while the *field of complex numbers* is denoted by $\mathbb{C}$.

- The inclusion '$\mathbb{R} \hookrightarrow \mathbb{C}$' is referred to the *field extension of* $\mathbb{R}$ *to* $\mathbb{C}$ by adding $\sqrt{-1}$, unless otherwise noted.

- The *real n-dimensional vector spaces* $\mathbb{R}^{\oplus n}$ vs. the *real n-manifold* $\mathbb{R}^n$;
similarly, the *complex r-dimensional vector space* $\mathbb{C}^{\oplus r}$ vs. the *complex r-fold* $\mathbb{C}^r$.

- All manifolds are paracompact, Hausdorff, and admitting a (locally finite) partition of unity. We adopt the *index convention for tensors* from differential geometry. In particular, the tuple coordinate functions on an n-manifold is denoted by, for example, $(y^1, \cdots y^n)$. The up-low index summation convention is always spelled out explicitly when used.

- '*differentiable*' = k-times differentiable (i.e. C^k) for some $k \in \mathbb{Z}_{\geq 1} \cup \infty$; '*smooth*' = C^∞; C^0 = *continuous* by standard convention.

- All the C^k-rings in this note can be assumed to be *finitely generated and finitely-near-point determined*. In particular, they are *finitely generated and germ-determined* ([Du], [M-R: Sec. I.4]) i.e. *fair* in the sense of [Joy: Sec. 2.4]. (Cf. [L-Y4].)

- For a C^k-subscheme Z of a C^k-scheme Y, Z_{red} denotes its associated reduced subscheme of Y by modding out all the nilpotent elements in $\mathcal{O}_Z$.

- The '*support*' *Supp* $(\mathcal{F})$ of a quasi-coherent sheaf $\mathcal{F}$ on a scheme Y in algebraic geometry or on a C^k-scheme in C^k-algebraic geometry means the *scheme-theoretical support* of $\mathcal{F}$ unless otherwise noted; $\mathcal{I}_Z$ denotes the *ideal sheaf* of a (resp. C^k-)subscheme of Z of a (resp. C^k-)scheme Y; $l(\mathcal{F})$ denotes the *length* of a coherent sheaf $\mathcal{F}$ of dimension 0.

- *coordinate-function index*, e.g. $(y^1, \cdots, y^n)$ for a real manifold vs. the *exponent of a power*, e.g. $a_0 y^r + a_1 y^{r-1} + \cdots + a_{r-1} y + a_r \in \mathbb{R}[y]$.

- *global section functor* $\Gamma(\,\cdot\,)$ on sheaves vs. *graph* Γ_f of a function f.

- '*d-manifold*' in the sense of '*d*erived manifold' vs. '*D-manifold*' in the sense of '*D*(irichlet)-brane that is supported on a manifold' vs. '*D-manifold*' in the sense of works [B-V-S1] and [B-V-S2] of Michael Bershadsky, Cumrun Vafa and Vladimir Sadov.

- The current Note D(11.2) continues the study in

  [L-Y1] *Azumaya-type noncommutative spaces and morphism therefrom: Polchinski's D-branes in string theory from Grothendieck's viewpoint*, arXiv:0709.1515 [math.AG] (D(1)).

  [L-L-S-Y] (with Si Li and Ruifang Song), *Morphisms from Azumaya prestable curves with a fundamental module to a projective variety: Topological D-strings as a master object for curves*, arXiv:0809.2121 [math.AG](D(2)).

  [L-Y3] *D-branes and Azumaya/matrix noncommutative differential geometry, I: D-branes as fundamental objects in string theory and differentiable maps from Azumaya/matrix manifolds with a fundamental module to real manifolds*, arXiv:1406.0929 [math.DG](D(11.1)).

  Notations and conventions follow these earlier works when applicable.

Outline

1 Differentiable maps from fermionic Azumaya/matrix branes: A primitive setting

In this section, we give a very primitive view of D-branes with fermions directly from the viewpoint of [L-Y3] (D(11.1)) that a bottommost ingredient to describe a D-brane in string theory as a dynamical object is the notion of a differentiable map from a matrix manifold (cf. the world-volume of coincident D-branes) to a real manifold (cf. the target space-time).

Bosonic fields and fermionic fields on the world-volume of coincident D-branes[1].

Fields on the world-volume of a D-brane are created by excitations of oriented open strings through their end-points that stick to the D-brane. Cf. FIGURE 1-1. When the open string

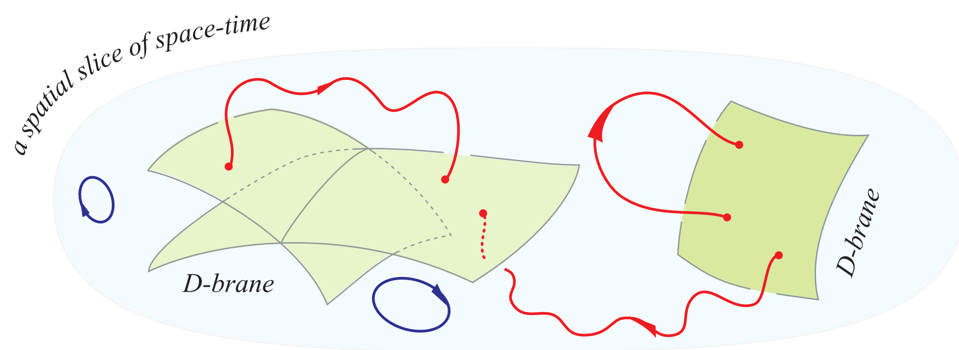

FIGURE 1-1. Fields on the world-volume of a D-brane are created by excitations of oriented open strings through their end-points that stick to the D-branes. The dynamics of these fields are dictated by the anomaly-free requirement of the conformal field theory on the open-string world-sheet ([Le]).

carries in addition fermionic degrees of freedom, the fields it creates on the D-brane world-volume include not only bosonic ones but also fermionic ones. Each of these fields is associated to an open string state $|\Lambda\rangle$ from a representation of the 2-dimensional superconformal algebra (associated to the open-string world-sheet theory) that is repackaged to a representation of Lorentz group under the requirement that the quantum field theory of these fields on the D-brane world-volume be Lorentz invariant.

When r-many simple D-branes coincide, the open string spectrum $\{|\Lambda\rangle\,|\,\Lambda\}$ on the common D-brane world-volume gets enhanced. There are three possible sectors of the newly re-organized spectrum of open string states:

(1) *From oriented open strings with both end-points stuck to the coincident D-brane* :
In this case, one has an enhancement

$$|\Lambda\rangle \;\Rightarrow\; |\Lambda; i, \bar{j}\rangle\,, \quad 1 \le i, j \le r\,.$$

The field ψ_Λ on the D-brane world-volume that is associated to $\{|\Lambda; i, \bar{j}\rangle\,|\,1 \le i, j \le r\}$ as a collection takes now $r \times r$-matrix-values.

[1] The contents of this theme are by now standard textbook materials under the theme on *quantization of closed or open strings and the spectrum closed or open strings create on the target*. The concise conceptual highlight here is only meant to make a passage to relate fields on the world-volume of coincident D-branes to sheaves of modules on a matrix manifold in the next theme. Unfamiliar mathematicians are referred to [B-B-Sc], [G-S-W], [Po2] for details.

(2) *When, for example, the whole target space-time itself is taken as a background simple D-brane or equivalently one of the two end-point of the oriented open string takes the Neumann boundary condition instead of the Dirichlet boundary condition*:

There are two sectors in this case:

$$|\Lambda\rangle \;\Rightarrow\; |\Lambda; i\rangle\,, \quad 1 \le i \le r\,, \qquad \text{and} \qquad |\Lambda\rangle \;\Rightarrow\; |\Lambda; \bar{j}\rangle\,, \quad 1 \le j \le r\,.$$

The former are created by oriented open strings with only the beginning end-point stuck to the D-brane world-volume; and the latter are created by oriented open strings with only the ending end-point stuck to the D-brane world-volume. The field ψ_Λ on the D-brane world-volume that is associated to $\{|\Lambda; i\rangle \,|\, 1 \le i \le r\}$ as a collection takes now $r \times 1$-matrix-values, i.e., column-vector-values. And the field ψ'_Λ on the D-brane world-volume that is associated to $\{|\Lambda; \bar{j}\rangle \,|\, 1 \le \bar{j} \le r\}$ as a collection takes now $1 \times r$-matrix-values, i.e., row-vector-values.

Cf. FIGURE 1-2.

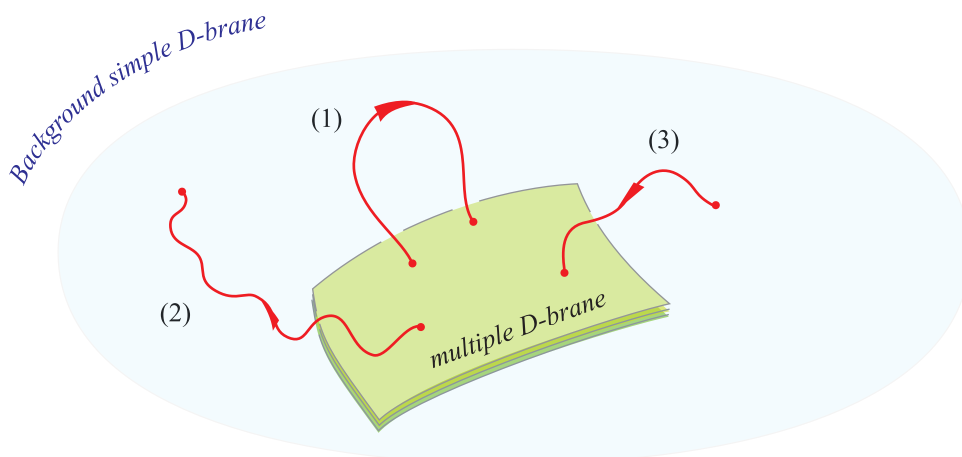

FIGURE 1-2. Three possible sectors of fields on the world-volume of coincident D-branes. They are created respectively by (1) oriented open strings with both end-points stuck to the D-brane world-volume, or (2) oriented open strings with only the beginning end-point stuck to the D-brane world-volume; or (3) oriented open strings with only the ending end-point stuck to the D-brane world-volume. Sector (1) is always there on the D-brane world-volume while Sectors (2) and (3) can arise only when there is a background D-brane world-volume in the space-time to which the other end of oriented open strings can stick. Fields in Sector (1) (resp. Sector (2), Sector (3)) are matrix-valued (resp. column-vector-valued, row-vector-valued).

Differentiable maps from matrix branes with fermions

As explained in [L-Y1] (D(1)), while originally the pair $(i, \bar{j})$ should be thought of as labeling elements in the Lie algebra $u(r)$, to bring geometry to the enhanced scalar field on the D-brane world-volume that describes the collective deformations of the coincident D-branes along the line of Grothendieck's Algebraic Geometry, it is more natural to embed $u(n)$ into the Lie algebra $gl(r, \mathbb{C})$ which now has the underlying unital associative algebra structure, namely the matrix ring $M_{r\times r}(\mathbb{C})$. Following this line, coincident D-branes are now collectively described by a differentiable map

$$\varphi\;:\;(X^{\!A\!z},\mathcal{E})\;:=\;(X,\mathcal{O}_X^{A\!z}:=\mathcal{E}nd_{\mathcal{O}_X^{\mathbb{C}}}(\mathcal{E}),\mathcal{E})\;\longrightarrow\;Y$$

from a matrix manifold $X^{\!A\!z}$ with a fundamental module $\mathcal{E}$ to the target space-time Y; cf. [L-Y3] (D(11.1)). The vector bundle associated to $\mathcal{E}$ plays the role of the Chan-Paton bundle on the D-brane world-volume.

The enhancement of fields on the D-brane world-volume due to coincidence of simple D-branes now takes the following form:

(1) *Field in Sector* (1) *corresponding to* $|\Lambda\rangle\Rightarrow\{|\Lambda;i,\bar{j}\rangle\,|\,1\le i,j\le r\}$:
$\mathcal{O}_X$-module $\mathcal{F}_\Lambda$ $\;\;\Rightarrow\;\;$ bi-$\mathcal{O}_X^{A\!z}$-module $\mathcal{G}_\Lambda\;:=\;\mathcal{E}\otimes_{\mathcal{O}_X^{\mathbb{C}}}\mathcal{F}_\Lambda\otimes_{\mathcal{O}_X}\mathcal{E}^\vee\simeq\mathcal{O}_X^{A\!z}\otimes_{\mathcal{O}_X^{A\!z}}\mathcal{F}_\Lambda$.

(2) *Field in Sector* (2) *corresponding to* $|\Lambda\rangle\Rightarrow\{|\Lambda;i\rangle\,|\,1\le i\le r\}$:
$\mathcal{O}_X$-module $\mathcal{F}_\Lambda$ $\;\;\Rightarrow\;\;$ left $\mathcal{O}_X^{A\!z}$-module $\mathcal{G}_\Lambda\;:=\;\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{F}_\Lambda$.

(3) *Field in Sector* (3) *corresponding to* $|\Lambda\rangle\Rightarrow\{|\Lambda;\bar{j}\rangle\,|\,1\le j\le r\}$:
$\mathcal{O}_X$-module $\mathcal{F}_\Lambda$ $\;\;\Rightarrow\;\;$ right $\mathcal{O}_X^{A\!z}$-module $\mathcal{G}_\Lambda\;:=\;\mathcal{F}_\Lambda\otimes_{\mathcal{O}_X^{\mathbb{C}}}\mathcal{E}^\vee$.

Here $\mathcal{E}^\vee:=\mathcal{H}om_{\mathcal{O}_X^{\mathbb{C}}}(\mathcal{E},\mathcal{O}_X^{\mathbb{C}})$ is the dual of $\mathcal{E}$. Note that the functor in Item (2) (resp. Item (3)) is the functor that appears in the Morita equivalence of the category of $\mathcal{O}_X^{\mathbb{C}}$-modules and the category of left (resp. right) $\mathcal{O}_X^{A\!z}$-modules.

Sections of $\mathcal{G}_\Lambda$ correspond to the field ψ_Λ on the world-volume of coincident D-branes in the previous theme. Here, ψ_Λ can be either bosonic or fermionic. The dynamics of differentiable map φ and that of sections of the various $\mathcal{G}_\Lambda$'s in general will influence each other through their equations of motion, which is a topic in its own right.

With this primitive setting in mind and as a motivation, we now proceed to study how the fermionic degrees of freedom on a D-brane world-volume can be united into the geometry of the D-brane world-volume — rendering it a matrix supermanifold with a fundamental module — and how the notion of differentiable maps from a matrix manifold can be promoted to the notion of differentiable maps from a matrix supermanifold.

Remark 1.1. [*reduction from* $M_{r\times r}(\mathbb{C})$ *to* $u(r)$]. Gauge theoretically, a reduction from the underlying Lie algebra $gl(r,\mathbb{C})$ of $M_{r\times r}(\mathbb{C})$ to the original $u(r)$ can be realized by introducing a Hermitian metric on the fundamental module $\mathcal{E}$. However, how this influences or constrains the notion of differential maps in our setting in [L-Y3] (D(11.1)) should be studied in more detail.

2 Algebraic geometry over super-C^k-rings

Basic notions and terminology from super-C^k-algebraic geometry required for the current note are introduced in this section. The setting given is guided by a formal $\mathbb{Z}/2$-graded extension of C^k-algebraic geometry and the goal to study fermionic D-branes later.

2.1 Superrings, modules, and differential calculus on superrings

We collect in this subsection the most basic notions in superrings, supermodules, and superdifferential calculus needed for the current note. Readers are referred to the thesis '*Superrings and supergroups*' [Wes] of Dennis Westra for further details and the foundation toward super-algebraic geometry in line with Grothendieck's Algebraic Geometry.

Superrings and modules over superrings

Definition 2.1.1. [superring]. A *superring* A is a $\mathbb{Z}/2$-graded $\mathbb{Z}/2$-commutative (unital associative) ring $A = A_0 \oplus A_1$ such that the multiplication $A \times A \to A$ satisfies

$$
\begin{array}{lll}
(\mathbb{Z}/2\text{-}\textit{graded}) & A_0A_0 \subset A_0\,, \quad A_0A_1 = A_1A_0 \subset A_1\,, \quad \text{and} \quad A_1A_1 \subset A_0\,, \\
(\mathbb{Z}/2\text{-}\textit{commutative}) & aa' = (-1)^{ii'}a'a \qquad \text{for } a \in A_i \text{ and } a' \in A_{i'},\ i, i' = 0, 1\,.
\end{array}
$$

A *morphism* between superrings (i.e. *superring-homomorphism*) is a $\mathbb{Z}/2$-grading-preserving ring-homomorphism of the underlying unital associative rings. The elements of A_0 are called *even*, the elements of A_1 are called *odd*, and an element that is either even or odd is said to be *homogeneous*. For a homogeneous element $a \in A$, denote by $|a|$ the $\mathbb{Z}/2$-*degree* or *parity* of a; $|a| = i$ if $a \in A_i$, for $i = 0$, 1.

An *ideal* of I of A is said to be $\mathbb{Z}/2$-*graded* if $I = (I \cap A_0) + (I \cap A_1)$. In this case, A induces a superring structure on the quotient ring A/I, with the $\mathbb{Z}/2$-grading given by $A/I = (A_0/(I \cap A_0)) \oplus (A_1/(I \cap A_1))$. The converse is also true; cf. Definition/Lemma 2.1.2.

Definition/Lemma 2.1.2. [$\mathbb{Z}/2$-graded ideal = supernormal ideal]. An ideal I of a superring A is called *supernormal* if A induces a superring structure on the quotient ring A/I. *In terms of this, I is $\mathbb{Z}/2$-graded if and only if I is supernormal.*

Proof. The only-if part is immediate. For the if part, let $\kappa : A \to A/I$ is the quotient-superring map and $a = a_0 + a_1 \in I = Ker(\kappa)$. If, say, $a_0 \notin I$, then both $\kappa(a_0)$ and $\kappa(a_1)$ are non-zero in A/I and hence have parity even and odd respectively since κ is a superring-homomorphism by the assumption. On the other hand, $\kappa(a) = \kappa(a_0) + \kappa(a_1) = 0$; thus, $\kappa(a_0) = -\kappa(a_1)$. Since $(A/I)_0 \cap (A/I)_1 = 0$, this implies that $\kappa(a_0) = \kappa(a_1) = 0$, which is a contradiction. This proves the lemma.

$\square$

Definition 2.1.3. [module over superring]. Let A be a superring. An A-*module* M is a *left module* over the unital associative ring underlying A that is endowed with a $\mathbb{Z}/2$-*grading* $M = M_0 \oplus M_1$ such that

$$
A_0M_0 \subset M_0\,, \quad A_1M_0 \subset M_1\,, \quad A_0M_1 \subset M_1\,, \quad \text{and} \quad A_1M_1 \subset M_0\,.
$$

The elements of M_0 are called *even*, the elements of M_1 are called *odd*, and an element that is either even or odd is said to be *homogeneous*. For a homogeneous element $m \in M$, denote by $|m|$ the $\mathbb{Z}/2$-*degree* or *parity* of m; $|m| = i$ if $m \in M_i$, for $i = 0$, 1.

For a superring A,

- *a left A-module is canonically a right A-module* by setting $ma := (-1)^{|m||a|}am$ for homogeneous elements $a \in A$ and $m \in M$ and then extending $\mathbb{Z}$-linearly to all elements.

For that reason, as in the case of commutative rings and modules, we don't distinguish a left-, right-, or bi-module for a module over a superring.

A *morphism* (or *module-homomorphism*) $h : M \to M'$ between A-modules is a right-module-homomorphism between the right-module over the unital associative ring underlying A; or equivalently a left-module-homomorphism between the left-module over the unital associative ring underlying A but with the sign rule applied to homogeneous components of h and homogeneous elements of A. Explicitly, h is said to be *even* if it preserves the $\mathbb{Z}/2$-grading or *odd* if it switches the $\mathbb{Z}/2$-grading; decompose h to $h = h_0 + h_1$ a summation of even and odd components, then $h_i(am) = (-1)^{i|a|} a h_i(m)$, $i = 0, 1$, for $a \in A$ homogeneous and $m \in M$.

Subject to the above sign rules when applicable, the notion of

- *submodule* $M' \hookrightarrow M$, (cf. *monomorphism*),
- *quotient module* $M \twoheadrightarrow M'$, (cf. *epimorphism*),
- *direct sum* $M \oplus M'$ of A-modules,
- *tensor product* $M \otimes_A M'$ of A-modules,
- *finitely generated*: if $A^{\oplus l} \twoheadrightarrow M$ exists for some l,
- *finitely presented*: if $A^{\oplus l'} \to A^{\oplus l} \to M \to 0$ is exact for some l, l'

are all defined in the ordinary way as in commutative algebra.

Differential calculus on a superring

Definition 2.1.4. [superderivation on superring]. Let A be a superring over another superring B with $B \to A$ the underlying superring-homomorphism. Then, a (*left*) *super-B-derivation* ζ on A *of fixed parity* $|\zeta| = 0$ or 1 is a map

$$\zeta \;:\; A \longrightarrow A$$

that satisfies

$$\begin{array}{ll} (\textit{left-}B\textit{-superlinearity}) & \zeta(ba + b'a') \;=\; (-1)^{|\zeta||b|} b\,\zeta(a) \,+\, (-1)^{|\zeta||b'|}\, b'\,\zeta(a')\,, \\ (\textit{super Leibniz rule}) & \zeta(aa') \;=\; \zeta(a)\, a' \,+\, (-1)^{|\zeta||a|}\, a\,\zeta(a') \end{array}$$

for all b, $b' \in B$ and a, $a' \in A$ homogeneous, and all the $\mathbb{Z}$-linear extensions of these relations. A (*left*) *super-B-derivation* ζ on A is a formal sum $\zeta = \zeta_0 + \zeta_1$ of a super-B-derivation ζ_0 on A of even parity and a super-B-derivation ζ_1 on A of odd parity.

Denote by $sDer_B(A)$ the set of all super-B-derivations on A. Then, $sDer_B(A)$ is $\mathbb{Z}/2$-graded by construction. Furthermore, if $\zeta \in sDer_B(A)$, then so does $a\zeta = (-1)^{|a||\zeta|}\zeta a$, with

$$(a\zeta)(\,\cdot\,) \;:=\; a(\zeta(\,\cdot\,)) \qquad \text{and} \qquad (\zeta a)(\,\cdot\,) \;:=\; (-1)^{|a||\,\cdot\,|}\,(\zeta(\,\cdot\,))a$$

for $a \in A$. Thus, *$sDer_B(A)$ is naturally a (bi-)A-module*, with $a \cdot \zeta := a\zeta$ and $\zeta \cdot a := \zeta a$ for $a \in A$ and $\zeta \in sDer_B(A)$.

Note that if ζ, $\zeta' \in sDer_B(A)$ (homogeneous), then so does their *super Lie bracket* (synonymously, *supercommutator*)

$$[\zeta, \zeta'] \;:=\; \zeta\zeta' \,-\, (-1)^{|\zeta||\zeta'|}\, \zeta'\zeta\,.$$

Thus, *$sDer_B(A)$ is naturally a super-Lie algebra.* Homogeneous elements in which satisfy the super-anti-commutativity identity and the super-Jacobi identity:

$$\begin{aligned} [\zeta, \zeta'] \;&=\; -\,(-1)^{|\zeta||\zeta'|}\,[\zeta', \zeta]\,, \\ [\zeta, [\zeta', \zeta'']] \;&=\; [[\zeta, \zeta'], \zeta''] \,+\, (-1)^{|\zeta||\zeta'|}\,[\zeta', [\zeta, \zeta'']]\,. \end{aligned}$$

When A is a k-algebra, we will denote $sDer_k(A)$ also by $sDer(A)$, with the ground field k understood.

Remark 2.1.5. [*equivalent condition*]. The *left-B-superlinearity condition* and the *super Leibniz rule condition* in Definition 2.1.4, are equivalent to

$$\begin{array}{ll} (\textit{right-}B\textit{-linearity}) & \zeta(ab+a'b') \;=\; \zeta(a)\,b \,+\, \zeta(a')\,b'\,, \\ (\textit{super Leibniz rule}') & \zeta(aa') \;=\; \zeta(a)\,a' \,+\, (-1)^{|a||a'|}\,\zeta(a')\,a \end{array}$$

respectively. Note that the parity $|\zeta|$ of ζ is removed in this equivalent form. The format in Definition 2.1.4 is more natural-looking while the above equivalent form is more convenient to use occasionally. Cf. Definition 2.1.7.

Remark 2.1.6. [*inner superderivation*]. Similarly to the commutative case, all the inner superderivations of a superring are zero.

Definition 2.1.7. [superderivation with value in module]. Let A be a superring over another superring B and M be an A-module. A $\mathbb{Z}$-linear map

$$d \,:\, A \,\rightarrow\, M$$

is called a *super-B-derivation with values in M* if d satisfies

$$\begin{array}{ll} (\textit{right-}B\textit{-linearity}) & d(ab+a'b') \;=\; d(a)\,b \,+\, d(a')\,b'\,, \\ (\textit{super Leibniz rule}') & d(aa') \;=\; d(a)\,a' \,+\, (-1)^{|a||a'|}\,d(a')\,a \end{array}$$

for a, $a' \in A$ homogeneous, and all $\mathbb{Z}$-linear extension of such identities. In particular, d is a B-module-homomorphism. d is said to be *even* if $d(A_0) \subset M_0$ and $d(A_1) \subset M_1$; and *odd* if $d(A_0) \subset M_1$ and $d(A_1) \subset M_0$. The set $sDer_B(A,M)$ of all super-B-derivations with values in M is naturally a ($\mathbb{Z}/2$-graded) bi-A-module, with the multiplication defined by

$$a\cdot d \,:\, a' \,\longmapsto\, a\,(d(a')) \qquad \text{and} \qquad d\cdot a \,:\, a' \,\longmapsto\, (-1)^{|a||a'|}\,(d(a'))\,a$$

for $a \in A$ and $d \in sDer_B(A,M)$ homogeneous, plus a $\mathbb{Z}$-linear extension.

Definition 2.1.8. [module of differentials of superring]. Continuing Definition 2.1.7. An A-module M with a super-B-derivation $d : A \rightarrow M$ is called the *cotangent module* of A if it satisfies the following universal property:

- For any A-module M' and super-B-derivation $d' : A \rightarrow M'$, there exists a unique homomorphism of A-modules $\psi : M \rightarrow M'$ such that $d' = \psi \circ d$.

$$\begin{tikzcd} A \arrow[rr, "d"] \arrow[dr, "d'"'] & & M \arrow[dl, dotted, "\psi"] \\ & M' & \end{tikzcd} .$$

(Thus, M is unique up to a unique A-module isomorphism.) We denote this M with $d : A \rightarrow M$ by $\Omega_{A/B}$, with the built-in super-B-derivation $d : A \rightarrow \Omega_{A/B}$ understood.

Remark 2.1.9. [*explicit construction of* $\Omega_{A/B}$]. The cotangent module $\Omega_{A/B}$ of a superring A over B can be constructed explicitly from the A-module generated by the set

$$\{d(a)\,|\,a\in A\}\,,$$

subject to the relations

$$\begin{array}{ll}
(\mathbb{Z}/2\text{-}grading) & |d(a)| \;=\; |a|\,,\\
(right\text{-}B\text{-}linearity) & d(ab+a'b') \;=\; d(a)\,b \,+\, d(a')\,b'\,,\\
(super\ Leibniz\ rule') & d(aa') \;=\; d(a)\,a' \,+\, (-1)^{|a||a'|}\,d(a')\,a\,,\\
(bi\text{-}A\text{-}module\ structure) & d(a)\,a' \;=\; (-1)^{|a||a'|}\,a'\,d(a)
\end{array}$$

for all b, $b'\in B$, a, $a'\in A$ homogeneous, plus a $\mathbb{Z}$-linear extension of these relations. Denote the image of $d(a)$ under the quotient by da. Then, by definition, the built-in map

$$\begin{array}{cccc}
d\;:\; & A & \longrightarrow & \Omega_{A/B}\\
 & a & \longmapsto & da
\end{array}$$

is a super-B-derivation from A to $\Omega_{A/B}$.

Remark 2.1.10. [*relation between* $sDer_B(A)$ *and* $\Omega_{A/B}$]. The universal property of $\Omega_{A/B}$ implies that there is an A-module-isomorphism $sDer_B(A,M)\simeq Hom_A(\Omega_{A/B},M)$ for any A-module M. In particular, for $M=A$, one has $sDer_B(A)\simeq Hom_A(\Omega_{A/B},A)$.

2.2 Super-C^k-rings, modules, and differential calculus on super-C^k-rings

Basic notions and terminology from super-C^k-algebra are introduced in this subsection. They serve the basis to construct super-C^k-schemes and quasi-coherent sheaves thereupon

Remark 2.2.1. [*on the setting in this subsection, alternative, and issue beyond*]. The setting in the current subsection allows the transcendental/nonalgebraic notion of C^k-rings to merge with the algebraic notion of superrings immediately. It leads to the notion of super-C^k-manifolds and super-C^k-schemes that are nothing but a sheaf-type super-thickening of ordinary C^k-manifolds and ordinary C^k-schemes; cf. Remark 2.3.16 and FIGURE 3-1. While mathematically these are not the most general kind of superspaces, physically they are broad enough to cover the superspaces and supermanifolds that appear in supersymmetric quantum field theory and superstring theory in most situations.

There are alternatives to our setting. A most fundamental one would be re-do the algebraic geometry over C^k-rings, using now the C^k-function rings $\coprod_{(p,q)} C^k(\mathbb{R}^{p|q})$, where $\mathbb{R}^{p|q}$ is the super-C^k-manifold whose coordinates $(x^1,\cdots,x^p;\theta^1,\cdots,\theta^q)$ have both commuting and anti-commuting variables. Different interpretations/settings to the evaluation of ordinary C^k-functions of $\mathbb{R}^p$ on p-tuples of supernumbers $\in\mathbb{R}^{1|q'}$ may lead to different classes of algebraic geometry over super-C^k-rings. In this sense, our setting is the simplest one. Other more general settings should be studied in their own right.

Super-C^k-rings

With Sec. 2.1 as background, we now build into the superring in question an additional C^k-ring structure on an appropriate subring of its even subring.

Definition 2.2.2. [superpolynomial ring over C^k-ring]. Let R be a C^k-ring. A *superpolynomial ring over* R is a (unital) associative ring over R of the following form

$$R[\theta^1,\,\cdots,\theta^s]\;:=\;\frac{R\langle\theta^1,\,\cdots,\theta^s\rangle}{(r\theta^\alpha-\theta^\alpha r\,,\,\theta^\beta\theta^\gamma+\theta^\gamma\theta^\beta\,|\,r\in R\,,\,1\le\alpha,\beta,\gamma\le s)}\,,$$

where

- $R\langle\theta^1,\,\cdots,\theta^s\rangle$ is the unital associative ring over R generated by the variables $\theta^1,\,\cdots,\theta^s$,
- $(r\theta^\alpha-\theta^\alpha r\,,\,\theta^\beta\theta^\gamma+\theta^\gamma\theta^\beta\,|\,r\in R\,,\,1\le\alpha,\beta,\gamma\le s)$ is the bi-ideal in $R\langle\theta^1,\,\cdots,\theta^s\rangle$ generated by the elements indicated.
- The $\mathbb{Z}/2$-grading is determined by specifying $|r|=0$, $|\theta^\alpha|=1$ for all $r\in R$ and $1\le\alpha\le s$ and the extension by the product rule $|\widehat{r}\widehat{r}'|=|\widehat{r}||\widehat{r}'|$ whenever applicable.

Underlying $R[\theta^1,\,\cdots\,\theta^s]$ as an algebra-extension of R is a *built-in split short exact sequence* of R-modules

$$\begin{tikzcd}0 \arrow[r] & (\theta^1,\,\cdots,\theta^s) \arrow[r] & R[\theta^1,\,\cdots,\theta^s] \arrow[r] & R \arrow[r] \arrow[l, dotted, bend left] & 0\,,\end{tikzcd}$$

where $R\to R[\theta^1,\,\cdots,\theta^s]$ is the built-in R-algebra inclusion map. It is also an exact sequence of $R[\theta^1,\,\cdots,\theta^s]$-modules in the sense of Definition 2.1.3.

Let V be a vector space over $\mathbb{R}$ of dimension s, spanned by $\{\theta^1,\,\cdots,\theta^s\}$, and $\bigwedge^\bullet V$ be the ($\mathbb{Z}/2$-graded $\mathbb{Z}/2$-commutative) *Grassmann/exterior algebra* associated to V. Then

$$R[\theta^1,\,\cdots,\theta^s]\;\simeq\;R\otimes_{\mathbb{R}}\textstyle\bigwedge^\bullet V$$

as $\mathbb{Z}/2$-graded $\mathbb{Z}/2$-commutative rings over R, with

$$R[\theta^1,\,\cdots,\theta^s]_0\;\simeq\;R\otimes_{\mathbb{R}}\textstyle\bigwedge^{even}V\qquad\text{and}\qquad R[\theta^1,\,\cdots,\theta^s]_1\;\simeq\;R\otimes_{\mathbb{R}}\textstyle\bigwedge^{odd}V\,.$$

.

Definition 2.2.3. [super-C^k-ring: split super-extension of C^k-ring]. Let R be a C^k-ring. A *split super-extension* $\widehat{R}$ *of* R is a (unital) associative ring $\widehat{R}$ over R that is equipped with a split short exact sequence of R-modules

$$\begin{tikzcd}0 \arrow[r] & M \arrow[r] & \widehat{R} \arrow[r] & R \arrow[r] \arrow[l, dotted, bend left] & 0\end{tikzcd}$$

such that

- There exists a superpolynomial ring $R[\theta^1,\,\cdots,\theta^s]$ over R, for some s, that can realize $\widehat{R}$ as its superring-quotient R-algebra

  $$R[\theta^1,\,\cdots,\theta^s]\twoheadrightarrow\widehat{R}$$

  in such a way that the following induced diagram commutes

  $$\begin{tikzcd}0 \arrow[r] & (\theta^1,\,\cdots,\theta^s) \arrow[r] \arrow[d] & R[\theta^1,\,\cdots,\theta^s] \arrow[r] \arrow[d] & R \arrow[r] \arrow[l, dotted, bend left] \arrow[d, equal] & 0\\ 0 \arrow[r] & M \arrow[r] & \widehat{R} \arrow[r] & R \arrow[r] \arrow[l, dotted, bend left] & 0\end{tikzcd}\quad.$$

We will call $\widehat{R}$ synonymously a *super-C^k-ring* over R. In particular, R itself is trivially a super-C^k-ring, with $M = 0$.

Definition 2.2.4. [homomorphism between super-C^k-rings]. Let R and S be C^k-rings, $\widehat{R}$ be a super-C^k-ring over R, and $\widehat{S}$ be a super-C^k-ring over S. A *super-C^k-ring-homomorphism* from $\widehat{R}$ to $\widehat{S}$ is a pair of superring-homomorphisms (cf. Definition 2.1.1)

$$\widehat{f}\;:\;\widehat{R}\;\longrightarrow\;\widehat{S}\qquad\text{and}\qquad f\;:\;R\;\longrightarrow\;S$$

such that

(1) f is a C^k-ring-homomorphism,

(2) $(\widehat{f}, f)$ is compatible with the underlying super-C^k-ring structure of $\widehat{R}$ and $\widehat{S}$; namely, the following diagram commutes

$$\begin{tikzcd}
\widehat{R} \arrow[r, "\widehat{f}"] \arrow[d] & \widehat{S} \arrow[d] \\
R \arrow[u, dotted, bend left] \arrow[r, "f"] & S \arrow[u, dotted, bend left]
\end{tikzcd}\quad .$$

For the simplicity of notations, we may denote the pair $(\widehat{f}, f) : (\widehat{R}, R) \to (\widehat{S}, S)$ also as $\widehat{f} : \widehat{R} \to \widehat{S}$. We say that a super-$C^k$-ring-homomorphism $\widehat{f} : \widehat{R} \to \widehat{S}$ is *injective* (resp. *surjective*) if both $\widehat{f}$ and f are injective (resp. surjective). In this case, $\widehat{f}$ is called a *super-C^k-ring-monomorphism* (resp. *super-C^k-ring-epimorphism*).

Definition 2.2.5. [ideal, super-C^k-normal ideal, super-C^k-quotient]. Let $\widehat{R}$ be a super-C^k-ring. An *ideal* $\widehat{I}$ of $\widehat{R}$ is an ideal $\widehat{I}$ of $\widehat{R}$ as a $\mathbb{Z}/2$-graded $\mathbb{R}$-algebra. $\widehat{I}$ is called *super-C^k-normal* if the super-C^k-ring structure on $\widehat{R}$ descends to a super-C^k-ring structure on the quotient $\mathbb{R}$-algebra $\widehat{R}/\widehat{I}$. In this case, $\widehat{I}$ must be a $\mathbb{Z}/2$-graded ideal of $\widehat{R}$ (cf. Lemma 2.1.2). $\widehat{R}/\widehat{I}$ with the induced super-C^k-ring structure is called a *super-C^k-quotient* of $\widehat{R}$ and one has the following commutative diagram

$$\begin{tikzcd}
\widehat{R} \arrow[r, "\widehat{q}"] \arrow[d] & \widehat{R}/\widehat{I} \arrow[d] \\
R \arrow[u, dotted, bend left] \arrow[r, "q"] & R/I \arrow[u, dotted, bend left]
\end{tikzcd}\quad ,$$

where $\widehat{q} : \widehat{R} \to \widehat{R}/\widehat{I}$ is the quotient map and $I := \widehat{I} \cap R$ is now a C^k-normal ideal of the C^k-ring R. Note that for $\widehat{I}$ C^k-normal, the quotient super-C^k-ring structure on $\widehat{R}/\widehat{I}$ is compatible with the quotient $\mathbb{R}$-algebra structure.

Definition 2.2.6. [localization of super-C^k-ring]. Let $\widehat{R}$ be a super-C^k-ring, with $R \hookrightarrow \widehat{R}$ the built-in inclusion, S be a subset of R, and $R[S^{-1}]$ be the localization of the C^k-ring R at S, with the built-in C^k-ring-homomorphism $R \to R[S^{-1}]$. The *localization* of $\widehat{R}$ at S, denoted by $\widehat{R}[S^{-1}]$, is the super-C^k-ring over $R[S^{-1}]$ defined by

$$\widehat{R}[S^{-1}]\;:=\;\widehat{R}\otimes_R R[S^{-1}]\,.$$

It goes with a built-in super-C^k-ring-homomorphism $\widehat{R} \to \widehat{R}[S^{-1}]$.

Modules over super-C^k-rings

Definition 2.2.7. [module over super-C^k-ring]. Let R be a C^k-ring and $\widehat{R}$ be a super-C^k-ring over R. Recall Definition 2.1.3. A *module* $\widehat{M}$ *over* $\widehat{R}$, or $\widehat{R}$*-module*, is a module over $\widehat{R}$ as a superring.

The notion of

- *homomorphism* $\widehat{M_1} \to \widehat{M_2}$ of $\widehat{R}$-modules,
- *submodule* $\widehat{M_1} \hookrightarrow \widehat{M_2}$, (cf. *monomorphism*),
- *quotient module* $\widehat{M_1} \twoheadrightarrow \widehat{M_2}$, (cf. *epimorphism*),
- *direct sum* $\widehat{M_1} \oplus \widehat{M_2}$ of $\widehat{R}$-modules,
- *tensor product* $\widehat{M_1} \otimes_{\widehat{R}} \widehat{M_2}$ of $\widehat{R}$-modules,
- *finitely generated*: if $\widehat{R}^{\oplus l} \twoheadrightarrow \widehat{M}$ exists for some l,
- *finitely presented*: if $\widehat{R}^{\oplus l'} \to \widehat{R}^{\oplus l} \to \widehat{M} \to 0$ is exact for some l, l'

are all defined as in Definition 2.1.3 for modules over a superring.

Denote by $\mathcal{M}od(\widehat{R})$ the category of modules over $\widehat{R}$.

Remark 2.2.8. [*module over super-C^k-ring vs. module over C^k-ring*]. Recall the built-in ring-homomorphism $R \to \widehat{R}$. Thus every $\widehat{R}$-module is canonically an R-module. The induced functor $\mathcal{M}od(\widehat{R}) \to \mathcal{M}od(R)$ is exact.

Definition 2.2.9. [localization of module over super-C^k-ring]. Let $\widehat{M}$ be a module over a super-C^k-ring $\widehat{R}$. Recall the built-in inclusion $R \hookrightarrow \widehat{R}$. Let $\widehat{R} \to \widehat{R}[S^{-1}]$ be the localization of $\widehat{R}$ at a subset $S \subset R$. Then, the *localization of* $\widehat{M}$ *at* S, denoted by $\widehat{M}[S^{-1}]$, is the $\widehat{R}[S^{-1}]$-module defined by

$$\widehat{M}[S^{-1}] \;:=\; \widehat{R}[S^{-1}] \otimes_{\widehat{R}} \widehat{M}\,.$$

By construction, it is equipped with an $\widehat{R}$-module-homomorphism $\widehat{M} \to \widehat{M}[S^{-1}]$.

Differential calculus on super-C^k-rings

Definition 2.2.10. [superderivation on super-C^k-ring]. Let $\widehat{R}$ be a super-C^k-ring over another super-C^k-ring $\widehat{S}$ with $\widehat{S} \to \widehat{R}$ the built-in super-C^k-ring-homomorphism. Then, a (*left*) *super-C^k-$\widehat{S}$-derivation* $\widehat{\Theta}$ on $\widehat{R}$ is a map

$$\widehat{\Theta}\;:\;\widehat{R}\;\longrightarrow\;\widehat{R}$$

that satisfies

$$\begin{array}{ll}
(\textit{right-}\widehat{S}\textit{-linearity}) & \widehat{\Theta}(\widehat{r}\widehat{s}+\widehat{r}'\widehat{s}') \;=\; \widehat{\Theta}(\widehat{r})\,\widehat{s} \,+\, \widehat{\Theta}(\widehat{r}')\,\widehat{s}'\,,\\
(\textit{super Leibniz rule}') & \widehat{\Theta}(\widehat{r}\widehat{r}') \;=\; \widehat{\Theta}(\widehat{r})\,\widehat{r}' \,+\, (-1)^{|\widehat{r}||\widehat{r}'|}\,\widehat{\Theta}(\widehat{r}')\,\widehat{r}
\end{array}$$

for all $\widehat{s}$, $\widehat{s}' \in \widehat{S}$, $\widehat{r}$, $\widehat{r}' \in \widehat{R}$ homogeneous, and the $\mathbb{R}$-linear extensions of these relations, and

(*chain rule*)

$$\widehat{\Theta}(h(r_1,\,\cdots,\,r_l)) \;=\; \partial_1 h(r_1,\,\cdots,\,r_l)\,\widehat{\Theta}(r_1) \;+\; \cdots \;+\; \partial_l h(r_1,\,\cdots,\,r_l)\,\widehat{\Theta}(r_l)$$

for all $h \in C^k(\mathbb{R}^l)$, $l \in \mathbb{Z}_{\geq 1}$, and $r_1, \cdots, r_l \in$ the C^k-ring $R \subset \widehat{R}$.

Denote by $sDer_{\widehat{S}}(\widehat{R})$ the set of all super-C^k-$\widehat{S}$-derivations on $\widehat{R}$. Then, $sDer_{\widehat{S}}(\widehat{R})$ is $\mathbb{Z}/2$-graded by construction. Furthermore, if $\widehat{\Theta} \in sDer_{\widehat{S}}(\widehat{R})$, then so does $\widehat{r}\widehat{\Theta} = (-1)^{|\widehat{r}||\widehat{\Theta}|}\widehat{\Theta}\widehat{r}$, with $(\widehat{r}\widehat{\Theta})(\,\cdot\,) := \widehat{r}(\widehat{\Theta}(\,\cdot\,))$ and $(\widehat{\Theta}\widehat{r})(\,\cdot\,) := (-1)^{|\widehat{r}||\,\cdot\,|}(\widehat{\Theta}(\,\cdot\,))\widehat{r}$, for $\widehat{r} \in \widehat{R}$. Thus, $sDer_{\widehat{S}}(\widehat{R})$ *is naturally a (bi-)$\widehat{R}$-module*, with $\widehat{r} \cdot \widehat{\Theta} := \widehat{r}\widehat{\Theta}$ and $\widehat{\Theta} \cdot \widehat{r} := \widehat{\Theta}\widehat{r}$ for $\widehat{\Theta} \in sDer_{\widehat{S}}(\widehat{R})$, $\widehat{r} \in \widehat{R}$.

Furthermore, if $\widehat{\Theta}$, $\widehat{\Theta}' \in sDer_{\widehat{S}}(\widehat{R})$ (homogeneous), then so does the *super Lie bracket*

$$[\widehat{\Theta}, \widehat{\Theta}'] \;:=\; \widehat{\Theta}\widehat{\Theta}' \,-\, (-1)^{|\widehat{\Theta}||\widehat{\Theta}'|}\,\widehat{\Theta}'\widehat{\Theta}$$

of $\widehat{\Theta}$ and $\widehat{\Theta}'$. Thus, $sDer_{\widehat{S}}(\widehat{R})$ *is naturally a super-Lie algebra.*

We will denote $sDer_{\mathbb{R}}(\widehat{R})$ also by $sDer(\widehat{R})$.

Remark 2.2.11. [*inner superderivation*]. Similarly to the commutative case, all the inner superderivations of a super-C^k-ring are zero.

Remark 2.2.12. [*relation to $Der_S(R)$*]. Note that the built-in inclusion $R \hookrightarrow \widehat{R}$ induces a natural $\widehat{R}$-module-homomorphism $sDer_{\widehat{S}}(\widehat{R}) \to Der_S(R) \otimes_R \widehat{R}$.

Definition 2.2.13. [superderivation with value in module]. Let $\widehat{R}$ be a super-C^k-ring over a super-C^k-ring $\widehat{S}$ and $\widehat{M}$ an $\widehat{R}$-module. An $\mathbb{R}$-linear map

$$d \;:\; \widehat{R} \;\to\; \widehat{M}$$

is called a *super-C^k-$\widehat{S}$-derivation with values in $\widehat{M}$*, if

$$\begin{array}{ll}
(\textit{right-}\widehat{S}\textit{-linearity}) & d(\widehat{r}\widehat{s} + \widehat{r}'\widehat{s}') \;=\; d(\widehat{r})\,\widehat{s} \,+\, d(\widehat{r}')\,\widehat{s}'\,, \\
(\textit{super Leibniz rule}') & d(\widehat{r}\widehat{r}') \;=\; d(\widehat{r})\,\widehat{r}' \,+\, (-1)^{|\widehat{r}||\widehat{r}'|}\,d(\widehat{r}')\,\widehat{r} \\
(\textit{chain rule}) & d(f(r_1, \cdots, r_n)) \;=\; \sum_{i=1}^n (\partial_i f)(r_1, \cdots, r_n) \cdot dr_i
\end{array}$$

for all $\widehat{s}$, $\widehat{s}' \in \widehat{S}$, $\widehat{r}$, $\widehat{r}' \in \widehat{R}$ homogeneous, $f \in \cup_n C^k(\mathbb{R}^n)$, and $r_i \in R$. Here $\partial_i f$ is the partial derivative of $f \in C^k(\mathbb{R}^n)$ with respect to the i-th coordinate of $\mathbb{R}^n$. In particular, d is a $\widehat{S}$-module-homomorphism. The set $sDer_{\widehat{S}}(\widehat{R}, \widehat{M})$ of all super-C^k-$\widehat{S}$-derivation with values in $\widehat{M}$ is naturally an $\widehat{R}$-module, with the multiplication defined by $\widehat{r} \cdot d : \widehat{r}' \mapsto \widehat{r}(d(\widehat{r}'))$ and $d \cdot \widehat{r} : \widehat{r}' \mapsto (-1)^{|\widehat{r}||\widehat{r}'|}(d(\widehat{r}'))\widehat{r}$ for $\widehat{r} \in \widehat{R}$.

Definition 2.2.14. [module of differentials of super-C^k-ring]. Let $\widehat{R}$ be a super-C^k-ring over $\widehat{S}$. An $\widehat{R}$-module $\widehat{M}$ with a super-C^k-$\widehat{S}$-derivation $d : \widehat{R} \to M$ is called the *C^k-cotangent module* of $\widehat{R}$ over $\widehat{S}$ if it satisfies the following universal property:

- For any $\widehat{R}$-module $\widehat{M}'$ and super C^k-$\widehat{S}$-derivation $d' : \widehat{R} \to \widehat{M}'$, there exists a unique homomorphism of $\widehat{R}$-modules $\psi : \widehat{M} \to \widehat{M}'$ such that $d' = \psi \circ d$.

$$\begin{tikzcd}
\widehat{R} \arrow[rr, "d"] \arrow[dr, "d'"'] & & \widehat{M} \arrow[dl, dotted, "\psi"] \\
& \widehat{M}' &
\end{tikzcd} .$$

(Thus, $\widehat{M}$ is unique up to a unique $\widehat{R}$-module isomorphism.) We denote this $\widehat{M}$ with $d:\widehat{R}\to\widehat{M}$ by $\Omega_{\widehat{R}/\widehat{S}}$, with the built-in super-C^k-$\widehat{S}$-derivation $d:\widehat{R}\to\Omega_{\widehat{R}/\widehat{S}}$ understood.

Remark 2.2.15. [*explicit construction of* $\Omega_{\widehat{R}/\widehat{S}}$]. The C^k-cotangent module $\Omega_{\widehat{R}/\widehat{S}}$ of $\widehat{R}$ over $\widehat{S}$ can be constructed explicitly from the $\widehat{R}$-module generated by the set

$$\{d(\widehat{r})\,|\,\widehat{r}\in\widehat{R}\}\,,$$

subject to the relations

$$\begin{array}{ll}
(\mathbb{Z}/2\text{-}grading) & |d(\widehat{r})|\;=\;|\widehat{r}|\,,\\
(right\text{-}\mathbb{R}\text{-}linearity) & d(\widehat{r}\widehat{s}+\widehat{r}'\widehat{s}')\;=\;d(\widehat{r})\,\widehat{s}\;+\;d(\widehat{r}')\,\widehat{s}'\,,\\
(super\ Leibniz\ rule') & d(\widehat{r}\widehat{r}')\;=\;d(\widehat{r})\,\widehat{r}'\;+\;(-1)^{|\widehat{r}||\widehat{r}'|}\,d(\widehat{r}')\,\widehat{r}\,,\\
(bi\text{-}\widehat{R}\text{-}module\ structure) & d(\widehat{r})\,\widehat{r}'\;=\;(-1)^{|\widehat{r}||\widehat{r}'|}\,\widehat{r}'\,d(\widehat{r})
\end{array}$$

for all $\widehat{s}$, $\widehat{s}'\in\widehat{S}$, $\widehat{r}$, $\widehat{r}'$, and

$$\begin{array}{ll}
(chain\ rule) & d(h(r_1,\,\cdots,\,r_s))\\
 & =\;\partial_1 h(r_1,\,\cdots,\,r_s)\,d(r_1)\;+\;\cdots\;+\;\partial_s h(r_1,\,\cdots,\,r_s)\,d(r_s)
\end{array}$$

for all $h\in C^k(\mathbb{R}^s)$, $s\in\mathbb{Z}_{\geq 1}$, and $r_1,\,\cdots,\,r_s\in R\subset\widehat{R}$. Denote the image of $d(\widehat{r})$ under the quotient by $d\widehat{r}$. Then, by definition, the built-in map

$$\begin{array}{cccc}
d & : & \widehat{R} & \longrightarrow & \Omega_{\widehat{R}/\widehat{S}}\\
 & & \widehat{r} & \longmapsto & d\widehat{r}
\end{array}$$

is a super-C^k-$\widehat{S}$-derivation from $\widehat{R}$ to $\Omega_{\widehat{R}/\widehat{S}}$.

Remark 2.2.16. [*relation to* $\Omega_{R/S}$]. Note that the built-in inclusion $R\hookrightarrow\widehat{R}$ induces a natural $\widehat{R}$-module-homomorphism $\Omega_{R/S}\otimes_R\widehat{R}\to\Omega_{\widehat{R}/\widehat{S}}$.

Remark 2.2.17. [*relation between* $sDer_{\widehat{S}}(\widehat{R})$ *and* $\Omega_{\widehat{R}/\widehat{S}}$]. The universal property of $\Omega_{\widehat{R}/\widehat{S}}$ implies that there is an $\widehat{R}$-module-isomorphism

$$sDer_{\widehat{S}}(\widehat{R})\;\longrightarrow\;Hom_{\widehat{R}}(\Omega_{\widehat{R}/\widehat{S}},\widehat{R})\,.$$

2.3 Super-C^k-manifolds, super-C^k-ringed spaces, and super-C^k-schemes

The notion of super-C^k-manifolds, super-C^k-ringed spaces, and super-C^k-schemes are introduced in this subsection. They are a super generalization of related notions from works of Eduardo Dubuc [Du], Dominic Joyce [Joy], Anders Kock [Ko], Ieke Moerdijk and Gonzalo E. Reyes [M-R], Juan Navarro González and Juan Sancho de Salas [NG-SdS] in C^∞-algebraic geometry or synthetic differential geometry. The presentation here proceeds particularly with [Joy] in mind.

Super-C^k-manifolds

The notion of '*supermanifolds*' to capture the geometry (either the space-time over which fermionic fields are defined or the symmetry group itself) behind supersymmetric quantum field theory was studied by various authors, including Majorie Batchelor [Bat], Felix Berezin and Dimitry Leites [B-L], Bryce DeWitt [DeW], Bertram Kostant [Kos], Alice Rogers [Rog], Mitchell Rothstein [Rot], leading to several inequivalent notions of 'supermanifolds'.

For this note, we follow the direction of Yuri Manin [Man] and Steven Shnider and Raymond Wells, Jr., [S-W]. In essence,

- *A supermanifold is a ringed space with the underlying space an ordinary manifold* $(X, \mathcal{O}_X)$, *as a* C^k*-scheme, but with the structure sheaf a* $\mathbb{Z}/2$*-graded* $\mathbb{Z}/2$*-commutative* $\mathcal{O}_X$*-algebra.*

Such setting (cf. the detail below) is both physically direct and compatible and mathematically in line with Grothendieck's Algebraic Geometry. It suits best for our purpose of generalization to the notion of matrix supermanifolds to describe coincident fermionic D-branes as maps therefrom.

Definition 2.3.1. [super-C^k-manifold]. Let $(X, \mathcal{O}_X)$ be a C^k-manifold of dimension m and $\mathcal{F}$ be a locally-free sheaf of $\mathcal{O}_X$-modules of rank s. Then

$$\widehat{\mathcal{O}}_X \;:=\; \textstyle\bigwedge^{\bullet}_{\mathcal{O}_X} \mathcal{F}$$

is a sheaf of superpolynomial rings over $\mathcal{O}_X$ (or equivalently a sheaf of Grassmann $\mathcal{O}_X$-algebras). The $\mathbb{Z}/2$-grading of $\widehat{\mathcal{O}}_X$ is given by setting

$$\widehat{\mathcal{O}}_X^{\,even} \;:=\; \textstyle\bigwedge^{even}_{\mathcal{O}_X} \mathcal{F} \qquad \text{and} \qquad \widehat{\mathcal{O}}_X^{\,odd} \;:=\; \textstyle\bigwedge^{odd}_{\mathcal{O}_X} \mathcal{F}\,.$$

The new ringed space

$$\widehat{X} \;:=\; (X, \widehat{\mathcal{O}}_X)$$

is called a *super-C^k-manifold*. The dimension m of X is called the *even dimension* of $\widehat{X}$ while the rank s of $\mathcal{F}$ is called the *odd dimension* of $\widehat{X}$. By construction, there is a *built-in split short exact sequence* of $\mathcal{O}_X$-modules (also as $\widehat{\mathcal{O}}_X$-modules)

$$0 \longrightarrow \widehat{\mathcal{I}}_X \longrightarrow \widehat{\mathcal{O}}_X \rightleftarrows \mathcal{O}_X \longrightarrow 0\,,$$

where $\widehat{\mathcal{I}}_X := \bigwedge^{\ge 1}_{\mathcal{O}_X} \mathcal{F}$. In terms of ringed spaces, one has thus

$$\widehat{X} \underset{\widehat{\iota}}{\overset{\widehat{\pi}}{\rightleftarrows}} X\;,$$

where $\widehat{\pi}$ is a dominant morphism and $\widehat{\iota}$ is an inclusion such that $\widehat{\pi}\circ\widehat{\iota} = Id_X$. Note that since in this case $\widehat{\mathcal{I}}_X$ is the nil-radical (i.e. the ideal sheaf of all nilpotent sections) of $\widehat{\mathcal{O}}_X$, $X = \widehat{X}_{\mathrm{red}} \subset \widehat{X}$. Cf. Definition 2.3.15.

A *morphism*

$$\hat{f} := (f, \widehat{f}^{\sharp}) \;:\; \widehat{X} := (X, \widehat{\mathcal{O}}_X) \;\longrightarrow\; \widehat{\mathcal{O}}_Y := (Y, \widehat{\mathcal{O}}_Y)$$

between super-C^k-manifolds is a C^k-map $f: X \to Y$ between C^k-manifolds together with a sheaf-homomorphism

$$\widehat{f}^{\sharp} \;:\; f^{-1}\widehat{\mathcal{O}}_Y \;\longrightarrow\; \widehat{\mathcal{O}}_X$$

that fits into the following commutative diagram

$$\begin{tikzcd}
f^{-1}\widehat{\mathcal{O}}_Y \arrow[r, "\widehat{f}^{\sharp}"] \arrow[d] & \widehat{\mathcal{O}}_X \arrow[d] \\
f^{-1}\mathcal{O}_Y \arrow[r, "f^{\sharp}"] \arrow[u, dotted, bend left] & \mathcal{O}_X \arrow[u, dotted, bend left]
\end{tikzcd} .$$

Here, the homomorphism $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ between sheaves of C^k-rings on X is induced by the C^k-map $f : X \to Y$ between C^k-manifolds. In terms of morphisms between ringed spaces, one has the commutative diagram:

$$\begin{tikzcd}
\widehat{X} \arrow[r, "\widehat{f}"] \arrow[d, shift left, "\widehat{\pi}"] & \widehat{Y} \arrow[d, shift left, "\widehat{\pi}"] \\
X \arrow[r, "f"] \arrow[u, shift left, "\widehat{\iota}"] & Y \arrow[u, shift left, "\widehat{\iota}"]
\end{tikzcd} .$$

We will call $\hat{f}$ also a *C^k-map*.

Remark 2.3.2. [*supermanifold/superspace in physics: spinor and issue of central charge*]. The above Definition 2.3.1 of super-C^k-manifolds gives a general mathematical setting. However, for a physical application some refinement or extension of the above definition is required:

(1) For a physical application to $N = 1$ *supersymmetric quantum field theories*, the C^k-manifold X is equipped with a Riemannian or Lorentzian metric and the sheaf $\mathcal{F}$ *is a sheaf of spinors* that arises *from* the associated bundle of the orthonormal-frame bundle to an *irreducible spinor representation* of the orthonormal group related to the metric.

(2) For a physical application to $N \geq 2$ *supersymmetric quantum field theories*, not only that the C^k-manifold X is equipped with a Riemannian or Lorentzian metric and the sheaf $\mathcal{F}$ *is a sheaf of spinors* that arises *from* the associated bundle of the orthonormal-frame bundle to *a direct sum of N-many irreducible spinor representations* of the orthonormal group related to the metric, one has but also a new issue of whether to include the (bosonic) extension to the structure sheaf $\widehat{\mathcal{O}}_X$ in Definition 2.3.1 that takes care also of the *central charges* in the $N \geq 2$ supersymmetry algebra; cf. [So], [W-B], and [W-O].

Such necessary refinement or extension should be made case by case to reflect physics. For the current note, our focus is on the notion of 'differentiable maps from a matrix-supermanifold to a real manifold'. The framework we develop (cf. Sec. 3 and Sec. 4) is intact once a refined or extended structure sheaf $\widehat{\mathcal{O}}_X$ for a supermanifold is chosen.

Remark 2.3.3. [*real spinor vs. complex spinor*]. Also to reflect physics presentation, it may be more convenient case by case to take $\mathcal{F}$ in Definition 2.3.1 to be an $\mathcal{O}_X^{\mathbb{C}}$-module, rather than an $\mathcal{O}_X$-module, when one constructs the structure sheaf $\widehat{\mathcal{O}}_X$. See, for example, Example 2.3.8 below. Again, our notion of 'differentiable maps from a matrix-supermanifold to a real manifold' remains intact.

The following sample list of superspaces is meant to give mathematicians a taste of the role of spinor representations in physicists' notion of a supermanifold beyond just a $\mathbb{Z}/2$-graded $\mathbb{Z}/2$-commutative manifold. See, for example, [Freed: Lecture 3] of Daniel Freed.

Example 2.3.4. [superspace $\mathbb{R}^{m|s}$ as super-C^k-manifold]. This is the supermanifold of topology $\mathbb{R}^m$ and function ring the superpolynomial ring $C^k(\mathbb{R}^m)[\theta^1, \cdots, \theta^s]$ over the C^k-ring $C^k(\mathbb{R}^m)$. In particular, $\mathbb{R}^{0|s}$ is called a *superpoint*.

Example 2.3.5. [$d = 1+1$, $N = (2,2)$ superspace]. The supermanifold $\mathbb{R}^{2|4}$ of the underlying space the $(1+1)$-dimensional Minkowski space-time $\mathbb{M}^{1+1}$ and of function ring the superpolynomial ring $C^\infty(\mathbb{R}^2)[\theta^1, \theta^2, \bar{\theta}^{\dot{1}}, \bar{\theta}^{\dot{2}}]$. Furthermore, each of θ^1 and θ^2 (resp. $\bar{\theta}^{\dot{1}}$ and $\bar{\theta}^{\dot{2}}$) is in the left (resp. right) Majorana-Weyl spinor representation of $SO(1,1)$. This is a basic superspace for a $d = 2$ superconformal field theory and a superstring theory.

Example 2.3.6. [$d = 2+1$, $N = 1$ superspace]. The supermanifold $\mathbb{R}^{3|2}$ of the underlying space the $(2+1)$-dimensional Minkowski space-time $\mathbb{M}^{2+1}$ and of function ring the superpolynomial ring $C^\infty(\mathbb{R}^3)[\theta^1, \theta^2]$. Furthermore, the tuple (θ^1, θ^2) is in the Majorana spinor representation of $SO(2,1)$ ($\simeq$ the fundamental representation of $SL(2,\mathbb{R})$). This is a basic superspace for a $d = 2+1$, $N = 1$ supersymmetric quantum field theory.

Example 2.3.7. [$d = 3$, $N = 1$ superspace]. The supermanifold $\mathbb{R}^{3|4}$ of the underlying space the 3-dimensional Euclidean space $\mathbb{E}^3$ and of function ring the superpolynomial ring $C^\infty(\mathbb{R}^3)[\theta^1, \theta^2, \theta^3, \theta^4]$. Furthermore, the tuple $(\theta^1, \theta^2, \theta^3, \theta^4)$ is in the pseudo-real spinor representation of $SO(3)$. Notice how the signature of the metric may influence the dimension of a minimal/irreducible spinor representation at the same manifold dimension; cf. Example 2.3.6.

Example 2.3.8. [$d = 3+1$, $N = 1$ superspace]. The supermanifold $\mathbb{R}^{4|4}$ of the underlying space the 3+1-dimensional Minkowski space-time $\mathbb{M}^{3+1}$ and of function ring the superpolynomial ring $C^\infty(\mathbb{R}^4)[\theta^1, \theta^2, \theta^3, \theta^4]$. Furthermore, the tuple $(\theta^1, \theta^2, \theta^3, \theta^4)$ is in the real Majorana spinor representation of $SO(3,1)$. In physics, it is more convenient to consider instead complex Weyl spinor representations of $SO(3,1)$ via the isomorphism $Spin(3,1) \simeq SL(2,\mathbb{C})$. In this case we take as the function ring $C^\infty(\mathbb{R}^4) \otimes_{\mathbb{R}} [\theta^1, \theta^2, \bar{\theta}^{\dot{1}}, \bar{\theta}^{\dot{2}}]^{\mathbb{C}}$, in which (θ^1, θ^2) and $(\bar{\theta}^{\dot{1}}, \bar{\theta}^{\dot{2}})$ are in complex Weyl spinor representations of opposite chirality. This is a basic superspace for a $d = 3+1$, $N = 1$ supersymmetric quantum field theory. See, for example, [W-B].

Example 2.3.9. [$d = 3+1$, $N = 2$ superspace]. The supermanifold $\mathbb{R}^{4|8}$ of the underlying space the 3+1-dimensional Minkowski space-time $\mathbb{M}^{3+1}$ and of function ring the superpolynomial ring $C^\infty(\mathbb{R}^4)[\theta^1, \theta^2, \theta^3, \theta^4, \theta'^1, \theta'^2, \theta'^3, \theta'^4]$. Furthermore, each of the tuples $(\theta^1, \theta^2, \theta^3, \theta^4)$ and $(\theta'^1, \theta'^2, \theta'^3, \theta'^4)$ is in the real Majorana representation of $SO(3,1)$. Here, we ignore the central charge in the $d = 3+1$, $N = 2$ super-Poincaré algebra. As in Example 2.3.8, for physics, it is more convenient to take as the function ring $C^\infty(\mathbb{R}^4) \otimes_{\mathbb{R}} [\theta^1, \theta^2, \bar{\theta}^{\dot{1}}, \bar{\theta}^{\dot{2}}, \theta'^1, \theta'^2, \bar{\theta}'^{\dot{1}}, \bar{\theta}'^{\dot{2}}]^{\mathbb{C}}$ through complex Weyl representations of $SO(3,1)$. This is a basic superspace for a $d = 3+1$, $N = 2$ supersymmetric quantum field theory.

Super-C^k-ringed spaces and super-C^k-schemes

Definition 2.3.10. [super-C^k-ringed space, C^k-map]. A *super-C^k-ringed space*

$$\widehat{X} \;=\; (X, \mathcal{O}_X, \widehat{\mathcal{O}}_X)$$

is a C^k-ringed space $(X, \mathcal{O}_X)$ together with a sheaf $\widehat{\mathcal{O}}_X$ of super-C^k-rings over $\mathcal{O}_X$. By construction, it has a built-in split short exact sequence of $\mathcal{O}_X$-modules (also as $\widehat{\mathcal{O}}_X$-modules)

$$
\begin{tikzcd}
0 \arrow[r] & \widehat{\mathcal{I}}_X \arrow[r] & \widehat{\mathcal{O}}_X \arrow[r] & \mathcal{O}_X \arrow[l, dotted, bend left] \arrow[r] & 0\,.
\end{tikzcd}
$$

Here, $\widehat{\mathcal{I}}_X := Ker(\widehat{\mathcal{O}}_X \to \mathcal{O}_X)$ is an ideal sheaf of $\hat{\mathcal{O}}_X$. The split short exact sequence defines the a pair of morphisms between the ringed spaces

$$
\begin{tikzcd}
\widehat{X} \arrow[r, shift left, "\widehat{\pi}"] & X \arrow[l, shift left, "\widehat{\iota}"]\ ,
\end{tikzcd}
$$

where $\widehat{\pi}$ is a dominant morphism and $\widehat{\iota}$ is an inclusion such that $\widehat{\pi}\circ\widehat{\iota} = Id_X$.

A *morphism*

$$
\hat{f} := (f, f^\sharp, \widehat{f^\sharp})\ :\ \widehat{X} := (X, \mathcal{O}_X, \widehat{\mathcal{O}}_X)\ \longrightarrow\ \widehat{\mathcal{O}}_Y := (Y, \mathcal{O}_Y, \widehat{\mathcal{O}}_Y)
$$

between super-C^k-ringed spaces is a morphism $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ between C^k-ringed spaces together with a sheaf-homomorphism

$$
\widehat{f^\sharp}\ :\ f^{-1}\hat{\mathcal{O}}_Y\ \longrightarrow\ \widehat{\mathcal{O}}_X
$$

that fits into the following commutative diagram

$$
\begin{tikzcd}
f^{-1}\widehat{\mathcal{O}}_Y \arrow[r, "\widehat{f^\sharp}"] \arrow[d] & \widehat{\mathcal{O}}_X \arrow[d] \\
f^{-1}\mathcal{O}_Y \arrow[r, "f^\sharp"] \arrow[u, dotted, bend left] & \mathcal{O}_X \arrow[u, dotted, bend left]
\end{tikzcd}
\quad .
$$

That is, a commutative diagram of morphisms between ringed spaces:

$$
\begin{tikzcd}
\widehat{X} \arrow[r, "\widehat{f}"] \arrow[d, shift left, "\widehat{\pi}"] & \widehat{Y} \arrow[d, shift left, "\widehat{\pi}"] \\
X \arrow[r, "f"] \arrow[u, shift left, "\widehat{\iota}"] & Y \arrow[u, shift left, "\widehat{\iota}"]
\end{tikzcd}
\quad .
$$

We will call $\hat{f}$ also a *C^k-map*.

Definition 2.3.11. [affine super-C^k-scheme]. Let $\widehat{R}$ be a super-C^k-ring with the structure ring-homomorphisms $\widehat{R} \rightleftarrows R$. The *affine super-$C^k$-scheme associated to* $\hat{R}$ is a super-C^k-ringed space $(X, \mathcal{O}_X, \widehat{\mathcal{O}}_X)$ defined as follows:

- $(X, \mathcal{O}_X)$ is the affine C^k-scheme associated to R.
- $\widehat{\mathcal{O}}_X$ is the quasi-coherent sheaf on X associated to $\widehat{R}$ as an R-module, as defined in C^k-algebraic geometry via localizations of $\widehat{R}$ at subsets of R.

By construction, $\widehat{\mathcal{O}}_X$ is a sheaf of super-C^k-rings over $\mathcal{O}_X$ and the ring-homomorphisms $\widehat{R} \rightleftarrows R$ induce a split short exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & \widehat{\mathcal{I}}_X \arrow[r] & \widehat{\mathcal{O}}_X \arrow[r] & \mathcal{O}_X \arrow[l, dotted, bend left] \arrow[r] & 0\,,
\end{tikzcd}
$$

which defines a pair of built-in morphisms

$$\widehat{X} \underset{\widehat{\iota}}{\overset{\widehat{\pi}}{\rightleftarrows}} X\,,$$

where $\widehat{\pi}$ is a dominant morphism and $\widehat{\iota}$ is an inclusion such that $\widehat{\pi}\circ\widehat{\iota} = Id_X$.

A *morphism* $(X, \mathcal{O}_X, \widehat{\mathcal{O}}_X) \to (Y, \mathcal{O}_Y, \widehat{\mathcal{O}}_Y)$ between affine super-C^k-schemes is defined to be a C^k-map $\widehat{f}: (X, \mathcal{O}_X, \widehat{\mathcal{O}}_X) \to (Y, \mathcal{O}_Y, \widehat{\mathcal{O}}_Y)$ of the underlying super-C^k-ringed spaces.

Remark 2.3.12. [*super-C^k-ring vs. affine super-C^k-scheme*]. For our purpose, we shall assume that all the C^k-rings in our discussion are finitely generated and germ-determined. In this case, the category of affine super-C^k-schemes is contravariantly equivalent to the category of super-C^k-rings. See, for example, [Joy: Proposition 4.15].

Definition 2.3.13. [super-C^k-scheme]. A super-C^k-ringed space $\widehat{X} := (X, \mathcal{O}_X, \widehat{\mathcal{O}}_X)$ is called a *super-C^k-scheme* if X admits an open-set covering $\{U_\alpha\}_{\alpha\in A}$ such that $(U_\alpha, \mathcal{O}_X|_{U_\alpha}, \widehat{\mathcal{O}}_X|_{U_\alpha})$ is an affine super-C^k-scheme for all $\alpha \in A$. By construction, $\widehat{\mathcal{O}}_X$ is a sheaf of super-C^k-rings over $\mathcal{O}_X$ with a built-in a split short exact sequence

$$0 \longrightarrow \widehat{\mathcal{I}}_X \longrightarrow \widehat{O}_X \rightleftarrows \mathcal{O}_X \longrightarrow 0\,,$$

which defines a pair of built-in morphisms

$$\widehat{X} \underset{\widehat{\iota}}{\overset{\widehat{\pi}}{\rightleftarrows}} X\,,$$

where $\widehat{\pi}$ is a dominant morphism and $\widehat{\iota}$ is an inclusion such that $\widehat{\pi}\circ\widehat{\iota} = Id_X$.

A *morphism* $(X, \mathcal{O}_X, \widehat{\mathcal{O}}_X) \to (Y, \mathcal{O}_Y, \widehat{\mathcal{O}}_Y)$ between super-C^k-schemes is defined to be a C^k-map $\widehat{f}: (X, \mathcal{O}_X, \widehat{\mathcal{O}}_X) \to (Y, \mathcal{O}_Y, \widehat{\mathcal{O}}_Y)$ of the underlying super-C^k-ringed spaces.

Remark 2.3.14. [*super-C^k-scheme vs. equivalence class of gluing systems of super-C^k-rings*]. Recall Remark 2.3.12. Under the assumption that all the C^k-rings in our discussion be finitely generated and germ-determined, the category of super-C^k-schemes is contravariantly equivalent to the category of equivalence classes of gluing systems of super-C^k-rings.

Definition 2.3.15. [super-C^k-normal ideal sheaf and super-C^k-subscheme]. Let $\widehat{X} := (X, \mathcal{O}_X, \widehat{\mathcal{O}}_X)$ be a super-C^k-scheme. A *super-C^k-normal ideal sheaf* $\widehat{\mathcal{I}}$ on $\widehat{X}$ is a sheaf of super-C^k-normal ideals of $\widehat{\mathcal{O}}_X$. In this case (and only in this case), $\widehat{\mathcal{I}}$ defines a *super-C^k-subschme* $\widehat{Z} := (Z, \mathcal{O}_Z, \widehat{\mathcal{O}}_Z)$ of $\widehat{X}$ with a built-in commutative diagram of $\widehat{\mathcal{O}}_X$-modules

$$\begin{tikzcd}
0 \arrow[r] & \widehat{\mathcal{I}} \arrow[r] & \widehat{\mathcal{O}}_X \arrow[r] \arrow[d, shift left] & \widehat{\mathcal{O}}_Z \arrow[r] \arrow[d, shift left] & 0 \\
0 \arrow[r] & \mathcal{I} := \widehat{\mathcal{I}}_0 \cap \mathcal{O}_X \arrow[r] \arrow[u, hook] & \mathcal{O}_X \arrow[r] \arrow[u, hook, shift left] & \mathcal{O}_Z \arrow[r] \arrow[u, hook, shift left] & 0
\end{tikzcd}$$

such that both horizontal sequences are exact. Here $\widehat{\mathcal{I}} = \widehat{\mathcal{I}}_0 + \widehat{\mathcal{I}}_1$ is the decomposition of $\widehat{\mathcal{I}}$ by its homogeneous components.

Remark 2.3.16. [*sheaf-type super-thickening*]. One may think of the graded-commutative scheme $\widehat{X}$ as a *sheaf-type super-thickening* of the underlying C^k-scheme X, and the morphism $\widehat{f}: \widehat{X} \to \widehat{Y}$ as a lifting of C^k-map $f: X \to Y$. Cf. FIGURE 3-1.

2.4 Sheaves of modules and differential calculus on a super-C^k-scheme

Basic notions of sheaves of modules and differential calculus on a super-C^k-scheme are collected in this subsection.

Sheaves of modules on a super-C^k-scheme

Definition 2.4.1. [sheaf of modules on super-C^k-scheme]. Let $\widehat{X} = (X, \mathcal{O}_X, \widehat{\mathcal{O}}_X)$ be a super-C^k-scheme. A *sheaf of modules* on $\widehat{X}$ (i.e. $\widehat{\mathcal{O}}_X$*-module*) is defined to be a sheaf $\widehat{\mathcal{F}}$ of modules on the ($\mathbb{Z}/2$-graded $\mathbb{Z}/2$-commutative) ringed space that underlies $\widehat{X}$. In particular, $\widehat{\mathcal{F}}$ is $\mathbb{Z}/2$-graded $\widehat{\mathcal{F}} = \widehat{\mathcal{F}}_0 \oplus \widehat{\mathcal{F}}_1$. Sections of $\widehat{\mathcal{F}}_0$ (resp. $\widehat{\mathcal{F}}_1$) is said to be *even* (resp. *odd*). Sections of either $\widehat{\mathcal{F}}_0$ or $\widehat{\mathcal{F}}_1$ is said to be *homogeneous*. The left-$\widehat{\mathcal{O}}_X$-module structure on $\widehat{\mathcal{F}}$ induces the right-$\widehat{\mathcal{O}}_X$-structure on $\widehat{\mathcal{F}}$ and vice versa; thus we won't distinguish left-, right-, or bi-module structures in our discussions.

The notion of

- *homomorphism* $\widehat{\mathcal{F}} \to \widehat{\mathcal{G}}$ of $\widehat{\mathcal{O}}_X$-modules,
- *submodule* $\widehat{\mathcal{G}} \hookrightarrow \widehat{\mathcal{F}}$, (cf. *monomorphism*),
- *quotient module* $\widehat{\mathcal{F}} \twoheadrightarrow \widehat{\mathcal{G}}$, (cf. *epimorphism*),
- *direct sum* $\widehat{\mathcal{F}} \oplus \widehat{\mathcal{G}}$ of $\widehat{\mathcal{O}}_X$-modules,
- *tensor product* $\widehat{\mathcal{F}} \otimes_{\widehat{\mathcal{O}}_X} \widehat{\mathcal{G}}$ of $\widehat{\mathcal{O}}_X$-modules,
- *finitely generated*: if $\widehat{\mathcal{O}}_X^{\oplus l} \twoheadrightarrow \widehat{\mathcal{F}}$ exists for some l,
- *finitely presented*: if $\widehat{\mathcal{O}}_X^{\oplus l'} \to \widehat{\mathcal{O}}_X^{\oplus l} \to \widehat{\mathcal{F}} \to 0$ is exact for some l, l'

are all defined in the ordinary way as in commutative algebraic geometry. A homomorphism $\widehat{\mathcal{F}} \to \widehat{\mathcal{G}}$ of $\widehat{\mathcal{O}}_X$-modules is said to be *even* (resp. *odd*) if it preserves (resp. switches) the parity of homogeneous sections.

Denote by $\mathcal{M}od(\widehat{X})$ the category of all $\widehat{\mathcal{O}}_X$-modules.

Definition 2.4.2. [push-forward and pull-back of sheaf of modules]. Let $\widehat{f}: \widehat{X} \to \widehat{Y}$ be a morphism between super-C^k-schemes and $\widehat{\mathcal{F}}$ (resp. $\widehat{\mathcal{G}}$) be a $\widehat{\mathcal{O}}_X$-module (resp. $\widehat{\mathcal{O}}_Y$-module). Then, the structure sheaf-of-rings homomorphism $\widehat{f}^\sharp: f^{-1}\widehat{\mathcal{O}}_Y \to \widehat{\mathcal{O}}_X$ renders $\widehat{\mathcal{F}}$ an $\widehat{\mathcal{O}}_Y$-module. It is called the *push-forward* of $\widehat{\mathcal{F}}$ by $\widehat{f}$ and is denoted by $\widehat{f}_*(\widehat{\mathcal{F}})$ or $\widehat{f}_*\widehat{\mathcal{F}}$. The inverse-image sheaf $f^{-1}\widehat{\mathcal{G}}$ of $\widehat{\mathcal{G}}$ under f is an $f^{-1}\widehat{\mathcal{O}}_Y$-module. Define the *pull-back* $\widehat{f}^*\widehat{\mathcal{G}}$ (or $\widehat{f}^*\widehat{\mathcal{G}}$) of $\widehat{\mathcal{G}}$ under $\widehat{f}$ to be the $\widehat{\mathcal{O}}_X$-module $f^{-1}\widehat{\mathcal{G}} \otimes_{f^{-1}\widehat{\mathcal{O}}_Y} \widehat{\mathcal{O}}_X$.

Let $\widehat{X} = (X, \mathcal{O}_X, \widehat{\mathcal{O}}_X)$ be an affine super-C^k-scheme associated to a super-C^k-ring $\widehat{R} \supset R$ and $\widehat{M}$ be an $\widehat{R}$-module. Then, the assignment $U \mapsto \widehat{M} \otimes_{\widehat{R}} \widehat{\mathcal{O}}_X(U)$, with the restriction map $Id_{\widehat{M}} \otimes \rho_{UV}$ for $V \subset U$, where $\rho_{UV}: \widehat{\mathcal{O}}_X(U) \to \widehat{\mathcal{O}}_X(V)$ is the restriction map of $\widehat{\mathcal{O}}_X$, is a presheaf on $\widehat{X}$. Let $\widehat{M}^\sim$ be its sheafification.

Definition 2.4.3. [quasi-coherent sheaf on affine super-C^k-scheme]. The sheaf $\widehat{M}^\sim$ of $\widehat{\mathcal{O}}_X$-modules on the affine super-C^k-scheme $\widehat{X}$ thus obtained from the $\widehat{R}$-module $\widehat{M}$ is called a *quasi-coherent sheaf* on $\widehat{X}$.

Definition 2.4.4. [quasi-coherent sheaf on super-C^k-scheme]. Let $\widehat{X} = (X, \mathcal{O}_X, \widehat{\mathcal{O}}_X)$ be a super-C^k-scheme. An $\widehat{\mathcal{O}}_X$-module $\widehat{\mathcal{F}}$ is said to be *quasi-coherent* if X admits an open-set covering $\{U_\alpha\}_{\alpha\in A}$ such that $(U_\alpha, \mathcal{O}_X|_{U_\alpha}, \widehat{\mathcal{O}}_X|_{U_\alpha})$ is an affine super-C^k-scheme and $\widehat{\mathcal{F}}|_{U_\alpha}$ is quasi-coherent in the sense of Definition 2.4.3 for all α.

Denote by $\mathcal{QC}oh(\widehat{X})$ the category of all quasi-coherent $\widehat{\mathcal{O}}_X$-modules.

Let $\widehat{X} = (X, \mathcal{O}_X, \widehat{\mathcal{O}}_X)$ be a super-C^k-scheme. Recall the built-in dominant morphism $\widehat{\pi}$: $\widehat{X} \to X$. The following immediate lemmas relate modules over $\widehat{X}$ and those over X:

Lemma 2.4.5. [quasi-coherent sheaf on super-C^k-scheme vs. on C^k-scheme]. *An $\widehat{\mathcal{O}}_X$-module $\widehat{\mathcal{F}}$ is quasi-coherent if and only if $\widehat{\pi}_*(\widehat{\mathcal{F}})$ is quasi-coherent on X. The push-forward functor*

$$\widehat{\pi}_* \;:\; \mathcal{M}od_{\widehat{X}} \longrightarrow \mathcal{M}od_X$$

is exact and takes $\mathcal{QC}oh(\widehat{X})$ to $\mathcal{QC}oh(X)$.

Lemma 2.4.6. [natural push-pull relation]. *Let $\widehat{f} : \widehat{X} \to \widehat{Y}$ be a morphism between super-C^k-schemes and $\widehat{\mathcal{F}}$ (resp. $\widehat{\mathcal{G}}$) be a $\widehat{\mathcal{O}}_X$-module (resp. $\widehat{\mathcal{O}}_Y$-module). Then, there are canonical isomorphisms*

$$f_*(\widehat{\pi}_*\widehat{\mathcal{F}}) \simeq \widehat{\pi}_*(\widehat{f}_*\widehat{\mathcal{F}}) \qquad \textit{and} \qquad f^*(\widehat{\pi}_*\widehat{\mathcal{G}}) \xrightarrow{\sim} \widehat{\pi}_*(\widehat{f}^*\widehat{\mathcal{G}})\,.$$

Differential calculus on a super-C^k-scheme

Definition 2.4.7. [tangent sheaf and cotangent sheaf]. Let $\widehat{X} = (X, \mathcal{O}_X, \widehat{\mathcal{O}}_X)$ be a super-C^k-scheme. (1) The presheaf on $\widehat{X}$ that associates to an open set $U \subset X$ the $\widehat{\mathcal{O}}_X(U)$-module $Der_{\mathbb{R}}(\widehat{\mathcal{O}}_X(U))$ is a quasi-coherent sheaf of $\widehat{\mathcal{O}}_X$-modules, denoted by $\mathcal{T}_*\widehat{X}$ and called interchangeably the *tangent sheaf of* $\widehat{X}$ or the *sheaf of super-C^k-derivations on* $\widehat{\mathcal{O}}_X$.

(2) The presheaf on $\widehat{X}$ that associates to an open set $U \subset X$ the $\widehat{\mathcal{O}}_X(U)$-module $\Omega_{\widehat{\mathcal{O}}_X(U)/\mathbb{R}}$ is a quasi-coherent sheaf of $\widehat{\mathcal{O}}_X$-modules, denoted by $\mathcal{T}^*\widehat{X}$ and called interchangeably the *cotangent sheaf of* $\widehat{X}$ or the *sheaf of differentials of* $\widehat{\mathcal{O}}_X$. By construction, there is a canonical even map $d : \widehat{\mathcal{O}}_X \to \mathcal{T}^*\widehat{X}$ as sheaves of $\mathbb{R}$-vector spaces on $\widehat{X}$.

It follows from the local study in Sec. 2.2 that there is a canonical isomorphism

$$\mathcal{T}_*\widehat{X} \longrightarrow \mathcal{H}om_{\widehat{\mathcal{O}}_X}(\mathcal{T}^*\widehat{X}, \widehat{\mathcal{O}}_X)$$

as $\widehat{\mathcal{O}}_X$-modules. Cf. Remark 2.2.17.

3 Azumaya/matrix super-C^k-manifolds with a fundamental module

Once the basics of super-C^k-rings are laid down (Sec. 2.2), the extension of the notion of super-C^k-rings to the notion of Azumaya/matrix super-C^k-rings proceeds in the same manner as the extension of the notion of C^k-rings to the notion of Azumaya/matrix C^k-rings in [L-Y3] (D(11.1)). After localizations at a subset in the center of the rings in question and then gluings from local to global, the extension of the notion of super-C^k-manifolds to the notion of

Azumaya/matrix super-C^k-manifolds with a fundamental module proceeds in the same manner as the extension of the notion of C^k-manifold to the notion of Azumaya/matrix C^k-manifolds with a fundamental module in ibidem. A brief review is given below for the introduction of terminology and notations we need and the completeness of the note.

Azumaya/matrix algebras over a super-C^k-ring, modules, and differential calculus

Let $\widehat{R}$ be a super-C^k-ring and $\widehat{R}^{\mathbb{C}} := \widehat{R} \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. The (*complex*) *matrix algebra* of rank r over $\widehat{R}^{\mathbb{C}}$ is the algebra $M_{r\times r}(\widehat{R}^{\mathbb{C}})$ of $r \times r$ matrices with entries elements in $\widehat{R}^{\mathbb{C}}$. The addition and the multiplication of elements of $M_{r\times r}(\widehat{R}^{\mathbb{C}})$ are defined through the matrix addition and the matrix multiplication in the ordinary way:

$$\left(\widehat{a}_{ij}\right)_{ij} + \left(\widehat{b}_{ij}\right)_{ij} := \left(\widehat{a}_{ij} + \widehat{b}_{ij}\right)_{ij} \qquad \text{and} \qquad \left(\widehat{a}_{ij}\right)_{ij} \cdot \left(\widehat{b}_{ij}\right)_{ij} := \left(\textstyle\sum_{k=1}^{r} \widehat{a}_{ik}\widehat{b}_{kj}\right)_{ij},$$

where $\widehat{a}_{ij} + \widehat{b}_{ij}$ and $\widehat{a}_{ik} \cdot \widehat{b}_{kj}$ are respectively the addition and the multiplication in $\widehat{R}^{\mathbb{C}}$. As an abstract (*complex*) *Azumaya algebra* over $\widehat{R}$,

$$M_{r\times r}(\widehat{R}^{\mathbb{C}}) \simeq \frac{\widehat{R}^{\mathbb{C}}\langle e_i^j \,|\, 1 \le i,j \le r \rangle}{(\,\widehat{r}e_i^j - e_i^j\widehat{r}\,,\, e_i^j e_{i'}^{j'} - \delta_{i'}^{j}\, e_i^{j'} \;|\; \widehat{r} \in \widehat{R}^{\mathbb{C}}\,,\, 1 \le i,j,i',j' \le r\,)}\,.$$

Here $\widehat{R}^{\mathbb{C}}\langle e_i^j \,|\, 1 \le i,j \le r\rangle$ is the unital associative algebra generated by $\widehat{R}^{\mathbb{C}}$ and the set $\{e_i^j \,|\, 1 \le i,j \le r\}$ and $(\,\widehat{r}e_i^j - e_i^j\widehat{r}\,,\, e_i^j e_{i'}^{j'} - \delta_{i'}^{j}\, e_i^{j'} \;|\; \widehat{r} \in \widehat{R}^{\mathbb{C}}\,,\, 1 \le i,j,i',j' \le r\,)$ is the bi-ideal thereof generated by the elements indicated. The $\mathbb{Z}/2$-grading of $M_{r\times r}(\widehat{R}^{\mathbb{C}})$ follows from the $\mathbb{Z}/2$-grading of $\widehat{R}^{\mathbb{C}}$ by assigning in addition the parity of e_i^j to be even. As $\mathbb{Z}/2$-graded $\mathbb{C}$-algebras, one has the isomorphism

$$M_{r\times r}(\widehat{R}^{\mathbb{C}}) \simeq M_{r\times r}(\mathbb{C}) \otimes_{\mathbb{C}} \widehat{R}^{\mathbb{C}}\,.$$

$M_{r\times r}(\widehat{R}^{\mathbb{C}})$ naturally represents on the free module $\widehat{R}^{\mathbb{C}}$-module $\widehat{F} := (\widehat{R}^{\mathbb{C}})^{\oplus r}$ of rank r, by the matrix multiplication to the left on a column vector. We will call $\widehat{F}$ the *fundamental module* of $M_{r\times r}(\widehat{R}^{\mathbb{C}})$. The dual $\widehat{F}^{\vee} := \mathit{Hom}_{\widehat{R}^{\mathbb{C}}}(\widehat{F}, \widehat{R}^{\mathbb{C}})$ of $\widehat{F}$, as a $\widehat{R}^{\mathbb{C}}$-module, is a right-$M_{r\times r}(\widehat{R}^{\mathbb{C}})$-module. Denote by $M_{r\times r}(\widehat{R}^{\mathbb{C}})$-$\mathcal{M}od$ the category of left-$M_{r\times r}(\widehat{R}^{\mathbb{C}})$-modules. Then there are natural functors

$$\begin{array}{ccc} \widehat{R}^{\mathbb{C}}\text{-}\mathcal{M}od & \longrightarrow & M_{r\times r}(\widehat{R}^{\mathbb{C}})\text{-}\mathcal{M}od \\ \widehat{M} & \longmapsto & \widehat{F} \otimes_{\widehat{R}^{\mathbb{C}}} \widehat{M} \end{array}$$

and

$$\begin{array}{ccc} M_{r\times r}(\widehat{R}^{\mathbb{C}})\text{-}\mathcal{M}od & \longrightarrow & \widehat{R}^{\mathbb{C}}\text{-}\mathcal{M}od \\ \widehat{N} & \longmapsto & \widehat{F}^{\vee} \otimes_{M_{r\times r}(\widehat{R}^{\mathbb{C}})} \widehat{N}\,. \end{array}$$

When $\widehat{R}$ is the function-ring of a super-C^k-manifold, these two functors render $\widehat{R}^{\mathbb{C}}$-$\mathcal{M}od$ and $M_{r\times r}(\widehat{R}^{\mathbb{C}})$-$\mathcal{M}od$ equivalent, called the *Morita equivalence.*

Let $\widehat{R}$ be a super-C^k-ring over another super-C^k-ring $\widehat{S}$. Then, under the isomorphism $M_{r\times r}(\widehat{R}^{\mathbb{C}}) \simeq M_{r\times r}(\mathbb{C}) \otimes_{\mathbb{C}} \widehat{R}^{\mathbb{C}}$ between $\mathbb{C}$-algebras,

$$\begin{array}{rcl} \mathit{sDer}_{\widehat{S}}(M_{r\times r}(\widehat{R}^{\mathbb{C}})) & \simeq & \mathit{Der}_{\mathbb{C}}(M_{r\times r}(\mathbb{C})) \otimes_{\mathbb{C}} \widehat{R}^{\mathbb{C}} \oplus \mathit{Id}_{r\times r} \otimes_{\mathbb{C}} \mathit{sDer}_{\widehat{S}}(\widehat{R})^{\mathbb{C}} \\ \text{and} \qquad \Omega_{M_{r\times r}(\widehat{R}^{\mathbb{C}})/\widehat{S}^{\mathbb{C}}} & \simeq & \Omega_{M_{r\times r}(\mathbb{C})/\mathbb{C}} \otimes_{\mathbb{C}} \widehat{R}^{\mathbb{C}} \oplus M_{r\times r}(\mathbb{C}) \otimes_{\mathbb{C}} \Omega^{\mathbb{C}}_{\widehat{R}/\widehat{S}}\,. \end{array}$$

Caution that the former is only a $\widehat{R}^{\mathbb{C}}$-module while the latter is an $M_{r\times r}(\widehat{R}^{\mathbb{C}})$-module.

Remark 3.1. [*center of* $M_{r\times r}(\widehat{R}^{\mathbb{C}})$]. Despite being only $\mathbb{Z}/2$-commutative, conceptually it is instructive to regard $\widehat{R}^{\mathbb{C}}$ as playing the role of the center of the matrix ring $M_{r\times r}(\widehat{R}^{\mathbb{C}})$.

Azumaya/matrix super-C^k-manifolds with a fundamental module

The notion of Azumaya/matrix manifolds ([L-Y3] (D(11.1))) and the notion of supermanifolds (Sec. 2.3) can be merged into the notion of Azumaya/matrix supermanifolds:

$$
\begin{tikzcd}
 & \{\,\mathit{C^k\text{-}manifolds}\,\} \arrow[dl, hook] \arrow[dr, hook] & \\
\{\,\mathit{Azumaya/matrix\ C^k\text{-}manifolds}\,\} \arrow[dr, hook] & & \{\,\mathit{super\text{-}C^k\text{-}manifolds}\,\} \arrow[dl, hook'] \\
 & \{\,\mathit{Azumaya/matrix\ super\text{-}C^k\text{-}manifolds}\,\} &
\end{tikzcd} .
$$

Definition 3.2. [Azumaya/matrix super-C^k-manifold with a fundamental module]. An *Azumaya* (or *matrix*) *super-C^k-manifold with a fundamental module* is the following tuple:

$$
(X, \mathcal{O}_X, \widehat{\mathcal{O}}_X, \widehat{\mathcal{O}}_X^{Az} := \mathcal{E}nd_{\widehat{\mathcal{O}}_X^{\mathbb{C}}}(\widehat{\mathcal{E}}), \mathcal{E}) \;=:\; (\widehat{X}^{Az}, \widehat{\mathcal{E}})\,,
$$

where

- $\widehat{X} := (X, \mathcal{O}_X, \widehat{\mathcal{O}}_X)$ is a super-C^k-manifold;
- $\mathcal{E}$ is a locally free $\mathcal{O}_X^{\mathbb{C}}$-module of finite rank, say, r;
- $\widehat{\mathcal{E}} \;:=\; \mathcal{E}\otimes_{\mathcal{O}_X}\widehat{\mathcal{O}}_X \;=\; \widehat{\pi}^*\mathcal{E}$, where $\widehat{X} \underset{\widehat{\pi}}{\overset{\widehat{\iota}}{\rightleftarrows}} X$ are the built-in morphisms for $\widehat{X}$.

Note that one has the canonical isomorphism

$$
\widehat{\mathcal{O}}_X^{Az} \;:=\; \mathcal{E}nd_{\widehat{\mathcal{O}}_X^{\mathbb{C}}}(\widehat{\mathcal{E}}) \;\simeq\; \mathcal{E}nd_{\mathcal{O}_X^{\mathbb{C}}}(\mathcal{E})\otimes_{\mathcal{O}_X}\widehat{\mathcal{O}}_X \;=\; \mathcal{O}_X^{Az}\otimes_{\mathcal{O}_X}\widehat{\mathcal{O}}_X =:\; \widehat{\pi}^*\mathcal{O}_X^{Az}\,.
$$

Built into the definition is the following commutative diagram of morphisms:

$$
\begin{tikzcd}
 & \widehat{X}^{Az} \arrow[dl, shift left, "\hat{\dot{\pi}}"] \arrow[dr, "\hat{\pi}^{Az}"] & \\
X^{Az} \arrow[ur, hook, shift left, "\hat{\dot{\iota}}"] \arrow[dr, "\pi^{Az}"'] & & \widehat{X} \arrow[dl, shift left, "\widehat{\pi}"] \\
 & X \arrow[ur, hook, shift left, "\widehat{\iota}"] &
\end{tikzcd} .
$$

FIGURE 3-1.

Remark 3.3. [*smearing of matrix-points*]. Conceptually, it is instructive to regards an Azumaya/ matrix super-C^k-manifold either as a smearing of unfixed matrix points over a super-C^k-manifold or as a smearing of unfixed matrix superpoint over a C^k-manifold.

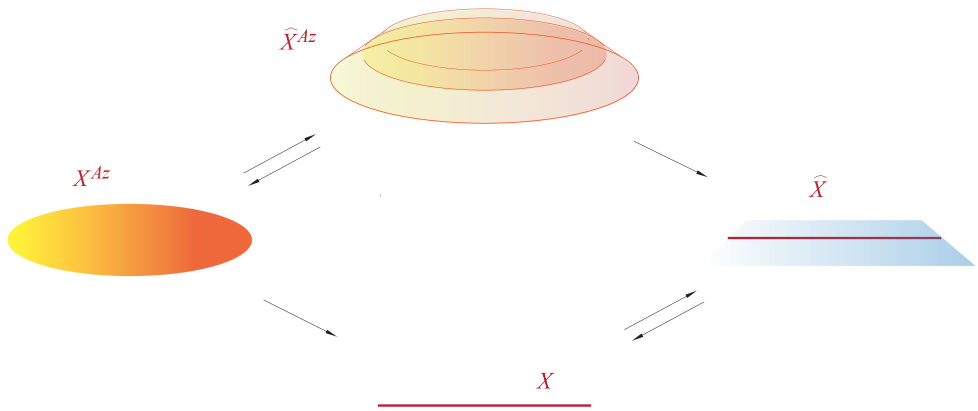

FIGURE 3-1. The built-in morphisms that underlie an Azumaya/matrix supermanifold $\widehat{X}^{Az}$. It is worth emphasizing that, as in the case of Azumaya/matrix manifolds, the superspace $\widehat{X}$ (and hecne X) should be regarded only as an auxiliary space, providing a topology underlying $\widehat{X}^{Az}$. The major object is the structure sheaf $\widehat{\mathcal{O}}_X^{Az}$, which should be thought of as a matrix-type noncommutayive cloud over $\widehat{X}$ and contains the geometrical contents that are relevant to D-branes. The built-in pair of morphisms $\widehat{X}^{Az} \rightleftarrows X^{Az}$ (resp. $\widehat{X} \rightleftarrows X$) means that $\widehat{X}^{Az}$ is a sheaf-type thickening of X^{Az} and contains X^{Az} as the zero-section, as indicated, (resp. $\widehat{X}$ is a sheaf-type thickening of X and contains X as the zero-section, as indicated). In the illustration, the bluish shade indicates a "*super*ization" while the orangish shade indicates "*matrix*ization"; and their combination is indicated by a bluish orangish shade.

Definition 3.4. [tangent sheaf]. Continuing the notation in Definition 3.2. The *tangent sheaf* $\mathcal{T}_*\widehat{X}^{Az}$ of $\widehat{X}^{Az}$ is the $\widehat{\mathcal{O}}_X$-module which assigns to each open $U \subset X$ the $\widehat{\mathcal{O}}_X(U)$-module $sDer_{\mathbb{C}}(\widehat{\mathcal{O}}_X^{Az}(U))$.

Definition 3.5. [cotangent sheaf]. Continuing the notation in Definition 3.2. The *cotangent sheaf* $\mathcal{T}^*\widehat{X}^{Az}$ of $\widehat{X}^{Az}$ is the $\widehat{\mathcal{O}}_X^{Az}$-module which assigns to each open $U \subset X$ the $\widehat{\mathcal{O}}_X^{Az}(U)$-module $\Omega_{\widehat{\mathcal{O}}_X^{Az}(U)/\mathbb{C}}$.

Through the Morita equivalence, general (left) $\widehat{\mathcal{O}}_X^{Az}$-modules on $\widehat{X}^{Az}$ can be obtained by the tensor $\widehat{\mathcal{E}} \otimes_{\widehat{\mathcal{O}}_X} \widehat{\mathcal{F}}$ of the fundamental module $\widehat{\mathcal{E}}$ with $\widehat{\mathcal{O}}_X$-modules $\widehat{\mathcal{F}}$.

Remarks on general endomorphism-ringed super-C^k-schemes and differential calculus thereupon

Let $\widehat{X} := (X, \mathcal{O}_X, \widehat{\mathcal{O}}_X)$ be a super-C^k-scheme, with $\mathcal{O}_X$ a sheaf of finitely-generated germ-determined C^k-rings, and $\widehat{\mathcal{F}}$ be a finitely-presented quasi-coherent $\widehat{\mathcal{O}}_X^{\mathbb{C}}$-module on $\widehat{X}$. Then, the sheaf $\mathcal{E}nd_{\widehat{\mathcal{O}}_X^{\mathbb{C}}}(\widehat{\mathcal{F}})$ of endomorphisms of $\widehat{\mathcal{F}}$ is a quasi-coherent $\widehat{\mathcal{O}}_X^{\mathbb{C}}$-module that is finitely presentable as well. Thus, if one assumes further that the built-in sheaf-of-rings homomorphism $\widehat{\mathcal{O}}_X^{\mathbb{C}} \to \mathcal{E}nd_{\widehat{\mathcal{O}}_X^{\mathbb{C}}}(\widehat{\mathcal{F}})$ is injective; i.e. $\widehat{\mathcal{F}}$ is supported on the whole $\widehat{X}$, and defines

$$\widehat{\mathcal{O}}_X^{nc} := \mathcal{E}nd_{\widehat{\mathcal{O}}_X^{\mathbb{C}}}(\widehat{\mathcal{F}}),$$

then $\widehat{\mathcal{O}}_X^{nc}$ is a sheaf of rings that is a finitely-presentable (noncommutative) algebraic extension of the sheaf of rings $\widehat{\mathcal{O}}_X^{\mathbb{C}}$. This defines a new ringed topological space

$$\widehat{X}^{nc} \;:=\; (X, \mathcal{O}_X, \widehat{\mathcal{O}}_X, \widehat{\mathcal{O}}_X^{nc})\,,$$

which may be named *endomorphism-ringed super-C^k-scheme with a fundamental module*. The Azumaya/matrix case $\widehat{X}^{Az}$ corresponds to the case $\mathcal{F}$ is in addition locally free. Notions or constructions for $\widehat{X}^{Az}$ that depend only on the finite-presentability or algebraicness of the extension $\widehat{\mathcal{O}}_X \hookrightarrow \widehat{\mathcal{O}}_X^{Az}$ generalize immediately to the general endomorphism-ringed space $\widehat{X}^{nc}$. In particular, the basics:

- the notion of $\widehat{X}^{nc}$ as an equivalence class of gluing systems of rings,

- the functor $\widehat{\mathcal{F}} \otimes_{\widehat{\mathcal{O}}_X} (\,\cdot\,) : \widehat{\mathcal{O}}_X\text{-}\mathcal{M}od \rightarrow \widehat{\mathcal{O}}_X^{nc}\text{-}\mathcal{M}od$ that turn an $\widehat{\mathcal{O}}_X$-module to a (left) $\widehat{\mathcal{O}}_X^{nc}$-module,

- the tangent sheaf $\mathcal{T}_*\widehat{X}^{nc}$ and the cotangent sheaf $\mathcal{T}^*\widehat{X}^{nc}$ of $\widehat{X}^{nc}$

can all be defined/establised. And it remains instructive to regard the whole $\widehat{\mathcal{O}}_X$, despite only $\mathbb{Z}/2$-commutative, as the center of $\widehat{\mathcal{O}}_X^{nc}$.

4 Differentiable maps from an Azumaya/matrix supermanifold with a fundamental module to a real manifold

With the preparations in Sec. 2 and Sec. 3, we now come to the main theme of the current note: the notion of '*differentiable map* from an Azumaya/matrix supermanifold with a fundamental module' that generalizes the setting in [L-Y3] (D(11.1)) for the case of Azumaya/matrix manifolds.

4.1 A local study: The affine case

The local study in this subsection is the foundation to the general notion of a differentiable map from an Azumaya/matrix supermanifold with a fundamental module to a real manifold.

From the aspect of function-rings and modules

Definition 4.1.1. [admissible homomorphism from C^k-ring to Azumaya/matrix super-C^k-ring]. Let

- $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open sets,

- E be a complex C^k vector bundle of rank r on U, for our purpose we may assume that E is trivial,

- $\widehat{U} \underset{\widehat{\pi}}{\overset{\widehat{\iota}}{\rightleftarrows}} U$ be a super-C^k-manifold supported on U associated to the super-C^k-polynomial ring $C^k(U)[\theta^1, \cdots, \theta^s]$, where $\theta^\alpha\theta^\beta + \theta^\beta\theta^\alpha = 0$ for $1 \le \alpha, \beta \le s$,

- $pr_{\widehat{U}} : \widehat{U} \times V \rightarrow \widehat{U}$, $pr_V : \widehat{U} \times V \rightarrow V$ be the projection maps, and

- $\widehat{E} := \widehat{\pi}^* E$ be the pull-back complex vector bundle on $\widehat{U}$.

Then the endomorphism algebra of $\widehat{E}$ over $\widehat{U}$ is isomorphic to the $r\times r$ matrix ring $M_{r\times r}(C^k(U)[\theta^1,\cdots,\theta^s])$ over $C^k(U)[\theta^1,\cdots,\theta^s]^{\mathbb{C}}$. With this identification, a ring-homomorphism

$$\widehat{\varphi}^\sharp\;:\;C^k(V)\;\longrightarrow\;M_{r\times r}(C^k(U)[\theta^1,\cdots,\theta^s]^{\mathbb{C}})$$

over $\mathbb{R}\hookrightarrow\mathbb{C}$ is said to be C^k-*admissible* if the following diagram of ring-homomorphisms

$$\begin{tikzcd}
M_{r\times r}(C^k(U)[\theta^1,\cdots,\theta^s]^{\mathbb{C}}) & C^k(V) \arrow[l, "\widehat{\varphi}^\sharp"'] \\
C^k(U)[\theta^1,\cdots,\theta^s] \arrow[u, hook, "\dot{\pi}^{Az,\sharp}"] &
\end{tikzcd}$$

extends to a commutative diagram of ring-homomorphisms (over $\mathbb{R}$ or $\mathbb{R}\hookrightarrow\mathbb{C}$ when applicable)

$$\begin{tikzcd}
M_{r\times r}(C^k(U)[\theta^1,\cdots,\theta^s]^{\mathbb{C}}) & & C^k(V) \arrow[ll, "\widehat{\varphi}^\sharp"'] \arrow[d, hook, "pr_V^\sharp"] \\
C^k(U)[\theta^1,\cdots,\theta^s] \arrow[u, hook, "\dot{\pi}^{Az,\sharp}"] \arrow[rr, hook, "pr_{\widehat{U}}^\sharp"'] & & C^k(U\times V)[\theta^1,\cdots,\theta^s] \arrow[llu, "\tilde{\widehat{\varphi}}^\sharp"']
\end{tikzcd}$$

such that the following two conditions are satisfied:

(1) The image of $\tilde{\widehat{\varphi}}^\sharp$

$$Im\,\tilde{\widehat{\varphi}}^\sharp\;\simeq\;\frac{C^k(U\times V)[\theta^1,\cdots,\theta^s]}{Ker\,\tilde{\widehat{\varphi}}^\sharp}$$

admits a quotient super-C^k-ring structure from the super-C^k-ring $C^k(U\times V)[\theta^1,\cdots,\theta^s]$.

(2) Regard $\dot{\pi}^{Az,\sharp}$ and $\widehat{\varphi}^\sharp$ now as ring-homomorphism to $Im\,\tilde{\widehat{\varphi}}^\sharp$; then, with respect to the super-C^k-ring structure in Condition (1), both ring-homomorphisms

$$\dot{\pi}^{Az,\sharp}\;:\;C^k(U)[\theta^1,\cdots,\theta^s]\;\longrightarrow\;Im\,\tilde{\widehat{\varphi}}^\sharp\,,$$
$$\widehat{\varphi}^\sharp\;:\;C^k(V)\;\longrightarrow\;Im\,\tilde{\widehat{\varphi}}^\sharp$$

are super-C^k-ring-homomorphisms.

Note that since $C^k(U\times V)[\theta^1,\cdots,\theta^s]$ is the push-out of $C^k(U)[\theta^1,\cdots,\theta^s]$ and $C^k(V)$ in the category of super-C^k-rings, $\tilde{\widehat{\varphi}}^\sharp$ is unique if exists. In this case, one may think of $Im\,\tilde{\widehat{\varphi}}^\sharp$ as *the super-C^k-ring generated by* $C^k(U)[\theta^1,\cdots,\theta^s]$ *and* $Im\,\widehat{\varphi}^\sharp$ in $M_{r\times r}(C^k(U)[\theta^1,\cdots,\theta^s]^{\mathbb{C}})$. In notation,

$$A_{\widehat{\varphi}}\;:=\;C^k(U)[\theta^1,\cdots,\theta^s]\langle Im\,\widehat{\varphi}^\sharp\rangle\;:=\;Im\,\tilde{\widehat{\varphi}}^\sharp\,.$$

Bringing both $A_{\widehat{\varphi}}$ and the module of sections of the fundamental vector bundle $\widehat{E}$ over $\widehat{U}$ into the picture, one now has the following full diagram for a C^k-admissible ring-homomorphism

$\widehat{\varphi}: C^k(V)\to M_{r\times r}(C^k(U)[\theta^1,\,\cdots,\theta^s]^{\mathbb{C}})$ over $\mathbb{R}\hookrightarrow\mathbb{C}$:

$$
\begin{tikzcd}
(C^k(U)[\theta^1,\,\cdots,\theta^s]^{\mathbb{C}})^{\oplus r} & & & \\
 & M_{r\times r}(C^k(U)[\theta^1,\,\cdots,\theta^s]^{\mathbb{C}}) \arrow[ul, squiggly] & & \\
 & A_{\widehat{\varphi}} := C^k(U)[\theta^1,\,\cdots,\theta^s]\langle Im\,\widehat{\varphi}^\sharp\rangle \arrow[uul, squiggly] \arrow[u, hook, "\sigma_{\widehat{\varphi}}^\sharp"'] & & C^k(V) \arrow[ull, "\widehat{\varphi}^\sharp"'] \arrow[ll, "f_{\widehat{\varphi}}^\sharp"] \arrow[d, hook, "pr_V^\sharp"] \\
 & C^k(U)[\theta^1,\,\cdots,\theta^s] \arrow[uuul, squiggly] \arrow[u, hook, "\pi_{\widehat{\varphi}}^\sharp"'] \arrow[rr, hook, "pr_{\widehat{U}}^\sharp"'] & & C^k(U\times V)[\theta^1,\,\cdots,\theta^s]\,. \arrow[ull, two heads, "\tilde{\widehat{\varphi}}^\sharp"]
\end{tikzcd}
$$

Remark 4.1.2. $[\,k=\infty\,]$. For the case $k=\infty$, Condition (1) in Definition 4.1.1 is always satisfied and, hence, redundant.

From the aspect of super-C^k-schemes and sheaves

Continuing the notations of the previous theme. Let

- $\mathcal{O}_U$ be the sheaf of C^k-functions on U, $\mathcal{O}_V$ be the sheaf of C^k-functions on V, $\widehat{\mathcal{O}}_U$ be the structure sheaf of $\widehat{U}$ as the affine super-C^k-scheme associated to the super-C^k-ring $C^k(U)[\theta^1,\,\cdots,\theta^s]$,
- $\mathcal{A}_{\widehat{\varphi}}$ be the $\widehat{\mathcal{O}}_U$-algebra associated to the $C^k(U)[\theta^1,\,\cdots,\theta^s]$-algebra $A_{\widehat{\varphi}}$,
- $\mathcal{E}$ be the sheaf of C^k-sections of E over U, $\widehat{\mathcal{E}}:=\widehat{\pi}^*\mathcal{E}$ the induced locally free $\widehat{\mathcal{O}}_U$-module,
- $\widehat{\mathcal{O}}_U^{Az}\;:=\;\mathcal{E}nd_{\widehat{\mathcal{O}}_U^{\mathbb{C}}}(\widehat{\mathcal{E}})$ the structure sheaf of the Azumaya/matrix super-C^k-manifold $\widehat{U}^{Az}$, which realizes $\widehat{\mathcal{E}}$ as the fundamental module on $\widehat{U}^{Az}$.

Let

$$\widehat{\varphi}^\sharp\;:\;C^k(V)\;\longrightarrow\;M_{r\times r}(C^k(U)[\theta^1,\,\cdots,\theta^s]^{\mathbb{C}})$$

be a C^k-admissible ring-homomorphism over $\mathbb{R}\hookrightarrow\mathbb{C}$ as in the previous theme. Then the full diagram of ring-homomorphisms associated to $\widehat{\varphi}$ in the previous theme can be translated into the following diagram of maps between spaces:

$$
\begin{tikzcd}
\widehat{\mathcal{E}} \arrow[dr, dotted] \arrow[ddr, dotted] \arrow[dddr, dotted] & & & \\
 & \widehat{U}^{Az} \arrow[ddrr, "\widehat{\varphi}"] \arrow[d, two heads, "\sigma_{\widehat{\varphi}}"] & & \\
 & \widehat{U}_{\widehat{\varphi}} \arrow[rr, "f_{\widehat{\varphi}}"'] \arrow[d, two heads, "\pi_{\widehat{\varphi}}"] \arrow[drr, hook, "\tilde{\widehat{\varphi}}"'] & & V \\
 & \widehat{U} & & \widehat{U}\times V\,. \arrow[ll, two heads, "pr_{\widehat{U}}"] \arrow[u, two heads, "pr_V"']
\end{tikzcd}
$$

Notice that in the above diagram, the maps $\pi_{\widehat{\varphi}}$, $f_{\widehat{\varphi}}$, and $\tilde{\widehat{\varphi}}$ are now maps between super-C^k-ringed spaces in the sense of Definition 2.3.10 while the 'maps' $\widehat{\varphi}$ and $\sigma_{\widehat{\varphi}}$ are only conceptual without real contents and are defined solely contravariantly through the ring-homomorphisms $\widehat{\varphi}^\sharp$ and $\sigma_{\widehat{\varphi}}^\sharp$ respectively.

Definition 4.1.3. [C^k-map from Azumaya/matrix super-C^k-manifold $\widehat{U}^{Az}$]. We shall call $\widehat{\varphi}:(\widehat{U}^{Az},\widehat{\mathcal{E}})\to V$ in the above diagram a *k-times differentiable map* (in brief, *C^k-map*) from the super-C^k-manifold with a fundamental module $(\widehat{U}^{Az},\widehat{\mathcal{E}})$ to the C^k-manifold V. Through the underlying admissible ring-homomorphism $\widehat{\varphi}^\sharp$ in the previous theme, $\widehat{\mathcal{E}}$ becomes an $\mathcal{O}_Y$-module, called the *push-forward* of $\widehat{\mathcal{E}}$ under $\widehat{\varphi}$ and denoted by $\widehat{\varphi}_*(\widehat{\mathcal{E}})$ or $\widehat{\varphi}_*\widehat{\mathcal{E}}$. The *image* of $\widehat{\varphi}:\widehat{U}^{Az}\to V$, in notation $Im\widehat{\varphi}$ or $\widehat{\varphi}(\widehat{U}^{Az})$, is the C^k-subscheme of V defined by the ideal $Ker(\widehat{\varphi})$ of $C^k(V)$.

Lemma 4.1.4. [image $Im\widehat{\varphi}$ vs. support $Supp(\widehat{\varphi}_*(\widehat{\mathcal{E}}))$]. *Continuing the notation in Definition 4.1.3. The image $Im(\widehat{\varphi})$ of the C^k-map $\widehat{\varphi}$ is identical to the C^k-scheme-theoretical support $Supp(\widehat{\varphi}_*(\widehat{\mathcal{E}}))$ of the push-forward $\widehat{\varphi}_*(\widehat{\mathcal{E}})$.*

Definition 4.1.5. [surrogate of $\widehat{U}^{Az}$ specified by $\widehat{\varphi}$]. Continuing the discussion. The super-C^k-scheme $\widehat{U}_{\widehat{\varphi}}$ together with the built-in C^k-maps

$$\begin{array}{ccc} \widehat{U}_{\widehat{\varphi}} & \xrightarrow{\;f_{\widehat{\varphi}}\;} & V \\ \Big\downarrow{\scriptstyle \pi_{\widehat{\varphi}}} & & \\ \widehat{U} & & \end{array}$$

is called the *surrogate of* (the noncommutative) $\widehat{U}^{Az}$ *specified by* $\widehat{\varphi}:\widehat{U}^{Az}\to V$. Caution that in general there is *no* C^k-map $\widehat{U}\to V$ that makes the diagram commute.

The role of the fundamental module $\widehat{\mathcal{E}}$

Let

$$\widehat{\varphi}\;:\;(\widehat{U}^{Az},\widehat{\mathcal{E}})\;\longrightarrow\;V$$

be a C^k-map defined by a C^k-admissible ring-homomorphism

$$\widehat{\varphi}^\sharp\;:\;C^k(V)\;\longrightarrow\;M_{r\times r}(C^k(U)[\theta^1,\,\cdots,\theta^s]^{\mathbb{C}})$$

over $\mathbb{R}\hookrightarrow\mathbb{C}$ after fixing a trivialization of $\mathcal{E}$ on U. Then the same argument as in [L-Y3: Sec. 5.2] (D(11.1)) implies by construction the following properties related to the fundamental module $\widehat{\mathcal{E}}$ on the super-C^k-manifold $\widehat{U}$:

(1) The fundamental module $\widehat{\mathcal{E}}:=\widehat{\pi}^*\mathcal{E}$, first on $\widehat{U}$, is also naturally an $\mathcal{A}_{\widehat{\varphi}}$-module on $\widehat{U}_{\widehat{\varphi}}$ and an $\widehat{\mathcal{O}}_U^{Az}$-module on $\widehat{U}^{Az}$. We will denote them all by $\widehat{\mathcal{E}}$.

(2) The $\mathcal{O}_V$-modules $\varphi_*(\widehat{\mathcal{E}})$ and $f_{\widehat{\varphi},*}(\widehat{\mathcal{E}})$ are canonically isomorphic.

Definition 4.1.6. [graph of $\widehat{\varphi}$]. The *graph* of the C^k-map $\widehat{\varphi}:(\widehat{U}^{Az},\widehat{\mathcal{E}})\to V$ is defined to be the $\mathcal{O}_{\widehat{U}\times V}$-module

$$\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}} \;:=\; \tilde{\widehat{\varphi}}_*(\widehat{\mathcal{E}})$$

on the product-space $\widehat{U}\times V$.

Definition 4.1.7. [C^k-admissible $\mathcal{O}_{\widehat{M}}$-module]. Let $\widehat{M}$ be a super-C^k-manifold. An $\mathcal{O}_{\widehat{M}}$-module $\widehat{\mathcal{F}}$ is said to be C^k*-admissible* if the annihilator ideal sheaf $Ker(\mathcal{O}_{\widehat{M}}\to \mathcal{E}nd_{\mathcal{O}_{\widehat{M}}}(\widehat{\mathcal{F}}))$ is super-C^k-normal. In this case, $Ker(\mathcal{O}_{\widehat{M}}\to \mathcal{E}nd_{\mathcal{O}_{\widehat{M}}}(\widehat{\mathcal{F}}))$ defines a super-C^k-subscheme structure for $Supp(\widehat{\mathcal{F}})$ in $\widehat{M}$; i.e., the quotient map of $\mathcal{O}_{\widehat{M}}$, as a sheaf of super-$C^k$-rings,

$$\mathcal{O}_{\widehat{M}} \;\longrightarrow\; \mathcal{O}_{Supp(\widehat{\mathcal{F}})} := \mathcal{O}_{\widehat{\mathcal{M}}}/Ker(\mathcal{O}_{\widehat{M}}\to \mathcal{E}nd_{\mathcal{O}_{\widehat{M}}}(\widehat{\mathcal{F}}))$$

induces a sheaf-of-super-C^k-rings structure on $\mathcal{O}_{Supp(\widehat{\mathcal{F}})}$.

Remark 4.1.8. [*case* $k=\infty$]. For a super-C^∞-manifold $\widehat{M}$, every $\mathcal{O}_{\widehat{M}}$-module $\widehat{\mathcal{F}}$ is C^∞-admissible.

Lemma 4.1.9. [basic properties of $\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}}$]. *The graph $\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}}$ of $\widehat{\varphi}$ has the following properties:*

(1) *$\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}}$ is a C^k-admissible $\mathcal{O}^{\mathbb{C}}_{\widehat{U}\times V}$-module; its super-$C^k$-scheme-theoretical support $Supp(\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}})$ is isomorphic to the surrogate $\widehat{U}_{\widehat{\varphi}}$ of $\widehat{U}^{Az}$ specified by $\widehat{\varphi}$. In particular, $\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}}$ is of relative dimension 0 over $\widehat{U}$*

(2) *There is a canonical isomorphism $\widehat{\mathcal{E}}\xrightarrow{\sim} pr_{\widehat{U},*}(\tilde{\widehat{\mathcal{E}}})$ of $\mathcal{O}^{\mathbb{C}}_{\widehat{U}}$-modules. In particular, $\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}}$ is flat over $\widehat{U}$, of relative complex length r.*

(3) *There is a canonical exact sequence of $\mathcal{O}^{\mathbb{C}}_{\widehat{U}\times V}$-modules*

$$pr^*_{\widehat{U}}(\widehat{\mathcal{E}}) \;\longrightarrow\; \tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}} \;\longrightarrow\; 0\,.$$

(4) *The $\widehat{\mathcal{O}}_V$-modules $pr_{V,*}(\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}})$ and $\widehat{\varphi}_*(\widehat{\mathcal{E}})$ are canonically isomorphic.*

Conversely, one has the following lemma of reconstruction, which follows the same argument as in [L-L-S-Y] (D(2)) for the algebraic case and [L-Y3] (D(11.1)) for the C^k-case:

Lemma 4.1.10. [reconstructing C^k-map via $\mathcal{O}^{\mathbb{C}}_{\widehat{U}\times Y}$-module]. *Let $\tilde{\widehat{\mathcal{E}}}$ be an $\mathcal{O}^{\mathbb{C}}_{\widehat{U}\times V}$-module that is C^k-admissible, and of relative dimension 0, of finite relative complex length, and flat over $\widehat{U}$. For the moment, we assume further that $pr_{\widehat{U},*}(\tilde{\widehat{\mathcal{E}}})$ is trivial. Let $\widehat{\mathcal{E}}:=pr_{\widehat{U},*}(\tilde{\widehat{\mathcal{E}}})$. Then $\tilde{\widehat{\mathcal{E}}}$ specifies a C^k-map $\widehat{\varphi}:(\widehat{U}^{Az},\widehat{\mathcal{E}})\to V$ whose graph $\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}}$ is canonically isomorphic to $\tilde{\widehat{\mathcal{E}}}$.*

Proof. Note that the $\mathcal{O}_V$-action on $\tilde{\widehat{\mathcal{E}}}$ via $pr_V^\sharp$ and the $\mathcal{O}^{\mathbb{C}}_{\widehat{U}}$-action on $\tilde{\widehat{\mathcal{E}}}$ via $pr_{\widehat{U}}^\sharp$ commute since the image of $pr_V^\sharp:\mathcal{O}_V\to\mathcal{O}_{\widehat{U}\times V}$ lies in the center of $\mathcal{O}_{\widehat{U}\times V}$. Thus, the $\mathcal{O}_V$-action on $\tilde{\widehat{\mathcal{E}}}$ via $pr_V^\sharp$ induces a C^k-admissible $\widehat{\varphi}^\sharp:\mathcal{O}_V\to\mathcal{E}nd_{\mathcal{O}^{\mathbb{C}}_{\widehat{U}}}(\widehat{\mathcal{E}})$, which defines a C^k-map
$\widehat{\varphi}:(U,\widehat{\mathcal{O}}^{Az}_U:=\mathcal{E}nd_{\mathcal{O}^{\mathbb{C}}_{\widehat{U}}}(\widehat{\mathcal{E}}),\widehat{\mathcal{E}})\to V$. □

Remark 4.1.11. [*when* $\widehat{U}=U$]. For $\mathcal{S}=0$ the zero-$\mathcal{O}_U$-module, $\widehat{U}=U$; and all the settings/objects/statements in this subsection for the super-C^k-case reduce to the corresponding settings/objects/statements in [L-Y3: Sec. 5.1 & Sec. 5.2] (D(11.1)) for the C^k case.

The induced C^k-map $\varphi:(U^{Az},\mathcal{E})\to V$

By the post-composition with $\hat{\imath}^\sharp$ of the built-in ring-homomorphisms of the matrix super-C^k-ring

$$\begin{tikzcd} M_{r\times r}(C^k(U)[\theta^1,\,\cdots\,,\theta^s]^{\mathbb{C}}) \arrow[r, shift left, two heads, "\hat{\imath}^\sharp"] & M_{r\times r}(C^k(U)^{\mathbb{C}}) \arrow[l, shift left, hook', "\hat{\pi}^\sharp"] \end{tikzcd}\,,$$

a ring-homomorphism

$$\widehat{\varphi}^\sharp\;:\;C^k(V)\;\longrightarrow\;M_{r\times r}(C^k(U)[\theta^1,\,\cdots\,,\theta^s]^{\mathbb{C}})$$

induces a ring-homomorphism

$$\varphi^\sharp\;:\;C^k(V)\;\longrightarrow\;M_{r\times r}(C^k(U)^{\mathbb{C}})\,.$$

It follows by construction that:

Lemma 4.1.12. [induced C^k-admissible ring-homomorphism]. *If $\widehat{\varphi}^\sharp$ is C^k-admissible, then $\varphi^\sharp$ is C^k-admissible as well.*

In this case, the C^k-map

$$\widehat{\varphi}\;:\;(\widehat{U}^{Az},\widehat{\mathcal{E}})\;:=\;(\widehat{U},\widehat{\mathcal{O}}_U^{Az}:=\mathcal{E}nd_{\widehat{\mathcal{O}}_U^{\mathbb{C}}}(\widehat{\mathcal{E}}),\widehat{\mathcal{E}})\;\longrightarrow\;(V,\mathcal{O}_V)$$

defined by $\widehat{\varphi}^\sharp$ induces a C^k-map

$$\varphi\;:\;(U^{Az},\mathcal{E})\;:=\;(U,\mathcal{O}_U^{Az}:=\mathcal{E}nd_{\mathcal{O}_U^{\mathbb{C}}}(\mathcal{E}),\mathcal{E})\;\longrightarrow\;(V,\mathcal{O}_V)$$

defined by $\varphi^\sharp$ that fits into the following commutative diagram

$$\begin{tikzcd} \widehat{U}^{Az} \arrow[r, "\widehat{\varphi}"] & V \\ U^{Az} \arrow[u, hook, "\hat{\imath}"] \arrow[ur, "\varphi"'] & \end{tikzcd}\qquad,$$

whose full detail is given in the commutative diagram below: (Cf. [L-Y3] (D(11.1)).)

$$
\begin{tikzcd}
\widehat{\mathcal{E}} = \widehat{\pi}^*\mathcal{E} \arrow[drrr, dotted] \arrow[ddrr, dotted] \arrow[dddr, dotted] & & & & & \\
 & & & \widehat{U}^{Az} \arrow[dl, "\sigma_{\widehat{\varphi}}"'] \arrow[drr, "\widehat{\varphi}"] \arrow[dddd, shift left, ->>, "\dot{\widehat{\pi}}"] & & \\
 & & \widehat{U}_{\widehat{\varphi}} \arrow[dl, "\pi_{\widehat{\varphi}}"'] \arrow[rrr, "f_{\widehat{\varphi}}"] \arrow[drr, hook, "\tilde{\widehat{\varphi}}"] \arrow[dddd, shift left, ->>, "\widehat{\pi}"] & & & V \arrow[dddd, equal] \\
 & \widehat{U} \arrow[dddd, shift left, ->>, "\widehat{\pi}"] & & & \widehat{U}\times V \arrow[lll, ->>, "pr_{\widehat{U}}"] \arrow[ur, "pr_V"'] \arrow[dddd, shift left, ->>, "\widehat{\pi}"] & \\
\mathcal{E} \arrow[drrr, dotted] \arrow[ddrr, dotted] \arrow[dddr, dotted] & & & & & \\
 & & & U^{Az} \arrow[uuuu, hook, shift left, "\dot{\widehat{\iota}}"] \arrow[dl, "\sigma_{\varphi}"] \arrow[drr, "\varphi"] & & \\
 & & U_{\varphi} \arrow[uuuu, hook, shift left, "\widehat{\iota}"] \arrow[dl, "\pi_{\varphi}"] \arrow[rrr, "f_{\varphi}"] \arrow[drr, hook, "\tilde{\varphi}"] & & & V \\
 & U \arrow[uuuu, hook, shift left, "\widehat{\iota}"] & & & U\times V \arrow[lll, ->>, "pr_U"] \arrow[ur, "pr_V"'] \arrow[uuuu, hook, shift left, "\widehat{\iota}"] &
\end{tikzcd}
\,.
$$

Further observation that $Ker\dot{\widehat{\iota}}^\sharp$ is a nilpotent bi-ideal of $M_{r\times r}(C^k(U)[\theta^1,\,\cdots,\theta^s]^{\mathbb{C}})$ implies then:

Lemma 4.1.13. [images differ by nilpotency]. *As C^k-subschemes of V,*

$$
(Im\,\widehat{\varphi})_{\mathrm{red}} \;=\; (Im\,\varphi)_{\mathrm{red}}\,.
$$

Remark 4.1.14. [*for general $\mathcal{E}$ and $\widehat{U}$*]. To make the discussion in this subsection more explicit and notationwise simpler, we choose the complex vector bundle E over U to be trivial (and trivialized) and the super-C^k-manifold $\widehat{U}$ to be of product type $U\times\widehat{p}$, where $\widehat{p}$ is a super-point. For general E and $\widehat{U}$,

$$
\mathcal{O}_{\widehat{U}} \;:=\; \widehat{\mathcal{O}}_U \;=\; \textstyle\bigwedge^\bullet\mathcal{S}
$$

for some locally free $\mathcal{O}_U$-module $\mathcal{S}$ associated to a vector bundle S on U of rank, say, s. Let

- $E^\vee$ be the dual complex bundle of E and
  $End_U(E) = E\otimes E^\vee$ be the bundle of (complex) endomorphisms of E.

Then

- $\widehat{\mathcal{O}}_U(U) \;=\; C^k(\bigwedge^\bullet S)$,
- $\widehat{\mathcal{E}}(U)$ is canonically isomorphic to $C^k(E)\otimes_{C^k(U)} C^k(\bigwedge^\bullet S)$,
- $\widehat{\mathcal{O}}_U^{Az}(U)$ is canonically isomorphic to $C^k(End_U(E))\otimes_{C^k(U)} C^k(\bigwedge^\bullet S)$.

And the argument in this subsection goes through with the following replacements:

- the $C^k(U)[\theta^1, \cdots \theta^s]$-module $(C^k(U)[\theta^1, \cdots, \theta^s]^{\mathbb{C}})^{\oplus r}$
  $\Longrightarrow$ the $C^k(\bigwedge^\bullet S)$-module $C^k(E)\otimes_{C^k(U)} C^k(\bigwedge^\bullet S)$;
- the $C^k(U)[\theta^1, \cdots, \theta^s]$-algebra $M_{r\times r}(C^k(U)[\theta^1, \cdots \theta^s]^{\mathbb{C}})$
  $\Longrightarrow$ the $C^k(\bigwedge^\bullet S)$-algebra $C^k(End_U(E))\otimes_{C^k(U)} C^k(\bigwedge^\bullet S)$.

4.2 Differentiable maps from an Azumaya/matrix supermanifold with a fundamental module to a real manifold

In Sec. 4.1 we see that locally the notion of a C^k-map from an Azumaya/matrix super-C^k-manifold is fundamentally the same as the notion of a C^k-map from an Azumaya/matrix C^k-manifold. The only difference is the ring involved, which may modify the exact presentation but not the underlying concept. It follows that

- *Gluing from local to global with respect to the topology of the (auxiliary) manifold X, all the settings/discussions in [L-Y3: Sec. 5.3] (D(11.1)) can be adapted without work to the current super-case.*

The essential details are given in this subsection for the completeness of discussion.

4.2.1 Aspect I [fundamental]: Maps as gluing systems of ring-homomorphisms

The notion of a differentiable map

$$\widehat{\varphi}\;:\;(\widehat{X},\widehat{\mathcal{O}}_X^{Az}:=\mathcal{E}nd_{\widehat{\mathcal{O}}_X^{\mathbb{C}}}(\widehat{\mathcal{E}}),\widehat{\mathcal{E}})\;\longrightarrow\;Y$$

from an Azumaya/matrix supermanifold with a fundamental module to a real manifold follows from the notion of '*morphisms between spaces*' studied in [L-Y1: Sec. 1.2 *A noncommutative space as a gluing system of rings*] (D(1)).

The fundamental aspect of C^k-maps from Azumaya manifolds

Definition 4.2.1.1. [gluing system of C^k-admissible ring-homomorphisms]. Let

- $(X,\mathcal{O}_X)$ be a C^k-manifold, with the structure sheaf $\mathcal{O}_X$ of C^k-functions on X,
- $\mathcal{E}$ be a locally free $\mathcal{O}_X^{\mathbb{C}}$-module of finite rank on X,
- $\widehat{X}$ be a super-C^k-manifold with the structure sheaf $\widehat{\mathcal{O}}_X:=\bigwedge^\bullet\mathcal{S}$ for some locally free $\mathcal{O}_X$-module $\mathcal{S}$ of finite rank on X,
- $\widehat{\mathcal{E}}:=\mathcal{E}\otimes_{\mathcal{O}_X}\widehat{\mathcal{O}}_X$,
- $(\widehat{X}^{Az},\widehat{\mathcal{E}}) := (\widehat{X},\widehat{\mathcal{O}}_X^{Az}:=\mathcal{E}nd_{\widehat{\mathcal{O}}_X^{\mathbb{C}}}(\widehat{\mathcal{E}}),\widehat{\mathcal{E}})$ be an Azumaya/matrix super-C^k-manifold with a fundamental module,
- $(Y,\mathcal{O}_Y)$ be a C^k-manifold, with the structure sheaf $\mathcal{O}_Y$ of C^k-functions on Y.

A *(contravariant) gluing system of C^k-admissible ring-homomorphisms* over $\mathbb{R}\hookrightarrow\mathbb{C}$ related to $(\widehat{X}^{Az},Y)$ consists of the following data:

- (*local charts on* $\widehat{X}^{Az}$) an open cover $\mathcal{U}=\{U_\alpha\}_{\alpha\in A}$ on X,

- (*local charts on* Y) an open cover $\mathcal{V}=\{V_\beta\}_{\beta\in B}$ on Y,

- a gluing system $\widehat{\Phi}^\sharp$ of C^k-admissible ring-homomorphisms from $\{C^k(V_\beta)\}_\beta$ to $\{C^k(\mathcal{E}nd_{\mathcal{O}^{\mathbb{C}}_{U_\alpha}}(\mathcal{E}_{U_\alpha}))\otimes_{C^k(U_\alpha)}C^k(\bigwedge^\bullet\mathcal{S}_{U_\alpha})\}_\alpha$ over $\mathbb{R}\hookrightarrow\mathbb{C}$, which consists of

  - (*specification of a target-chart for each local chart on* $\widehat{X}^{Az}$) a map $\sigma:A\to B$,
  - (*differentiable map from charts on* $\widehat{X}^{Az}$ *to charts on* Y) a C^k-admissible ring-homomorphism over $\mathbb{R}\hookrightarrow\mathbb{C}$

  $$\widehat{\phi}^\sharp_{\alpha,\sigma(\alpha)}\;:\;C^k(V_{\sigma(\alpha)})\;\longrightarrow\;C^k(\mathcal{E}nd_{\mathcal{O}^{\mathbb{C}}_{U_\alpha}}(\mathcal{E}_{U_\alpha}))\otimes_{C^k(U_\alpha)}C^k(\textstyle\bigwedge^\bullet\mathcal{S}_{U_\alpha})$$

  for each $\alpha\in A$

  that satisfy

  - (*gluing/identification of maps at overlapped charts on* $\widehat{X}^{Az}$) for each pair $(\alpha_1,\alpha_2)\in A\times A$,

  (G1) $(\widehat{\phi}_{\alpha,\sigma(\alpha_1)})_*(\widehat{\mathcal{E}}_{U_{\alpha_1}\cap U_{\alpha_2}})$ is completely supported in $V_{\sigma(\alpha_1)}\cap V_{\sigma(\alpha_2)}\subset V_{\sigma(\alpha_1)}$,

  (G2) recall the C^k-admissible ring-homomorphism over $\mathbb{R}\hookrightarrow\mathbb{C}$

$$\widehat{\phi}^\sharp_{\alpha_1\alpha_2,\,\sigma(\alpha_1)\sigma(\alpha_2)}\;:\;C^k(V_{\sigma(\alpha_1)}\cap V_{\sigma(\alpha_2)})\;\longrightarrow\;C^k(\mathcal{E}nd_{\mathcal{O}^{\mathbb{C}}_{U_{\alpha_1}\cap U_{\alpha_2}}}(\mathcal{E}_{U_{\alpha_1}\cap U_{\alpha_2}}))\otimes_{C^k(U_{\alpha_1}\cap U_{\alpha_2})}C^k(\textstyle\bigwedge^\bullet\mathcal{S}_{U_{\alpha_1}\cap U_{\alpha_2}})$$

  induced by $\phi_{\alpha_1,\sigma(\alpha_1)}$, then

$$\widehat{\phi}^\sharp_{\alpha_1\alpha_2,\,\sigma(\alpha_1)\sigma(\alpha_2)}\;=\;\widehat{\phi}^\sharp_{\alpha_2\alpha_1,\,\sigma(\alpha_2)\sigma(\alpha_1)}\,.$$

Definition 4.2.1.2. [equivalent systems]. A gluing system $(\mathcal{U}',\mathcal{V}',\widehat{\Phi}'^\sharp)$ is said to be a *refinement* of another gluing system $(\mathcal{U},\mathcal{V},\widehat{\Phi}^\sharp)$, in notation $(\mathcal{U}',\mathcal{V}',\widehat{\Phi}'^\sharp)\preccurlyeq(\mathcal{U},\mathcal{V},\widehat{\Phi}^\sharp)$, if

- $\mathcal{U}'=\{U'_{\alpha'}\}_{\alpha'\in A'}$ is a refinement of $\mathcal{U}=\{U_\alpha\}_{\alpha\in A}$, with a map $\tau:A'\to A$ that labels inclusions $U'_{\alpha'}\hookrightarrow U_{\tau(\alpha')}$; similarly, $\mathcal{V}'=\{V'_{\beta'}\}_{\beta'\in B'}$ is a refinement of $\mathcal{V}=\{V_\beta\}_{\beta\in B}$, with a map $\upsilon:B'\to B$ that labels inclusions $V'_{\beta'}\hookrightarrow V_{\upsilon(\beta')}$; the maps between the index sets A, B, A', and B' satisfy the commutative diagram

$$\begin{array}{ccc} A' & \xrightarrow{\;\sigma'\;} & B' \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\upsilon} \\ A & \xrightarrow{\;\sigma\;} & B \end{array}\qquad.$$

- the C^k-admissible ring-homomorphism

$$\widehat{\phi}'^\sharp_{\alpha',\,\sigma'(\alpha')}\;:\;C^k(V_{\sigma'(\alpha')})\;\longrightarrow\;C^k(\mathcal{E}nd_{\mathcal{O}^{\mathbb{C}}_{U_{\alpha'}}}(\mathcal{E}_{U_{\alpha'}}))\otimes_{C^k(U_{\alpha'})}C^k(\textstyle\bigwedge^\bullet\mathcal{S}_{U_{\alpha'}})$$

  in $\widehat{\Phi}'^\sharp$ coincides with the C^k-admissible ring-homomorphism

$$C^k(V_{\sigma'(\alpha')})\;\longrightarrow\;C^k(\mathcal{E}nd_{\mathcal{O}^{\mathbb{C}}_{U_{\alpha'}}}(\mathcal{E}_{U_{\alpha'}}))\otimes_{C^k(U_{\alpha'})}C^k(\textstyle\bigwedge^\bullet\mathcal{S}_{U_{\alpha'}})$$

  induced by

$$\widehat{\phi}^\sharp_{\tau(\alpha'),\,\sigma(\tau(\alpha'))}\;=\;\widehat{\phi}^\sharp_{\tau(\alpha'),\,\upsilon(\sigma'(\alpha'))}\;:\;C^k(V_{\upsilon(\sigma'(\alpha'))})\;\longrightarrow\;C^k(\mathcal{E}nd_{\mathcal{O}^{\mathbb{C}}_{U_{\tau(\alpha')}}}(\mathcal{E}_{U_{\tau(\alpha')}}))\otimes_{C^k(U_{\tau(\alpha')})}C^k(\textstyle\bigwedge^\bullet\mathcal{S}_{U_{\tau(\alpha')}})$$

  in $\widehat{\Phi}^\sharp$ from the inclusions $U_{\alpha'}\hookrightarrow U_{\tau(\alpha')}$ and $V_{\sigma'(\alpha')}\hookrightarrow V_{\upsilon(\sigma'(\alpha'))}$.

Two gluing systems $(\mathcal{U}_1, \mathcal{V}_1, \widehat{\Phi}_1^\sharp)$ and $(\mathcal{U}_2, \mathcal{V}_2, \widehat{\Phi}_2^\sharp)$ are said to be *equivalent* if they have a common refinement.

Definition 4.2.1.3. [differentiable map as equivalence class of gluing systems]. We denote an equivalence class of contravariant gluing systems of C^k-admissible ring-homomorphisms compactly as

$$\varphi^\sharp \;:\; \mathcal{O}_Y \;\longrightarrow\; \widehat{\mathcal{O}}_X^{Az} \;:=\; \mathcal{E}nd_{\,\widehat{\mathcal{O}}_X^{\mathbb{C}}}(\widehat{\mathcal{E}})\,.$$

This defines a *k-times differentiable map* (i.e., *C^k-map*)

$$\widehat{\varphi} \;:\; (\widehat{X}, \widehat{\mathcal{O}}_X^{Az} := \mathcal{E}nd_{\,\widehat{\mathcal{O}}_X^{\mathbb{C}}}(\widehat{\mathcal{E}}), \widehat{\mathcal{E}}) \;\longrightarrow\; Y\,.$$

The C^k-admissible ring-homomorphism

$$\widehat{\phi}^\sharp_{\alpha,\sigma(\alpha)} \;:\; C^k(V_{\sigma(\alpha)}) \;\longrightarrow\; C^k(\mathcal{E}nd_{\,\mathcal{O}^{\mathbb{C}}_{U_\alpha}}(\mathcal{E}_{U_\alpha}))\otimes_{C^k(U_\alpha)} C^k(\textstyle\bigwedge^\bullet \mathcal{S}_{U_\alpha})$$

renders $C^k(\mathcal{E}_{U_\alpha})\otimes_{C^k(U_\alpha)} C^k(\bigwedge^\bullet \mathcal{S}_{U_\alpha})$ a $C^k(V_{\sigma(\alpha)})$-module. Passing to germs of C^k-sections (with respect to the topology of X), this defines a sheaf of $\mathcal{O}_{V_{\sigma(\alpha)}}$-modules, denoted by $(\widehat{\phi}_{\alpha,\,\sigma(\alpha)})_*(\widehat{\mathcal{E}}_{U_\alpha})$. The following lemma/definition follows by construction:

Lemma/Definition 4.2.1.4. [push-forward $\widehat{\varphi}_*(\widehat{\mathcal{E}})$ under $\widehat{\varphi}$]. *The collection of sheaves on local charts $\{(\widehat{\phi}_{\alpha,\,\sigma(\alpha)})_*(\widehat{\mathcal{E}}_{U_\alpha})\}_{\alpha\in A}$ glue to a sheaf of $\mathcal{O}_Y$-modules on Y. It is independent of the contravariant gluing system of C^k-admissible ring-homomorphisms over $\mathbb{R}\hookrightarrow\mathbb{C}$ that represents $\widehat{\varphi}$.* It is called the *push-forward of $\widehat{\mathcal{E}}$ under $\widehat{\varphi}$* and is denoted by $\widehat{\varphi}_*(\widehat{\mathcal{E}})$.

Lemma/Definition 4.2.1.5. [surrogate of $\widehat{X}^{Az}$ specified by $\widehat{\varphi}$]. (Cf. Definition 4.1.5.) *The collection of local surrogates $\widehat{U}_{\varphi_{\alpha,\sigma(\alpha)}}$ of $\widehat{U}_\alpha^{Az}$ specified by $\widehat{\varphi}_{\alpha,\sigma(\alpha)}$ glue to a super-C^k-scheme over $\widehat{X}$. It is independent of the contravariant gluing system of C^k-admissible ring-homomorphisms over $\mathbb{R}\hookrightarrow\mathbb{C}$ that represents $\widehat{\varphi}$.* It is called the *surrogate of $\widehat{X}^{Az}$ specified by $\widehat{\varphi}$*; in notation, $\widehat{X}_{\widehat{\varphi}}$.

Readers are referred to [L-Y3: Remark 5.3.1.8] (D(11.1)), with a straightforward adaptation to the current super-case, for three conceptually important remarks on Definition 4.2.1.3.

The equivalent affine setting

Recall that the C^k-manifolds $(X,\mathcal{O}_X)$ and $(Y,\mathcal{O}_Y)$ are affine C^k-schemes associated respectively to the C^k-rings $C^k(X)$ and $C^k(Y)$ in the context of C^k-algebraic geometry. Observe also that

- *As an $\mathcal{O}_X$-module, the sheaf $\widehat{\mathcal{O}}_X^{Az}$ of $\mathcal{O}_X^{\mathbb{C}}$-algebras is quasi-coherent. Explicitly, it is the quasi-coherent sheaf on the affine C^k-scheme $(X,\mathcal{O}_X)$ associated to the $C^k(X)^{\mathbb{C}}$-module $C^k(\mathcal{E}nd_{\,\mathcal{O}_X^{\mathbb{C}}}(\mathcal{E}))\otimes_{C^k(X)} C^k(\bigwedge^\bullet \mathcal{S})$.*

This implies that

Lemma/Definition 4.2.1.6. [**C^k-map in affine setting**]. *The equivalence class*

$$\widehat{\varphi}^\sharp \;:\; \mathcal{O}_Y \;\longrightarrow\; \widehat{\mathcal{O}}_X^{Az} := \mathcal{E}nd_{\widehat{\mathcal{O}}_X^{\mathbb{C}}}(\widehat{\mathcal{E}})$$

of gluing systems of C^k-admissible ring-homomorphisms over $\mathbb{R} \hookrightarrow \mathbb{C}$ in Definition 4.2.1.3 defines a C^k-admissible ring-homomorphism, still denoted by $\widehat{\varphi}^\sharp$,

$$\widehat{\varphi}^\sharp \;:\; C^k(Y) \;\longrightarrow\; C^k(\widehat{X}^{Az}) := C^k(\mathcal{E}nd_{\mathcal{O}_X^{\mathbb{C}}}(\mathcal{E})) \otimes_{C^k(X)} C^k(\bigwedge^\bullet \mathcal{S})$$

over $\mathbb{R} \hookrightarrow \mathbb{C}$. Conversely, any C^k-admissible ring-homomorphism $\widehat{\varphi}^\sharp : C^k(Y) \to C^k(\widehat{X}^{Az})$ over $\mathbb{R} \to \mathbb{C}$ defines an equivalence class $\widehat{\varphi}^\sharp : \mathcal{O}_Y \to \widehat{\mathcal{O}}_X^{Az}$ of contravariant gluing systems of C^k-admissible ring-homomorphisms over $\mathbb{R} \hookrightarrow \mathbb{C}$ associated to $(\widehat{X}^{Az}, Y)$. It follows that the notion of a C^k-map

$$\widehat{\varphi} \;:\; (X, \widehat{\mathcal{O}}_X^{Az} := \mathcal{E}nd_{\widehat{\mathcal{O}}_X^{\mathbb{C}}}(\widehat{\mathcal{E}}), \widehat{\mathcal{E}}) \;\longrightarrow\; Y\,.$$

in Definition 4.2.1.3 can be equivalently defined by a C^k-admissible ring-homomorphism $\widehat{\varphi}^\sharp : C^k(Y) \to C^k(\widehat{X}^{Az})$ over $\mathbb{R} \hookrightarrow \mathbb{C}$.

Remark 4.2.1.7. [*when* $\widehat{X} = X$]. For $\mathcal{S} = 0$ the zero-$\mathcal{O}_X$-module, $\widehat{X} = X$; and all the settings/objects/statements in this subsection for the super-C^k-case in the current subsection reduce to the corresponding settings/objects/statements in [L-Y3: Sec. 5.3.1] (D(11.1)) for the C^k case. And hence similarly, Sec. 4.2.2, Sec. 4.2.3, Sec. 4.2.4 of the current note to [L-Y3: Sec. 5.3.2, Sec. 5.3.3, Sec. 5.3.4] (D(11.1)).

The induced C^k-map $\varphi : (X^{Az}, \mathcal{E}) \to Y$

Continuing the notation in Definition 4.2.1.1. Let

$$\Big(\mathcal{U} = \{U_\alpha\}_{\alpha\in A}\,,\; \mathcal{V} = \{V_\beta\}_{\beta\in B}\,, \\ \widehat{\Phi}^\sharp = \big(\, \sigma : A \to B\,,\; \{\widehat{\phi}^\sharp_{\alpha,\sigma(\alpha)} : C^k(V_{\sigma(\alpha)}) \to C^k(\mathcal{E}nd_{\mathcal{O}^{\mathbb{C}}_{U_\alpha}}(\mathcal{E})) \otimes_{C^k(U_\alpha)} C^k(\bigwedge^\bullet \mathcal{S}_{U_\alpha})\}_{\alpha\in A}\,\big)\Big)$$

be a contravariant gluing system of C^k-admissible ring-homomorphisms over $\mathbb{R} \hookrightarrow \mathbb{C}$ associated to $(\widehat{X}^{Az}, Y)$. Recall the surjective ring-homomorphism

$$\hat{\iota}^\sharp \;:\; C^k(\mathcal{E}nd_{\mathcal{O}^{\mathbb{C}}_{U_\alpha}}(\mathcal{E})) \otimes_{C^k(U_\alpha)} C^k(\bigwedge^\bullet \mathcal{S}_{U_\alpha}) \;\longrightarrow\; C^k(\mathcal{E}nd_{\mathcal{O}^{\mathbb{C}}_{U_\alpha}}(\mathcal{E}))$$

for every $\alpha \in A$ and let

$$\phi^\sharp_{\alpha,\sigma(\alpha)} \;:=\; \hat{\iota} \circ \widehat{\phi}^\sharp_{\alpha,\sigma(\alpha)}\,.$$

Then,

$$\Big(\mathcal{U} = \{U_\alpha\}_{\alpha\in A}\,,\; \mathcal{V} = \{V_\beta\}_{\beta\in B}\,, \\ \Phi^\sharp = \big(\, \sigma : A \to B\,,\; \{\phi^\sharp_{\alpha,\sigma(\alpha)} : C^k(V_{\sigma(\alpha)}) \to C^k(\mathcal{E}nd_{\mathcal{O}^{\mathbb{C}}_{U_\alpha}}(\mathcal{E}))\}_{\alpha\in A}\,\big)\Big)$$

becomes a contravariant gluing system of C^k-admissible ring-homomorphisms over $\mathbb{R} \hookrightarrow \mathbb{C}$ associated to (X^{Az}, Y). Furthermore, if $(\mathcal{U}_1, \mathcal{V}_1, \widehat{\Phi}_1^\sharp)$ and $(\mathcal{U}_2, \mathcal{V}_2, \widehat{\Phi}_2^\sharp)$ are equivalent, then, so are their associated gluing systems $(\mathcal{U}_1, \mathcal{V}_1, \Phi_1^\sharp)$ and $(\mathcal{U}_2, \mathcal{V}_2, \Phi_2^\sharp)$. It follows that

Proposition 4.2.1.8. [induced C^k-map $\varphi : (X^{Az}, \mathcal{E}) \to Y$]. *$\widehat{\varphi}^\sharp : \mathcal{O}_Y \to \widehat{\mathcal{O}}_X^{Az}$ defines an accompanying $\varphi^\sharp : \mathcal{O}_Y \to \mathcal{O}_X^{Az}$ through the post-composition with $\dot{\widehat{\iota}}^\sharp : \widehat{\mathcal{O}}_X^{Az} \to \mathcal{O}_X^{Az}$; that is, a commutative diagram*

$$
\begin{tikzcd}
\widehat{\mathcal{O}}_X^{Az} \arrow[d, "\dot{\widehat{\iota}}^\sharp"'] & & \mathcal{O}_Y \arrow[ll, "\widehat{\varphi}^\sharp"'] \arrow[dll, "\varphi^\sharp"] \\
\mathcal{O}_X^{Az} & &
\end{tikzcd}\quad,
$$

or, equivalently, a commutative diagram of C^k-maps

$$
\begin{tikzcd}
\widehat{X}^{Az} \arrow[rr, "\widehat{\varphi}"] & & Y \\
X^{Az} \arrow[u, hook, "\dot{\widehat{\iota}}"] \arrow[urr, "\varphi"'] & &
\end{tikzcd}\quad,
$$

whose full detail is given in the commutative diagram below: (Cf. Diagrams after Lemma 4.1.12.)

$$
\begin{tikzcd}[column sep=small]
\widehat{\mathcal{E}} = \widehat{\pi}^*\mathcal{E} \arrow[drrr, dotted] \arrow[ddrr, dotted] \arrow[dddr, dotted] & & & & & \\
& & & \widehat{X}^{Az} \arrow[dl, "\sigma_{\widehat{\varphi}}"'] \arrow[drr, "\widehat{\varphi}"] \arrow[dddd, shift left, "\dot{\widehat{\pi}}"] & & \\
& & \widehat{X}_{\widehat{\varphi}} \arrow[rrr, "f_{\widehat{\varphi}}"] \arrow[drr, hook, "\tilde{\widehat{\varphi}}"] \arrow[dl, "\pi_{\widehat{\varphi}}"'] \arrow[dddd, shift left, "\widehat{\pi}"] & & & Y \arrow[dddd, equal] \\
& \widehat{X} \arrow[dddd, shift left, "\widehat{\pi}"] & & & \widehat{X} \times Y \arrow[lll, two heads, "pr_{\widehat{X}}"] \arrow[ur, "pr_Y"'] \arrow[dddd, shift left, "\widehat{\pi}"] & \\
\mathcal{E} \arrow[drrr, dotted] \arrow[ddrr, dotted] \arrow[dddr, dotted] & & & & & \\
& & & X^{Az} \arrow[uuuu, hook, shift left, "\dot{\widehat{\iota}}"] \arrow[dl, "\sigma_{\varphi}"] \arrow[drr, "\varphi"] & & \\
& & X_{\varphi} \arrow[uuuu, hook, shift left, "\widehat{\iota}"] \arrow[rrr, "f_{\varphi}"] \arrow[drr, hook, "\tilde{\varphi}"] \arrow[dl, "\pi_{\varphi}"] & & & Y \\
& X \arrow[uuuu, hook, shift left, "\widehat{\iota}"] & & & X \times Y \arrow[lll, two heads, "pr_X"] \arrow[ur, "pr_Y"] \arrow[uuuu, hook, shift left, "\widehat{\iota}"] &
\end{tikzcd}\quad.
$$

Since $Ker\dot{\widehat{\iota}}^\sharp$ is a nilpotent ideal sheaf of $\widehat{\mathcal{O}}_X^{Az}$ on $\widehat{X}^{Az}$, one has

Corollary 4.2.1.9. *As C^k-subschemes of Y,*

$$
(Im\,\widehat{\varphi})_{\rm red} \;=\; (Im\,\varphi)_{\rm red}\,.
$$

4.2.2 Aspect II: The graph of a differentiable map

Similar to the studies [L-L-S-Y: Sec. 2.2] (D(2)) and [L-Y2: Sec. 2.2] (D(6)) in the algebro-geometric setting and [L-Y3: Sec. 5.3] (D(11.1)) in the C^k-algebro-geometric/synthetic-differential-topological setting, the graph of a differentiable map $\widehat{\varphi}:(X,\widehat{\mathcal{O}}_X^{Az}:=\mathcal{E}nd_{\widehat{\mathcal{O}}_X^{\mathbb{C}}}(\widehat{\mathcal{E}}),\widehat{\mathcal{E}})\to Y$ is a sheaf $\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}}$ of $\mathcal{O}^{\mathbb{C}}_{\widehat{X}\times Y}$-modules on the super-$C^k$-manifold $\widehat{X}\times Y$ with special properties. And $\widehat{\varphi}$ can be recovered from its graph. Details are given below for the current super-synthetic-differential-topological setting.

Graph of a differentiable map $\widehat{\varphi}:(\widehat{X}^{Az},\widehat{\mathcal{E}})\to Y$

It follows from the local study in Sec. 4.1 that an equivalence class

$$\widehat{\varphi}^{\sharp}\;:\;\mathcal{O}_Y\;\longrightarrow\;\widehat{\mathcal{O}}_X^{Az}:=\mathcal{E}nd_{\widehat{\mathcal{O}}_X^{\mathbb{C}}}(\widehat{\mathcal{E}})$$

of gluing systems of C^k-admissible ring-homomorphisms over $\mathbb{R}\hookrightarrow\mathbb{C}$ extends canonically to an equivalence class

$$\tilde{\widehat{\varphi}}^{\sharp}\;:\;\mathcal{O}_{\widehat{X}\times Y}\;\longrightarrow\;\widehat{\mathcal{O}}_X^{Az}$$

of gluing systems of ring-homomorphisms over $\mathbb{R}\hookrightarrow\mathbb{C}$ that defines canonically to a map

$$\tilde{\widehat{\varphi}}\;:\;(X,\widehat{\mathcal{O}}_X^{Az}:=\mathcal{E}nd_{\widehat{\mathcal{O}}_X^{\mathbb{C}}}(\widehat{\mathcal{E}}),\widehat{\mathcal{E}})\;\longrightarrow\;\widehat{X}\times Y\,,$$

making the following diagram commute:

$$\begin{tikzcd}
(X,\widehat{\mathcal{O}}_X^{Az}:=\mathcal{E}nd_{\widehat{\mathcal{O}}_X^{\mathbb{C}}}(\widehat{\mathcal{E}}),\widehat{\mathcal{E}}) \arrow[r, "\tilde{\widehat{\varphi}}"] \arrow[dr, "\widehat{\varphi}"'] & \widehat{X}\times Y \arrow[d, "pr_Y"] \\
 & Y
\end{tikzcd}.$$

Here $pr_Y:\widehat{X}\times Y\to Y$ is the projection map to Y.

Definition 4.2.2.1. [graph of $\widehat{\varphi}$]. The *graph* of a C^k-map $\widehat{\varphi}:(\widehat{X}^{Az},\widehat{\mathcal{E}})\to Y$ is a sheaf $\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}}$ of $\mathcal{O}^{\mathbb{C}}_{\widehat{X}\times Y}$-modules, defined by

$$\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}}\;:=\;\tilde{\widehat{\varphi}}_*(\widehat{\mathcal{E}})\,.$$

The following basic properties of $\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}}$ follow directly from the local study in Sec. 4.1:

Lemma 4.2.2.2. [basic properties of $\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}}$]. *The graph $\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}}$ of $\widehat{\varphi}$ has the following properties:*

(1) *$\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}}$ is a C^k-admissible $\mathcal{O}^{\mathbb{C}}_{\widehat{X}\times Y}$-module; its super-$C^k$-scheme-theoretical support $Supp\,(\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}})$ is isomorphic to the surrogate $\widehat{X}_{\widehat{\varphi}}$ of $\widehat{X}^{Az}$ specified by $\widehat{\varphi}$. In particular, $\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}}$ is of relative dimension 0 over $\widehat{X}$*

(2) *There is a canonical isomorphism $\widehat{\mathcal{E}}\xrightarrow{\sim}pr_{\widehat{X},*}(\tilde{\widehat{\mathcal{E}}})$ of $\mathcal{O}^{\mathbb{C}}_{\widehat{X}}$-modules. In particular, $\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}}$ is flat over $\widehat{X}$, of relative complex length r.*

(3) *There is a canonical exact sequence of* $\mathcal{O}^{\mathbb{C}}_{\widehat{X}\times Y}$*-modules*

$$pr_{\widehat{X}}^*(\widehat{\mathcal{E}}) \longrightarrow \tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}} \longrightarrow 0\,.$$

(4) *The* $\widehat{\mathcal{O}}_Y$*-modules* $pr_{Y,*}(\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}})$ *and* $\widehat{\varphi}_*(\widehat{\mathcal{E}})$ *are canonically isomorphic.*

Lemma 4.2.2.3. [presentation of graph of $\widehat{\varphi}$]. *Continuing the notation. The graph* $\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}}$ *of* $\widehat{\varphi}$ *admits a presentation given by a natural isomorphism*

$$\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}} \;\simeq\; pr_{\widehat{X}}^*(\widehat{\mathcal{E}}) \Big/ \Big((pr_Y^*(f) - pr_{\widehat{X}}^*(\widehat{\varphi}^\sharp(f))\,:\, f\in C^k(Y))\cdot pr_{\widehat{X}}^*(\widehat{\mathcal{E}}) \Big)\;.$$

Remark 4.2.2.4. [*presentation of* $\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}}$ *in local trivialization of* $\widehat{\mathcal{E}}$].
Note that with respect to a local trivialization $\mathbb{C}^{\oplus r}\otimes_{\mathbb{R}}(C^k(U)\otimes_{\mathbb{R}} C^k(\bigwedge^\bullet \mathcal{S}_U))$ of $\widehat{\mathcal{E}}$ and, hence, a local trivialization

$$\mathbb{C}^{\oplus r}\otimes_{\mathbb{R}} C^k(U\times Y)\otimes_{\mathbb{R}} C^k(\textstyle\bigwedge^\bullet \mathcal{S}_U) \;\simeq\; \mathbb{C}^{\oplus r}\otimes_{\mathbb{R}}(C^k(U)\otimes_{\mathbb{R}} C^k(\bigwedge^\bullet \mathcal{S}_U))\otimes_{C^k} C^k(Y))\,,$$

where $\otimes_{C^k}$ is the C^k-push-out, of $pr_{\widehat{X}}^*(\widehat{\mathcal{E}})$ on $\widehat{X}\times Y$ restricted to over $\widehat{U}\subset\widehat{X}$, the subsheaf $(pr_Y^*(f) - pr_{\widehat{X}}^*(\widehat{\varphi}^\sharp(f))\,:\, f\in C^k(Y))\cdot pr_{\widehat{X}}^*(\widehat{\mathcal{E}})$ in the above lemma is generated (as an $\mathcal{O}^{\mathbb{C}}_{\widehat{X}\times Y}$-module) by elements of the form

$$v\otimes f \;-\; (\widehat{\varphi}^\sharp(f)(v))\otimes 1\,,\qquad f\in C^k(Y)\,.$$

Here, v represents an $r\times 1$ column-vector with coefficients in $C^k(U)\otimes_{\mathbb{R}} C^k(\bigwedge^\bullet \mathcal{S}_U)$.

Recovering $\widehat{\varphi}:(\widehat{X}^{Az},\widehat{\mathcal{E}})\to Y$ from an $\mathcal{O}^{\mathbb{C}}_{\widehat{X}\times Y}$-module

Conversely, let $\widehat{X}=(X,\widehat{\mathcal{O}}_X:=\bigwedge^\bullet\mathcal{S})$ be a super-C^k-manifold, Y be a C^k-manifold, and $\tilde{\widehat{\mathcal{E}}}$ be a sheaf of $\mathcal{O}^{\mathbb{C}}_{\widehat{X}\times Y}$-modules with the following properties:

(M1) The annihilator ideal sheaf $Ker(\mathcal{O}_{\widehat{X}\times Y}\to \mathcal{E}nd_{\mathcal{O}_{\widehat{X}\times Y}}(\tilde{\widehat{\mathcal{E}}}))$ is super-C^k-normal in $\mathcal{O}_{\widehat{X}\times Y}$; thus, $Supp(\tilde{\widehat{\mathcal{E}}})$ is a super-C^k-subscheme of the super-C^k-manifold $\widehat{X}\times Y$. Assume that $Supp(\tilde{\widehat{\mathcal{E}}})$ is of relative dimension 0 over $\widehat{X}$.

(M2) The push-forward $\widehat{\mathcal{E}}:=pr_{\widehat{X},*}(\tilde{\widehat{\mathcal{E}}})$ is a locally free $\mathcal{O}^{\mathbb{C}}_{\widehat{X}}$-module of finite rank, say, r.

Then

$$(X,\widehat{\mathcal{O}}_X^{Az}:=\mathcal{E}nd_{\widehat{\mathcal{O}}_X^{\mathbb{C}}}(\widehat{\mathcal{E}}),\widehat{\mathcal{E}})$$

is an Azumaya/matrix super-C^k-manifold with a fundamental module and $\tilde{\widehat{\mathcal{E}}}$ defines an equivalence class

$$\widehat{\varphi}^\sharp:\mathcal{O}_Y \longrightarrow \widehat{\mathcal{O}}_X^{Az}$$

of contravariant gluing systems of C^k-admissible ring-homomorphisms over $\mathbb{R}\hookrightarrow\mathbb{C}$ related to $(\widehat{X}^{Az},Y)$ as follows:

(1) Let $\widehat{U} \subset \widehat{X}$ be an open set from an atlas of X such that $pr_Y(Supp\,(\tilde{\widehat{\mathcal{E}}}_{\widehat{U}}))$ is contained in an open set $V \subset Y$ that lies in an atlas of Y. Here, we treat $\tilde{\widehat{\mathcal{E}}}$ also as a sheaf over $\widehat{X}$ and $\tilde{\widehat{\mathcal{E}}}_{\widehat{U}} := \tilde{\widehat{\mathcal{E}}}|_{\widehat{U}\times Y}$ is the restriction of $\tilde{\widehat{\mathcal{E}}}$ to over $\widehat{U}$.

(2) Let $f \in C^k(V)$. Then the multiplication by $pr_Y^*(f) \in C^k(\widehat{U}\times V)$ induces an endomorphism $\tilde{\widehat{\alpha}}_f : \tilde{\widehat{\mathcal{E}}}_{\widehat{U}} \to \tilde{\widehat{\mathcal{E}}}_{\widehat{U}}$ as an $\mathcal{O}^{\mathbb{C}}_{\widehat{U}\times V}$-module. Since $\widehat{U}\times V \supset Supp\,(\tilde{\widehat{\mathcal{E}}}_{\widehat{U}})$ and $pr_Y^*(f)$ lies in the center of $\mathcal{O}_{\widehat{U}\times Y}$,

$$\widehat{\alpha}_f \;:=\; pr_{\widehat{X}*}(\tilde{\widehat{\alpha}}_f)$$

defines in turn a C^k-endomorphism of the $\widehat{\mathcal{O}}_U^{\mathbb{C}}$-module $\widehat{\mathcal{E}}_U$; i.e. $\widehat{\alpha}_f \in \widehat{\mathcal{O}}_X^{Az}(U)$. This defines a ring-homomorphism $\widehat{\varphi}^\sharp : C^k(V) \to \widehat{\mathcal{O}}_X^{Az}(U)$ over $\mathbb{R} \hookrightarrow \mathbb{C}$, with $f \mapsto \widehat{\alpha}_f$. By construction, $\widehat{\varphi}^\sharp$ is C^k-admissible.

(3) Compatibility of the system of C^k-admissible ring-homomorphisms $\widehat{\varphi}^\sharp : C^k(V) \to \widehat{\mathcal{O}}_X^{Az}(U)$ over $\mathbb{R} \hookrightarrow \mathbb{C}$ with gluings follows directly from the construction.

In this way, $\tilde{\widehat{\mathcal{E}}}$ defines a C^k-map $\widehat{\varphi} : (\widehat{X}^{Az}, \widehat{\mathcal{E}}) \to Y$.

By construction, the graph $\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}}$ of the C^k-map $\widehat{\varphi}$ associated to $\tilde{\widehat{\mathcal{E}}}$ is canonically isomorphic to $\tilde{\widehat{\mathcal{E}}}$. This gives an equivalence of the two notions/categories:

$$\boxed{C^k\text{-maps } \widehat{\varphi} : (\widehat{X}^{Az}, \widehat{\mathcal{E}}) \to Y} \;\Longleftrightarrow\; \boxed{\mathcal{O}^{\mathbb{C}}_{\widehat{X}\times Y}\text{-modules } \tilde{\widehat{\mathcal{E}}} \text{ that satisfy (M1) and (M2)}}$$

(FIGURE 4-2-2-1.)

4.2.3 Aspect III: From maps to the stack of D0-branes

Aspect II of a C^k-map $\widehat{\varphi} : (\widehat{X}^{Az}, \widehat{\mathcal{E}}) \to Y$ discussed in Sec. 4.2.2 brings out a third aspect of $\widehat{\varphi}$, which we now explain.

An Azumaya/matrix supermanifold $\widehat{X}^{Az}$ as a smearing of unfixed Azumaya/matrix points over the underlying supermanifold $\widehat{X}$

We shall illuminate this in three steps. First, recall the following example in complex analysis:

Example 4.2.3.1. [real contour γ in complex line $\mathbb{C}^1$]. ([L-Y3 : Example 5.3.3.1] (D(11.1)).) A differentiable contour in the complex line $\mathbb{C}^1$ with complex coordinate $z = x + \sqrt{-1}y$ is a differentiable map

$$\gamma \;=\; \gamma_x + \sqrt{-1}\gamma_y \;:\; [0,1] \;\longrightarrow\; \mathbb{C}^1\,.$$

There is no issue about this, if γ is treated as a map between point-sets with a manifold structure: from the interval $[0,1]$ to the underlying real 2-space $\mathbb{R}^2$ of $\mathbb{C}^1$ with coordinates (x,y). However, in terms of function rings, some care needs to be taken. While there is a built-in ring-homomorphism $\mathbb{R} \hookrightarrow \mathbb{C}$ over $\mathbb{R}$, there exists no ring-homomorphism $\mathbb{C} \to \mathbb{R}$ with $0 \mapsto 0$ and $1 \mapsto 1$. If follows that there is no ring-homomorphism $\gamma^\sharp : C^k(\mathbb{C}^1)^{\mathbb{C}} \to C^k([0,1])$, where $C^k(\mathbb{C}^1)^{\mathbb{C}}$ is the algebra of complex-valued C^k-functions on $\mathbb{C}^1$. To remedy this, one should first complexify $C^k([0,1])$ to

$$C^k([0,1])^{\mathbb{C}} \;:=\; C^k([0,1]) \otimes_{\mathbb{R}} \mathbb{C}\,;$$

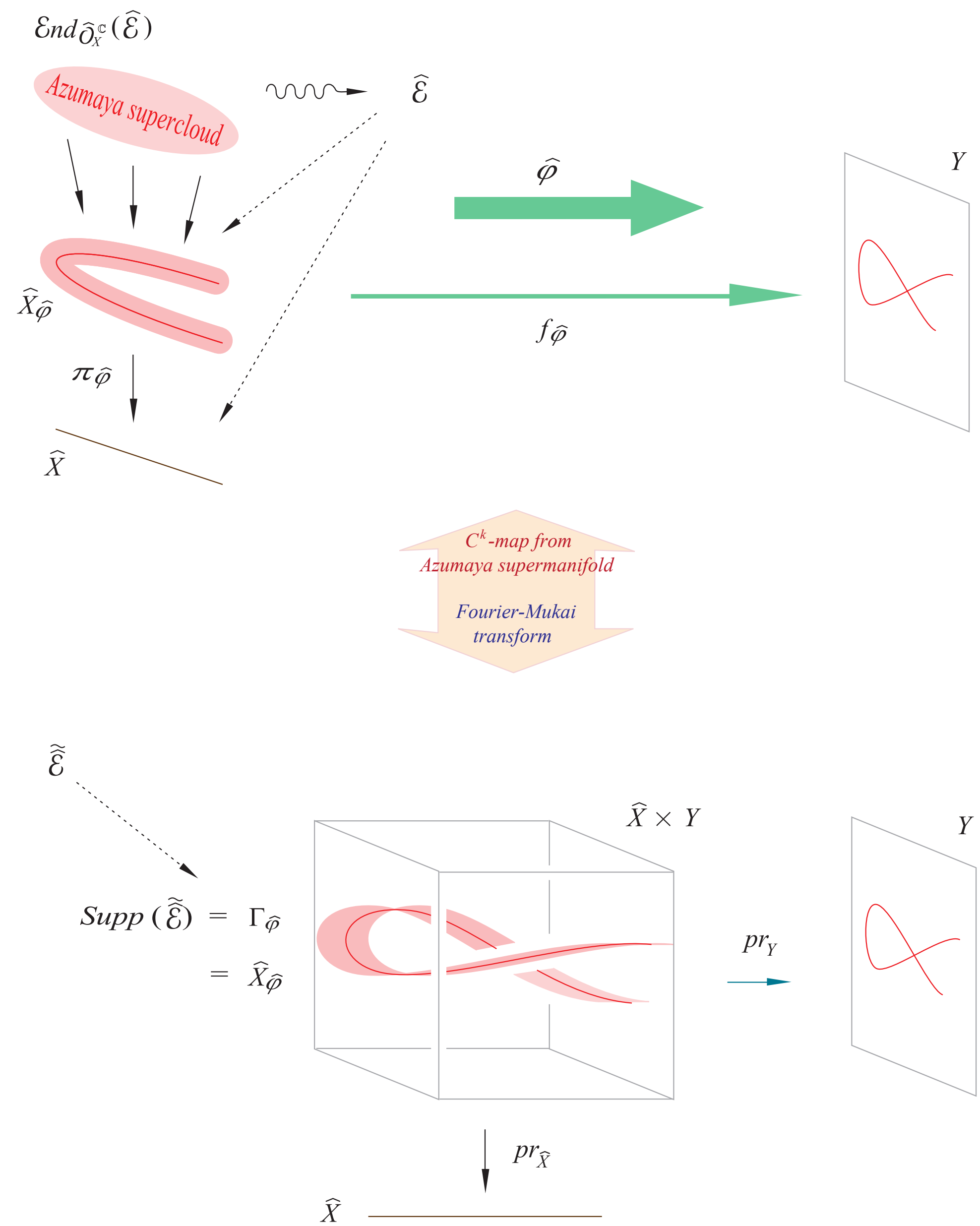

FIGURE 4-2-2-1. The equivalence between a C^k-map $\widehat{\varphi}$ from an Azumaya/matrix super-C^k-manifold with a fundamental module $(\widehat{X}, \widehat{\mathcal{O}}_X^{Az} := \mathcal{E}nd_{\widehat{\mathcal{O}}_X^{\mathbb{C}}}(\widehat{\mathcal{E}}), \widehat{\mathcal{E}})$ to a C^k-manifold Y and a special kind of Fourier-Mukai transform $\tilde{\widehat{\mathcal{E}}} \in \mathcal{M}od^{\mathbb{C}}(\widehat{X} \times Y)$ from $\widehat{X}$ to Y. Here, $\mathcal{M}od^{\mathbb{C}}(\widehat{X} \times Y)$ is the category of $\mathcal{O}^{\mathbb{C}}_{\widehat{X}\times Y}$-modules.

then there is a well-defined algebra-homomorphism over $\mathbb{C}$

$$\gamma^\sharp\;:\; C^k(\mathbb{C}^1)^{\mathbb{C}} \longrightarrow C^k([0,1])^{\mathbb{C}}$$

by the pull-back of functions via γ. Here comes the guiding question:

Q. *What is the geometric meaning of the above algebraic operation?*

The answer comes from an input to differential topology from algebraic geometry.

By definition, a point with function field $\mathbb{R}$ is an $\mathbb{R}$-*point* while a point with function field $\mathbb{C}$ is a $\mathbb{C}$-*point*. Topologically they are the same but algebraically they are different, as already indicated by

$$\mathbb{R} \hookrightarrow \mathbb{C}\,, \quad \text{while} \quad \mathbb{C} \not\rightarrow \mathbb{R}\,,$$

which means algebrao-geometrically, concerning the existence of a map from one to the other,

$$\mathbb{C}\text{-}point \longrightarrow \mathbb{R}\text{-}point\,, \quad \text{while} \quad \mathbb{R}\text{-}point \not\rightarrow \mathbb{C}\text{-}point\,.$$

By replacing $C^k([0,1])$ by its complexification $C^k([0,1])^{\mathbb{C}}$, we promote each original $\mathbb{R}$-points on $[0,1]$ to a $\mathbb{C}$-point. In other words, we smear $\mathbb{C}$-points along the interval $[0,1]$. The map γ now simply specifies a C^k $[0,1]$-family of $\mathbb{C}$-points on $\mathbb{C}^1$ by associating to each $\mathbb{C}$-point on $[0,1]$ a $\mathbb{C}$-point on $\mathbb{C}^1$, which is now allowed algebro-geometrically. This concludes the example

$\square$

Let p^{Az} be a point with function ring isomorphic to the endomorphism algebra $End_{\mathbb{C}}(\mathbb{C}^{\oplus r})$ Then, recall from [L-Y3] (D(11.1)) that by exactly the same reasoning and geometric pictures as in Example 4.2.3.1, with $(\,\cdots)\otimes_{\mathbb{R}}\mathbb{C}$ replaced by $(\,\cdots)\otimes_{\mathbb{R}} End_{\mathbb{C}}(E)$ locally where E is a $\mathbb{C}$-vector space and $(\,\cdots)$ is the C^k-ring in question, one has

Azumayanized manifold $(X, \mathcal{O}_X\otimes_{\mathbb{R}} End_{\mathbb{C}}(\mathbb{C}^{\oplus r}))$	$\Longleftrightarrow$	the smearing of *fixed* p^{Az}'s along X
general Azumaya manifold $(X^{Az}, \mathcal{E})$	$\Longleftrightarrow$	a smearing of *unfixed* p^{Az}'s along X

Finally, by the same reasoning but with $(\,\cdots)$ above replaced by the super-C^k-ring in question, one has completely analogously

Azumayanized supermanifold $(X, \widehat{\mathcal{O}}_X\otimes_{\mathbb{R}} End_{\mathbb{C}}(\mathbb{C}^{\oplus r}))$	$\Longleftrightarrow$	the smearing of *fixed* p^{Az}'s along $\widehat{X}$
general Azumaya supermanifold $(\widehat{X}^{Az}, \widehat{\mathcal{E}})$	$\Longleftrightarrow$	a smearing of *unfixed* p^{Az}'s along $\widehat{X}$

Remark 4.2.3.2. [*another smearing*]. Though for our purpose the above viewpoint is preferred, there is however a second viewpoint:

- Let $\widehat{p}^{Az}$ be an Azumaya/matrix superpoint with function ring $End_{\mathbb{C}}(\mathbb{C}^{\oplus r})\otimes_{\mathbb{R}} \bigwedge^\bullet(\mathbb{R}^{\oplus s})$. Then $\widehat{X}^{Az}$ can be regarded as smearing unfixed $\widehat{p}^{Az}$'s along X as well.

A C^k-map $\widehat{\varphi}:(\widehat{X}^{Az},\widehat{\mathcal{E}})\to Y$ as smearing D0-branes on Y along $\widehat{X}$

To press on along this line, we have to list two objects that are studied in algebraic geometry and yet their counter-objects are much less known/studied in differential topology/geometry:

(1) [*Quot-schemes*] Grothendieck's *Quot*-scheme $Quot_Y^r((\mathcal{O}_Y^{\mathbb{C}})^{\oplus r})$ of 0-dimensional quotient sheaves of $(\mathcal{O}_Y^{\mathbb{C}})^{\oplus r}$ of complex-length r. This is the parameter space of differentiable maps from the fixed Azumaya point $(p^{Az},\mathbb{C}^{\oplus r})$ to Y; cf. [L-Y3: Sec. 3, Lemma/Definition 5.3.1.9, Sec. 5.3.2] (D(11.1)). In other words, it parameterizes D0-branes $\mathcal{F}$ on Y (where $\mathcal{F}$ is a complex 0-dimensional sheaf on Y of complex length r) that is decorated with an isomorphism $\mathbb{C}^{\oplus r}\xrightarrow{\sim} C^k(\mathcal{F})$ over $\mathbb{C}$.

(2) [*Quotient stacks*] The general linear group $GL_r(\mathbb{C})$ acts on $Quot_Y^r(\mathcal{O}_Y^{\mathbb{C}})^{\oplus r})$ by its tautological action on the $\mathbb{C}^{\oplus r}$-factor in the canonical isomorphism $(\mathcal{O}_Y^{\mathbb{C}})^{\oplus r}\simeq\mathcal{O}_Y\otimes_{\mathbb{R}}\mathbb{C}^{\oplus r}$. This defines a quotient stack $[Quot_Y^r((\mathcal{O}_Y^{\mathbb{C}})^{\oplus r})/GL_r(\mathbb{C})]$, which now parameterizes differentiable maps φ from unfixed Azumaya points (p^{Az},E), where E is a $\mathbb{C}$-vector space of rank r, to Y. In other words, $[Quot_Y^r((\mathcal{O}_Y^{\mathbb{C}})^{\oplus r})/GL_r(\mathbb{C})]$ is precisely the moduli stack $\mathfrak{M}_r^{0^{Az^f}}(Y)$ of D0-branes of complex length r on Y, realized as complex 0-dimensional sheaves on Y of complex length r via push-forwards $\varphi_*(E)$, from [L-Y3: Definition 5.3.1.5] (D(11.1)); cf. Definition 4.2.1.3 and Remark 4.2.1.7.

Recall from Sec. 4.2.2 that a differentiable map $\widehat{\varphi}:(\widehat{X}^{Az},\widehat{\mathcal{E}})\to Y$ is completely encoded by its graph $\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}}$ on $\widehat{X}\times Y$. Over any super-C^k-subscheme $\widehat{Z}\subset\widehat{X}$, $\tilde{\widehat{\mathcal{E}}}_{\widehat{\varphi}}|_{\widehat{Z}\times Y}$ is simply a flat $\widehat{Z}$-family of 0-dimensional $\mathcal{O}_Y^{\mathbb{C}}$-modules of complex length r. Despite missing the details of these parameter "spaces", it follows from their definition as functors or sheaves of groupoids over the category of super-C^k-schemes, with suitable Grothendieck topology, that

$$\boxed{C^k\text{-maps }\widehat{\varphi}:(\widehat{X}^{Az},\widehat{\mathcal{E}})\to Y}\quad\Longleftrightarrow\quad\boxed{\text{admissible maps }\widehat{X}\to\mathfrak{M}_r^{0^{Az^f}}(Y)}$$

This matches perfectly with the picture of an Azumaya/matrix supermanifold $(\widehat{X}^{Az},\widehat{\mathcal{E}})$ as a smearing of unfixed Azumaya/matrix points p^{Az} along $\widehat{X}$ since, then, a map $\widehat{X}^{Az}\to Y$ is nothing but an $\widehat{X}$-family of maps $p^{Az}\to Y$, which is exactly the map $\widehat{X}\to\mathfrak{M}_r^{0^{Az^f}}(Y)$. Cf. FIGURE 4-2-3-1.

4.2.4 Aspect IV: From associated $GL_r(\mathbb{C})$-equivariant maps

Again, we list a parallel issue in differential topology that remains to be understood:

(1) [*Fibered product*] The notion of the fibered product of stratified singular spaces with a structure sheaf and its generalization to stacks needs to be developed.

Subject to this missing detail, from the very meaning of a quotient stack, it is natural to anticipate that any natural definition of the notion of fibered product should lift a map

$$\widehat{X}\;\longrightarrow\;\mathfrak{M}_r^{0^{Az^f}}(Y)\;=\;[Quot_Y^r((\mathcal{O}_Y^{\mathbb{C}})^{\oplus r})/GL_r(\mathbb{C})]$$

lift to a $GL_r(\mathbb{C})$-equivariant map

$$P_{\widehat{X}}\;\longrightarrow\;Quot_Y^r((\mathcal{O}_Y^{\mathbb{C}})^{\oplus r})\,,$$

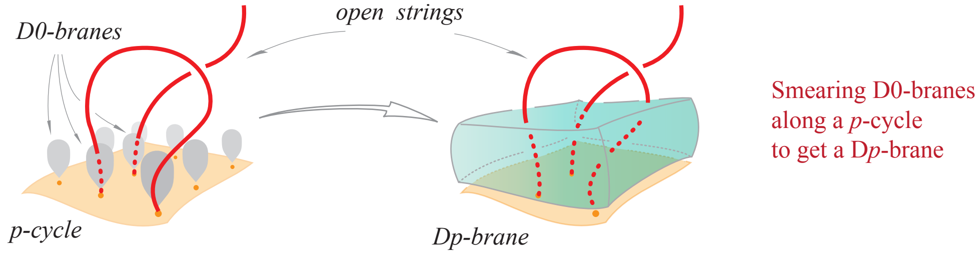

FIGURE 4-2-3-1. ([L-L-S-Y: FIGURE 3-1-1].) The original stringy operational definition of D-branes as objects in the target space(-time) Y of fundamental strings where end-points of open-strings can and have to stay suggests that smearing D0-branes along a (real) p-dimensional submanifold X in Y renders X a Dp-brane. Such a smearing can be generalized to the supercase and is realized as a map from the supermanifold $\widehat{X}$ (over and containing X) to the stack $\mathfrak{M}^{D0}(Y)$ of D0-branes on Y. In the figure, the Chan-Paton sheaf $\mathcal{E}$ that carries the index information on the end-points of open strings is indicated by a shaded cloud.

where $P_{\widehat{X}}$ is a principal $GL_r(\mathbb{C})$-bundle over $\widehat{X}$ from the fibered product

$$P_{\widehat{X}} \;:=\; \widehat{X}\times_{\mathfrak{M}_r^{0^{Az^f}}(Y)} Quot_Y^r((\mathcal{O}_Y^{\mathbb{C}})^{\oplus r})$$

Conversely, any latter map should define a former map. Together with Aspect III in Sec. 4.2.3, this gives a correspondence

$$\boxed{C^k\text{-maps } \widehat{\varphi}:(\widehat{X}^{Az},\widehat{\mathcal{E}})\to Y}$$

$$\Longleftrightarrow\;\boxed{\text{admissible } GL_r(\mathbb{C})\text{-equivariant maps } P_{\widehat{X}}\to Quot_Y^r((\mathcal{O}_Y^{\mathbb{C}})^{\oplus r})}$$

4.3 Remarks on differentiable maps from a general endomorphism-ringed super-C^k-scheme with a fundamental module to a real supermanifold

Recall the last theme '*Remarks on general endomorphism-ringed super-C^k-schemes and differential calculus thereupon*' of Sec. 3, in which the notion of *endomorphism-ringed super-C^k-schemes with a fundamental module* is introduced, which generalizes the notion of Azumaya/matrix super-C^k-schemes with a fundamental module. Similar notion of 'differentiable maps from an endomorphism-ringed super-C^k-scheme with a fundamental module to a real supermanifold' is readily there from a generalization of the discussions in Sec. 4.1 and Sec. 4.2.

Consider the local study first. Let

- $R\simeq C^k(\mathbb{R}^n)/I$ (i.e. R a finitely generated C^k-ring) and
  $\widehat{R}\rightleftarrows R$ be a super-C^k-ring over R,
- $S=C^k(V)$ where V be an open set of $\mathbb{R}^n$ and
  $\widehat{S}\rightleftarrows S$ be a superpolynomial ring over S.

Then

- The *push-out* $R \otimes_{C^k} S$ of R and S exists in the category of C^k-rings with $R \otimes_{C^k} S \simeq C^k(\mathbb{R}^n \times V)/I$ and *with the built-in* C^k*-ring-homomorphisms* $pr_1^\sharp : R \hookrightarrow R \otimes_{C^k} S$ *and* $pr_2^\sharp : S \hookrightarrow R \otimes_{C^k} S$ coincident with pulling back of functions via the projections pr_1 and pr_2 of the product space to its factors. Here $I \hookrightarrow C^k(\mathbb{R}^n \times V)$ via the canonical inclusion $C^k(\mathbb{R}^n) \subset C^k(\mathbb{R}^n \times V)$.

- The *push-out* $\widehat{R} \otimes_{sC^k} \widehat{S}$ of $\widehat{R}$ and $\widehat{S}$ exists in the category of super-C^k-rings *with the built-in super-*C^k*-ring-homomorphisms* $\widehat{pr}_1^\sharp : \widehat{R} \hookrightarrow \widehat{R} \otimes_{sC^k} \widehat{S}$ *and* $\widehat{pr}_2^\sharp : \widehat{S} \hookrightarrow \widehat{R} \otimes_{sC^k} \widehat{S}$.

- One has a built-in commutative diagram of morphisms

$$
\begin{tikzcd}
 & & \widehat{S} \arrow[dl, hook', "\widehat{pr}_2^\sharp"'] \arrow[dd, shift left] \\
\widehat{R} \arrow[r, hook, "\widehat{pr}_1^\sharp"] \arrow[dd, shift left] & \widehat{R} \otimes_{sC^k} \widehat{S} \arrow[dd, shift left] & \\
 & & S \arrow[dl, hook', "pr_2^\sharp"'] \arrow[uu, shift left] \\
R \arrow[r, hook, "pr_1^\sharp"] \arrow[uu, shift left] & R \otimes_{C^k} S \arrow[uu, shift left] &
\end{tikzcd}
\quad .
$$

Let

- $\widehat{M}$ be a finitely generated $\widehat{R}^{\mathbb{C}}$-module and

- $End_{\widehat{R}^{\mathbb{C}}}(\widehat{M})$ be the $\widehat{R}^{\mathbb{C}}$-algebra of $\widehat{R}^{\mathbb{C}}$-endomorphisms of $\widehat{M}$.

We *assume that the built-in* $\widehat{R}$*-algebra-homomorphism* $\widehat{R} \to End_{\widehat{R}^{\mathbb{C}}}(\widehat{M})$ *is injective.* We will denote this inclusion by $\widehat{\pi}^\sharp$. Then, the key notion in the whole setting is the following:

Definition 4.3.1. [C^k-admissible superring-homomorphism]. (Cf. [L-Y3: Definition 5.1.2] (D(11.1)) and Definition 4.1.1 in Sec. 4.1.) A superring-homomorphism

$$
\widehat{\varphi}^\sharp \;:\; \widehat{S} \;\longrightarrow\; End_{\widehat{R}^{\mathbb{C}}}(\widehat{M})
$$

over $\mathbb{R} \hookrightarrow \mathbb{C}$ is said to be C^k*-admissible* if the arrow $\widehat{\varphi}^\sharp$ extends to the following commutative diagram of superring-homomorphisms

$$
\begin{tikzcd}
End_{\widehat{R}^{\mathbb{C}}}(\widehat{M}) & & \widehat{S} \arrow[ll, "\widehat{\varphi}^\sharp"'] \arrow[d, hook, "\widehat{pr}_2^\sharp"] \\
\widehat{R} \arrow[u, hook, "\widehat{\pi}^\sharp"] \arrow[rr, hook, "\widehat{pr}_1^\sharp"'] & & \widehat{R} \otimes_{sC^k} \widehat{S} \arrow[ull, "\tilde{\widehat{\varphi}}^\sharp"']
\end{tikzcd}
$$

such that

- $Ker(\tilde{\widehat{\varphi}}^\sharp)$ is super-C^k-normal and hence $Im\,\tilde{\widehat{\varphi}}^\sharp$ can be equipped with a quotient super-C^k-ring structure from that of $\widehat{R} \otimes_{sC^k} \widehat{S}$ via $\tilde{\widehat{\varphi}}^\sharp$.

- With respect to the super-C^k-ring structure on $Im\,\tilde{\widehat{\varphi}}^\sharp$, $\widehat{\varphi}^\sharp$ is a super-C^k-ring-homomorphism as a superring-homomorphism $\widehat{S} \to Im\,\tilde{\widehat{\varphi}}^\sharp$.

Moving on to the global study. Let $\widehat{X} := (X, \mathcal{O}_X, \widehat{\mathcal{O}}_X)$ be a super-C^k-scheme and $\widehat{\mathcal{F}}$ be a finitely generated quasi-coherent $\widehat{\mathcal{O}}_X^{\mathbb{C}}$-module. We shall *assume that the built-in $\widehat{\mathcal{O}}_X$-algebra-homomorphism $\widehat{\mathcal{O}}_X \to \mathcal{E}nd_{\widehat{\mathcal{O}}_X^{\mathbb{C}}}(\widehat{\mathcal{F}})$ is injective.* Consider the endomorphism-ringed super-C^k-scheme with a fundamental module

$$(\widehat{X}^{nc}, \widehat{\mathcal{F}}) \;:=\; (\widehat{X}, \widehat{\mathcal{O}}_X^{nc} := \mathcal{E}nd_{\widehat{\mathcal{O}}_X^{\mathbb{C}}}(\widehat{\mathcal{F}}), \widehat{\mathcal{F}})\,.$$

Definition 4.3.2. [differentiable map]. (Cf. [L-Y3: Definition 5.3.1.5] (D(11.1)) and Definition 4.2.1.3 in Sec. 4.2.1.) Let $\widehat{Y}$ be a super-C^k-manifold. A *k-times differentiable map* (i.e. *C^k-map*)

$$\widehat{\varphi} \;:\; (\widehat{X}^{nc}, \widehat{\mathcal{F}}) \;\longrightarrow\; \widehat{Y}$$

is defined contravariantly as an *equivalence class of gluing systems of C^k-admissible superring-homomorphisms*, in notation,

$$\widehat{\varphi}^{\sharp} \;:\; \widehat{\mathcal{O}}_Y \;\longrightarrow\; \widehat{\mathcal{O}}_X^{nc}\,,$$

exactly as in Definition 4.2.1.1, Definition 4.2.1.2, and Definition 4.2.1.3 in Sec. 4.2.1.

As in [L-Y3: Sec. 5.3.1] (D(11.1)) and Sec. 4.2 for the case of C^k-maps from an Azumaya/matrix super-C^k-manifold to a C^k-manifold, one has the following well-defined basic notions:

- the *surrogate* $\widehat{X}_{\widehat{\varphi}}$ of $\widehat{X}^{nc}$ specified by $\widehat{\varphi}$,
- the *push-forward* $\widehat{\varphi}_*\widehat{\mathcal{F}}$ of $\widehat{\mathcal{F}}$ to $\widehat{Y}$,
- the *graph* $\tilde{\widehat{\mathcal{F}}}_{\widehat{\varphi}}$ of $\widehat{\varphi}$, which is an $\widehat{\mathcal{O}}_{X\times Y}^{\mathbb{C}}$-module on $\widehat{X} \times \widehat{Y}$.

One has now *Aspect I* and *Aspect II* for φ. When $\widehat{\mathcal{F}}$ is in addition locally free and $\widehat{Y} = Y$, one recovers the notion of C^k-map in Sec. 4.2.1 and has in addition Aspect III and Aspect IV.

Recall the built-in inclusion $\widehat{\imath}: X \hookrightarrow \widehat{X}$ and let

$$\mathcal{F} \;:=\; \widehat{\imath}^*\widehat{\mathcal{F}}$$

be the restriction of $\widehat{\mathcal{F}}$ to X. Then, $\widehat{\varphi}$ induces a C^k-map

$$\varphi \;:\; (X^{nc}, \mathcal{F}) := (X, \mathcal{O}_X^{nc} := \mathcal{E}nd_{\mathcal{O}_X^{\mathbb{C}}}(\mathcal{F}), \mathcal{F}) \;\longrightarrow\; Y$$

with a built-in commutative diagrams of morphisms

$$
\begin{tikzcd}
\widehat{\mathcal{F}} \arrow[rrrd, dotted] \arrow[rrdd, dotted] \arrow[rddd, dotted] & & & & & \\
& & & \widehat{X}^{nc} \arrow[dl, "\sigma_{\widehat{\varphi}}"'] \arrow[drr, "\widehat{\varphi}"] \arrow[dddd, shift left, two heads, "\dot{\widehat{\pi}}"] & & \\
& & \widehat{X}_{\widehat{\varphi}} \arrow[rrr, "f_{\widehat{\varphi}}"] \arrow[drr, hook, "\tilde{\widehat{\varphi}}"] \arrow[dl, "\pi_{\widehat{\varphi}}"'] \arrow[dddd, shift left, two heads, "\widehat{\pi}"] & & & \widehat{Y} \arrow[dddd, shift left, two heads, "\widehat{\pi}"] \\
& \widehat{X} \arrow[dddd, shift left, two heads, "\widehat{\pi}"] & & & \widehat{X}\times\widehat{Y} \arrow[lll, "pr_{\widehat{X}}"] \arrow[ur, "pr_{\widehat{Y}}"'] \arrow[dddd, shift left, two heads, "\widehat{\pi}"] & \\
\mathcal{F} := \widehat{\iota}^{*}\widehat{\mathcal{F}} \arrow[rrrd, dotted] \arrow[rrdd, dotted] \arrow[rddd, dotted] & & & & & \\
& & & X^{nc} \arrow[dl, "\sigma_{\varphi}"] \arrow[drr, "\varphi"] \arrow[uuuu, shift left, hook, "\dot{\widehat{\iota}}"] & & \\
& & X_{\varphi} \arrow[rrr, "f_{\varphi}"] \arrow[drr, hook, "\tilde{\varphi}"] \arrow[dl, "\pi_{\varphi}"] \arrow[uuuu, shift left, hook, "\widehat{\iota}"] & & & Y \arrow[uuuu, shift left, hook, "\widehat{\iota}"] \\
& X \arrow[uuuu, shift left, hook, "\widehat{\iota}"] & & & X \times Y \arrow[lll, "pr_X"] \arrow[ur, "pr_Y"'] \arrow[uuuu, shift left, hook, "\widehat{\iota}"] & .
\end{tikzcd}
$$

as in Proposition 4.2.1.8.

5 A glimpse of super D-branes, as dynamical objects, and the Higgs mechanism in the current setting

We give in this section a glimpse of super D-branes, as dynamical objects in string theory, and the Higgs mechanism on D-branes in the current setting. It serves to give readers a taste of applications to string theory and a bridge to sequels of the current note.

5.1 Fermionic D-branes as fundamental/dynamical objects in string theory

There are two versions of fermionic (fundamental, either open or closed) strings:

(1) *Ramond-Neveu-Schwarz* (*RNS*) *fermionic string*, for which world-sheet spinors are manifestly involved ([N-S] of André Neveu and John Schwarz and [Ra] of Pierre Ramond);

(2) *Green-Schwarz* (*GS*) *fermionic string*, for which space-time spinors are manifestly involved ([G-S] of Michael Green and John Schwarz).

Mathematicians are referred particularly to [G-S-W: Chap. 4 & Chap. 5] of Green, Schwarz, and Witten for thorough explanations. Once having the notion of differentiable maps from Azumaya/matrix manifold to a real manifold ([L-Y3] (D(11.1))) and its super-extension (Sec. 4.2

of the current note), it takes no additional work to give a prototypical definition of *fermionic D-branes* in the style of either Ramond-Neveu-Schwarz or Green-Schwarz fermionic string once one understands the meaning of such fermionic strings from the viewpoint of Grothendieck's Algebraic Geometry.

Ramond-Neveu-Schwarz fermionic string and Green-Schwarz fermionic string from the viewpoint of Grothendieck's Algebraic Geometry

This discussion in this theme follows [G-S-W: Chap. 4 & Chap. 5] (with possibly some mild change of notations to be compatible with the current note) and [Ha: Chap. II]. Let $\mathbb{M}^{(d-1)+1}$ be the d-dimensional Minkowski space-time with coordinates $y := (y^\mu)_\mu = (y^0, y^1, \cdots, y^{d-1})$ and $\Sigma \simeq \mathbb{R}^1 \times S^1$ or $\mathbb{R}^1 \times [0, 2\pi]$ be a string world-sheet with coordinates $\sigma := (\sigma^0, \sigma^1)$.

(*a*) *Ramond-Neveu-Schwarz* (*RNS*) *fermionic string*

In this setting, there are both bosonic (world-sheet scalar) fields $y^\mu(\sigma)$ and fermionic (world-sheet spinor) fields $\psi^\mu(\sigma)$ on the string world-sheet Σ for $\mu = 0, 1, \cdots, d-1$. The former collectively describe a map $f: \Sigma \to \mathbb{M}^{(d-1)+1}$ and the latter as its superpartner.

Consider the supermanifold $\widehat{\Sigma}$ that have the same topology as Σ but with additional Grassmann coordinates $\theta := (\theta^A)_A = (\theta^1, \theta^2)$ forming 2-component Majorana spinor on Σ. Then, after adding auxiliary (nondynamical) fields $B^\mu(\sigma)$ to the world-sheet, these fields on Σ can be grouped to superfields:(Cf. [G-S-W: Sec. 4.1.2; Eq. (4.1.16)].)

$$Y^\mu(\sigma) \;=\; y^\mu(\sigma) \,+\, \bar{\theta}\psi^\mu(\sigma) \,+\, \frac{1}{2}\,\bar{\theta}\theta\, B^\mu(\sigma)\,.$$

From the viewpoint of Grothendieck's Algebraic Geometry, a map $\widehat{f}: \widehat{\Sigma} \to \mathbb{M}^{(d-1)+1}$ is specified contravariantly by a homomorphism

$$\begin{array}{cccc} \widehat{f}^\sharp & : & C^\infty(\mathbb{M}^{(d-1)+1}) & \longrightarrow & C^\infty(\widehat{\Sigma}) \\ & & y^\mu & \longmapsto & \widehat{f}^\sharp(y^\mu) \end{array}$$

of the function rings in question. Since $C^\infty(\widehat{\Sigma}) = C^\infty(\Sigma)[\theta^1, \theta^2]$ a superpolynomial ring over $C^\infty(\Sigma)$, $\hat{f}^\sharp(y^\mu)$ must be of the form

$$\widehat{f}^\sharp(y^\mu) \;=\; f^\mu(\sigma) \,+\, \bar{\theta}\psi^\mu(\sigma) \,+\, \frac{1}{2}\,\bar{\theta}\theta\, B^\mu(\sigma)\,,$$

which is exactly the previous quoted expression [G-S-W: Eq. (4.1.16)]. In conclusion,

- *A Ramond-Neveu-Schwarz fermionic string moving in a Minkowski space-time* $\mathbb{M}^{(d-1)+1}$ *as studied in* [G-S-W: Chap. 4] *can be described by a map* $\widehat{f}: \widehat{\Sigma} \to \mathbb{M}^{(d-1)+1}$ *in the sense of Grothendieck's Algebraic Geometry.*

(*b*) *Green-Schwarz* (*GS*) *fermionic string*

In this setting, in addition to the ordinary bosonic (world-sheet scalar) fields $y^\mu(\sigma)$, $\mu = 0, 1, \cdots, d-1$, on Σ that collectively describe a map $f: \Sigma \to \mathbb{M}^{(d-1)+1}$, there are also a set of *world-sheet scalar yet mutually anticommuting* fields $\theta^{Aa}(\sigma)$, $A = 1, \cdots, N$ and $a = 1, \cdots, s$, on Σ. Here s is the dimension of a spinor representation of the Lorentz group $SO(d-1, 1)$ of the target Minkowski space-time $\mathbb{M}^{(d-1)+1}$.

Differential geometrically intuitively, one would think of these (world-sheet scalar) fields on Σ collectively as follows:

- Let $\widehat{\mathbb{M}}^{(d-1)+1}$ be a superspace with coordinates the original coordinates $y := (y^\mu)_\mu$ of $\mathbb{M}^{(d-1)+1}$ and additional anticommuting coordinates θ^{Aa}, $A = 1, \cdots, N$ and $a = 1, \cdots, s$, such that each tuple $(\theta^{A1}, \cdots, \theta^{As})$, $A = 1, \cdots, N$, is in a spinor representation of the Lorentz group $SO(d-1,1)$, the symmetry of the space-time $\mathbb{M}^{(d-1)+1}$. Note that $\widehat{\mathbb{M}}^{(d-1)+1} \simeq \mathbb{R}^{d|Ns}$ as supermanifolds.

- The collection $(y^\mu(\sigma), \theta^{Aa}(\sigma))_{\mu,A,a}$ of (world-sheet scalar) fields on Σ describe collectively a map $\widehat{f} : \Sigma \to \widehat{\mathbb{M}}^{(d-1)+1}$. In other words, a Green-Schwarz fermionic string moving in $\mathbb{M}^{(d-1)+1}$ is described by a map from an ordinary world-sheet to a super-Minkowski space-time.

However, *algebraic geometrically* some revision to this naive differential geometric picture has to be made.

- One would like a contravariant equivalence between spaces and their function ring:

$$\widehat{f} \;:\; \Sigma \;\longrightarrow\; \widehat{\mathbb{M}}^{(d-1)+1}$$

  with
$$\begin{array}{cccc} \widehat{f}^\sharp & : & C^\infty(\mathbb{M}^{(d-1)+1})[\theta^{Aa} \,:\, 1 \le A \le N,\, 1 \le a \le s] & \longrightarrow & C^\infty(\Sigma) \\ & & y^\mu & \longmapsto & y^\mu(\sigma) \\ & & \theta^{Aa} & \longmapsto & ? \end{array} .$$

  Here, $C^\infty(\mathbb{M}^{(d-1)+1})[\theta^{Aa} \,:\, 1 \le A \le N,\, 1 \le a \le s]$ is the superpolynomial ring over the C^∞-ring $C^\infty(\mathbb{M}^{(d-1)+1})$ with anticummuting generators in $\{\theta^{Aa}\}_{A,a}$.

- The natural candidate for $\widehat{f}(\theta^{Aa})$ is certainly the world-sheet scalar field $\theta^{Aa}(\sigma)$ regarded as an element in the function-ring of Σ. However, the anticommuting nature of fields θ^{Aa}, $1 \le A \le N$ and $1 \le a \le s$, among themselves forbids them to lie in $C^\infty(\Sigma)$.

- The way out of this from the viewpoint of Grothendieck's Algebraic Geometry is to extend the world-sheet Σ also to a superworld-sheet $\widehat{\Sigma}$ with the function ring the superpolynomial ring $C^\infty(\Sigma)[\theta'^{Aa} \,:\, 1 \le A \le N,\, 1 \le a \le s]$.

- One now has a well-defined super-C^∞-ring-homomorphism

$$\begin{array}{ccccc} \widehat{f}^\sharp & : & C^\infty(\mathbb{M}^{(d-1)+1})[\,\theta^{Aa} \,:\, A,\, a\,] & \longrightarrow & C^\infty(\Sigma)[\,\theta'^{Aa} \,:\, A,\, a\,] \\ & & y^\mu & \longmapsto & y^\mu(\sigma) \\ & & \theta^{Aa} & \longmapsto & \theta^{Aa}(\sigma) \end{array} .$$

- Furthermore, since all the fields $\theta^{Aa}(\sigma)$ are dynamical, in comparison with the setting for the RNS fermionic string, it is reasonable to require in addition that

$$\widehat{f}^\sharp(\theta^{Aa}) \;=\; \theta^{Aa}(\sigma) \;\in\; Span_{C^\infty(\Sigma)}\{\,\theta'^{Aa} \,|\, A, a\,\}\,.$$

In conclusion,

- *Assuming the notation from the above discussion. A Green-Schwarz fermionic string moving in a Minkowski space-time* $\mathbb{M}^{(d-1)+1}$ *as studied in* [G-S-W: Chap. 5] *can be described* *in the sense of Grothendieck's Algebraic Geometry by a map* $\widehat{f}:\widehat{\Sigma}\rightarrow\widehat{\mathbb{M}}^{(d-1)+1}$*, defined by a super-*$C^\infty$*-ring-homomorphism*

$$\begin{array}{cccc}\widehat{f}^\sharp & : & C^\infty(\mathbb{M}^{(d-1)+1})[\,\theta^{Aa}\,:\,A,\,a\,] & \longrightarrow & C^\infty(\Sigma)[\,\theta'^{Aa}\,:\,A,\,a\,]\\ & & y^\mu & \longmapsto & y^\mu(\sigma)\\ & & \theta^{Aa} & \longmapsto & \theta^{Aa}(\sigma)\end{array}$$

such that

$$\widehat{f}^\sharp(\theta^{Aa})\;=\;\theta^{Aa}(\sigma)\;\in\;Span_{C^\infty(\Sigma)}\{\,\theta'^{Aa}\,|\,A,a\,\}\,.$$

Fermionic D-branes as dynamical objects à la RNS or GS fermionic strings

Terminology 5.1.1. [*Azumaya/matrix super-*C^k*-manifold associated to* $(\mathcal{S},\mathcal{E})$]. Let X be a C^k-manifold, $\mathcal{S}$ be a locally free $\mathcal{O}_X$-module of finite rank, and $\mathcal{S}$ be a locally free $\mathcal{O}_X^{\mathbb{C}}$-module of finite rank. For convenience, introduce the following terminologies:

- $\widehat{X}:=(X,\widehat{\mathcal{O}}_X:=\bigwedge^\bullet\mathcal{S})$ be the *supermanifold generated by* $\mathcal{S}$ on X, denote $\widehat{\mathcal{E}}:=\mathcal{E}\otimes_{\mathcal{O}_X}\widehat{\mathcal{O}}_X$,
- $X^{Az}:=(X,\mathcal{O}_X^{Az}:=\mathcal{E}nd_{\mathcal{O}_X^{\mathbb{C}}}(\mathcal{E}))$ be the *Azumaya/matrix manifold associated to* $\mathcal{E}$ on X, $\widehat{X}^{Az}:=(\widehat{X},\widehat{\mathcal{O}}_X^{Az}:=\mathcal{E}nd_{\widehat{\mathcal{O}}_X^{\mathbb{C}}}(\widehat{\mathcal{E}})\simeq\mathcal{O}_X^{Az}\otimes_{\mathcal{O}_X}\widehat{\mathcal{O}}_X)$ be the *Azumaya/matrix supermanifold specified by the pair* $(\mathcal{S},\mathcal{E})$ on X, (and $\widehat{\mathcal{E}}$ is the fundamental module of $\widehat{X}^{Az}$).

Definition-Prototype 5.1.2. [fermionic D-branes à la a RNS fermionic string]. Let

- Y be a C^k-manifold (e.g. a space-time with a Lorentzian metric, or a Euclidean space-time from Wick rotation, or a Riemannian internal space in a compactification of superstring theory background).

Then, a *fermionic D-brane in* Y *in the style of a Ramond-Neveu-Schwarz fermionic string* consists of the following data:

$$(X,\,\mathcal{S},\,\mathcal{E},\,\widehat{\varphi}:(\widehat{X}^{Az},\widehat{\mathcal{E}})\rightarrow Y)\,,$$

where

- X is a C^k-manifold (with a Riemannian or Lorentzian structure, depending on the context),
- $\mathcal{S}$ is a (finite) direct sum of sheaves of spinors on X,
- $\mathcal{E}$ is a locally free $\mathcal{O}_X^{\mathbb{C}}$-module of some finite rank r,
- $(\widehat{X}^{Az},\widehat{\mathcal{E}})$ is the Azumaya/matrix super-C^k-manifold with a fundamental module specified by $(\mathcal{S},\mathcal{E})$ on X and $\widehat{\varphi}:(\widehat{X}^{Az},\widehat{\mathcal{E}})\rightarrow Y$ is a C^k-map from $(\widehat{X}^{Az},\widehat{\mathcal{E}})$ to Y, as defined in Definition 4.2.1.3.

Definition-Prototype 5.1.3. [fermionic D-branes à la a GS fermionic string]. Let

- Y be a C^k-manifold (e.g. a space-time with a Lorentzian metric, or a Euclidean space-time from Wick rotation, or a Riemannian internal space in a compactification of superstring theory background),

- $\mathcal{S}_Y$ be a (finite) direct sum of sheaves of spinors on Y,
- $\widehat{Y} := (Y, \widehat{\mathcal{O}}_Y := \bigwedge^{\bullet} \mathcal{S}_Y)$ be the super-C^k-manifold generated by $\mathcal{S}_Y$ on Y.

Then, a *fermionic D-brane in Y in the style of a Green-Schwarz fermionic string* consists of the following data:

$$(X,\, \mathcal{S},\, \mathcal{E},\, \widehat{\varphi} : (\widehat{X}^{Az}, \widehat{\mathcal{E}}) \to \widehat{Y})\,,$$

where

- X is a C^k-manifold (with a Riemannian or Lorentzian structure, depending on the context),
- $\mathcal{S}$ is a locally free $\mathcal{O}_X$-module of the same rank as that of the $\mathcal{O}_Y$-module $\mathcal{S}_Y$,
- $\mathcal{E}$ is a locally free $\mathcal{O}_X^{\mathbb{C}}$-module of some finite rank r,
- $(\widehat{X}^{Az}, \widehat{\mathcal{E}})$ is the Azumaya/matrix super-C^k-manifold with a fundamental module specified by $(\mathcal{S}, \mathcal{E})$ on X and $\widehat{\varphi} : (\widehat{X}^{Az}, \widehat{\mathcal{E}}) \to \widehat{Y}$ is a C^k-map from $(\widehat{X}^{Az}, \widehat{\mathcal{E}})$ to $\widehat{Y}$, as defined in Definition 4.3.2, such that
$$\varphi^{\sharp}(\mathcal{S}_Y) \;\subset\; \mathcal{O}_X^{Az} \otimes_{\mathcal{O}_X} \mathcal{S}\,.$$

Remark 5.1.4. [*action functional for fermionic D-brane*]. The supersymmetric action functional of either type of fermionic D-branes in the above prototypical definitions remains to be worked out and understood. Cf. Sec. 6.

5.2 The Higgs mechanism on D-branes vs. deformations of maps from a matrix brane

Throughout the series of works in this project, we have in a few occasions brought out the term 'Higgsing/un-Higgsing' of D-branes in the study of deformations of maps/morphisms from an Azumaya/matrix scheme or manifold with a fundamental module; cf. [L-Y1: Sec. 2.2] and [L-Y2: Example 2.3.2.11]. In particlular, recall FIGURE 5-2-0-1. A closer look at the link of the two can now be made.

5.2.1 The Higgs mechanism in the Glashow-Weinberg-Salam model

To manifest the parallel setting in our situation, we highlight in this subsubsection the relevant classical part of the Higgs mechanism in the Glashow-Weinberg-Salam model for leptons (here, electron and muon and their corresponding neutrinos) that breaks the gauge symmetry from $SU(2)\times U(1)_Y$ (the gauge symmetry for the electroweak interaction) to $U(1)_{em}$ (the gauge symmetry for the electromagnetic interaction). Here, both $U(1)_Y$ and $U(1)_{em}$ are isomorphic to the $U(1)$ group; the different labels Y and *em* indicates that $U(1)_{em}$ is a subgroup of $SU(2)\times U(1)_Y$ that is different from the factor $U(1)_Y$ in the product. *em* here stands for '*e*lectro*m*agnetic'. Mathematicians are referred to [P-S: Sec. 20.2] and also [I-Z: Sec. 12.6], [Mo: Chap. 3], [Ry: Sec. 8.5] for a complete discussion that takes care also of quantum-field-theoretical issues such as renormalizability of and anomalies in the theory.

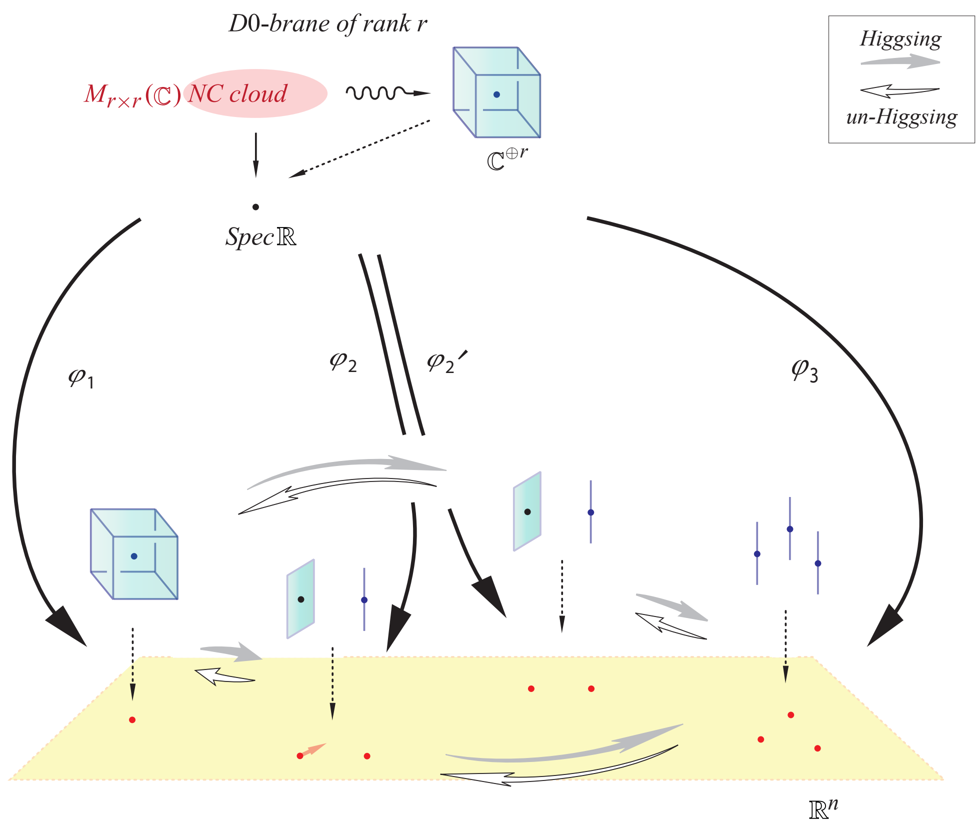

FIGURE 5-2-0-1. (Cf. [L-Y2: FIGURE 2-1-1] (D(6)).) Readers are referred to [L-Y3: FIGURE 3-4-1, caption] (D(11.1)) for explanations.

The Lagrangian density of the Glashow-Weinberg-Salam model

This is a 4-dimensional quantum field theory on the Minkowski space-time $\mathbb{M}^{3+1}$ (with coordinates $x := (x^\mu)_{\mu=0,1,2,3}$), whose Lagrangian density is given by

$$\mathcal{L} \;=\; \mathcal{L}_{fermion} + \mathcal{L}_{gauge} + \mathcal{L}_{Higgs} + \mathcal{L}_{Yukawa}\,,$$

where

- $\mathcal{L}_{fermion} \;=\; \bar{E}_L(i\not{D})E_L + \bar{e}_R(i\not{D})e_R + \bar{Q}_L(i\not{D})Q_L + \bar{u}_R(i\not{D})u_R + \bar{d}_R(i\not{D})d_R,$
- $\mathcal{L}_{gauge} \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$
- $\mathcal{L}_{Higgs} \;=\; (D_\mu\phi)^\dagger(D^\mu\phi) + \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2,$
- $\mathcal{L}_{Yukawa} \;=\; -\lambda_e\bar{E}_L\cdot\phi e_R - \lambda_d\bar{Q}_L\cdot\phi d_R - \lambda_u\epsilon^{ab}\bar{Q}_{La}\phi_b^\dagger u_R +$ (hermition conjugates) consists of the cubic interaction terms that are linear in Higgs-field components and quadratic in other matter-field components.

The part in this expression that is most relevant to us is explained/reviewed below: (Assuming up-down-repeated-dummy-index summation convention.)

(1) The Lie algebra $su(2)$ takes the basis $\{\tau_1, \tau_2, \tau_3\}$ from Pauli matrices σ_1, σ_2, σ_3 with

$$\tau_1 \;=\; \frac{1}{2}\sigma_1 \;=\; \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tau_2 \;=\; \frac{1}{2}\sigma_2 \;=\; \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \tau_3 \;=\; \frac{1}{2}\sigma_3 \;=\; \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix};$$

$A = A^a_\mu(x)\tau_a dx^\mu$ is the $su(2)$ gauge field in the adjoint representation of $su(2)$, $B = B_\mu(x)dx^\mu$ is the $U(1)_Y$ gauge field, $F = F_{\mu\nu}(x)dx^\mu\wedge dx^\nu$ the field strength (i.e. curvature) of the $su(2)\oplus u(1)_Y$-valued 1-form field $A\oplus B$.

(2) The $su(2)\oplus u(1)_Y$ representation-theoretical contents of all the fields in the model are given by the following table:

field in $su(2)$-rep components	$su(2)\oplus u(1)_Y$ representation	name
gauge fields		
$A_\mu \;=\; \begin{pmatrix} A^1_\mu \\ A^2_\mu \\ A^3_\mu \end{pmatrix}$	$(3,0)$	gauge boson gauge boson gauge boson
B_μ	(1,0)	gauge boson
matter fields		
$E_L \;=\; \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}$	$(2,-\frac{1}{2})$	neutrino (left-handed) electron (left-handed)
$Q_L \;=\; \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$(2,\frac{1}{6})$	quark (left-handed) quark (left-handed)
e_R	$(1,-1)$	electron (right-handed)
u_R	$(1,\frac{2}{3})$	quark (right-handed)
d_R	$(1,-\frac{1}{3})$	quark (right-handed)
Higgs fields		
$\phi \;=\; \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$	$(2,\frac{1}{2})$	Higgs field

In the above table, matter fields ν_{eL}, e_L^-, u_L, d_L (resp. e_R, u_R, d_R) are described by left-handed (resp. right-handed) Weyl spinor fields on $\mathbb{M}^{3+1}$.

- For a field ψ belonging to a representation T of $su(2)$ with $u(1)_Y$ charge Y, the covariant derivative $D_\mu\psi$ of ψ is given by

$$D_\mu\psi \;=\; (\partial_\mu - igA_\mu^a T_a - ig'YB_\mu)\,\psi\,.$$

  For example, for the complex 2-component Higgs field ϕ,

$$D_\mu\phi \;=\; \left(\partial_\mu - igA_\mu^a(x)\tau_a - \frac{i}{2}g'B_\mu(x)\right)\phi\,.$$

  Similarly, for the covariant derivative of E_L, Q_L, e_R, u_R, d_R.

- g, g', μ^2, λ, and λ_e, λ_d, λ_u in $\mathcal{L}$ are real-valued coupling constants of the model.

- $\mathcal{L}_{fermion}$ as given is the standard gauge-invariant Lagrangian density without a potential for massless fermions.

Formally and classically, one has massless fermions (ν_{eL}, e_L, u_L, d_L, e_R, u_R, d_R), massless gauge bosons (A_μ^a, B_μ), and tachyonic Higgs bosons (ϕ^+, ϕ^0) to begin with.

Remark 5.2.1.1. [*quark and strong interaction*]. The above model contains quarks u_L, u_R, d_L, d_R. However, only their involvement with the electroweak interaction is considered. The Standard Model, which takes into account also the strong interaction, enlarges the gauge group from the current $SU(2)\times U(1)_Y$ to $SU(2)_L\times U(1)_Y\times SU(3)_c$. On the other hand, as long as the purpose of comparison to Higgs mechanism on D-branes in our setting is concerned, one can remove all the quarks in the model and consider only the truncated theory

$$\begin{aligned}\mathcal{L}' \;&=\; \mathcal{L}'_{fermion} + \mathcal{L}_{gauge} + \mathcal{L}_{Higgs} + \mathcal{L}'_{Yukawa}\\ &=\; \bar{E}_L(i\not{D})E_L + \bar{e}_R(i\not{D})e_R \;-\; \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\\ &\quad\; +(D_\mu\phi)^\dagger(D^\mu\phi) \;+\mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \;-\lambda_e\bar{E}_L\cdot\phi e_R \;+\; (\text{hermition conjugates})\,.\end{aligned}$$

Spontaneous gauge-symmetry breaking from a spontaneous settling-down to a vacuum of the Higgs field

The vacuum manifold for the Higgs field ϕ is given by the locus in the ϕ-space $\simeq \mathbb{C}^2$ on which the potential $V(\phi)$ takes its minimum:

$$\{(z^1,z^2)\,|\,|z^1|^2+|z^2|^2 \;=\; \mu^2/(2\lambda)\}\;\subset\;\mathbb{C}^2\,.$$

Let

$$v \;=\; \sqrt{\mu^2/\lambda}\;>\;0\,.$$

Then, up to a gauge transformation, we may assume that the Higgs field "condenses" to (i.e. takes its value only at) the *vacuum expectation value* (*VEV*) $(0, v/\sqrt{2})$. This reduces the gauge group from the original $SU(2)\times U(1)_Y$ to the subgroup $Stab\,((0,v/\sqrt{2}))$ (the *stabilizer* or *isotropy group* of $(0,v/\sqrt{2})$) consisting of all elements of $SU(2)\times U(1)_Y$ that leave $(0,v/\sqrt{2})$ fixed. Or equivalently in terms of Lie algebras, it reduces $su(2)\oplus u(1)$ to the sub-Lie algebra $Ann\,((0,v/\sqrt{2}))$ (the *annihilator* of $(0,v/\sqrt{2})$) consisting of all elements in $su(2)\oplus u(1)_Y$ that annihilate $(0,v/\sqrt{2})$:

$$u(1)_{em} \;:=\; \{\,(c^1\tau_1+c^2\tau_2+c^3\tau_3,c^4)\,|\,c^1=c^2=0,\;c^3=c^4\,\}\;\subset\; su(2)\oplus u(1)_Y\,.$$

The original model $\mathcal{L}$ descends to a new quantum field theory on fields that fluctuate around the Higgs vacuum $(0, v/\sqrt{2})$ (a procedure that influences only the Higgs field in the current model) and in terms of the induced representations of the residual $u(1)_{em}$ (a procedure that reorganizes and may regroup and influence all the fields in the model):

(1) A general Higgs field ϕ around the VEV $\frac{1}{\sqrt{2}}\binom{0}{v}$ can be expressed as

$$\phi(x) \;=\; U(x)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+h(x) \end{pmatrix}$$

under a gauge transformation $U(x)$ with value in $SU(2)\times U(1)_Y$, where $h(x)$ is a real-valued scalar field on $\mathbb{M}^{3+1}$ with $\langle h(x)\rangle = 0$. Thus, the only physical degree of freedom from ϕ after gauge-symmetry breaking is h.

(2) The $u(1)_{em}$ representation-theoretical contents of all the fields around vacuum in the model after the symmetry breaking are given by the following table. Each is specified by its $u(1)_{em}$-charge.

field in $u(1)_{em}$-rep	$u(1)_{em}$-charge	name
($u(1)_{em}$) gauge field		
$A_\mu^{em} \;:=\; \frac{1}{\sqrt{g^2+g'^2}}(g'A_\mu^3 + gB_\mu)$	0	gauge boson
massive vector fields		
$W_\mu^+ \;:=\; \frac{1}{\sqrt{2}}(A_\mu^1 - iA_\mu^2)$	1	vector boson
$W_\mu^- \;:=\; \frac{1}{\sqrt{2}}(A_\mu^1 + iA_\mu^2)$	-1	vector boson
$Z_\mu^0 \;:=\; \frac{1}{\sqrt{g^2+g'^2}}(gA_\mu^3 - g'B_\mu)$	0	vector boson
matter fields		
$E_L \;=\; \binom{\nu_{eL}}{e_L}$	0 -1	neutrino (left-handed) electron (left-handed)
$Q_L \;=\; \binom{u_L}{d_L}$	$2/3$ $-1/3$	quark (left-handed) quark (left-handed)
e_R	-1	electron (right-handed)
u_R	$2/3$	quark (right-handed)
d_R	$-1/3$	quark (right-handed)
Higgs field		
h	0	Higgs field

Note in particular that the $u(1)_{em}$-charge of both Weyl spinors e_L and e_R on $\mathbb{M}^{3+1}$ is -1, reaffirming that they correspond to electrons of different chiralities moving in the Minkowski space-time. Note also that all quarks have fractional $u(1)_{em}$-charge.

This gives the full field contents of the model after the gauge-symmetry breaking in terms of representations of the $U(1)_{em}$ symmetry left.

Mass generation to fermions and gauge bosons after the symmetry breaking

Rewrite the Lagrangian density $\mathcal{L}$ now in terms of fields after the gauge-symmetry breaking. Then, up to an overall constant,

$$\begin{aligned}\mathcal{L}_{Higgs} &= \tfrac{1}{2}(\partial_\mu h)^2 - \mu^2 h^2 + \tfrac{g^2v^2}{4} W_\mu^- W_\mu^+ + \tfrac{(g^2+g'^2)v^2}{4}(Z_\mu^0)^2 \\ &\qquad\qquad\qquad + (\,\text{higher order terms in } h,\, W_\mu^-,\, W_\mu^+,\, Z_\mu^0\,) \\ \mathcal{L}_{Yukawa} &= -\tfrac{\lambda_e v}{\sqrt{2}}\bar{E}_L\cdot e_R - \tfrac{\lambda_d v}{\sqrt{2}}\bar{Q}_L\cdot d_R - \tfrac{\lambda_u v}{\sqrt{2}}\bar{u}_L u_R \\ &\qquad\qquad\qquad + (\,\text{higher order terms and hermition conjugates}\,)\,.\end{aligned}$$

Note that the kinetic terms for h in $\mathcal{L}_{Higgs}$, W_μ^-, W_μ^+, Z_μ^0 in $\mathcal{L}_{gauge}$, e_L, e_R, d_L, d_R, u_L, u_R in $\mathcal{L}_{fermion}$ are all in the standard form. It follows that classically the mass of particles associated to h, W_μ^-, W_μ^+, Z_μ^0 (resp. e_L, e_R, d_L, d_R, u_L, u_R) are read off from $\mathcal{L}_{Higgs}$ (resp. $\mathcal{L}_{Yukawa}$) as

$$m_h = \sqrt{2}\mu\,, \qquad m_{W_\mu^-} = m_{W_\mu^+} = \frac{gv}{2}\,, \qquad m_{Z_\mu^0} = \frac{\sqrt{g^2+g'^2}\,v}{2}\,,$$

$$m_{e_L} = m_{e_R} = \frac{\lambda_e v}{\sqrt{2}}\,, \qquad m_{d_L} = m_{d_R} = \frac{\lambda_d v}{\sqrt{2}}\,, \qquad m_{u_L} = m_{u_R} = \frac{\lambda_u v}{\sqrt{2}}\,.$$

To recap in words,

(1) Before gauge-symmetry breaking, $\mathcal{L}_{Higgs}$ is the only part in $\mathcal{L}$ that contains quartic terms that are quadratic in gauge fields and quadratic in Higgs field. After gauge-symmetry breaking, such terms create a mass term for the gauge fields that correspond to broken gauge symmetry, rendering them massive vector bosons in the symmetry-broken theory.

(2) Before gauge-symmetry breaking, $\mathcal{L}_{Yukawa}$ is the only part in $\mathcal{L}$ that contains cubic terms that are quadratic in fermionic fields and linear in Higgs fields. After gauge-symmetry breaking, such terms create a mass term for the fermionic fields, rendering them massive fermions in the symmetry-broken theory.

Remark 5.2.1.2. [*mathematical reflection*: *principal bundle, representation, associated bundle, reduction of gauge group, induced representation*]. In mathematical terms, let

- $P_{Lorentz}$ be the principle Lorentz-frame bundle (with group $SO(3,1)$ and trivialized by the flat Levi-Civita connection associated to the space-time metric) over $\mathbb{M}^{3+1}$,
- P_G be a principle G-bundle (trivial in the above case) with group $G = SU(2)\times U(1)_Y$.

The various tensor products of irreducible representations V_{ρ^L} of $SO(3,1)$ and irreducible representations V_{ρ^G} of G give rise to various associated vector bundles $E_{\rho^L\otimes\rho^G}$ of the $SO(3,1)\times G$-principle bundle $P_{Lorentz}\times_{\mathbb{M}^{3+1}} P_G$ whose sections corresponds to various fields $\psi_{\rho^L\otimes\rho^G}$ on the space-time $\mathbb{M}^{3+1}$. ρ_L determines the spin (e.g. bosons vs. fermions) of $\psi_{\rho^L\otimes\rho^G}$ while ρ^G distinguishes other particle features (e.g. electrons vs. quarks). A choice of a collection of such representations and a choice of a gauge invariant Lagrangian density $\mathcal{L}$ for the fields corresponding to these representations together give a model of particle physics.

When some of the fields take their VEV, the gauge group is reduced to the stabilizer subgroup $H\subset G$ of the VEV and a principal subbundle $P_H\subset P_G$ (and hence $P_{Lorentz}\times_{\mathbb{M}^{3+1}} P_H \subset P_{Lorentz}\times_{\mathbb{M}^{3+1}} P_G$) is selected. Original fields expanding around VEV assume naturally induced representations from G to its subgroup H. Re-writing $\mathcal{L}$ in terms of fields corresponding to these induced representations gives the classical picture of the Higgs mechanism.

Mathematicians should be aware that this is the easy part. It is the contents at the quantum level that take works. And for the Glashow-Weinberg-Salam model, such quantum-field-theoretical conclusions are experimentally justified, up to possibly higher order corrections.

5.2.2 The Higgs mechanism on the matrix brane world-volume

While the full detail of the Higgs mechanism that fits our setting can be produced only after the action functional for differentiable maps from a matrix brane with fields is introduced, the basic structure that initiates the mechanism and the associated gauge-symmetry-breaking pattern are readily there in the setting. Indeed, the following two steps

(1) Recall Remark 5.2.1.2 and consider the associated Lie-algebra bundle $E_{\mathfrak{G}}$ of P_G from the Adjoint representation G on its Lie algebra $\mathfrak{G}$. Note that all G-modules are naturally $\mathfrak{G}$-modules from the induced endomorphisms.

(2) Promote the Lie algebra $\mathfrak{G}$ to a (unital) associative algebra $\mathcal{A}$, the Lie-algebra bundle $E_{\mathfrak{G}}$ to an associative-algebra bundle $E_{\mathcal{A}}$ and all $\mathfrak{G}$-modules to $\mathcal{A}$-modules; and do the same reduction procedure as in Sec. 5.2.1, with the sub-Lie algebra of $\mathfrak{G}$ specified by a VEV replaced by an appropriate subalgebra of $\mathcal{A}$.

give the essential formal classical picture of the Higgs mechanism on D-branes in our setting. (And this is why we call it Higgs mechanism in our setting after all.) We now explain the details, assuming that an unspecified action functional for D-branes in our setting with fields thereupon is given (cf. Sec. 6).

Candidates for the Higgs fields

There are two classes of fields on a matrix brane that can serve as the Higgs fields:

(1) *Differentiable maps from the matrix brane to the target space(-time).*

This can be traced back to the origin of this project ([L-Y1] (D(1))). Coincident D-branes in the space-time exhibit, in addition to enhanced gauge symmetries, an enhanced matrix-valued scalar field on the D-brane world-volume that describes the deformations of the D-branes collectively. Cf. [Po2: vol. I, Sec. 8.7] of Joseph Polchinski.

(2) *Differentiable sections on the fundamental bundle E or its dual $E^{\vee}$.*

Such fields can occur when a dynamical D-brane is immersed in a non-dynamical background D-brane in the space-time, in the same spirit as [Do-G: Sec. 5] of Michael Douglas and Gregory Moore. Cf. FIGURE 1-2 and its caption.

Remark 5.2.2.1. [*soft lower-dimensional D-branes in a hard higher-dimensional D-brane*]. Recall that the tension T_p of a Dp-brane in a target-space-time of a superstring is given by

$$T_p \;=\; 2\pi \left(\frac{T}{2\pi}\right)^{\frac{p+1}{2}} ,$$

where T is the tension of the superstring. Thus, in the regime of the superstring theory where $T \gg 2\pi$, higher-dimensional D-branes would have much larger tension than the lower-dimensional ones and it makes sense to consider a soft/dynamical D-branes immersed in a hard/background D-branes.

Remark 5.2.2.2. [*comparison with Glashow-Weinberg-Salam model*]. If the Glashow-Weinberg-Salam model is realized as a gauge theory on the world-volume of coincident D3-branes, then the Higgs field ϕ in the model would correspond to a section of the Chan-Paton bundle E.

Reduction induced by a VEV of Higgs field: Transverse fluctuations and the new field contents

The notion of '*reduction*' induced by a VEV of Higgs field:

- a '*transverse fluctuation*' of a Higgs field around its vacuum expectation value (VEV) and the resulting '*new field contents*' under the "*induced symmetry-breaking*' from the VEV of the Higgs field

are naturally built into the setting as follows.

Case (*a*): *Differentiable map from the brane world-volume as Higgs field*

Let $\varphi_0 : (X^{Az}, \mathcal{E}) \to Y$ be a C^k-map defined by a C^k-admissible $\varphi_0^\sharp : \mathcal{O}_Y \to \mathcal{O}_X^{Az}$, and

$$\begin{array}{ccc}
\mathcal{O}_X^{Az} = \mathcal{E}nd_{\mathcal{O}_X^{\mathbb{C}}}(\mathcal{E}) & & \\
\big\uparrow & & \\
\mathcal{A}_{\varphi_0} \;:=\; \mathcal{O}_X\langle Im\,\varphi_0^\sharp\rangle & \xleftarrow{\quad\varphi_0^\sharp\quad} & \mathcal{O}_Y \\
\big\uparrow & & \\
\mathcal{O}_X & &
\end{array}$$

be the underlying diagram. Let $\mathcal{C}(\mathcal{A}_{\varphi_0})$ be the *commutant sheaf* of $\mathcal{A}_{\varphi_0}$ in $\mathcal{O}_X^{Az}$, defined by

$$\mathcal{C}(\mathcal{A}_{\varphi_0})(U) \;:=\; \{a \in \mathcal{O}_X^{Az}(U) \,|\, [a, a'] = 0 \text{ for all } a' \in \mathcal{A}_{\varphi_0}(U)\}$$

for open sets U of X. Here, $[a, a'] := aa' - a'a$. Then, $\mathcal{C}(\mathcal{A}_{\varphi_0})$ is an $\mathcal{O}_X^{\mathbb{C}}$-subalgebra of $\mathcal{O}_X^{Az}$, which contains $\mathcal{A}_{\varphi_0}$ since $\mathcal{A}_{\varphi_0}$ is commutative.

Consider

- the class of C^k-maps $\varphi : (X^{Az}, \mathcal{E}) \to Y$ with the constraint that $Im\,\varphi^\sharp \subset \mathcal{C}(\mathcal{A}_{\varphi_0})$.

A C^k-map φ in this class has the property that $\mathcal{A}_\varphi \subset \mathcal{C}(\mathcal{A}_{\varphi_0})$ as well. It follows that such $\varphi^\sharp : \mathcal{O}_Y \to \mathcal{O}_X^{Az}$ induces an equivalence class of gluing systems of ring-homomorphisms

$$\underline{\varphi}^\sharp \;:\; \mathcal{O}_Y \;\longrightarrow\; \mathcal{C}(\mathcal{A}_{\varphi_0})$$

and, equivalently, φ induces a map

$$\underline{\varphi} \;:\; (X, \mathcal{C}(\mathcal{A}_{\varphi_0})) \;\longrightarrow\; Y\,.$$

By construction, φ factors through $\underline{\varphi}$ as indicated in the following commutative diagram:

$$\begin{tikzcd}
(X, \mathcal{O}_X^{Az}, \mathcal{E}) \arrow[r, "\varphi"] \arrow[d] & Y \\
(X, \mathcal{C}(\mathcal{A}_{\varphi_0}), \mathcal{E}) \arrow[ur, "\underline{\varphi}"'] &
\end{tikzcd},$$

where

- $(X, \mathcal{C}(\mathcal{A}_{\varphi_0}))$ is a noncommutative space with the underlying topology X and the structure sheaf $\mathcal{C}(\mathcal{A}_{\varphi_0})$;

- $\mathcal{E}$ is now regarded as the $\mathcal{C}(\mathcal{A}_{\varphi_0})$-module via the built-in inclusion $\mathcal{C}(\mathcal{A}_{\varphi_0}) \subset \mathcal{O}_X^{Az}$;

- the surjection $(X, \mathcal{O}_X^{Az}) \to (X, \mathcal{C}(\mathcal{A}_{\varphi_0}))$ is defined by the built-in inclusion $\mathcal{C}(\mathcal{A}_{\varphi_0}) \hookrightarrow \mathcal{O}_X^{Az}$.

Note that, also by construction,

- *φ and $\underline{\varphi}$ have identical surrogates, i.e., $X_\varphi = X_{\underline{\varphi}}$, and identical C^k-maps $f_\varphi : X_\varphi \to Y$ and $f_{\underline{\varphi}} : X_{\underline{\varphi}} \to Y$ under the above identification of surrogates.*

With Sec. 5.2.1 in mind, it is natural to interpret such φ as a transverse fluctuation of Higgs field around φ_0 in the current case. Moreover,

- *Note that $\mathcal{C}(\mathcal{A}_{\varphi_0})$ is an $\mathcal{O}_X^{\mathbb{C}}$-subalgebra of $\mathcal{O}_X^{Az}$ of lower rank as $\mathcal{O}_X^{\mathbb{C}}$-modules. It is in this sense that φ_0 "breaks the original symmetry" in the current context.*

All $\mathcal{O}_X^{Az}$-modules can be regarded also as $\mathcal{C}(\mathcal{A}_{\varphi_0})$-modules naturally.

As for the generation of masses by φ_0, as long as there are terms $\mathcal{L}_{Yukawa}$ and $\mathcal{L}_{Higgs}$ in the action functional for fields on X^{Az} that are parallel to like terms of the same notation in the Glashow-Weinberg-Salam model in Sec. 5.2.1, φ_0 would play such role.

Case (b): Section of fundamental module as Higgs field

Let ξ_0 be a nowhere-zero C^k-section of E. Since fiberwise the $Aut(E)$-orbit of ξ_0 is open in E, there is no transverse fluctuation of the Higgs field in the current case. (However, see Remark 5.2.2.3.) Nevertheless, ξ_0, regarded now as the VEV of the Higgs field in the current case, remains to have effect both on symmetry-breaking and on generation of masses, as we now explain.

Define the *null-sheaf* $\mathcal{N}(\xi_0)$ of ξ_0 in $\mathcal{O}_X^{Az}$ to be the sheaf on X defined by

$$\mathcal{N}(\xi_0)(U) \;:=\; \left\{ a \in \mathcal{O}_X^{Az}(U) \,|\, a \cdot \xi_0|_U \;=\; 0 \right\}$$

for open sets U of X. Then, $\mathcal{N}(\xi_0)$ is an $\mathcal{O}_X^{\mathbb{C}}$-module that is multiplicatively closed. But it is not an $\mathcal{O}_X^{\mathbb{C}}$-algebra since it has no unit element for the multiplication. To remedy this, consider

$$\mathcal{N}^+(\xi_0) \;:=\; \mathcal{N}(\xi_0) + \mathcal{O}_X^{\mathbb{C}} \;\subset\; \mathcal{O}_X^{Az}$$

This is now an $\mathcal{O}_X^{\mathbb{C}}$-subalgebra of $\mathcal{O}_X^{Az}$. Note that $\mathcal{N}(\xi_0) \cap \mathcal{O}_X^{\mathbb{C}} = 0$ in $\mathcal{O}_X^{Az}$; thus, $\mathcal{N}^+(\xi_0) \simeq \mathcal{N}(\xi_0) \oplus \mathcal{O}_X^{\mathbb{C}}$.

Consider

- the class of C^k-maps $\varphi : (X^{Az}, \mathcal{E}) \to Y$ with the constraint that $Im\,\varphi^\sharp \subset \mathcal{N}^+(\xi_0)$.

A C^k-map φ in this class has the property that $\mathcal{A}_\varphi \subset \mathcal{N}^+(\xi_0)$ as well. It follows that such $\varphi^\sharp : \mathcal{O}_Y \to \mathcal{O}_X^{Az}$ induces an equivalence class of gluing systems of ring-homomorphisms

$$\underline{\varphi}^\sharp \;:\; \mathcal{O}_Y \;\longrightarrow\; \mathcal{N}^+(\xi_0)$$

and, equivalently, φ induces a map

$$\underline{\varphi} \;:\; (X, \mathcal{N}^+(\xi_0)) \;\longrightarrow\; Y\,.$$

By construction, φ factors through $\underline{\varphi}$ as indicated in the following commutative diagram:

$$\begin{tikzcd}
(X, \mathcal{O}_X^{Az}, \mathcal{E}) \arrow[r, "\varphi"] \arrow[d] & Y \\
(X, \mathcal{N}^+(\xi_0), \mathcal{E}) \arrow[ur, "\underline{\varphi}"'] &
\end{tikzcd} \qquad ,$$

where

- $(X, \mathcal{N}^+(\xi_0))$ is a noncommutative space with the underlying topology X and the structure sheaf $\mathcal{N}^+(\xi_0)$;
- $\mathcal{E}$ is now regarded as the $\mathcal{N}^+(\xi_0)$-module via the built-in inclusion $\mathcal{N}^+(\xi_0) \subset \mathcal{O}_X^{Az}$;
- the surjection $(X, \mathcal{O}_X^{Az}) \to (X, \mathcal{N}^+(\xi_0))$ is defined by the built-in inclusion $\mathcal{N}^+(\xi_0) \hookrightarrow \mathcal{O}_X^{Az}$.

Note that, also by construction,

- *φ and $\underline{\varphi}$ have identical surrogates, i.e., $X_\varphi = X_{\underline{\varphi}}$, and identical C^k-maps $f_\varphi : X_\varphi \to Y$ and $f_{\underline{\varphi}} : X_{\underline{\varphi}} \to Y$ under the above identification of surrogates.*

On the other hand,

- *Since ξ_0 is nowhere-zero, $\mathcal{N}^+(\xi_0)$ is an $\mathcal{O}_X^{\mathbb{C}}$-subalgebra of $\mathcal{O}_X^{Az}$ of lower rank as $\mathcal{O}_X^{\mathbb{C}}$-modules. It is in this sense that ξ_0 "breaks the original symmetry" in the current context.*

As for the generation of masses by ξ_0, the situation is the same as in the previous case: as long as there are terms $\mathcal{L}_{Yukawa}$ and $\mathcal{L}_{Higgs}$ in the action functional for fields on X^{Az} that are parallel to like terms of the same notation in the Glashow-Weinberg-Salam model in Sec. 5.2.1, such role of ξ_0 remain intact.

Remark 5.2.2.3. [*reduction of structure group from $GL(r, \mathbb{C})$ to $U(r)$*]. The notion of 'transverse fluctuation' of the Higgs field in the current Case (b) will be postponed to the future. It is better addressed after a Hermitian metric on the Chan-Paton bundle E is introduced and the details of the reduction $M_{r\times r}(\mathbb{C}) \Rightarrow GL(n, \mathbb{C}) \Rightarrow U(r)$ in line with our setting are understood. Cf. Remark 1.1.

The following toy model of a dynamical Azumaya/matrix brane with fermions illustrates the further issue of the generation of masses from the VEV of a Higgs field. It is a modification (by a potential) of a truncated and simplified version of an action functional motivated by Matrix Theory in the sense of Thomas Banks, Willy Fischler, Stephen Shenker, and Leonard Susskind [B-F-S-S]. See, for example, [T-vR1], [T-vR2], [T-vR3] for some related study in curved space-time.

Example 5.2.2.4. [Higgs mechanism on Azumaya/matrix brane with fermion]. Let

- $X = \mathbb{R}^1$ be the real line as a C^∞-manifold with coordinate t, and $\mathcal{O}_X := \mathcal{O}_{\mathbb{R}^1}$ be the structure sheaf of C^∞-functions on $\mathbb{R}^1$;
- $\mathcal{E} \simeq \mathcal{O}_{\mathbb{R}^1} \otimes_{\mathbb{R}} \mathbb{C}^{\oplus r}$ be a free $\mathcal{O}_{\mathbb{R}^1}^{\mathbb{C}}$-module of rank r; for concreteness, we assume that $\mathcal{E}$ is trivialized;
- $(X^{Az}, \mathcal{E}) := (\mathbb{R}^{1,Az}, \mathcal{E}) := (\mathbb{R}^1, \mathcal{O}_{\mathbb{R}^1}^{Az} := \mathcal{E}nd_{\mathcal{O}_{\mathbb{R}^1}^{\mathbb{C}}}(\mathcal{E}), \mathcal{E})$ be an Azumaya/matrix real line with a fundamental module and
- $Y = \mathbb{M}^{(d-1)+1}$ be the d-dimensional Minkowski space-time, as a C^∞-manifold with coordinates $(y^a)_a := (y^0, y^1, \cdots, y^{d-1})$ and a flat Lorentzian metric $ds^2 = -(dy^0)^2 + (dy^1)^2 + \cdots + (dy^{d-1})^2$.

Consider a 1-dimensional quantum field theory on the matrix real line $\mathbb{R}^{1,Az}$ with fields:

- *Bosonic*:

- C^∞*-maps* $\varphi : \mathbb{R}^{1,Az} \to \mathbb{M}^{(d-1)+1}$ from the matrix real line to the Minkowski space-time; recall that φ is defined by a C^∞-admissible ring-homomorphism

$$\begin{array}{ccccc} \varphi^\sharp & : & C^\infty(\mathbb{M}^{(d-1)+1}) & \longrightarrow & M_{r\times r}(C^\infty(\mathbb{R}^1)) \\ & & y^a & \longmapsto & Y^a(t)\,, \qquad a = 0,\,1,\,\cdots,d-1\,; \end{array}$$

- *gauge fields* $A(t)\,dt\,, A(t) \in M_{r\times r}(C^\infty(\mathbb{R}^1))\,,$ on the fundamental module $\mathcal{E}$; recall that its induced connection on $\mathcal{O}_{\mathbb{R}^1}^{Az}$ is simply the inner derivation $[A(t),\,\cdot\,]\,dt$ and note that A is nondynamical since its curvature vanishes;

· *Fermionic*:

- $\mathcal{O}_{\mathbb{R}^1}^{Az}$*-valued spinor fields* $\Theta^\alpha \in (\mathcal{O}_{\mathbb{R}^1}^{Az} \otimes_{\mathcal{O}_{\mathbb{R}^1}} \mathcal{S})(\mathbb{R}^1) = M_{r\times r}(C^\infty(S))$ on $\mathbb{R}^1$, $\alpha = 1,\,\cdots,N$, where
  - · S is the spinor bundle on $\mathbb{R}^1$ (with flat metric $(dt)^2$) of rank 1,
  - · $\mathcal{S}$ is the sheaf of $\mathcal{O}_{\mathbb{R}^1}$-modules associated to S, and
  - · N is the dimension of a spinor representation of $SO(d-1,1)$;

  note that since $\mathcal{S} \simeq \mathcal{O}_{\mathbb{R}^1}$ as $\mathcal{O}_{\mathbb{R}^1}$-modules, $M_{r\times r}(C^\infty(S)) \simeq M_{r\times r}(C^\infty(\mathbb{R}^1))$;

and Lagrangian: (Up-low-repeated-dummy-index summation rule is assumed.)

$$\begin{aligned} \mathcal{L}(\varphi,\Theta,A) \;:=\; T_0\, Tr \Big\{ & -D_tY_aD_tY^a + V(Y^0,Y^1,\cdots,Y^{d-1}) \\ & + \Theta_\alpha D_t\Theta^\alpha - c_0\Theta^\alpha\gamma_{a,\alpha\beta}[\,Y^a,\Theta^\beta\,] \Big\}\,, \end{aligned}$$

where

· T_0 and c_0 are constants (depending on the string tension and the string coupling constant when the setting is fitted into string theory);

· $D_tY^a = \partial_tY^a + [A,Y^a]$ and $D_t\Theta^\alpha = \partial_t\Theta^\alpha + [A,\Theta^\alpha]$ are covariant derivatives of Y^a and Θ^α respectively;

· γ_a, $a = 0,\,1,\,\cdots,d-1$, are the γ-matrices for $\mathbb{M}^{(d-1)+1}$.

As given, this quantum field theory on $\mathbb{R}^{1,Az}$ has massless fermions.

In comparison to the Lagrangian density for the Glashow-Weinberg-Salam model in Sec. 5.2.1, one has immediately that

$$\begin{array}{lllllll} \mathcal{L}_{fermion} & = & \Theta_\alpha D_t\Theta^\alpha\,, & \qquad & \mathcal{L}_{Higgs} & = & -D_tY_aD_tY^a + V(Y^0,Y^1,\cdots,Y^{d-1})\,, \\ \mathcal{L}_{gauge} & = & 0\,, & & \mathcal{L}_{Yukawa} & = & -\,c_0\Theta^\alpha\gamma_{a,\alpha\beta}[\,Y^a,\Theta^\beta\,]\,. \end{array}$$

With the C^∞-maps $\varphi : \mathbb{R}^{1,Az} \to \mathbb{M}^{(d-1)+1}$ serving as Higgs fields of the current toy model and the Higgs mechanism in the Glashow-Weinberg-Salam model and the discussion of open strings and D-branes in [Po: vol. I: Sec. 8.6 and Sec. 8.7] of Polchinski in mind, one expects thus:

Claim 5.2.2.4.1. [mass generation of fermion]. *A vacuum expectation value VEV φ_0 of φ may generate mass for some of the fermions in the model. Furthermore, the mass the VEV φ_0 generates for fermions in the model may depend on the distance between the connected or irreducible components of the image brane $\varphi_0(\mathbb{R}^{1,Az})$ in the space-time $\mathbb{M}^{(d-1)+1}$.*

To justify this, observe that

Lemma 5.2.2.4.2. [commutator with diagonal matrix]. *Let $r = r_1 + \cdots + r_l$ be a positive-integer decomposition of r, $Id_{r_i\times r_i}$ be the $r_i \times r_i$ identity matrix, and*

$$M^{(r_1,\cdots,r_l)}_{(\lambda_1,\cdots\lambda_l)} := \begin{bmatrix} \lambda_1\, Id_{r_1\times r_1} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_l\, Id_{r_l\times r_l} \end{bmatrix}$$

with the i-th block in the diagonal being $\lambda_i \cdot Id_{r_i\times r_i}$ and all other entries being 0. Let

$$B = [B_{ij}]_{ij,\,l\times l} \in M_{r\times r}(\mathbb{C})$$

be an $r\times r$-matrix in the $l\times l$ block-matrix form, where $B_{ij}\in M_{r_i\times r_j}(\mathbb{C})$ is an $r_i\times r_j$-matrix. Then, the commutator

$$[M^{(r_1,\cdots,r_l)}_{(\lambda_1,\cdots,\lambda_l)}\,,\, B] \;=\; [(\lambda_i-\lambda_j)\,B_{ij}]_{ij,\,l\times l}\,.$$

Suppose now that the potential $V(\varphi) := V(Y^0, Y^1, \cdots, Y^{d-1})$ is chosen so that it takes a VEV at a C^∞-map $\varphi_0 : \mathbb{R}^{1,Az} \to \mathbb{M}^{(d-1)+1}$ defined by

$$\begin{array}{cccc} \varphi_0^\sharp & : & C^\infty(\mathbb{M}^{(d-1)+1}) & \longrightarrow & M_{r\times r}(C^\infty(\mathbb{R}^1)) \\ & & y^a & \longmapsto & Y_0^a(t)\,, \qquad a = 0,\,1,\,\cdots,d-1 \end{array}$$

with

$$Y_0^0(t) \;=\; t\cdot Id_{r\times r} \qquad \text{and} \qquad Y_0^a(t) \;=\; M^{(r_1,\cdots,r_l)}_{(\lambda_1^a,\cdots,\lambda_l^a)} \quad \text{for } a = 1,\cdots,d-1\,,$$

for some positive-integer decomposition $r = r_1 + \cdots + r_l$ of r independent of a. Recall from [L-Y3: Sec. 3] (D(11.1)) that all the λ_i^a must be real and they correspond to the y^a-coordinate of the i-th components of $\varphi_0(\mathbb{R}^{1,Az})$ in $\mathbb{M}^{(d-1)+1}$. As elements in $M_{r\times r}(C^\infty(S))$, express

$$\Theta^\beta \;=\; [\Theta^\beta_{ij}]_{ij,\,l\times l}$$

in the $l\times l$ block-matrix form. Then, after formally expanding φ around the VEV φ_0, rewriting the Lagrangian $\mathcal{L}$ as done in the Glashow-Weinberg-Salam model, and applying Lemma 5.2.2.4.2, one concludes that

$$\mathcal{L}_{Yukawa} \;=\; -\,c_0\;\Theta^\alpha\gamma_{a,\alpha\beta}[(\lambda_i^a-\lambda_j^a)\Theta^\beta_{ij}]_{ij,\,l\times l} \;+\; (\text{higher order terms})\,.$$

Assume that there is at least one a such that λ_i^a are distinct for all i. Then one has now a nontrivial mass-matrix for fermions that depends only on the set

$$\{\lambda_i^a-\lambda_j^a\,|\, a = 1,\cdots,d-1\,;\, 1\le i,j\le l\}$$

that describes the relative position/distance of connected/irreducible components of $\varphi_0(\mathbb{R}^{1,Az})$ in $\mathbb{M}^{(d-1)+1}$. This justifies the claim and concludes the example. Cf. FIGURE 5-2-2-4-1.

Further details of the Higgs mechanism on D-branes in our setting should be re-picked up after the details of the action functional is understood; cf. Sec. 6.

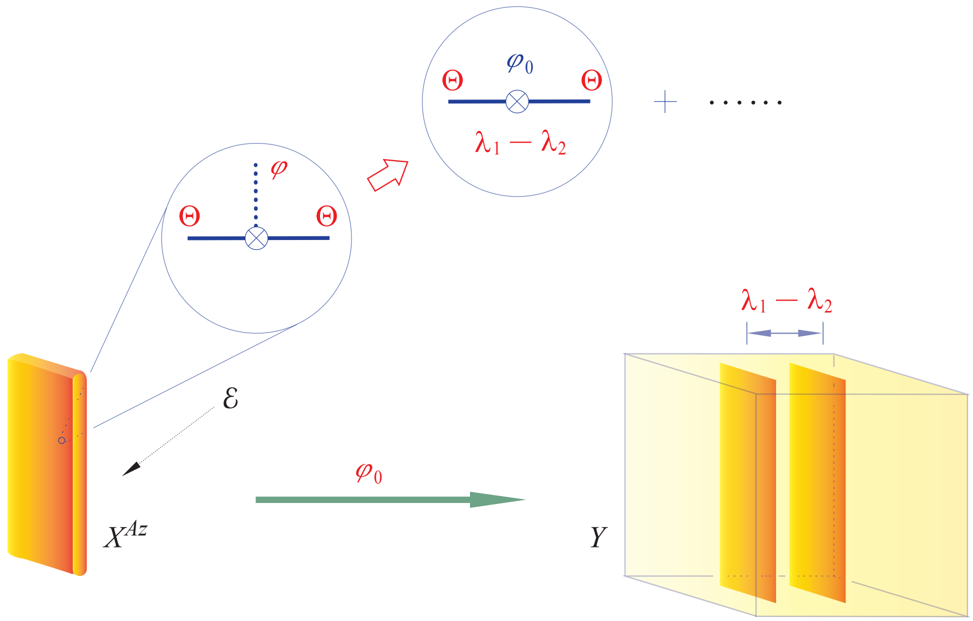

FIGURE 5-2-2-4-1. Generation of mass for fermions on the D-brane world-volume $(X^{Az}, \mathcal{E})$, a matrix manifold with a fundamental module, through the Higgs mechanism that takes C^∞-maps $\varphi : X^{Az} \to Y$ as Higgs field. In the FIGURE, this process is indicated by the arrow $\Rightarrow$ that transforms a $\Theta\varphi\Theta$ Feynman diagram to a series of diagrams with the lowest-order term the propagator diagram for fermions. The geometry of the image configuraion $Im(\varphi_0) \subset Y$ of X^{Az} under a VEV φ_0 determines the mass of fermions Θ through the Yukawa coupling terms in the actional functional, as indicated by the relevant Feynman diagram, with fermions in solid line —— and Higgs fields in dashed line ······. Such Feynman diagrams may be thought of as reflecting the scattering in X^{Az} of particles associated to these fields on X^{Az}. The factor $(\lambda_1 - \lambda_2)$ for fermion mass terms after Higgsing is only meant to be schematic, indicating its dependence on the distance "$\lambda_1 - \lambda_2$" of components of the image $Im(\varphi_0)$ of X^{Az} in Y.

6 Where we are, and some more new directions

Recall the following guiding question (cf. [L-Y1: Sec. 2.2] (D(1))):

Q. *What is a D-brane intrinsically?*

that initiated our D-brane project. Following the line of Grothendieck's theory of schemes for modern algebraic geometry, [L-Y1] (D(1)) provided a proto-typical setting for dynamical D-branes in the common realm of string theory and algebraic geometry, as *maps/morphisms from an Azumaya/matrix scheme with a fundamental module*. Its equivalent settings were realized in [L-L-S-Y] (D(2), with Si Li and Ruifang Song). Seven years later, [L-Y3] (D(11.1)) and the current note D(11.2) together brought into play the notion of differentiable rings from synthetic differential geometry and algebraic geometry over differentiable rings, and extends such settings to the common realm of string theory and differential/symplectic geometry, as *differentiable maps from an Azumaya/matrix manifold with a fundamental module* with similar equivalent settings. Before moving on, it is instructive to pause here, as a conclusion of this note, with a reflection on where we are now — in comparison with how string theory began — and a sample list of new themes/directions naturally brought out from the study.

First, another guiding question:

Q. *How does string theory begin?*

Physically and historically, it began with the attempt to understand hadrons (particles that interacts through the strong interaction). However, as you open any textbook on string theory, another answer from another aspect may immediately come to you:

A. *Mathematically*, string theory begins with the notion of a *differentiable map from a string or the world-sheet of a string* (open or closed, with or without world-sheet fermions) *to a space-time.*

For example, [B-B-Sc: Sec. 2.2], [G-S-W: vol. 1: Sec. 1.3.2], [Joh: Sec. 2.2], [Po2: vol. I: Sec. 1.2], [Zw: Chap. 6]. Indeed, replacing 'string' (resp. 'world-sheet') with any physical object (resp. 'world-volume'), the same answer should work for any dynamical object moving in a space-time; in particular, D-branes. And that's what we just completed and that's exactly where we are:

- *After [L-Y3] (D(11.1)) and the current note D(11.2), we are now at the beginning/entrance of a theory of D-branes — purely bosonic or with fermionic fields and supersymmetry — as dynamical objects moving in a space-time.* FIGURE 6-1.

With the above comparison to the history of string theory in mind, in [L-Y3] (D(11.1)) we bring out a sample of five new directions all related to or motivated by D-branes in string theory that the notion of differentiable maps from matrix manifolds may play a role. Some more immediate new directions include:

(1) The *Dirac-Born-Infeld term*, the *Chern-Simons term*, as well as any other term, and their supersymmetric generalization in the full action functional for coincident D-branes from the aspect of functionals for maps from a matrix manifold with various bosonic and fermionic fields thereupon.

(2) *Synthetic/C^k-algebraic symplectic geometry.*

(3) *Synthetic/C^k-algebraic calibrated geometry.*

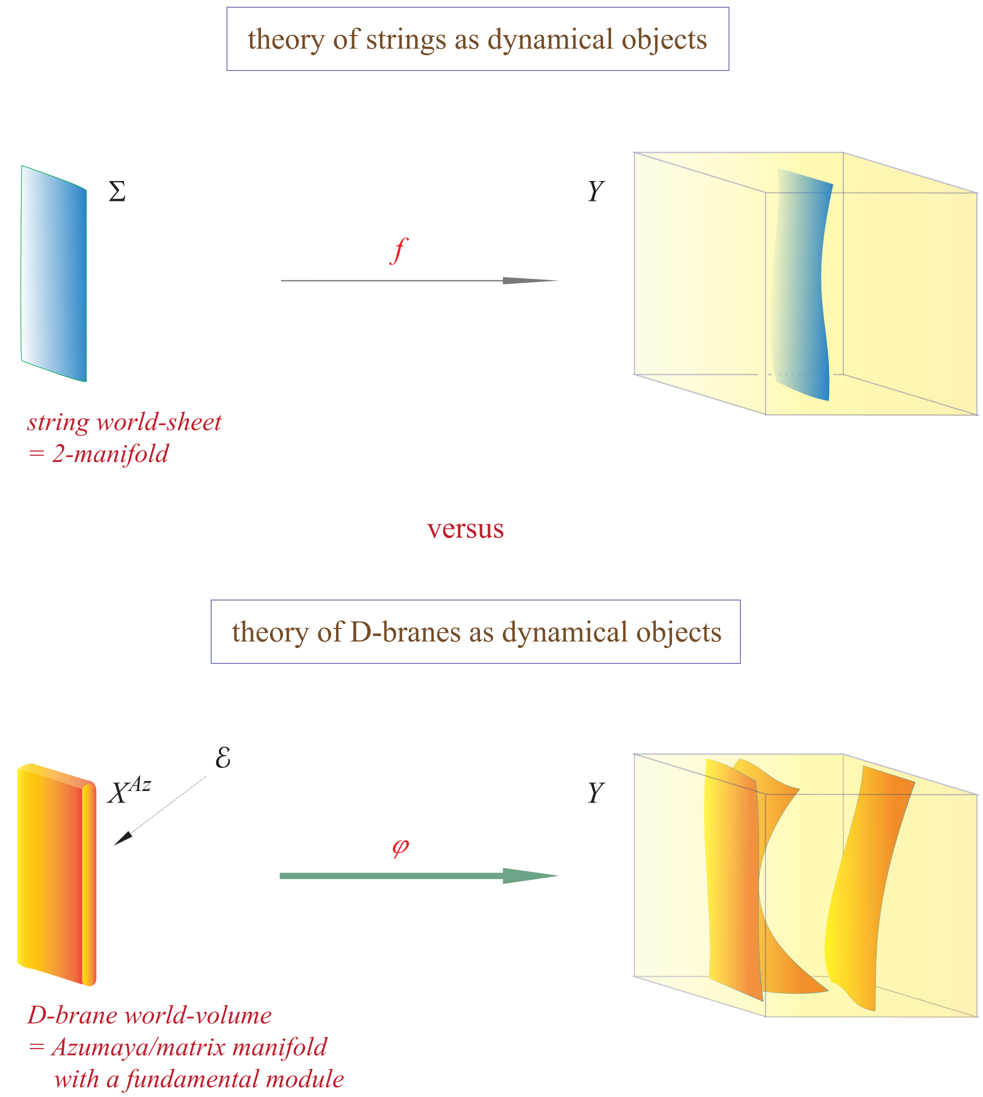

FIGURE 6-1. The mathematical starting point of string theory with string as a dynamical object moving in a space-time (cf. [G-S-W: Sec. 1.3, FIGURE 1.3] of Green, Schwarz, and Witten) vs. the mathematical starting point of D-brane theory with D-brane as a dynamical object moving in a space-time: The former begins with the notion of 'differentiable maps $f : \Sigma \to Y$ from a string world-sheet to the space-time' while the latter begins with the notion of 'differentiable maps $\varphi : (X^{Az}, \mathcal{E}) \to Y$ from a matrix manifold (i.e. the D-brane world-volume) with a fundamental module to the space-time'. *Unlike* the string world-sheet Σ, X^{Az} carries a matrix-type "noncommutative cloud" over its underlying topology. Under a differentiable map φ as defined in [L-Y3: Definition 5.3.1.5] (D(11.1), (cf. Definition 4.2.1.3 of the current note) the image $\varphi(X^{Az})$ can behavior in a complicated way. In particular, it could be disconnected or carry some nilpotent fuzzy structure. See also FIGURE 5-2-0-1.

(4) A *new matrix theory* based on complex matrices of real eigenvalues.

From the *string-theory point of view*,

- Theme (1) is *the next* guiding theme: Only when one is able to give a string-theory-compatible action functional on differentiable maps from an Azumaya/matric manifold with a fundamental module to a space-time can one begin to address physics of D-branes as fundamental objects in their own right in string theory .

From the *mathematical point of view*,

- Theme (2) and Theme (3) have been missing in symplectic/calibrated geometry when Lagrangian or calibrated submanifolds and their deformations/collidings were studied. Whatever the reason they are overlooked, we now provide a motivation to study them from the new aspect of dynamical D-branes, cf. [L-Y3: Sec. 7.2] (D(11.1)). This should bring the study of Lagrangian/calibrated submanifolds (possibly supporting a decorated sheaf) to a footing closer to that of Hilbert- or Quot-schemes or moduli of coherent or (semi-)stable sheaves in algebraic geometry.

From *both mathematical and physical aspects*,

- Theme (4), the new matrix theory — as a theory for differentiable maps from an Azumaya/matrix point with a fundamental module to a space — could provide one with a starting point before attacking questions for general Azumaya/matrix manifolds.

With the notion/framework of differentiable maps from a matrix manifold-or-supermanifold to a space-time (or superspace-time) in place, the stage has just been set.

References

[Arg1] P.C. Argyres, *Introduction to supersymmetry*, course Phys 661, Cornell University, fall 1996; notes obtainable from `http://www.physics.uc.edu/~argyres/661/`

[Arg2] ———, *Supersymmetric effective actions in four dimensions*, in *Non-perturbative aspects of strings, branes and supersymmetry*, M. Duff, E. Sezgin, C. Pope, B. Greene, J. Louis, K.S. Narain, S. Randjbar-Daemi, G. Thompson eds., World Scientific, 1999.

[Arg3] ———, *Non-perturbative dynamics of four-dimensional supersymmetric field theories*, in *Conformal Field Theory: New Non-Perturbative Methods in String and Field Theory*, Y. Nutku, C. Saclioglu, T. Turgut eds., Westview Press, 2000.

[Arg4] ———, *An introduction to global supersymmetry*, course Phys 661, Cornell University, fall 2000; notes obtainable from `http://www.physics.uc.edu/~argyres/661/`

[Art] M. Artin, *On Azumaya algebras and finite dimensional representations of rings*, *J. Alg.* **11** (1969), 532–563.

[As] M. Asano, *Non-commutative branes in D-brane backgrounds*, *Int. J. Mod. Phys.* **A17** (2002), 4733–4748. (arXiv:hep-th/0106253)

[Bac] C.P. Bachas, *Lectures on D-branes*, in *Duality and supersymmetric theories*, D.I. Olive and P.C. West eds., 414–473, Publ. Newton Inst., Cambridge Univ. Press, 1999. (arXiv:hep-th/9806199)

[Bat] M. Batchelor, *The structure of supermanifolds*, *Trans. Amer. Math. Soc.* **253** (1979), 329–338.

[B-B-Sc] K. Becker, M. Becker, and J.H. Schwarz, *String theory and M-theory - a modern introduction*, Cambridge Univ. Press, 2007.

[B-B-St] K. Becker, M. Becker, and A. Strominger, *Fivebranes, membranes and non-perturbative string theory*, *Nucl. Phys.* **B456** (1995), 130–152. (arXiv:hep-th/9507158)

[B-D] M. Berkooz and M.R. Douglas, *Five-branes in M(atrix) theory*, *Phys. Lett.* **B395** (1997), 196–202. (arXiv:hep-th/9610236)

[B-F-S-S] T. Banks, W. Fischer, S.H. Shenker, and L. Susskind, *M theory as a matrix model: A conjecture*, *Phys. Rev.* **D55** (1997), 5112–5128. (arXiv:hep-th/9610043)

[B-G-V] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Grund. math. Wiss. 298, Springer, 1992.

[B-H] J. Baez and J. Huerta, *The algebra of grand unified theories*, *Bull. Amer. Math. Soc.* **47** (2010), 483–552. (arXiv:0904.1556 [hep-th])

[B-L] F.A. Berezin and D.A. Leites,, *Supermanifolds*, *Soviet. Math. Dokl.* **16** (1975), 1218–1222.

[B-M-R-S] J. Brodzki, V. Mathai, J. Rosenberg, and R.J. Szabo, *D-branes, RR-fields and duality on noncommutative manifolds*, *Commun. Math. Phys.* **277** (2008), 643–706. (arXiv:hep-th/0607020)

[B-P] N. Berkovits, V. Pershin, *Supersymmetric Born-Infeld from the pure spinor formalism of the open superstring*, *J. High Energy Phys.* **0301** (2003) 023. (arXiv:hep-th/0205154)

[B-S] R.L. Bryant and E. Sharpe, *D-branes and spin^c structures*, *Phys. Lett.* **B450** (1999), 353–357. (arXiv:hep-th/9812084)

[B-T1] E. Bergshoeff and P.K. Townsend, *Super D-branes*, *Nucl. Phys.* **B490** (1997), 145–162. (arXiv:hep-th/9611173)

[B-T2] ———, *Super D-branes revisited*, *Nucl. Phys.* **B531** (1998), 226–238. (arXiv:hep-th/9804011)

[C-H-S] C.S. Chu, P.S. Howe, and E. Sezgin, *Strings and D-branes with boundaries*, *Phys. Lett.* **B428** (1998), 59–67. (arXiv:hep-th/9801202)

[DeW] B. DeWitt, *Supermanifolds*, 2nd ed., Cambridge Univ. Press, 1992.

[Dor1] H. Dorn, *Nonabelian gauge field dynamics on matrix D-branes*, *Nucl. Phys.* **B494** (1997), 105–118. (arXiv:hep-th/9612120)

[Dor2] ———, *Non-Abelian gauge field dynamics on matrix D-branes in curved space and two-dimensional σ-models*, *Fortsch. Phys.* **47** (1999), 151–157. (arXiv:hep-th/9712057)

[Dou] M.R. Douglas, *Branes within branes*, in *Strings, branes and dualities*, L. Baulieu, P. Di Francesco, M.R. Douglas, V.A. Kazakov, M. Picco, and P. Windey eds., 267–275, Kluwer Acad. Publ., 1999. (arXiv:hep-th/9512077)

[Du] E.J. Dubuc, *C^∞-schemes*, *Amer. J. Math.* **103** (1981), 683–690.

[D-E-F-J-K-M-M-W] P. Deligne, P. Etingof, D.S. Freed, L.C. Jeffrey, D. Kazhdan, J.W. Morgan, D.R. Morrison, and E. Witten eds., *Quantum fields and strings: A course for mathematicians*, vol. 1 and vol. 2, Amer. Math. Soc. and Inst. Advanced Study, 1999.

[Di-G] S. Dimopoulos and H. Georgi, *Softly broken supersymmetry and* $SU(5)$, *Nucl. Phys.* **B193** (1981), 150–174.

[Do-G] M.R. Douglas and G.W. Moore, *D-branes, quivers, and ALE instantons*, arXiv:hep-th/9603167.

[DV-K-M] M. Dubois-Violette, R. Kerner, and J. Madore, *Super matrix geometry*, *Class. Quantum Grav.* **8** (1991), 1077–1089.

[D-V1] R. Dijkgraaf and C. Vafa, *Matrix models, topological strings, and supersymmetric gauge theories*, *Nucl. Phys.* **B644** (2002), 3–20. (arXiv:hep-th/0206255)

[D-V2] ——, *On geometry and matrix models*, *Nucl. Phys.* **B644** (2002), 21–39. (arXiv:hep-th/0207106)

[D-V3] ——, *Toda theories, matrix models, topological strings, and N=2 gauge systems*, arXiv:0909.2453 [hep-th].

[Ei] D. Eisenbud, *Commutative algebra - with a view toward algebtaic geometry*, GTM 150, Springer-Verlag, 1995.

[E-H] D. Eisenbud and J. Harris, *The geometry of schemes*, GTM 197, Springer, 2000.

[Freed] D.S. Freed, *Five lectures on supersymmetry*, Amer. Math. Soc., 1999.

[Freund] P.G.O. Freund, *Introduction to supersymmetry*, Cambridge Univ. Press, 1986.

[G-G] H. Georgi and S.L. Glashow, *Unity of all elementary-particle forces*, *Phys. Rev. Lett.* **32** (1974), 438–441.

[G-G-R-S] S.J. Gates, Jr., M.T. Grisaru, M. Rocček, and W. Siegel, *Superspace – One thousand and one lessons in supersymmetry*, Frontiers Phys. Lect. Notes Ser. 58, Benjamin/Cummings Publ. Co., Inc., 1983.

[G-S] M.B. Green and J.H. Schwarz, *Supersymmetrical dual string theory*, *Nucl. Phys.* **B181** (1981), 502–530.

[G-S-W] M.B. Green, J.H. Schwarz, and E. Witten, *Superstring theory, vol. 1: Introduction*; *vol. 2: Loop amplitudes, anomalies, and phenomenology*, Cambridge Univ. Press, 1987.

[G-W] K. Gottfried and V.F. Weisskopf, *Concepts of particle physics*, vol. I and vol. II, Oxford Univ. Press, 1986.

[Ha] R. Hartshorne, *Algebraic geometry*, GTM 52, Springer, 1977.

[H-W] P.-M. Ho and Y.-S. Wu, *Noncommutative geometry and D-branes*, *Phys. Lett.* **B398** (1997), 52–60. (arXiv:hep-th/9611233)

[I-Z] C. Itzykson and J.-B. Zuber, *Quantum field theory*, McGraw-Hill, 1985.

[Joh] C.V. Johnson, *D-branes*, Cambridge Univ. Press, 2003.

[Joy] D. Joyce, *Algebraic geometry over* C^∞*-rings*, arXiv:1001.0023v4 [math.AG].

[Ko] A. Kock, *Synthetic differential geometry*, 2nd ed., London Math. Soc. Lect. Note Ser. 333, Cambridge Univ. Press, 2006.

[Kos] B. Kostant, *Graded manifolds, graded Lie theory, and prequantization*, *Differential geometrical methods in mathematical physics*, K. Bleuler, A. Reetz eds., 177–306, Lect. Notes Math. 570, Springer, 1977.

[Le] R.G. Leigh, *Dirac-Born-Infeld action from Dirichlet* σ*-model*, *Mod. Phys. Lett.* **A4** (1989), 2767–2772.

[Liu] C.-H. Liu, *Azumaya noncommutative geometry and D-branes - an origin of the master nature of D-branes*, lecture given at the workshop *Noncommutative algebraic geometry and D-branes*, December 12 – 16, 2011, organized by Charlie Beil, Michael Douglas, and Peng Gao, at Simons Center for Geometry and Physics, Stony Brook University, Stony Brook, NY; arXiv:1112.4317 [math.AG].

[L-M] H.B. Lawson, Jr. and M.-L. Michelsohn, *Spin geometry*, Princeton Math. Ser. 38, Princeton Univ. Press, 1989.

[L-M-S] F. Lizzi, N.E. Mavromatos, and R.J. Szabo, *Matrix sigma models for multi-D-brane dynamics*, *Mod. Phys. Lett.* **A13** (1998), 829–842. (arXiv:hep-th/9711012)

[L-Y1] C.-H. Liu and S.-T. Yau, *Azumaya-type noncommutative spaces and morphism therefrom: Polchinski's D-branes in string theory from Grothendieck's viewpoint*, arXiv:0709.1515 [math.AG]. (D(1))

[L-L-S-Y] S. Li, C.-H. Liu, R. Song, S.-T. Yau, *Morphisms from Azumaya prestable curves with a fundamental module to a projective variety: Topological D-strings as a master object for curves*, arXiv:0809.2121 [math.AG]. (D(2))

[L-Y2] ———, *D-branes and Azumaya noncommutative geometry: From Polchinski to Grothendieck*, arXiv:1003.1178 [math.SG]. (D(6))

[L-Y3] C.-H. Liu and S.-T. Yau, *D-branes and Azumaya/matrix noncommutative differential geometry, I: D-branes as fundamental objects in string theory and differentiable maps from Azumaya/matrix manifolds with a fundamental module to real manifolds*, arXiv:1406.0929 [math.DG]. (D(11.1))

[L-Y4] ———, manuscript in preparation.

[Man] Y.I. Manin, *Gauge field theory and complex geometry*, Springer, 1988.

[Mat] H. Matsumura, *Commutative ring theory*, translated from Japanese by M. Reid, Cambridge Univ. Press, 1986.

[Mo] R.N. Mohapatra, *Unification and supersymmetry – the frontier of quark-lepton physics*, 2nd ed., GTCP, Springer-Verlag, 1992.

[My1] R.C. Myers, *Dielectric-branes*, *J. High Energy Phys.* **9912** (1999) 022. (arXiv:hep-th/9910053)

[My2] ———, *Nonabelian D-branes and noncommutative geometry*, *J. Math. Phys.* **42** (2001), 2781–2797. (arXiv:hep-th/0106178)

[M-R] I. Moerdijk and G.E. Reyes, *Models for smooth infinitesimal analysis*, Springer, 1991.

[M-S1] N.E. Mavromatos and R.J. Szabo, *Matrix D-brane dynamics, logarithmic operators and quantization of noncommutative spacetime*, *Phys. Rev.* **D59** (1999), 104018. (arXiv:hep-th/9808124)

[M-S2] ———, *Non-linear Schrödinger dynamics of matrix D-branes*, *Int. J. Mod. Phys.* **A16** (2001), 209–250. (arXiv:hep-th/9909129)

[N-S] A. Neveu and J.H. Schwarz, *Factorizable dual model of pions*, *Nucl. Phys.* **B31** (1971), 86–112.

[NG-SdS] J.A. Navarro González and J.B. Sancho de Salas, *C^∞-differentiable spaces*, Lect. Notes Math. 1824, Springer, 2003.

[O-O-Y] H. Ooguri, Y. Oz, and Z. Yin, *D-branes on Calabi-Yau spaces and their mirrors*, *Nucl. Phys.* **B477** (1996), 407–430. (hep-th/9606112)

[Po1] J. Polchinski, *Lectures on D-branes*, in "*Fields, strings, and duality*", TASI 1996 Summer School, Boulder, Colorado, C. Efthimiou and B. Greene eds., World Scientific, 1997. (arXiv:hep-th/9611050)

[Po2] ———, *String theory*, vol. I: *An introduction to the bosonic string*; vol. II: *Superstring theory and beyond*, Cambridge Univ. Press, 1998.

[P-S] M.E. Peskin and D.V. Schroeder, *An introduction to quantum field theory*, Addison-Wesley Publ. Co.. 1995.

[Ra] P. Ramond, *Dual theory for free fermions*, *Phys. Rev.* **D3** (1971), 2415–2418.

[Rog] A. Rogers, *A global theory of supermanifolds*, *J. Math. Phys.* **21** (1980), 1352–1365.

[Ros] G.G. Ross, *Grand unified theories*, Frontier Phys. 60, Benjamin/Cummings Publ. Co., Inc., 1985.

[Rot] M. Rothstein, *The axioms of supermanifolds and a new structure arising from them*, *Trans. Amer. Math. Soc.* **297** (1986), 159–180.

[Ry] L.H. Ryder, *Quantum field theory*, Cambridge Univ. Press, 1985.

[Sa] N. Sakai, *Naturalness in supersymmetric Guts*, *Z. Phys.* **C11** (1981), 153–157.

[Sé] E. Sérié, *Théories de jauge en géométrie non commutative et généralisation du modèle de Born-Infeld*, Ph.D. thesis at Université Paris 6, September 2005.

[So] M. F. Sohnius. *Supersymmetry and central charges*, *Nucl. Phys.* **B138** (1978), 109–121.

[St] M.J. Strassler, *An unorthodox introduction to supersymmetric gauge theory*, in *Strings, branes and extra dimensions*, TASI 2001, S.S. Gubser and JD. Lykken eds., 561–638, World Scientific, 2004; arXiv:hep-th/0309149.

[Sz1] R.J. Szabo, *Superconnections, anomalies and non-BPS brane charges*, *J. Geom. Phys.* **43** (2002), 241–292. (arXiv:hep-th/0108043)

[Sz2] ———, *An introduction to string theory and D-brane dynamics*, Imperial College Press, 2004.

[S-W] S. Shnider and R.O. Wells, Jr., *Supermanifolds, super twister spaces and super Yang-Mills fields*, Séminaire Math. Supér. 106. Press. Univ. Montréal, 1989.

[Ta] W. Taylor IV, *Lectures on D-branes, gauge theory and M(atrices)*, arXiv:hep-th/9801182.

[Ts] A.A. Tseytlin, *On non-abelian generalization of Born-Infeld action in string theory*, *Nucl. Phys.* **B 501** (1997), 41–52. (arXiv:hep-th/9701125)

[T-vR1] W. Taylor IV and M. van Raamsdonk, *Supergravity currents and linearized interactions for Matrix Theory configurations with fermionic backgrounds*, *J. High Energy Phys.* **9904** (1999) 013. (arXiv:hep-th/9812239)

[T-vR2] ———, *Multiple D0-branes in weakly curved backgrounds*, *Nucl. Phys.* **B558** (1999), 63–95. (arXiv:hep-th/9904095)

[T-vR3] ———, *Multiple Dp-branes in weak background fields*, *Nucl. Phys.* **B573** (2000), 703–734. (arXiv:hep-th/9910052)

[Wa] C.T.C. Wall, *Graded Brauer groups*, *J. Reine Angew. Math.* **213** (1963/1964), 187–199.

[Wei] S. Weinberg, *The quantum theory of fields*, vol. III, *Supersymmetry*, Cambridge Univ. Press, 2000.

[Wes] D.B. Westra, *Superrings and supergroups*, Ph.D. thesis, Universität Wien, October 2009.

[Wi1] E. Witten, *Mirror manifolds and topological field theory.* in *Essays on mirror manifolds*, S-T. Yau ed., 120–158, International Press, 1992; reprinted in *Mirror symmetry I*, 121–160, S.-Y. Yau ed., Amer. Math. Soc. & International Press, 1998.

[Wi2] ———, *Phases of $N = 2$ theories in two dimensions*, *Nucl. Phys.* **B403** (1993), 159–222. (arXiv:hep-th/9301042)

[Wi3] ———, *Bound states of strings and p-branes*, *Nucl. Phys.* **B460** (1996), 335–350. (arXiv:hep-th/9510135)

[Wi4] ———, *Notes on supermanifolds and integration*, arXiv:1209.2199 [hep-th].

[Wu] L. Wulff, *Strings, boundary fermions and coincident D-branes*, Ph.D. thesis, Stockholm University, Sweden, 2007.

[W-B] J. Wess and J. Bagger, *Supersymmetry and supergravity*, 2nd ed., Princeton Ser. Phys., Princeton Univ. Press, 1992.

[W-O] E. Witten and D. Olive, *Supersymmetry algebras that include topological charges*, *Phys. Lett.* **B78** (1978), 97–101.

[Zw] B. Zwiebach, *A first course in string theory*, Cambridge Univ. Press, 2004.

chienhao.liu@gmail.com, chienliu@math.harvard.edu;
yau@math.harvard.edu