FINITENESS OF SUBFAMILIES OF CALABI-YAU N-FOLDS OVER CURVES WITH MAXIMAL LENGTH OF YUKAWA-COUPLING

KEFENG LIU, ANDREY TODOROV, SHING-TUNG YAU, AND KANG ZUO

Dedicated to the memory of E. Viehweg

Abstract. We give a simplified and more algebraic proof of the finiteness of the families of Calabi-Yau n-folds with non-vanishing of Yukawa-coupling over a fixed base curve and with fixed degeneration locus. We also give a generalization of this result. Our method is variation of Hodge structure and poly-stability of Higgs bundles.

1. **Introduction**

In this short note, we give a simplified proof of the main theorem in [LTYZ03], namely the set of subfamilies of Calabi-Yau n-folds with non-vanishing of Yukawa-coupling over the fixed base curve and with fixed degeneration locus is finite. The method is based on variation of Hodge structure and poly-stability of Higgs bundles. We hope this approach will be more understandable to algebraic geometers, particularly to people working on Hodge theory. Note that if we consider the subset of sub-families passing through the maximal nilpotent degeneration locus, the finiteness of such families was proved by Zhang Yi in his PhD thesis [Zh04]. However, in general a rigid family does not need passing through the maximal nilpotent degeneration locus.

Very recently J. C. Rohde [rohde09] and later A. Garbagnati and B. Van Geemen [Gar-Gee09] have found some universal families of Calabi-Yau 3-folds without maximal nilpotent degeneration locus. Moreover the Yukawa-coupling vanishes. To cover more general cases, in this note we also formulate a more general criterion for the rigidity of sub-families of those universal families with vanishing Yukawa-coupling. We use the notion of the length of Yukawa coupling attached to a subfamily introduced in [VZ03] and show that the set of subfamilies of Calabi-Yau n-folds over the fixed base curve and with fixed degeneration locus, whose Yukawa-coupling have the same length as the length of the universal family, is finite.

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2. Universal Family over Moduli Space with Level $N-$Structure

Let $\mathcal{M}_h(\mathbb{C})$ denote the set of isomorphic classes of minimal polarized manifolds F with fixed Hilbert polynomial h, and let $\mathcal{M}_h$ be the corresponding moduli functor, i.e.

$$\mathcal{M}_h(U) = \left\{ \begin{array}{c} (f : V \to U, \mathcal{L}); f \text{ smooth and} \\ (f^{-1}(u), \mathcal{L}|_{f^{-1}}(u)) \in \mathcal{M}_h(\mathbb{C}), \text{ for all } u \in U \end{array} \right\}$$

By Viehweg [V95] there exists a quasi-projective coarse moduli scheme M_h for $\mathcal{M}_h$. Fixing a projective manifold $\bar{U}$ and the complement U of a normal crossing divisor, we want to consider

$$\mathbf{H}_{\bar{\mathbf{U}},\mathbf{U}} = \left\{ \begin{array}{c} \varphi : (\bar{U}, U) \to (\bar{M}_h, M_h) \quad \text{induced} \\ \text{by non} - \text{isotrivial families}\, f : X \to U \end{array} \right\}.$$

Since M_h is just a coarse moduli scheme, it is not clear whether $\mathbf{H}$ has a scheme structure. However, since we are considering here the moduli space of polarized Calabi-Yau n-folds, all $F \in \mathcal{M}(\mathbb{C})$ admit locally injective Torelli map for the Hodge structure on $H^n(F, \mathbb{Z})$. Hence by Theorem 10.3, f) in [Popp77](see also [Sz98]) there exists a fine moduli scheme M_h^N with a sufficiently large level structure N on $H^n(F, \mathbb{Z})$, which is finite over M_h. By abuse of notations, we will replace $\mathcal{M}_h$ by the moduli functor of polarized manifolds with a level N structure, and choose some smooth compactification $\bar{M}_h$, such that $\bar{M}_h \setminus M_h =: S$ is a normal crossing divisor.. Then $\mathbf{H}_{\bar{\mathbf{U}},\mathbf{U}}$ parameterizes all morphisms from $\varphi : (\bar{U}, U) \to (\bar{M}_h, M_h)$, hence it is a scheme. Moreover there exists a universal family $f : X \to \mathbf{H}_{\bar{\mathbf{U}},\mathbf{U}} \times U$.

3. Boundedness of Families of Polarized Manifolds

Theorem 1. *Assume* $\dim \bar{U} = 1$, *then* $\mathbf{H}_{\bar{\mathbf{U}},\mathbf{U}}$ *is of finite type.*

Theorem 1 is known since long time for many cases and due to many people: the case of curves is due to Arakelov and Parshin, and the case of abelian varieties is due to Faltings [Fa83]. Yau [Yau78] proved a generalization of Schwarz lemma which implies the Arakelov inequalities and also that Arakelov equality holds for a family of curves of genus bigger or equal to two if and only if the VHS split. This result was obtained earlier by Viehweg and K. Zuo using the technique of Higgs bundles. The Schwarz lemma of Yau was used by Jost-Yau [JY93] to give another proof of Falting's theorem in [Fa83]. C. Peters [Pe90] has proven the boundedness for general period maps. Recently this kind of boundedness result has been generalized to polarized manifolds without assuming the local injectivity of Torelli map. The case of surfaces of general type is due to Bedulev-Viehweg [BV00], the case of surfaces of Kodaira dimension ≤ 1 is due to Oguiso-Viehweg [OV01] and the case the semi positive canonical line bundle is due do Viehweg-Zuo [VZ02]. Very recently, Kovace and Lieblich [KL06] have shown the boundedness for canonically polarized manifolds over higher dimensional bases.

As the case of Calabi-Yau manifolds is a sub-case of the semi-positive canonical line bundle, for reader's convenience we reproduce the proof for Theorem 1, which can be found in ([VZ02], Theorem 6.2).

Proof of Theorem 1.:Viehweg [V95] has constructed a series of nef invertible sheaves λ_ν on M_h such that λ_ν are ample $\nu >> 1$ and are natural in the sense $\varphi^*\lambda_\nu = \det f_*\omega_{X/U}^\nu$ for $\varphi : U \to M_h$ induced by a family $f : X \to U$. He has shown furthermore [V06] that λ_ν extends to nef invertible sheaves $\bar{\lambda}_\nu$ on $\bar{M}_h$ which are again natural in the sense that if $\varphi : \bar{U} \to \bar{M}_h$ is induced by a semi stable family $\bar{f} : \bar{X} \to \bar{U}$ then

$$\varphi^* \bar{\lambda}_\nu = \det \bar{f}_* \omega^\nu_{\bar{X}/\bar{U}}.$$

For a point $(\varphi : (\bar{U}, U) \to (\bar{M}_h, M_h)) \in \mathbf{H}_{\bar{\mathbf{U}},\mathbf{U}}$, let $f : X \to U$ denote the pull-back of the universal family via φ. Assume $\dim \bar{U} = 1$ and let $S := \bar{U} \setminus U$, $s := \#S$, F the generic fibre of f and $n := \dim F$ then one finds the Arakelov inequality for f [VZ01] and [VZ02]

$$\deg(\varphi^* \bar{\lambda}_\nu) \leq \nu \cdot \mathrm{rank} f_* \omega^\nu_{X/U} \cdot (n \cdot \deg \Omega^1_{\bar{U}}(\log S) + s) \cdot e(\omega^\nu_F),$$

where $e(\omega^\nu_F))$ is the holomorphic Euler characteristic of ω^ν_F. Fixing a $\nu >> 1$, then $\bar{\lambda}_\nu$ is nef on $\bar{M}_h$, ample on M_h and $\deg(\varphi^* \bar{\lambda}_\nu)$ is particularly bounded above by a number independent of φ. Hence $\mathbf{H}_{\bar{\mathbf{U}},\mathbf{U}}$ is of finite type. Theorem 1 is proved.

4. Higgs Bundles and Yukawa-Couplings Attached to Families, Rigidity and Finiteness

Let $\mathbb{V}$ denote the local system $R^n f_{univ*} \mathbb{Z}_X$ attached to the universal family $f_{univ} : X \to M_h$. Then the local monodromies around each component of S are quasi-unipotent. By taking another finite cover of $\bar{M}_h$, which is étale over M_h we may assume the local monodromies around all components of S are unipotent. $\mathbb{V}$ carries a polarized $\mathbb{Z}-$variation of Hodge structure of weight-n with the Hodge filtration

$$F^n \subset \cdots F^0 = \mathbb{V} \otimes \mathcal{O}_U =: V,$$

satisfying the Griffiths's transversality. The grading $F^p/F^{p+1} =: E^{p,n-p}$ with the projection Gauss-Manin connection $\theta^{p,q} : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_{M_h}$ defines the correspondence between $\mathbb{V}$ and its Higgs bundle

$$(E := \bigoplus_{p+q=n} E^{p,q}, \theta := \bigoplus_{p+q=n} \theta^{p,q}), \quad \theta \wedge \theta = 0.$$

The following properties are known

$$E^{p,q} \simeq R^q f_{univ*} \Omega^p_{X/M_h},$$

and the Higgs map

$$\theta^{p,q} : R^q f_{univ*} \Omega^p_{X/M_h} \to R^{q+1} f_{univ*} \Omega^{p-1}_{X/M_h} \otimes \Omega^1_{M_h}$$

coincides with the Kodaria-Spencer maps of f_{univ} on $R^q f_{univ*} \Omega^p_{X/M_h}$. By Deligne (E, θ) admits a canonical extension, such that the extended θ takes the value in $\Omega^1_{\bar{M}_h}(\log S)$. The extension is denoted again by (E, θ).

For a point $(\varphi : (\bar{U}, U) \to (\bar{M}_h, M_h)) \in \mathbf{H}_{\bar{\mathbf{U}},\mathbf{U}}$, let $f : X \to U$ denote the pull-back of the universal family via φ. Then the pull-back $\varphi^*(\mathbb{V})$ is the VHS $\mathbb{V}_f$ attached to f. Let (E_f, θ_f) be the Higgs bundle corresponding to $\mathbb{V}_f$, then $E_f = \varphi^* E$, and θ_f is equal to $d\varphi(\theta)$ defined as

$$d\varphi(\theta) : E_f \stackrel{\varphi^*\theta}{\to} E_f \otimes \varphi^* \Omega^1_{\bar{M}_h}(\log S) \stackrel{d\varphi}{\to} E_f \otimes \Omega^1_{\bar{U}}(\log S)$$

Definition 1. *The n-times iterated Kodaira-Spencer map*

$$\theta_f^n : E_f^{n,0} \stackrel{\theta_f^{n,0}}{\to} E_f^{n-1,1} \otimes \Omega_U^1 \stackrel{\theta_f^{n-1,1}}{\to} \cdots \stackrel{\theta_f^{1,n-1}}{\to} E_f^{0,n} \otimes S^n \Omega_{\bar{U}}^1(\log S)$$

is called the Yukawa-coupling θ_f^n *attached to the family* $f : X \to U$. *We will say that the Yukawa-coupling attached the* f *does not vanish, if* $\theta_f^n \neq 0$.

In general, following [VZ03] we consider the l-times iterated Kodaira-Spencer map attached to the family $f : X \to U$ for $1 \leq l \leq n$.

$$\theta_f^l : E_f^{n,0} \stackrel{\theta_f^{n,0}}{\to} E_f^{n-1,1} \otimes \Omega_U^1 \stackrel{\theta_f^{n-1,1}}{\to} \cdots \stackrel{\theta_f^{n-l+1,l-1}}{\to} E_f^{n-l,l} \otimes S^l \Omega_{\bar{U}}^1(\log S).$$

Definition 2. *The length of the Yukawa-coupling attached to a family* $f : X \to U$ *is the maximal* l *such that* $\theta_f^l \neq 0$. *This number is an invariant of the family* $f : X \to U$ *and will be denoted by* $l(\theta_f)$.

Remark 1. *The length of the Yuwawa-coupling attached to a sub family can be in general smaller than the Yuwawa-coupling attached to the universal family.*

The length of the universal family of Calabi-Yau quintic hyper surfaces in P^4 *is equal to 3. By* [VZ03] *there do exist sub-families of Calabi-Yau quintic hypersurfaces with the length of Yukawa-coupling equal to 1, or 2.*

Very recently Rohde [rohde09] *has found a universal family of Calabi-Yau 3-folds over a complex ball quotient, which does not have maximal nilpotent degeneration points. (see also* [Gar-Gee09]*). Moreover the length of the Yukawa-coupling of this universal family is equal to 1.*

Definition 3. *We will say that the family* $f : X \to U$ *has the Yukawa-coupling with the maximal length if* $l(\theta_f) = l(\theta_{f_{univ}})$.

Theorem 2. *Assume* θ_f^n *attached to a family* $f : X \to U$ *does not vanish, then* f *is rigid, i.e. the component of* $\mathbf{H}_{\bar{\mathbf{U}},\mathbf{U}}$ *containing* $\varphi : (\bar{U}, U) \to (\bar{M}_h, M_h)$ *induced by* $f : X \to U$ *is zero-dimensional.*

Regarding the existence of universal families of Calabi-Yau n-folds whose Yukawa-coupling does vanish, i.e. the length of the Yukawa-coupling is smaller than n we have the following corresponding criterion for the rigidity.

Theorem 3. *Assume that the length of the Yukawa-coupling attached to a family* $f : X \to U$ *is equal to the length of the Yukawa-coupling attached the universal family, then* f *is rigid.*

Theorem 2 and 3 combined together with Theorem 1 on the boundedness of $\mathbf{H}_{\bar{\mathbf{U}},\mathbf{U}}$ imply

Theorem 4. *Assume* $\dim U = 1$, *then the subset*

$$\mathbf{H}_{\bar{\mathbf{U}},\mathbf{U}}{}^0 := \{(\varphi : (\bar{U}, U) \to (\bar{M}_h, M_h)) \in \mathbf{H}_{\bar{\mathbf{U}},\mathbf{U}} \,|\, \theta_f^n \neq 0\} \subset \mathbf{H}_{\bar{\mathbf{U}},\mathbf{U}}$$

is finite.

Theorem 4 can be generalized as follows:

Theorem 5. *Assume* $\dim U = 1$, *then the subset*

$$\mathbf{H}_{\bar{\mathbf{U}},\mathbf{U}}{}^0 := \{(\varphi : (\bar{U}, U) \to (\bar{M}_h, M_h)) \in \mathbf{H}_{\bar{\mathbf{U}},\mathbf{U}} \,|\, l(\theta_f) = l(\theta_{f_{univ}})\} \subset \mathbf{H}_{\bar{\mathbf{U}},\mathbf{U}}$$

is finite.

As a corollary from Theorem 5 we have

Theorem 6. *If* $l(\theta_{f_{univ}}) = 1$ *(for example, J. C. Rohde's universal family) then* $\mathbf{H}_{\bar{\mathbf{U}},\mathbf{U}}$ *is finite.*

5. Decomposition of (E_f, θ_f) Induced by Infinitesimal Deformations and Proofs of Theorems 2, 3, 4 and 5

The universal logarithmic Higgs bundle

$$\theta : E \to E \otimes \Omega^1_{\bar{M}_h}(\log S)$$

attached to the universal family $f_{univ} : X \to M_h$ tautologically defines the universal Higgs map

$$\theta : T_{\bar{M}_h}(-\log S) \to End(E),$$

which coincides with the differential of the period map of the universal family. By the Griffiths's transversality θ is of the Hodge type (-1,1), i.e. for any local vector field v of $T_{\bar{M}_h}(-\log S)$ then $\theta_v(E^{p,q}) \subset E^{p-1,q+1}$.

A non-trivial infinitesimal deformation of $\varphi : (\bar{U}, U) \to (\bar{M}_h, M_h)$ is a non-zero section

$$\mathcal{O}_{\bar{U}} \xrightarrow{t} \varphi^* T_{\bar{M}_h}(-\log S),$$

which does not factor through $d\varphi : T_{\bar{U}}(-\log S) \to \varphi^* T_{\bar{M}_h}$.

The composition of t with the pull-back of the universal Higgs map

$$\varphi^*\theta_t : \mathcal{O}_{\bar{U}} \xrightarrow{t} \varphi^* T_{\bar{M}_h}(-\log S) \xrightarrow{\varphi^*\theta} End(E_f)$$

corresponds to the induced infinitesimal deformation of the period map of the family $f : X \to U$.

Lemma 1. ***i)*** *$\varphi^*\theta_t$ is an endomorphism of the Higgs bundle (E_f, θ_f).*
ii) *Let $K := \ker(\varphi^*\theta_t) \subset E_f$. Then K is a Higgs subbundle and there exists a decomposition of the Higgs bundle (E_f, θ_f)*

$$(E_f, \theta_f) = (K, \theta_f|_K) \oplus (K^\perp, \theta_f|_{K^\perp}),$$

where $K^\perp$ is orthogonal to K w.r.t the Hodge metric.

Proof of Lemma 1: We give two proofs for i). First proof of **i)**: By Theorem 3.2, (i) in [Pe90] $\varphi^*\theta_t$ is induced by a flat endomorphism

$$\alpha : \mathbb{V}_f \otimes \mathbb{C} \to \mathbb{V}_f \otimes \mathbb{C}$$

of the Hodge type (-1,1), i.e. $\alpha \in End(V_f)$, commutes with the Gauss-Manin connection, and $\alpha(F_f^p) \subset F_f^{p-1}$. The induced endomorphism by α on the gradings $E = \bigoplus_{p+q=n} E_f^{p,q}$ coincides with $\varphi^*\theta_t$. Note that the Higgs map θ_f is defined by the projection of the Gauss-Manin connection of the gradings, the commutativity of α with the Gauss-Manin connection implies the commutativity of $\varphi^*\theta_t$ with θ_f, i.e., $\varphi^*\theta_t$ is an endomorphism of Higgs bundles

$$\varphi^*\theta_t : (E_f, \theta_f) \to (E_f, \theta_f).$$

The second proof of **i)** goes back to [Z00], and is much simpler. Consider the universal Higgs map on $\bar{M}_h$

$$\theta : E \to E \otimes \Omega^1_{\bar{M}_h}(\log S), \qquad \theta \wedge \theta = 0 \text{ in } End(E) \otimes \Omega^2_{\bar{M}_h}(\log S).$$

If $e_1, \cdots, e_m$ is a local basis of $\Omega^1_{\bar{M}_h}(\log S)|_W$ over some open set W, then

$$\theta = \sum \theta_i e_i,$$

where θ_i are endomorphisms of $E|_W$ The condition $\theta \wedge \theta = 0$ means that θ_i commutes with one another. Given a local vector field

$$v = \sum c_i e_i^* \in T_{\bar{M}_h}(-\log S)|_W,$$

the image of v under the map

$$\theta : T_{\bar{M}_h}(-\log S)|_W \to End(E|_W)$$

defines an endomorphism

$$\theta_v = \sum c_i \theta_i : E_W \to E_W,$$

then θ_v commutes with

$$\theta : E_W \to E_W \otimes \Omega^1_{\bar{M}_h}(\log S)|_W.$$

Pulling back this commutativity via φ, we see that $\varphi^*\theta_t$ commutes with $\varphi^*\theta$, hence commutes with $d\varphi(\theta) = \theta_f$. □

As for **ii)** By [Si90], Section 4, it is known that the Hodge metric on $(E_f, \theta_f)|_U$ is Hermitian-Yang-Mills and by [Si90], Theorem 5, the canonical extended Higgs bundle, (E_f, θ_f) is Higgs poly-stable of slope zero. By i) $\varphi^*\theta_t$ is an endomorphism of (E_f, θ_f), $\ker(\varphi^*\theta_t)(= K)$ and $\mathrm{im}(\varphi^*\theta_t)$ are Higgs subsheaves.

Applying the Higgs-poly-stability to K and $\mathrm{im}(\varphi^*\theta_t)$ one finds

$$\deg K \leq 0, \quad \deg \mathrm{im}(\varphi^*\theta_t) \leq 0.$$

Since

$$\deg K + \deg \mathrm{im}(\varphi^*\theta_t) = \deg(E_f) = 0,$$

$\deg K = 0$. Again by applying the Higgs poly-stability on K, K defines a splitting

$$(E_f, \theta_f) = (K, \theta_f|_K) \oplus (K^\perp, \theta_f|_{K^\perp}),$$

such that $K^\perp$ is orthogonal to K w.r.t. the Hodge metric. ii) is complete. □

Lemma 2. ***i)*** *$E_f^{0,n} \subset K$.* ***ii)*** *$E_f^{n,0} \subset K^\perp$.*

Proof of Lemma 2: Since $\varphi^*\theta_t$ is of the Hodge type (-1,1),

$$\varphi^*\theta_t(E_f^{0,n}) \subset E_f^{-1,n+1} = 0.$$

i) is proven. □

As for ii). We note first that the Kodaira-Spencer map

$$\theta : T_{M_h} \otimes E^{n,0} \to E^{n-1,1}$$

for the universal family of Calabi-Yau manifolds is an isomorphism. Hence, particularly, the sheaf morphism $\varphi^*\theta_t : E_f^{n,0} \to E_f^{n-1,1}$ is injective.

Write any local section

$$s \in E_f = \bigoplus_{p+q=n} E_f^{p,q}$$

in the form

$$s = \sum_{p+q=n} s^{p,q},$$

then $\varphi^*\theta_t$ maps $s^{p,q}$ into $E_f^{p-1,q+1}$. Hence if $\varphi^*\theta_t(s) = 0$, then all $\varphi^*\theta_t(s^{p,q}) = 0$. Thus, the injectivity of $\varphi^*\theta_t$ on $E_f^{n,0}$ implies that $s^{n,0} = 0$ if $s \in K$, i.e.

$$K \subset E_f^{n-1,1} \oplus \cdots \oplus E_f^{0,n}.$$

Since the Hodge decomposition

$$E_f = \bigoplus_{p+q=n} E_f^{p,q}$$

is orthogonal w.r.t. the Hodge metric, $E_f^{n,0}$ is orthogonal to K. ii) is complete.

Proof of Theorem 2: If the component of $\mathbf{H}_{\bar{\mathbf{U}},\mathbf{U}}$ contains a point

$$\varphi : (\bar{U}, U) \to (\bar{M}_h, M_h)$$

induced by $f : X \to U$ with $\theta_f^n \neq 0$ is positive dimensional. Then by Lemma 5 a non-trivial infinitesimal deformation of φ induces a decomposition

$$(E_f, \theta_f) = (K, \theta_f|_K) \oplus (K^\perp, \theta_f|_{K^\perp}).$$

By lemma 6

$$E_f^{0,n} \subset K, \quad E_f^{n,0} \subset K^\perp.$$

Since $K^\perp$ is θ_f-invariant,

$$\theta_f^n(E^{n,0}) \subset K^\perp \otimes S^n(\Omega^1_{\bar{U}}(\log S).$$

On the other hand, $\theta_f^n(E_f^{n,0}) \subset E_f^{0,n} \otimes S^n(\Omega^1_{\bar{U}}(\log S)$, which is contained in

$$K \otimes S^n(\Omega^1_{\bar{U}}(\log S).$$

Since $K \cap K^\perp = \{0\}, \theta_f^n(E^{n,0}) = 0$. A contradiction to the assumption $\theta_f^n \neq 0$. The proof of Theorem 2 is complete. □

The proof of Theorem 3 is similar to the proof of Theorem 2. The only difference is that Lemma 2 has to be reformulated in the following form:

We denote

$$K^{n-l,l} := \mathrm{Ker}(\theta_{f_{univ}} : E^{n-l,l} \to E^{n-l-1,l+1} \otimes \Omega^1_{\bar{M}_h}(\log S)).$$

Lemma 3. ***i)*** $\varphi^* K^{n-l,l} \subset K$. ***ii)*** $E^{n,0} \subset K^\perp$.

Proof of Lemma 3: The proof of part ii) of Lemma 3 is exactly the same as the proof of part ii) Lemma 2. So we need to prove only part i). The definition of $K^{n-l,l}$ implies

$$\varphi^*\theta_{f_{univ}} : \varphi^* T_{\bar{M}_h}(-\log S) \otimes \varphi^* K^{n-l,l} \to 0$$

and since $t : \bar{U} \to \varphi^* T_{\bar{M}_h}(-\log S)$, we have

$$\varphi^*\theta_t(\varphi^* K^{n-l,l}) = \varphi^*\theta_{f_{univ}}(t \otimes \varphi^* K^{n-l,l}) = 0.$$

This shows i). □

Proof of Theorem 3: Since $l(\theta_{f_{univ}}) = l$, we have $\theta^{l+1}_{f_{univ}} = 0$. This implies

$$\theta^l_{f_{univ}}(S^l T_{\bar{M}_h}(-\log S) \otimes E^{n,0}) \subset K^{n-l,l}.$$

Pulling back this inclusion to $\bar{U}$ via $\varphi : \bar{U} \to \bar{M}_h$ and by part i) of Lemma 3 one obtains

$$\theta_f^l(S^l T_{\bar{U}}(-\log S) \otimes E_f^{n,0}) \subset \varphi^* \theta_{f_{univ}}^l(S^l T_{\bar{M}_h}(-\log S) \otimes E^{n,0}) \subset \varphi^* K^{n-l,l} \subset K.$$

On the other hand, since $K^\perp$ is θ_f-invariant and $E_f^{n,0} \subset K^\perp$

$$\theta_f^i(S^i T_{\bar{U}}(-\log S) \otimes E^{n,0}) \subset K^\perp$$

for $0 \le i \le n$. For $i = l$ this inclusion together with the inclusion above implies that

$$\theta_f^l(S^l T_{\bar{U}}(-\log S) \otimes E^{n,0}) = 0$$

which contradicts to $\theta_f^l \neq 0$. The proof of Theorem 3 is complete. □

Remark 2. *In the proof of the above theorems on the rigidity, what we need is the existence of a polarized complex VHS over the moduli space such that the first Hodge bundle is a line bundle and the first Higgs map is an isomorphism. Such type VHS is called Calabi-Yau like VHS. In* [VZ03] *it is shown that the universal family of hypersurfaces in P^n of degree $\geq n+1$ carries a Calabi-Yau like VHS. Hence, all theorems here holds true for those subfamilies.*

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UCLA, Department of Mathematics, Los Angeles, CA 90095, and Center of Mathematical Sciences, Zheijiang University

UC, Department of Mathematics, Santa Cruz, CA 95064, Institute of Mathematics, Bulgarian Academy of Sciences

Harvard University, Department of Mathematics, Cambridge, Mass. 02138

University Mainz, Department of Mathematics, Mainz 55099