

**Nontrivial Azumaya noncommutative schemes, morphisms therefrom,  
and their extension by the sheaf of algebras of differential operators:  
D-branes in a  $B$ -field background à la Polchinski-Grothendieck Ansatz**

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**Abstract**

In this continuation of [L-Y1], [L-L-S-Y], [L-Y2], and [L-Y3] (arXiv:0709.1515 [math.AG], arXiv:0809.2121 [math.AG], arXiv:0901.0342 [math.AG], arXiv:0907.0268 [math.AG]), we study D-branes in a target-space(-time) with a fixed  $B$ -field background  $(Y, \alpha_B)$  along the line of the Polchinski-Grothendieck Ansatz, explained in [L-Y1] and further extended in the current work. We focus first on the gauge-field-twist effect of  $B$ -field to the Chan-Paton module on D-branes. Basic properties of the moduli space of D-branes, as morphisms from Azumaya schemes with a twisted fundamental module to  $(Y, \alpha_B)$ , are given. For holomorphic D-strings, we prove a valuation-criterion property of this moduli space. The setting is then extended to take into account also the deformation-quantization-type noncommutative geometry effect of  $B$ -field to both the D-brane world-volume and the superstring target-space(-time)  $Y$ . This brings the notion of twisted  $\mathcal{D}$ -modules that are realizable as twisted locally-free coherent modules with a flat connection into the study. We use this to realize the notion of both the classical and the quantum spectral covers as morphisms from Azumaya schemes with a fundamental module (with a flat connection in the latter case) in a very special situation. The 3rd theme (subtitled “Sharp vs. Polchinski-Grothendieck”) of Sec. 2.2 is to be read with the work [Sh3] (arXiv:hep-th/0102197) of Sharp while Sec. 5.2 (subtitled less appropriately “Dijkgraaf-Holland-Sulkowski-Vafa vs. Polchinski-Grothendieck”) is to be read with the related sections in [D-H-S-V] (arXiv:0709.4446 [hep-th]) and [D-H-S] (arXiv:0810.4157 [hep-th]) of Dijkgraaf, Hollands, Sulkowski, and Vafa.

**Key words:** Azumaya scheme, Azumaya structure,  $B$ -field, D-brane, D-string,  $\mathcal{D}$ -module, deformation quantization, gerbe, Higgs/spectral pair, moduli stack, morphism, Polchinski-Grothendieck Ansatz, quantum spectral curve, sheaf of algebras of differential operators, twisted sheaf, valuation criterion.

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*Chien-Hao Liu dedicates this work to his mother, Mrs. Pin-Fen Lin Liu,  
and siblings (resp. siblings-in-law)  
Hsiu-Chuan Liu (Yi-Shou Chao), Chien-Ying Liu (Tracy Tse Chui),  
Alice Hsiu-Hsiang Liu (Tom Cheng-Wei Chang), Hsiu-Luan Liu,  
and to the memory of his father, Mr. Han-Tu Liu<sup>†</sup> [<sup>†</sup>deceased 1986],  
at a special family moment.*

## 0. Introduction and outline.

Since the work [Pol1] of Polchinski, D-branes have become a central object of study in superstring theory. It has also motivated numerous related works on the mathematical side. (Cf. References of [L-Y1], [L-L-S-Y], [L-Y2], [L-Y3] for a brief list relevant to the project.)

### Azumaya structure on D-branes as an origin of D-brany phenomena.

The emergence of an Azumaya structure on a D-brane world-volume follows directly from a comparison of (1) *the behavior of the open-string-induced field on the D-brane that governs its deformation* and (2) *Grothendieck's contravariant equivalence of algebras and local geometries* ([Pol2: vol. I, Sec. 8.7], [L-Y1: Sec. 2], and Sec. 2.1).<sup>1</sup> If we turn the history around to take such a structure as a fundamental definition of D-branes, and consider morphisms from such objects to a string-target-space(-time), then we see that basic D-brane phenomena, e.g., Higgsing/un-Higgsing of gauge field theory on D-brane world-volume, deformation and resolution of a singular Calabi-Yau space via a D-brane probe, can be reproduced; [L-Y1], [L-L-S-Y], [L-Y2], and [L-Y3]. This is an indication that Azumaya structure is fundamentally/solidly carved into a D-brane world-volume as part of its substantial building structures. This gives an Azumaya origin of many D-brany phenomena.

### $B$ -field and its effect on fields and geometry in string theory.

A  $B$ -field on a space-time  $Y$  is a connection on a gerbe  $\mathcal{Y}$  over  $Y$ . It can be presented as a Čech 0-cochain  $(B_i)_i$  of local 2-forms  $B_i$  with respect to a cover  $\mathcal{U} = \{U_i\}_i$  on  $Y$  such that on  $U_i \cap U_j$ ,  $B_i - B_j = d\Lambda_{ij}$  for some real 1-forms  $\Lambda_{ij}$  that satisfies  $\Lambda_{ij} + \Lambda_{jk} + \Lambda_{ki} = -\sqrt{-1}d \log \alpha_{ijk}$  on  $U_i \cap U_j \cap U_k$ , where  $(\alpha_{ijk})_{ijk}$  is a Čech 2-cocycle of  $U(1)$ -valued functions on  $Y$ . In the algebro-geometric language,  $(\alpha_{ijk})_{ijk}$  is given by a presentation of an equivalence class  $\alpha_B \in \check{C}_{\acute{e}t}^2(Y, \mathcal{O}_Y^*)$  of étale Čech 2-cocycles with values in  $\mathcal{O}_Y^*$ . Through its coupling to the open-string current on an open-string world-sheet with boundary on a D-brane world-volume  $X \subset Y$ , a background  $B$ -field on  $Y$  induces<sup>2</sup> a twist to the gauge field  $A$  on the Chan-Paton vector bundle  $E$  on  $X$  that renders  $E$  itself a twisted vector bundle with the twist specified by  $\alpha_B|_X \in \check{C}_{\acute{e}t}^2(X, \mathcal{O}_X^*)$ . (Cf. [Al], [Br], [Ch], [F-W], [Hi2], [Ka], and [Wi1].) Furthermore, the 2-point functions on the open-string world-sheet with boundary on  $X$  indicate that the D-brane world-volume is deformed to a deformation-quantization type noncommutative geometry in a way that is governed by the  $B$ -field (and the space-time metric). (Cf. [C-H1, C-H2], [C-K], [Schmo], and [S-W].)

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<sup>1</sup>Azumaya structure on D-brane world-volume has already been brought to string theorists' attention in late 1990s, for example, from the effect of the background  $B$ -field on the open-string target-space-time, as explained in [Ka: Sec. 1.2] of Kapustin. However, it should be noted that *the emergence of Azumaya structures on a D-brane world-volume is at a more fundamental level than this. It is purely an open-string-induced effect that is enforced on a coincident D-brane world-volume whether or not there is a supersymmetry in the field theory on the D-brane world-volume (or on the open-string world-sheet with boundary on this brane-world-volume, or on the space-time) or a B-field background on the space-time.* Rather, the latter extra SUSY requirement or  $B$ -field data comes to constraint the class of Azumaya structures that can occur on the D-brane world-volume. For example, in complex geometry language we are discussing holomorphic Azumaya algebra over a complex manifold. This comes from a supersymmetry constraint.  $B$ -field will then select further a class of such Azumaya algebras, cf. [Ka] and Sec. 2.

<sup>2</sup>Unfamiliar mathematicians are highly recommended to read [Zw] of Zwiebach for a very down-to-earth explanation of this.

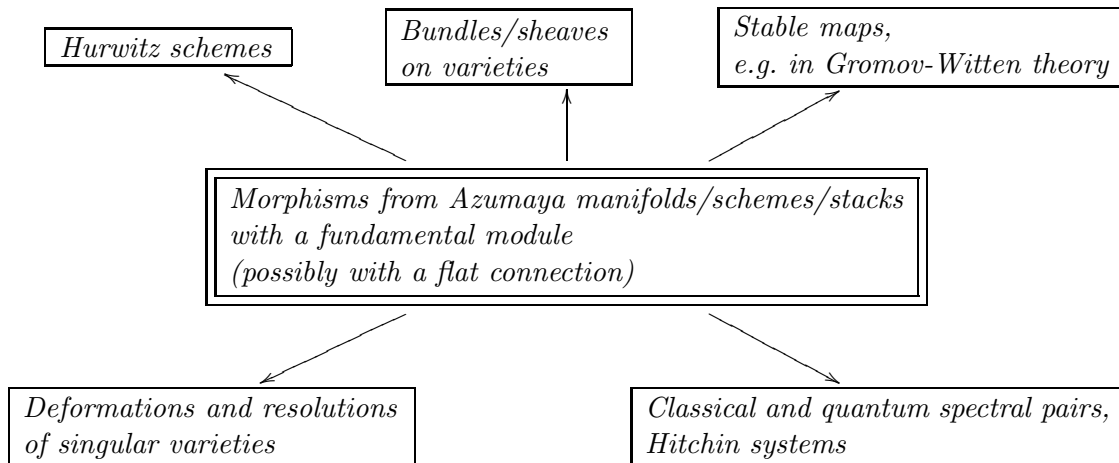
*Remark 0.1. [open-string world-sheet anomaly].* There is a further effect on D-branes that arises from the global world-sheet anomaly on the open-string world-sheet with the boundary on the D-brane world-volume ([F-W]). This anomaly effect is ignored in the current work. See ibidem, [C-K-S], [Ka], and [K-S] for more discussions.

**D-brane as a master object in superstring theory vs. morphism from Azumaya schemes with a fundamental module as a master object in geometry.**

In this continuation of [L-Y1], [L-L-S-Y], [L-Y2], and [L-Y3], we study D-branes in a fixed  $B$ -field background  $(Y, \alpha_B)$  along the line of the Polchinski-Grothendieck Ansatz, explained in [L-Y1] and further extended in the current work. We focus first on the twist effect of  $B$ -field to the Chan-Paton module on D-branes. Basic properties of the moduli space of D-branes, as morphisms from Azumaya schemes with a twisted fundamental module to  $(Y, \alpha_B)$ , are given. For holomorphic D-strings, we prove a valuation-criterion property of this moduli space. The setting is then extended to take into account also the deformation-quantization effect of  $B$ -field to both the D-brane world-volume and the target-space  $Y$ . This brings the notion of twisted  $\mathcal{D}$ -modules that are realizable as twisted locally-free coherent modules with a flat connection into the study. We use this to realize the notion of both the classical and the quantum spectral covers as morphisms from Azumaya schemes with a fundamental module (with a flat connection in the latter case) in a very special situation. The 3rd theme (subtitled “*Sharp vs. Polchinski-Grothendieck*”) of Sec. 2.2 is to be read with the work [Sh3] of Sharp while Sec. 5.2 (subtitled less appropriately “*Dijkgraaf-Holland-Sulkowski-Vafa vs. Polchinski-Grothendieck*”) is to be read with the related sections in [D-H-S-V] and [D-H-S] of Dijkgraaf, Hollands, Sulkowski, and Vafa. From this, we see once again:

- *the master nature of morphisms from Azumaya schemes with a fundamental module in geometry in parallel to the master nature of D-branes in superstring theory.*

This would be highly surprising/un-anticipated on the mathematics side if not because of the Polchinski-Grothendieck Ansatz, which realizes morphisms from Azumaya manifolds/schemes/stacks with a fundamental module as the *lowest level* presentation of D-branes, and superstring theory dictates the master nature of such an object. Together with [L-L-S-Y] (D(2)), [L-Y2] (D(3)), and [L-Y3] (D(4)), the following diagram of unity emerges:



It is anticipated that this is only a part of a to-be-understood complete diagram of unity in geometry in view of the ubiquity of D-branes in superstring theory.

**Convention.** Standard notations, terminology, operations, facts in (1) physics aspects of D-branes ; (2) algebraic geometry and stacks can be found respectively in (1) [Pol2], [Jo], and [Zw]; (2) [Ha] and [L-MB].

- All schemes are Noetherian over  $\mathbb{C}$  unless otherwise noted..
- *B-field* (in the sense of quantum field theory) vs. *base* scheme  $B$  vs. D-branes of *type B*.
- D-branes for *Dirichlet*-branes vs.  $\mathcal{D}$ -modules for modules of the sheaf  $\mathcal{D}$  of algebras of *differential* operators.
- The word “*twist/twisting*” has two different meanings: (1) in the sense of twisted sheaves as a presentation of sheaves on gerbes and (2) the operation of tensoring by (usually) a (twisted or ordinary in the sense of (1)) line bundle.

## Outline.

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# 1 Gerbes, twisted sheaves, and Azumaya algebras over a scheme.

To fix terminology and notations, essential definitions of gerbes and twisted sheaves are given in this section. Readers are referred to [Br: Chap. 5], [Că: Chap. 1], [Lie1: Chap. 2], [Mi: Chap. IV] and also [Ch], [Gir], [Hi2] for further details and to, e.g., [Sh2] and [C-K-S] to get a glimpse of gerbes and twisted sheaves in string theory. We will assume that the schemes  $X$  and  $Y$  in the following discussions are quasi-projective<sup>3</sup> over  $\mathbb{C}$ .

## 1.1 Gerbes and twisted sheaves over a scheme.

### Gerbes over a scheme and coherent sheaves thereupon.

Let  $X$  be a (Noetherian) scheme (over  $\mathbb{C}$ ). Given a stack  $\mathcal{S}$  over the category  $\mathit{Scheme}/X$  of schemes over  $X$ , we will denote the groupoid  $\mathcal{S}(U)$  assigned by  $\mathcal{S}$  to a  $(U \rightarrow X) \in \mathit{Scheme}/X$  also by  $\mathcal{S}_U$ . An element  $s \in \mathcal{S}_U$  will be called a *section* of  $\mathcal{S}$  over  $U$ .  $s$  defines a morphism  $s : U \rightarrow \mathcal{S}$ , and conversely. Thus, we will denote  $s \in \mathcal{S}_U$  and  $s : U \rightarrow \mathcal{S}$  interchangeably. We will equip  $\mathit{Scheme}/X$  with the fppf topology unless otherwise noted. This induces a topology on a stack over  $\mathit{Scheme}/X$ .

**Definition 1.1.1. [gerbe over  $X$ ].** A *gerbe over  $X$*  is a stack  $\mathcal{X}$  over  $\mathit{Scheme}/X$  that has the following two properties:

- (1) *étale local existence of a section*: For any  $U \rightarrow X$ , there exists an étale cover  $U' \rightarrow U$  of  $U$  such that  $\mathcal{X}_{U'}$  is nonempty.
- (2) *sections étale locally isomorphic*: For any  $U \rightarrow X$  and  $s_1, s_2 \in \mathcal{X}_U$ , there exists an étale cover  $p : U' \rightarrow U$  of  $U$  such that  $p^*s_1 \simeq p^*s_2$  in  $\mathcal{X}_{U'}$ .

We will denote a gerbe  $\mathcal{X}$  over  $X$  also by  $\mathcal{X}/X$  to manifest the *underlying scheme*<sup>4</sup>  $X$ , particularly when there are different underlying schemes involved in the discussion.

**Lemma/Definition 1.1.2. [sheaf of automorphism groups and its right action].** Given a gerbe  $\mathcal{X}/X$ , the assignment  $s \in \mathcal{X}_U \mapsto \text{Aut}(s) = \text{Mor}(s, s) \subset \text{Mor}(\mathcal{X}_U)$  is a sheaf  $\mathcal{A}(\mathcal{X})$  on the stack  $\mathcal{X}$ . We will call  $\mathcal{A}(\mathcal{X})$  the *sheaf of automorphism groups* on  $\mathcal{X}$ . Let  $\mathcal{F}$  be a sheaf on  $\mathcal{X}$ . Then the operation of pulling-back by automorphisms defines a natural right group action<sup>5</sup>  $\mu : \mathcal{F} \times \mathcal{A}(\mathcal{X}) \rightarrow \mathcal{F}$  of  $\mathcal{A}(\mathcal{X})$  on  $\mathcal{F}$ .

Let  $\mathcal{X}$  be a gerbe over  $X$  and  $\mathcal{O}_X^*$  be the sheaf of invertible elements of  $\mathcal{O}_X$ . Denote the pull-back of  $\mathcal{O}_X^*$  to  $\mathcal{X}$  via the structure morphism  $\mathcal{X} \rightarrow X$  also by  $\mathcal{O}_X^*$ . This is the sheaf on  $\mathcal{X}$  that assigns to each  $s \in \mathcal{X}_U$  the (multiplicative) abelian group  $\mathcal{O}_U^*(U)$ .

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<sup>3</sup>This technical assumption is imposed to render the étale Čech cohomology  $\check{H}_{\text{ét}}^*(X, \mathcal{F})$  and the étale cohomology  $H_{\text{ét}}^*(X, \mathcal{F})$  identical for  $\mathcal{F}$  a sheaf on  $X_{\text{ét}}$ .

<sup>4</sup>The reason we call  $X$  the *underlying scheme* of the gerbe  $\mathcal{X}$  is that when  $\mathcal{X}$  arises as the moduli stack of a moduli problem of a class of objects, the scheme  $X$  becomes the coarse moduli space of the moduli problem.  $X$  parameterizes all the  $\mathbb{C}$ -points of  $\mathcal{X}$  while  $\mathcal{X}$  encodes in addition the data of automorphisms of the objects these  $\mathbb{C}$ -points represent. See Definition/Lemma 1.2.3 for an example.

<sup>5</sup>Explicitly, let  $s \in \mathcal{X}_U$ ,  $f \in \mathcal{F}(s)$ , and  $h \in \text{Aut}(s)$ . Then,  $\mu(f, h) = h^*f \in \mathcal{F}(s)$ .

**Definition 1.1.3. [gerbe with band  $\mathcal{O}_X^*$ ].** A gerbe over  $X$  with band  $\mathcal{O}_X^*$  is a gerbe  $\mathcal{X}/X$  with an isomorphism  $\mathcal{O}_X^* \xrightarrow{\sim} \mathcal{A}(\mathcal{X})$ .

**Lemma 1.1.4. [gerbe as algebraic stack].** ([Lie1: Lemma 2.2.1.1].) Let  $\mathcal{X}$  be a gerbe over  $X$  with band  $\mathcal{O}_X^*$ . Then  $\mathcal{X}$  is an (Noetherian) algebraic stack<sup>6</sup> over Scheme/ $X$ .

An atlas for such an  $\mathcal{X}/X$  is given by an étale cover  $U \rightarrow X$  with the property that for any  $x \in X$ , there is a connected component  $U_x$  of  $U$  that gives an étale neighborhood of  $x \in X$  such that  $\mathcal{X}_{U_x}$  is nonempty.

The notion of a (Cartesian) coherent sheaf  $\mathcal{F}$  on a gerbe  $\mathcal{X}/X$  with band  $\mathcal{O}_X^*$  is then defined as that for algebraic stacks, given in [L-MB]. Its (stacky) support  $Supp \mathcal{F}$ , defined by the annihilator ideal sheaf  $Ker(\mathcal{O}_{\mathcal{X}} \rightarrow \mathcal{E}nd_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}))$  of  $\mathcal{F}$ , is a closed substack of  $\mathcal{X}$ .

### Twisted sheaves à la Căldăraru.

Given an étale cover  $p : U^{(0)} := \coprod_{i \in I} U_i \rightarrow X$  of  $X$ , we will adopt the following notations:

$$\begin{aligned} \cdot U_{ij} &:= U_i \times_X U_j =: U_i \cap U_j, \quad U_{ijk} := U_i \times_X U_j \times_X U_k =: U_i \cap U_j \cap U_k; \\ \cdot \dots &\rightrightarrows U^{(2)} := U \times_X U \times_X U \\ &\rightrightarrows U^{(1)} := U \times_X U \xrightarrow{p_1, p_2} U^{(0)} \xrightarrow{p} X \end{aligned}$$

are the projection maps from fibered products as indicated; the restriction of these projections maps to respectively  $U_{ijk}$  and  $U_{ij}$  will be denoted the same;

- the pull-back of an  $\mathcal{O}_{U_i}$ -module  $\mathcal{F}_i$  on  $U_i$  to  $U_{ij}$ ,  $U_{ji}$ ,  $U_{ijk}$ ,  $\dots$  via compositions of these projection maps will be denoted by  $\mathcal{F}_i|_{U_{ij}}$ ,  $\mathcal{F}_i|_{U_{ji}}$ ,  $\mathcal{F}_i|_{U_{ijk}}$ ,  $\dots$  respectively.

**Definition 1.1.5. [ $\alpha$ -twisted  $\mathcal{O}_X$ -module on an étale cover of  $X$ ].** ([Că: Definition 1.2.1].) Let  $\alpha \in \check{C}_{\acute{e}t}^2(X, \mathcal{O}_X^*)$  be a Čech 2-cocycle in the étale topology of  $X$ . An  $\alpha$ -twisted  $\mathcal{O}_X$ -module on an étale cover of  $X$  is a triple

$$\mathcal{F} = (\{U_i\}_{i \in I}, \{\mathcal{F}_i\}_{i \in I}, \{\phi_{ij}\}_{i, j \in I})$$

that consists of the following data

- an étale cover  $p : U := \coprod_{i \in I} U_i \rightarrow X$  of  $X$  on which  $\alpha$  can be represented as a 2-cocycle:
$$\alpha = \{ \alpha_{ijk} : \alpha_{ijk} \in \Gamma(U_{ijk}, \mathcal{O}_X^*) \text{ with } \alpha_{jkl} \alpha_{ikl}^{-1} \alpha_{ijl} \alpha_{ijk}^{-1} = 1 \text{ on } U_{ijkl} \text{ for all } i, j, k, l \in I \},$$
such a cover will be called an  $\alpha$ -admissible étale cover of  $X$ ;
- $\mathcal{F}_i$  is a sheaf of  $\mathcal{O}_{U_i}$ -modules on  $U_i$ ;
- (gluing data)  $\phi_{ij} : \mathcal{F}_i|_{U_{ij}} \rightarrow \mathcal{F}_j|_{U_{ij}}$  is an  $\mathcal{O}_{U_{ij}}$ -module isomorphism that satisfies
  - $\phi_{ii}$  is the identity map for all  $i \in I$ ;

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<sup>6</sup>I.e. Artin stack.

- (2)  $\phi_{ij} = \phi_{ji}^{-1}$  for all  $i, j \in I$ ;  
(3) (*twisted cocycle condition*)  $\phi_{ki} \circ \phi_{jk} \circ \phi_{ij}$  is the multiplication by  $\alpha_{ijk}$  on  $\mathcal{F}_i|_{U_{ijk}}$ .

$\mathcal{F}$  is said to be *coherent* (resp. *quasi-coherent*, *locally free*) if  $\mathcal{F}_i$  is a coherent (resp. quasi-coherent, locally free)  $\mathcal{O}_{U_i}$ -module for all  $i \in I$ . A *homomorphism*

$$h : \mathcal{F} = (\{U_i\}_{i \in I}, \{\mathcal{F}_i\}_{i \in I}, \{\phi_{ij}\}_{i, j \in I}) \longrightarrow \mathcal{F}' = (\{U_i\}_{i \in I}, \{\mathcal{F}'_i\}_{i \in I}, \{\phi'_{ij}\}_{i, j \in I})$$

between  $\alpha$ -twisted  $\mathcal{O}_X$ -modules on the étale cover  $p$  of  $X$  is a collection  $\{h_i : \mathcal{F}_i \rightarrow \mathcal{F}'_i\}_{i \in I}$ , where  $h_i$  is an  $\mathcal{O}_{U_i}$ -module homomorphism, such that  $\phi'_{ij} \circ h_i = h_j \circ \phi_{ij}$  for all  $i, j \in I$ . In particular,  $h$  is an *isomorphism* if all  $h_i$  are isomorphisms. Denote by  $\mathcal{M}od(X, \alpha, p)$  the category of  $\alpha$ -twisted  $\mathcal{O}_X$ -modules on the étale cover  $p : U^{(0)} \rightarrow X$  of  $X$ .

Given an  $\alpha$ -twisted sheaf  $\mathcal{F}$  on the étale cover  $p : U \rightarrow X$  of  $X$ . let  $p' : U' \rightarrow X$  be an étale refinement of  $p : U \rightarrow X$ . Then  $\alpha$  can be represented also on  $p' : U' \rightarrow X$  and  $\mathcal{F}$  on  $p$  defines an  $\alpha$ -twisted  $\mathcal{O}_X$ -module  $\mathcal{F}'$  on  $p'$  via the pull-back under the built-in étale cover  $U' \rightarrow U$  of  $U$ . This defines an equivalence of categories:

$$\mathcal{M}od(X, \alpha, p) \longrightarrow \mathcal{M}od(X, \alpha, p').$$

([Că: Lemma 1.2.3, Lemma 1.2.4, Remark 1.2.5].)

**Definition 1.1.6.** [ $\alpha$ -twisted  $\mathcal{O}_X$ -module on  $X$ ]. An  $\alpha$ -twisted  $\mathcal{O}_X$ -module on  $X$  is an equivalence class  $[\mathcal{F}]$  of  $\alpha$ -twisted  $\mathcal{O}_X$ -modules  $\mathcal{F}$  on étale covers of  $X$ , where the equivalence relation is generated by étale refinements and descents by étale covers of  $X$  on which  $\alpha$  can be represented. An  $\mathcal{F}' \in [\mathcal{F}]$  is called a *representative* of the  $\alpha$ -twisted  $\mathcal{O}_X$ -module  $[\mathcal{F}]$ . For simplicity of terminology, we will also call  $\mathcal{F}'$  directly an  $\alpha$ -twisted  $\mathcal{O}_X$ -module on  $X$ .

Cf. [Că: Corollary 1.2.6 and Remark 1.2.7].

Standard notions of  $\mathcal{O}_X$ -modules, in particular

- the *scheme-theoretic support*  $\text{Supp } \mathcal{E}$ ,
- the *dimension*  $\dim \mathcal{E}$ , and
- *flatness over a base*  $S$

of an  $\alpha$ -twisted sheaf  $\mathcal{E}$  on  $X$  (or on  $X/S$ ) are defined via a(ny) presentation of  $\mathcal{E}$  on an  $\alpha$ -admissible étale cover  $U \rightarrow X$ .

Standard operations on  $\mathcal{O}_\bullet$ -modules apply to twisted  $\mathcal{O}_\bullet$ -modules on appropriate admissible étale covers by applying the operations component by component over the cover. These operations apply then to twisted  $\mathcal{O}_X$ -modules as well: They are defined on representatives of twisted sheaves in such a way that they pass to each other by pull-back and descent under étale refinements of admissible étale covers. In particular:

**Proposition 1.1.7.** [**basic operations on twisted sheaves**]. ([Că: Proposition 1.2.10].) (1) Let  $\mathcal{F}$  and  $\mathcal{G}$  be an  $\alpha$ -twisted and a  $\beta$ -twisted  $\mathcal{O}_X$ -module respectively, where  $\alpha, \beta \in \check{C}_{\text{ét}}^2(X, \mathcal{O}_X^*)$ . Then  $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$  is an  $\alpha\beta$ -twisted  $\mathcal{O}_X$ -module and  $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$  is an  $\alpha^{-1}\beta$ -twisted  $\mathcal{O}_X$ -module. In particular, if  $\mathcal{F}$  and  $\mathcal{G}$  are both  $\alpha$ -twisted  $\mathcal{O}_X$ -modules, then  $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$  descends to an (ordinary/untwisted)  $\mathcal{O}_X$ -module, still denoted by  $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ , on  $X$ .



(2) Let  $f : X \rightarrow Y$  be a morphism of schemes/ $\mathbb{C}$  and  $\alpha \in \check{C}_{\text{ét}}^2(Y, \mathcal{O}_Y^*)$ . Note that an  $\alpha$ -admissible étale cover of  $Y$  pulls back to an  $f^*\alpha$ -admissible étale cover of  $X$  under  $f$ , through which the pull-back and push-forward of a related twisted sheaf can be defined. If  $\mathcal{F}$  is an  $\alpha$ -twisted  $\mathcal{O}_Y$ -module on  $Y$ , then  $f^*\mathcal{F}$  is an  $f^*\alpha$ -twisted  $\mathcal{O}_X$ -module on  $X$ . If  $\mathcal{F}$  is an  $f^*\alpha$ -twisted  $\mathcal{O}_X$ -module on  $X$ , then  $f_*\mathcal{F}$  is an  $\alpha$ -twisted  $\mathcal{O}_Y$ -module on  $Y$ .

## 1.2 (General) Azumaya algebras over a scheme.

**Definition 1.2.1.** [Azumaya algebra of rank  $r$  over  $X$ ]. An Azumaya algebra  $\mathcal{A}$  of rank<sup>7</sup>  $r$  over a scheme  $X/\mathbb{C}$  is a locally free  $\mathcal{O}_X$ -algebra such that its fiber  $\mathcal{A} \otimes_{\mathcal{O}_X} k(x)$  at each closed point  $x \in X$  is isomorphic to  $\text{End}(\mathbb{C}^r)$  (= the  $r \times r$  matrix algebra  $M_r(\mathbb{C})$  over  $\mathbb{C}$ ) as  $\mathbb{C}$ -algebras.

**Proposition 1.2.2.** [local trivialization of Azumaya algebra]. ([Mi: IV, Proposition 2.1].) Let  $\mathcal{A}$  be a sheaf of  $\mathcal{O}_X$ -algebras on  $X$ . The following statements are equivalent:

- (1)  $\mathcal{A}$  is an Azumaya algebra of rank  $r$  on  $X$ ;
- (2) there is an étale cover  $U \rightarrow X$  such that  $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_U \simeq \text{End}_{\mathcal{O}_U}(\mathcal{O}_U^{\oplus r}) (= M_r(\mathcal{O}_U))$ ;
- (3) there is a flat cover  $U \rightarrow X$  such that  $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_U \simeq \text{End}_{\mathcal{O}_U}(\mathcal{O}_U^{\oplus r})$ .

**Definition/Lemma 1.2.3.** [gerbe associated to Azumaya algebra]. Given an Azumaya algebra  $\mathcal{A}$  over  $X$ , the stack of trivializations of fibers of  $\mathcal{A}$ , defined by the assignment

$U \in \text{Scheme}_X \mapsto$  the category with

- *objects:*  
the pairs  $(\mathcal{E}, a)$ , where  $\mathcal{E}$  is a locally free  $\mathcal{O}_U$ -module and  $a$  is an isomorphism  $\text{End}_{\mathcal{O}_U}(\mathcal{E}) \xrightarrow{\sim} \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_U$  of  $\mathcal{O}_U$ -algebras,
- *morphism:*  
a morphism  $(\mathcal{E}_1, a_1) \rightarrow (\mathcal{E}_2, a_2)$  is an isomorphism  $h : \mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}_2$  such that the induced  $h_* : \text{End}_{\mathcal{O}_U}(\mathcal{E}_1) \xrightarrow{\sim} \text{End}_{\mathcal{O}_U}(\mathcal{E}_2)$  satisfies  $a_2 \circ h_* = a_1$ .

is a gerbe, denoted by  $\mathcal{X}_{\mathcal{A}}$ , over  $X$  with band  $\mathcal{O}_X^*$ . We will call it the *gerbe over  $X$  associated to  $\mathcal{A}$* . The collection of objects  $(\mathcal{E}, a)$  define a locally-free coherent  $\mathcal{O}_{\mathcal{X}_{\mathcal{A}}}$ -module  $\mathcal{F}$  on  $\mathcal{X}_{\mathcal{A}}$ . We will call it the *tautological fundamental module* on  $\mathcal{X}_{\mathcal{A}}$ . The pull-back of  $\mathcal{A}$  on  $X$  to  $\mathcal{X}_{\mathcal{A}}$  is canonically isomorphic to  $\text{End}_{\mathcal{O}_{\mathcal{X}_{\mathcal{A}}}}(\mathcal{F})$ . We will call this the *tautological Azumaya algebra* over  $\mathcal{X}_{\mathcal{A}}$ .

Given an  $\alpha$ -twisted locally-free  $\mathcal{O}_X$ -module  $\mathcal{E}$  of rank  $r$ , it follows from Proposition 1.1.7 that  $\mathcal{A} := \text{End}_{\mathcal{O}_X}(\mathcal{E})$  is an Azumaya algebra over  $X$ . Let  $U \rightarrow X$  be an  $\alpha$ -admissible étale cover of  $X$ . Then a presentation of  $\mathcal{E}$  on  $U$  corresponds to a morphism  $s : U \rightarrow \mathcal{X}_{\mathcal{A}}$ . In terms of this,  $\mathcal{E} = s^*\mathcal{F}$ , where  $\mathcal{F}$  is the tautological fundamental module on  $\mathcal{X}_{\mathcal{A}}$ .

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<sup>7</sup>Note that there are two conventions in the literature: rank as an  $\mathcal{O}_X$ -module vs. rank as an  $\mathcal{O}_X$ -algebra. Here we take the latter convention.

**Definition/Lemma 1.2.4. [Brauer group  $Br(\bullet)$ ].** Two Azumaya algebras,  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , are said to be (*stably*) *equivalent* if there exist locally-free coherent  $\mathcal{O}_X$ -modules,  $\mathcal{E}_1$  and  $\mathcal{E}_2$ , such that  $\mathcal{A}_1 \otimes_{\mathcal{O}_X} \text{End}_{\mathcal{O}_X}(\mathcal{E}_1) \simeq \mathcal{A}_2 \otimes_{\mathcal{O}_X} \text{End}_{\mathcal{O}_X}(\mathcal{E}_2)$  as  $\mathcal{O}_X$ -algebras. Denote the equivalence class of  $\mathcal{A}$  by  $[\mathcal{A}]$ . Then, *the set of equivalence classes of Azumaya algebras over  $X$  form an abelian group, in notation  $Br(X)$ , under  $[\mathcal{A}_1] \cdot [\mathcal{A}_2] := [\mathcal{A}_1 \otimes_{\mathcal{O}_X} \mathcal{A}_2]$ , identity =  $[\mathcal{O}_X]$ , and  $[\mathcal{A}]^{-1} := [\mathcal{A}^\circ]$ , where  $\mathcal{A}^\circ$  is the opposite algebra<sup>8</sup> of  $\mathcal{A}$ .  $Br(X)$  is called the *Brauer group* of  $X$ .*

The set of isomorphism classes of Azumaya algebras of rank  $r$  over  $X$  is given by the étale cohomology group  $H_{\acute{e}t}^1(X, PGL_r(\mathbb{C}))$ . The exact sequence

$$1 \longrightarrow \mathcal{O}_X^* \longrightarrow GL_r(\mathcal{O}_X) \longrightarrow PGL_r(\mathcal{O}_X) \longrightarrow 1$$

of sheaves on  $X_{\acute{e}t}$  defines an exact sequence of pointed-sets

$$\cdots \longrightarrow H_{\acute{e}t}^1(X, \mathcal{O}_X^*) \longrightarrow H_{\acute{e}t}^1(X, GL_r(\mathcal{O}_X)) \longrightarrow H_{\acute{e}t}^1(X, PGL_r(\mathcal{O}_X)) \xrightarrow{d} H_{\acute{e}t}^2(X, \mathcal{O}_X^*).$$

**Theorem 1.2.5.** [ $Br(X) \subset H_{\acute{e}t}^2(X, \mathcal{O}_X^*)$ ]. ([Mi: IV, Theorem 2.5].) *The connecting homomorphism  $H_{\acute{e}t}^1(X, PGL_r(\mathcal{O}_X)) \xrightarrow{d} H_{\acute{e}t}^2(X, \mathcal{O}_X^*)$  induces a canonical injective group-homomorphism  $Br(X) \hookrightarrow H_{\acute{e}t}^2(X, \mathcal{O}_X^*)$ .*

## 2 Azumaya geometry and D-branes à la Polchinski-Grothendieck Ansatz revisited: the twist from a $B$ -field background.

### 2.1 Polchinski-Grothendieck Ansatz revisited with the étale topology.

#### Polchinski-Grothendieck Ansatz: Azumaya-type noncommutativity on D-branes.

Recall how the Polchinski-Grothendieck Ansatz for D-branes is reached in [L-Y1: Sec. 2.2].<sup>9</sup> Consider a D-brane (or a D-brane world-volume) in a space(-time) that is geometrically realized as an embedded submanifold  $f : Z \hookrightarrow M$  in an open-string target space(-time)  $M$ . The boundary of open-string world-sheets are mapped to  $f(Z)$  in  $M$ . Through this, open strings induce additional structures on  $Z$ , including a Chan-Paton bundle on  $Z$  that supports the gauge field created from the vibrations of open-strings with end-points on  $f(Z)$ . Let  $\xi := (\xi^a)_a$  be local coordinates on  $Z$  and  $X := (X^a; X^\mu)_{a,\mu}$  be local coordinates on  $M$  such that the embedding  $f : Z \hookrightarrow M$  is locally expressed as

$$X = X(\xi) = (X^a(\xi); X^\mu(\xi))_{a,\mu} = (\xi^a, X^\mu(\xi))_{a,\mu};$$

i.e.,  $X^a$ 's (resp.  $X^\mu$ 's) are local coordinates along (resp. transverse to)  $f(Z)$  in  $M$ . This choice of local coordinates removes redundant degrees of freedom of the map  $f$ , and  $X^\mu = X^\mu(\xi)$  can be regarded as (scalar) fields on  $Z$  that collectively describes the positions/shapes/fluctuations

<sup>8</sup>I.e. the Azumaya algebra with the same  $\mathcal{O}_X$ -module  $\mathcal{A}$  but with the reversed product  $\cdot'$  defined by  $a_1 \cdot' a_2 := a_2 \cdot a_1$  in the Azumaya algebra  $\mathcal{A}$ .

<sup>9</sup>Readers are referred to Polchinski [Pol2: vol. I, Sec. 8.7] and [L-Y1: Sec. 2.2] for more thorough discussions and comparison. In this theme, we use as close notation to Polchinski as possible for a direct comparison. For all other parts of the work, we will use the more standard  $X \rightarrow Y$  to represent a D-brane (or D-brane world-volume)  $X$  that is mapped to a target-space(-time)  $Y$ .

of  $Z$  in  $M$  locally. Here, both  $\xi^a$ 's,  $X^a$ 's, and  $X^\mu$ 's are  $\mathbb{R}$ -valued. The gauge field on  $Z$  is locally given by the connection 1-form  $A = \sum_a A_a(\xi)d\xi^a$  of a  $U(1)$ -bundle on  $Z$ .

When  $r$ -many such D-branes  $Z$  are coincident, from the associated massless spectrum of (oriented) open strings with both end-points on  $f(Z)$  one can draw the conclusion that

- (1) The gauge field  $A = \sum_a A_a(\xi)d\xi^a$  on  $Z$  is enhanced to  $u(r)$ -valued.
- (2) Each scalar field  $X^\mu(\xi)$  on  $Z$  is also enhanced to matrix-valued.

Property (1) says that there is now a  $U(r)$ -bundle on  $Z$ . To understand Property (2), one has two perspectives:

- (A1) [*coordinate tuple as point*] A tuple  $(\xi^a)_a$  (resp.  $(X^a; X^\mu)_{a,\mu}$ ) represents a point on the world-volume  $Z$  of the D-brane (resp. on the target space-time  $M$ ).
- (A2) [*local coordinates as generating set of local functions*] Each local coordinate  $\xi^a$  of  $Z$  (resp.  $X^a, X^\mu$  of  $M$ ) is a local function on  $Z$  (resp. on  $M$ ) and the local coordinates  $\xi^a$ 's (resp.  $X^a$ 's and  $X^\mu$ 's) together form a generating set of local functions on the world-volume  $Z$  of the D-brane (resp. on the target space-time  $M$ ).

While Aspect (A1) leads one to the anticipation of a noncommutative space from a noncommutatization of the target space-time  $M$  when probed by coincident D-branes, Aspect (A2) of Grothendieck leads one to a different/dual conclusion: a noncommutative space from a noncommutatization of the world-volume  $Z$  of coincident D-branes, as follows.

Denote by  $\mathbb{R}\langle \xi^a \rangle_a$  (resp.  $\mathbb{R}\langle X^a; X^\mu \rangle_{a,\mu}$ ) the local function ring on the associated local coordinate chart on  $Z$  (resp. on  $M$ ). Then the embedding  $f : Z \rightarrow M$ , locally expressed as  $X = X(\xi) = (X^a(\xi); X^\mu(\xi))_{a,\mu} = (\xi^a; X^\mu(\xi))$ , is locally contravariantly equivalent to a ring-homomorphism

$$f^\# : \mathbb{R}\langle X^a; X^\mu \rangle_{a,\mu} \longrightarrow \mathbb{R}\langle \xi^a \rangle_a, \quad \text{generated by } X^a \longmapsto \xi^a, X^\mu \longmapsto X^\mu(\xi).$$

When  $r$ -many such D-branes are coincident,  $X^\mu(\xi)$ 's become  $M_r(\mathbb{C})$ -valued. Thus,  $f^\#$  is promoted to a new local ring-homomorphism:

$$\hat{f}^\# : \mathbb{R}\langle X^a; X^\mu \rangle_{a,\mu} \longrightarrow M_r(\mathbb{C}\langle \xi^a \rangle_a), \quad \text{generated by } X^a \longmapsto \xi^a \cdot \mathbf{1}, X^\mu \longmapsto X^\mu(\xi).$$

Under Grothendieck's contravariant local equivalence of function rings and spaces,  $\hat{f}^\#$  is equivalent to saying that we have now a map  $\hat{f} : Z_{\text{noncommutative}} \rightarrow M$ . Thus, the D-brane-related noncommutativity in Polchinski's treatise re-read from the viewpoint of Grothendieck implies the following ansatz:

**Polchinski-Grothendieck Ansatz [D-brane: noncommutativity].** *The world-volume of a D-brane carries a noncommutative structure locally associated to a function ring of the form  $M_r(R)$ , where  $r \in \mathbb{Z}_{\geq 1}$  and  $M_r(R)$  is the  $r \times r$  matrix ring over  $R$ .*

Note that  $R$  can be either commutative or noncommutative, cf. Remark 5.1.9.

Cf. [L-L-S-Y: FIGURE 1-2].

## Polchinski-Grothendieck Ansatz with the étale topology adaptation.

In the *smooth differential-geometric setting* of Polchinski, the word “*locally*” in the ansatz means “locally in the  $C^\infty$ -topology”. This can be generalized to adapt the ansatz to fit various settings: “locally” in the *analytic* (resp. *Zariski*) *topology* for the *holomorphic* (resp. *algebraic-geometric*) *setting*. These are enough to study D-branes in a space(-time) without a background  $B$ -field. The Azumaya structure sheaf  $\mathcal{O}_Z^{Az}$  that encodes the matrix-type noncommutative structure on  $Z$  in these cases is then of the form  $\text{End}_{\mathcal{O}_Z}(\mathcal{E})$  with  $\mathcal{E}$  the Chan-Paton module, a locally free  $\mathcal{O}_Z$ -module of rank  $r$  on which the Azumaya  $\mathcal{O}_Z$ -algebra  $\mathcal{O}_Z^{Az}$  acts tautologically as a simple/fundamental (left)  $\mathcal{O}_Z^{Az}$ -module. This is the case studied in the previous part [L-Y1], [L-L-S-Y], [L-Y2], [L-Y3] of the project.  $\mathcal{O}_Z^{Az}$  in this case corresponds to the zero-class in the Brauer group  $Br(Z)$  of  $Z$ .

On pure mathematical ground, one can further adapt the ansatz for  $Z$  equipped with any Grothendieck topology/site. On string-theoretic ground, as recalled in Sec. 0, when a background  $B$ -field  $B$  on  $M$  is turned on, the Chan-Paton module  $\mathcal{E}$  on  $Z$  becomes twisted and is no longer an honest sheaf of  $\mathcal{O}_Z$ -modules on  $Z$ . The interpretation of “*locally*” in the ansatz in the sense of (small) *étale topology* on  $Z$  becomes forced upon us. This corresponds to the case when the Azumaya structure sheaf  $\mathcal{O}_Z^{Az}$  on  $Z$  represents a non-zero class in  $Br(Z)$ .

We now turn to the algebraic-geometric aspect of D-branes, following *Polchinski-Grothendieck Ansatz* but *with this étale topology adaptation* on the D-brane or D-brane world-volume.

## 2.2 D-branes in a $B$ -field background as morphisms from Azumaya schemes with a twisted fundamental module.

Recall the twisting effect of  $B$ -field from string theory highlighted in Introduction. We now study D-branes in a  $B$ -field background along the line of the adapted Polchinski-Grothendieck Ansatz. Except the additional involvement of étale topology, twisted sheaves, and the matching of twists, the setting/study in [L-Y1: Sec. 1] and [L-L-S-Y: Sec. 2] carries over directly to the current situation.

### D-branes in a $B$ -field background.

**Definition 2.2.1.** [Azumaya scheme with a fundamental module]. An *Azumaya scheme with a fundamental module in class  $\alpha$*  is a tuple

$$(X^{Az}, \mathcal{E}) := (X, \mathcal{O}_X^{Az} = \text{End}_{\mathcal{O}_X}(\mathcal{E}), \mathcal{E}),$$

where  $X = (X, \mathcal{O}_X)$  is a (Noetherian) scheme (over  $\mathbb{C}$ ),  $\alpha \in \check{C}_{\text{ét}}^2(X, \mathcal{O}_X^*)$  represents a class  $[\alpha] \in Br(X) \subset H_{\text{ét}}^2(X, \mathcal{O}_X^*)$ , and  $\mathcal{E}$  is a locally-free coherent  $\alpha$ -twisted  $\mathcal{O}_X$ -module on  $X$ . A *commutative surrogate* of  $(X^{Az}, \mathcal{E})$  is a scheme  $X_{\mathcal{A}} := \mathbf{Spec} \mathcal{A}$ , where  $\mathcal{O}_X \subset \mathcal{A} \subset \text{End}_{\mathcal{O}_X}(\mathcal{E})$  is an inclusion sequence of commutative  $\mathcal{O}_X$ -subalgebras of  $\text{End}_{\mathcal{O}_X}(\mathcal{E})$ . Let  $\pi : X_{\mathcal{A}} \rightarrow X$  be the built-in dominant finite morphism. Then  $\mathcal{E}$  is tautologically a  $\pi^*\alpha$ -twisted  $\mathcal{O}_{X_{\mathcal{A}}}$ -module on  $X_{\mathcal{A}}$ , denoted by  $\mathcal{O}_{X_{\mathcal{A}}}$ . We say that  $X^{Az}$  is an *Azumaya scheme of rank  $r$*  if  $\mathcal{E}$  has rank  $r$  and that it is a *nontrivial* (resp. *trivial*) *Azumaya scheme* if  $[\alpha] \neq 0$  (resp.  $[\alpha] = 0$ ).

*Remark 2.2.2.* [the twisted sheaf  $\mathcal{O}_{X_A} \mathcal{E}$  on  $X_A$ ]. Explicitly, let  $p : U \rightarrow X$  be an  $\alpha$ -admissible étale cover of  $X$  on which the  $\alpha$ -twisted  $\mathcal{O}_X$ -module  $\mathcal{E}$  is represented as an ordinary  $\mathcal{O}_U$ -module and  $p_A : U_A := U \times_X X_A \rightarrow X_A$  be the pull-back étale cover of  $X_A$ :

$$\begin{array}{ccc} U_A & \xrightarrow{p_A} & X_A \\ \tilde{\pi} \downarrow & & \downarrow \pi \\ U & \xrightarrow{p} & X \end{array} \quad .$$

Then  $\mathcal{O}_U \subset p^* \mathcal{A} \subset p^* \mathcal{O}_X^{Az} = \text{End}_{\mathcal{O}_U}(\mathcal{E}) = \mathcal{O}_U^{Az}$  is a sequence of  $\mathcal{O}_U$ -subalgebra inclusions and  $\tilde{\pi} : U_A \rightarrow U$  is nothing but the commutative surrogate  $\mathbf{Spec}(p^* \mathcal{A}) \rightarrow U$  of  $U^{Az} = (U, \mathcal{O}_U^{Az})$ . In terms of this,  $\mathcal{E}$  is canonically a  $p^* \mathcal{A} = \mathcal{O}_{U_A}$ -module, which defines then the  $\pi^* \alpha$ -twisted  $\mathcal{O}_{X_A}$ -module  $\mathcal{O}_{X_A} \mathcal{E}$  on  $X_A$  in the above definition.

*Remark 2.2.3.* [noncommutative-space viewpoint]. It is instructive to think of  $\mathcal{O}_X^{Az}$  as the structure sheaf  $\mathcal{O}$  of a noncommutative space  $\text{Space } \mathcal{O}_X^{Az}$ ,  $\text{Space } \mathcal{O}_U^{Az}$  as an étale cover of  $\text{Space } \mathcal{O}_X^{Az}$ ,  $\mathcal{E}$  as an ordinary sheaf of (left) modules on  $\text{Space } \mathcal{O}_U^{Az}$  that defines a (left) twisted  $\mathcal{O}$ -module on  $\text{Space } \mathcal{O}_X^{Az}$ , and that there is a dominant morphism  $\text{Space } \mathcal{O}_X^{Az} \rightarrow X_A$ , under which  $\mathcal{E}$  on  $\text{Space } \mathcal{O}_X^{Az}$  is pushed forward to the  $\pi^* \alpha$ -twisted  $\mathcal{O}_{X_A}$ -module  $\mathcal{O}_{X_A} \mathcal{E}$  on  $X_A$ .

Let  $Y$  be a (commutative, Noetherian) scheme/ $\mathbb{C}$  and  $\alpha_B \in \check{C}_{\text{ét}}^2(Y, \mathcal{O}_Y^*)$  be the étale Čech cocycle associated to a fixed  $B$ -field on  $Y$ . Then a proto-typical definition of D-branes (of B/holomorphic type) in  $(Y, B)$  is given by morphisms  $\varphi : (X^{Az}, \mathcal{E}) \rightarrow (Y, \alpha_B)$ , defined as follows.

**Definition 2.2.4.** [morphism from Azumaya scheme with fundamental module to  $B$ -field background]. Let  $(X^{Az}, \mathcal{E})$  be an Azumaya scheme with a fundamental module in the class  $\alpha \in \check{C}_{\text{ét}}^2(X, \mathcal{O}_X^*)$ . Then, a *morphism* from  $(X^{Az}, \mathcal{E})$  to  $(Y, \alpha_B)$ , in notation  $\varphi : (X^{Az}, \mathcal{E}) \rightarrow (Y, \alpha_B)$ , is a pair

$$(\mathcal{O}_X \subset \mathcal{A}_\varphi \subset \mathcal{O}_X^{Az}, f_\varphi : X_\varphi := \mathbf{Spec} \mathcal{A}_\varphi \rightarrow Y),$$

where

- $\mathcal{A}_\varphi$  is a commutative  $\mathcal{O}_X$ -subalgebra of  $\mathcal{O}_X^{Az}$ ,
- $f_\varphi : X_\varphi \rightarrow Y$  is a morphism of (commutative) schemes,

that satisfies the following properties:

- (1) (*minimal property of  $X_\varphi$* ) there exists no  $\mathcal{O}_X$ -subalgebra  $\mathcal{O}_X \subset \mathcal{A}' \subset \mathcal{A}_\varphi$  such that  $f_\varphi$  factors as the composition of morphisms  $X_\varphi \rightarrow \mathbf{Spec} \mathcal{A}' \rightarrow Y$ ;
- (2) (*matching of twists on  $X_\varphi$* ) let  $\pi_\varphi : X_\varphi \rightarrow X$  be the built-in finite dominant morphism, then  $\pi_\varphi^* \alpha = f_\varphi^* \alpha_B$  in  $\check{C}_{\text{ét}}^2(X_\varphi, \mathcal{O}_{X_\varphi}^*)$ .

$X_\varphi$  is called the *surrogate of  $X^{Az}$  associated to  $\varphi$* . Condition (2) implies that  $\varphi_* \mathcal{E} := f_{\varphi*}(\mathcal{O}_{X_\varphi} \mathcal{E})$  is an  $\alpha_B$ -twisted  $\mathcal{O}_Y$ -module on  $Y$ , supported on  $\text{Im}(\varphi) := \varphi(X^{Az}) := f_\varphi(X_\varphi)$ , where the last is the usual scheme-theoretic image of  $X_\varphi$  under  $f_\varphi$ .<sup>10</sup>

Given two morphisms  $\varphi_1 : (X_1^{Az}, \mathcal{E}_1) \rightarrow (Y, \alpha_B)$  and  $\varphi_2 : (X_2^{Az}, \mathcal{E}_2) \rightarrow (Y, \alpha_B)$ , a *morphism*  $\varphi_1 \rightarrow \varphi_2$  from  $\varphi_1$  to  $\varphi_2$  is a pair  $(h, \tilde{h})$ , where

<sup>10</sup>In other words, a morphism from  $(X^{Az}, \mathcal{E})$  to  $(Y, \alpha_B)$  is a usual morphism  $\varphi : X^{Az} \rightarrow Y$  from the (possibly nontrivial) Azumaya scheme  $X^{Az}$  to  $Y$  subject to the twist-matching Condition (2) so that  $\varphi_* \mathcal{E}$  remains a twisted sheaf in a way that is compatible with the  $B$ -field background on  $Y$ .

- $h : X_1 \rightarrow X_2$  is an isomorphism of schemes with  $h^*\alpha_2 = \alpha_1$ , where  $\alpha_i$  is the underlying class of  $\mathcal{E}_i$  in  $\check{C}_{\acute{e}t}^2(X_i, \mathcal{O}_{X_i}^*)$ ;
- $\tilde{h} : \mathcal{E}_1 \xrightarrow{\sim} h^*\mathcal{E}_2$  be an isomorphism of twisted sheaves on  $X_1$  that satisfies
  - $\tilde{h} : \mathcal{A}_{\varphi_1} \xrightarrow{\sim} h^*\mathcal{A}_{\varphi_2}$ ,
  - the following diagram commutes

$$\begin{array}{ccc}
X_{\varphi_2} & & \\
\tilde{h} \downarrow & \searrow^{f_{\varphi_2}} & \\
X_{\varphi_1} & \xrightarrow{f_{\varphi_1}} & Y .
\end{array}$$

Here, we denote both of the induced isomorphisms,  $\mathcal{O}_{X_1}^{Az} \xrightarrow{\sim} h^*\mathcal{O}_{X_2}^{Az}$  and  $\mathcal{A}_{\varphi_1} \xrightarrow{\sim} h^*\mathcal{A}_{\varphi_2}$ , of  $\mathcal{O}_{X_1}$ -algebras still by  $\tilde{h}$  and  $\hat{h} : X_{\varphi_2} \xrightarrow{\sim} X_{\varphi_1}$  is the scheme-isomorphism associated to  $\tilde{h} : \mathcal{A}_{\varphi_1} \xrightarrow{\sim} h^*\mathcal{A}_{\varphi_2}$ .

This defines the category  $Morphism_{Azf}(Y, \alpha_B)$  of morphisms from Azumaya schemes with a fundamental module to  $(Y, \alpha_B)$ .

**Definition 2.2.5. [D-brane and Chan-Paton module].** Following the previous Definition,  $\varphi(X^{Az})$  is called the *image D-brane* on  $(Y, \alpha_B)$  and  $\varphi_*\mathcal{E}$  the *Chan-Paton module* on the image D-brane. Similarly, for *image D-brane world-volume* if  $X$  is served as a (Wicked-rotated) D-brane world-volume.

*Remark 2.2.6. [fundamental vs. solitonic D-brane].* The setting here treats D-branes in string theory more as a fundamental/soft extended object. For *solitonic/hard* D-brane in space-time, one may require in addition that  $f_\varphi : X_\varphi \rightarrow Y$  be an embedding.

### Azumaya without Azumaya and morphisms without morphisms.

Similar to the case of trivial Azumaya curves studied in [L-L-S-Y: Sec. 2], Definition 2.2.4 has an equivalent version in terms of twisted sheaves on the related product space as follows.

Given a morphism  $\varphi : (X^{Az}, \mathcal{E}) \rightarrow (Y, \alpha_B)$  as in Definition 2.2.4, the minimal property of the surrogate  $X_\varphi$  of  $X^{Az}$  associated to  $\varphi$  implies that  $(\pi_\varphi, f_\varphi) : X_\varphi \rightarrow X \times Y$  embeds  $X_\varphi$  in  $X \times Y$  as a subscheme  $\Gamma_\varphi$ . Let  $pr_1 : X \times Y \rightarrow X$  and  $pr_2 : X \times Y \rightarrow Y$  be the projection maps. Then the  $\pi_\varphi^*\alpha$ -twisted  $\mathcal{O}_{X_\varphi}$ -module  $\mathcal{O}_{X_\varphi}\mathcal{E}$  on  $X_\varphi$  is pushed forward to a  $pr_1^*\alpha$ -twisted  $\mathcal{O}_{X \times Y}$ -module  $\tilde{\mathcal{E}}$  on  $X \times Y$  that is supported on  $\Gamma_\varphi$ . The matching condition of twists on  $\varphi$  says that  $pr_1^*\alpha = pr_2^*\alpha_B$  on  $\Gamma_\varphi$ . By construction,  $\tilde{\mathcal{E}}$  on  $X \times Y$  is flat over  $X$  with relative length  $r$ .

Conversely, given a  $(\alpha, \alpha_B) \in \check{C}_{\acute{e}t}^2(X, \mathcal{O}_X^*) \times \check{C}_{\acute{e}t}^2(Y, \mathcal{O}_Y^*)$  as before and a coherent  $pr_1^*\alpha$ -twisted  $\mathcal{O}_{X \times Y}$ -module  $\tilde{\mathcal{E}}$  on  $X \times Y$  that satisfies the following two conditions:

- $\tilde{\mathcal{E}}$  is flat over  $X$  with relative length  $r$ ;
- $pr_1^*\alpha = pr_2^*\alpha_B$  on  $Supp \tilde{\mathcal{E}} =: \Gamma$ .

Then,  $\mathcal{E} := pr_{1*}\tilde{\mathcal{E}}$  is an  $\alpha$ -twisted  $\mathcal{O}_X$ -module on  $X$ . This defines an Azumaya scheme with a fundamental module, i.e.  $(X, \mathcal{O}_X^{Az} := \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}), \mathcal{E})$ , over  $X$  in the class  $\alpha$ . The defining  $\mathcal{O}_\Gamma$ -algebra homomorphism  $\mathcal{O}_\Gamma \rightarrow \mathcal{E}nd_{\mathcal{O}_\Gamma}(\tilde{\mathcal{E}})$  realizes  $\mathcal{O}_\Gamma$  as an  $\mathcal{O}_X$ -algebra  $\mathcal{A}$  that fits into  $\mathcal{O}_X \subset \mathcal{A} \subset \mathcal{O}_X^{Az}$  canonically. By construction,  $X_{\mathcal{A}} := \mathbf{Spec} \mathcal{A} \simeq \Gamma$  canonically and the restriction  $X \xleftarrow{pr_1} \Gamma \xrightarrow{pr_2} Y$  of the projection maps defines morphisms  $X \xleftarrow{\pi} X_{\mathcal{A}} \xrightarrow{f} Y$  that satisfies both Condition (1) (minimal property) and Condition (2) (matching of twists) in Definition 2.2.4.  $f$  thus defines a morphism  $\varphi : (X^{Az}, \mathcal{E}) \rightarrow (Y, \alpha_B)$  with  $X_\varphi = X_{\mathcal{A}}$  and  $f_\varphi = f$ .

Let  $Coh_{product^0}(Y, \alpha_B)$  be the category with objects coherent twisted modules  $\tilde{\mathcal{E}}$  on a product  $X \times Y$  that is flat over  $X$  with relative dimension 0 and satisfies Condition (b) above. A *morphism*  $\tilde{\mathcal{E}}_1 \rightarrow \tilde{\mathcal{E}}_2$  from  $\tilde{\mathcal{E}}_1$  on  $X_1 \times Y$  to  $\tilde{\mathcal{E}}_2$  on  $X_2 \times Y$  is a pair  $(h, \tilde{h})$  where

- $h : X_1 \rightarrow X_2$  is an isomorphism of schemes with  $h^*\alpha_2 = \alpha_1$ , where  $\alpha_i$  is the underlying class in  $\check{C}_{\acute{e}t}^2(X_i, \mathcal{O}_{X_i}^*)$  in question;
- denote the induced isomorphism  $X_1 \times Y \xrightarrow{\sim} X_2 \times Y$  also by  $h$ , then  $\tilde{h} : \tilde{\mathcal{E}}_1 \xrightarrow{\sim} h^*\tilde{\mathcal{E}}_2$  is an isomorphism of  $\alpha_1$ -twisted  $\mathcal{O}_{X_1}$ -modules on  $X_1$ .

The discussion above defines two functors

$$Morphism_{Azf}(Y, \alpha_B) \begin{array}{c} \xleftarrow{F} \\ \xrightarrow{G} \end{array} Coh_{product^0}(Y, \alpha_B).$$

**Lemma 2.2.7. [Azumaya without Azumaya, morphisms without morphisms].**  $(F, G)$  defines an equivalence of categories  $Morphism_{Azf}(Y, \alpha_B)$  and  $Coh_{product^0}(Y, \alpha_B)$ .

**The description in terms of morphisms from Azumaya gerbes with a fundamental module to a target gerbe.**<sup>11</sup>

Given an Azumaya algebra  $\mathcal{A} = \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$  over  $X$ , where  $\mathcal{E}$  is an  $\alpha$ -twisted locally free  $\mathcal{O}_X$ -module defined on an  $\alpha$ -admissible étale cover  $p : U \rightarrow X$  of  $X$ , recall the  $\mathcal{O}_X^*$ -gerbe  $\mathcal{X}_{\mathcal{A}}$  over  $X$  and the tautological fundamental module  $\mathcal{F}$  on  $\mathcal{X}_{\mathcal{A}}$ . Then  $\mathcal{E}$  defines an atlas  $\check{p} : U \rightarrow \mathcal{X}_{\mathcal{A}}$  of the algebraic stack  $\mathcal{X}_{\mathcal{A}}$ , with  $\check{p}^*\mathcal{F} = \mathcal{E}$ . The pull-back  $\check{A}$  of  $\mathcal{A}$  to  $\mathcal{X}_{\mathcal{A}}$  is canonically isomorphic to  $\mathcal{O}_{\mathcal{X}_{\mathcal{A}}}^{Az} := \mathcal{E}nd_{\mathcal{O}_{\mathcal{X}_{\mathcal{A}}}}(\mathcal{F})$  on  $\mathcal{X}_{\mathcal{A}}$ . The latter defines the tautological trivial Azumaya structure on  $\mathcal{X}_{\mathcal{A}}$  with the tautological fundamental module  $\mathcal{F}$ .

- [notation] We will denote the gerbe  $\mathcal{X}_{\mathcal{A}}$  by  $\mathcal{X}$  and the Azumaya algebraic stack with a fundamental module  $(\mathcal{X}_{\mathcal{A}}, \mathcal{O}_{\mathcal{X}_{\mathcal{A}}}^{Az}, \mathcal{F})$  by  $(\mathcal{X}^{Az}, \mathcal{F})$  for simplicity in the following discussion.

The notion of *surrogates* of an Azumaya scheme generalizes directly to that of an Azumaya algebraic stack and the notion of *morphisms* from a trivial Azumaya scheme with a fundamental module generalizes directly - after the combination with the notion of morphisms of algebraic stacks - to that from a trivial Azumaya algebraic stack with a fundamental module.

<sup>11</sup>This theme is written with the work [Sh3] of Eric Sharpe, particularly [Sh3: Sec. 6.3 *D-brane “bundles”*] and [Sh3: Sec. 7 *Conclusions*] concerning *the equivalence of “turning on the B-field” and “compactifying on a generalized space (i.e. a gerbe or a sheaf on a gerbe)”*, also in mind and, hence, goes with a hidden subtitle: *Sharpe vs. Polchinski-Grothendieck*. Readers are highly recommended to read ibidem alongside. We thank him for comments on [L-Y1] and sharing with us his insights on various subtle issues in string theory in fall 2007.

In our case, let  $\varphi : (X^{\text{Az}}, \mathcal{E}) \rightarrow (Y, \alpha_B)$  be a morphism, specified by a pair

$$(\mathcal{O}_X \subset \mathcal{A}_\varphi \subset \mathcal{O}_X^{\text{Az}} := \text{End}_{\mathcal{O}_X}(\mathcal{E}), f_\varphi : X_\varphi := \mathbf{Spec} \mathcal{A}_\varphi \rightarrow Y).$$

Let  $\mathcal{X}_\varphi = \mathcal{X} \times_X X_\varphi$ . This is a gerbe over  $X_\varphi$  in the class  $\pi_\varphi^* \alpha$ , with an atlas

$$\check{p}_\varphi : U_\varphi := U_{\mathcal{A}_\varphi} := U \times_X X_\varphi \longrightarrow \mathcal{X}_\varphi$$

induced by  $p_\varphi := \pi_\varphi^* p : U_\varphi \rightarrow X_\varphi$ . The  $\mathcal{O}_X$ -subalgebra  $\mathcal{A}_\varphi$  of  $\mathcal{O}_X^{\text{Az}}$  induces an  $\mathcal{O}_X$ -subalgebra  $\check{\mathcal{A}}_\varphi$  of  $\mathcal{O}_X^{\text{Az}}$ . This defines a surrogate  $\mathcal{X}_{\check{\mathcal{A}}_\varphi}$  of  $\mathcal{X}^{\text{Az}}$  that is precisely  $\mathcal{X}_\varphi$ . Let  $\mathcal{Y} = \mathcal{Y}_{\alpha_B}$  be an  $\mathcal{O}_Y$ -gerbe over  $Y$  that represents  $\alpha_B$  and  $\check{q} : V \rightarrow \mathcal{Y}$  be an atlas of  $\mathcal{Y}$  with the following properties:

- (1) the underlying  $q : V \rightarrow Y$  gives an  $\alpha_B$ -admissible étale cover of  $Y$ ;
- (2)  $\underline{\text{Isom}}(\check{q}, \check{q}) := V \times_{\mathcal{Y}} V$  has a global section over  $V \times_Y V$ .

(Note that  $\mathcal{Y}$  is non-empty on each connected component of  $V$ .) Let  $p'_\varphi : U'_\varphi \rightarrow X_\varphi$  be an étale refinement of  $p_\varphi$  of  $X_\varphi$  so that:

- (1)  $p'_\varphi$  refines also the étale cover  $f_\varphi^* q : X_\varphi \times_Y V \rightarrow X_\varphi$  of  $X_\varphi$ ;
- (2)  $\underline{\text{Isom}}(\check{p}'_\varphi, \check{p}'_\varphi) := U'_\varphi \times_{\mathcal{X}_\varphi} U'_\varphi$  has a global section over  $U'_\varphi \times_{X_\varphi} U'_\varphi$ , where  $\check{p}'_\varphi : U'_\varphi \rightarrow \mathcal{X}_\varphi$  is the new atlas of  $\mathcal{X}_\varphi$  associated to the refinement  $U'_\varphi \rightarrow U_\varphi$ .

Then, one has the following diagram

$$\begin{array}{ccc} U'_\varphi & \xrightarrow{\hat{f}_\varphi} & V \\ p'_\varphi \downarrow & & \downarrow q \\ X_\varphi & \xrightarrow{f_\varphi} & Y, \end{array}$$

where  $\hat{f}_\varphi$  is the composition  $U'_\varphi \rightarrow X_\varphi \times_Y V \rightarrow V$ .

Fix a global section of the  $\mathcal{O}_{X_\varphi}^*$ -torsor  $U'_\varphi \times_{\mathcal{X}_\varphi} U'_\varphi$  over  $U'_\varphi \times_{X_\varphi} U'_\varphi$  and a global section of the  $\mathcal{O}_Y^*$ -torsor  $V \times_{\mathcal{Y}} V$  over  $V \times_Y V$ . This trivializes the  $\mathcal{O}_\bullet^*$ -torsors  $(U'_\varphi \times_{\mathcal{X}_\varphi} U'_\varphi)/(U'_\varphi \times_{X_\varphi} U'_\varphi)$  and  $(V \times_{\mathcal{Y}} V)/(V \times_Y V)$ , the  $\mathcal{O}_\bullet^* \times \mathcal{O}_\bullet^*$ -torsors  $(U'_\varphi \times_{\mathcal{X}_\varphi} U'_\varphi \times_{\mathcal{X}_\varphi} U'_\varphi)/(U'_\varphi \times_{X_\varphi} U'_\varphi \times_{X_\varphi} U'_\varphi)$  and  $(V \times_{\mathcal{Y}} V \times_{\mathcal{Y}} V)/(V \times_Y V \times_Y V)$ ,  $\dots$ , etc. and it follows that  $\hat{f}_\varphi$  lifts to a commutative diagram of multi-arrows:<sup>12</sup>

$$\begin{array}{ccc} \vdots & \xrightarrow{\hat{f}_\varphi^{(\bullet)}} & \vdots \\ \Downarrow & & \Downarrow \\ U'_\varphi \times_{\mathcal{X}_\varphi} U'_\varphi \times_{\mathcal{X}_\varphi} U'_\varphi & \xrightarrow{\hat{f}_\varphi^{(2)}} & V \times_{\mathcal{Y}} V \times_{\mathcal{Y}} V \\ \Downarrow & & \Downarrow \\ U'_\varphi \times_{\mathcal{X}_\varphi} U'_\varphi & \xrightarrow{\hat{f}_\varphi^{(1)}} & V \times_{\mathcal{Y}} V \\ \Downarrow & & \Downarrow \\ U'_\varphi & \xrightarrow{\hat{f}_\varphi} & V \\ p'_\varphi \downarrow & & \downarrow q \\ X_\varphi & \xrightarrow{f_\varphi} & Y \end{array}$$

<sup>12</sup>Here, it is understood that a commutative diagram applies only to a square in the tower with same-type projection maps for its vertical arrows. E.g.  $\hat{f}_\varphi^{(1)} \circ p'_{\varphi,13} = q_{13} \circ \hat{f}_\varphi^{(2)}$ .



that covers - indeed trivialized trivial torsor over - the  $\hat{f}_\varphi$ -induced tautological tower

$$\begin{array}{ccc}
 \vdots & \xrightarrow{\quad} & \vdots \\
 \Downarrow & & \Downarrow \\
 U'_\varphi \times_{X_\varphi} U'_\varphi \times_{X_\varphi} U'_\varphi & \xrightarrow{\quad} & V \times_Y V \times_Y V \\
 \Downarrow & & \Downarrow \\
 U'_\varphi \times_{X_\varphi} U'_\varphi & \xrightarrow{\quad} & V \times_Y V \\
 \Downarrow & & \Downarrow \\
 U'_\varphi & \xrightarrow{\quad \hat{f}_\varphi \quad} & V \\
 p'_\varphi \downarrow & & \downarrow q \\
 X_\varphi & \xrightarrow{\quad f_\varphi \quad} & Y .
 \end{array}$$

A standard decent-data argument implies then that:

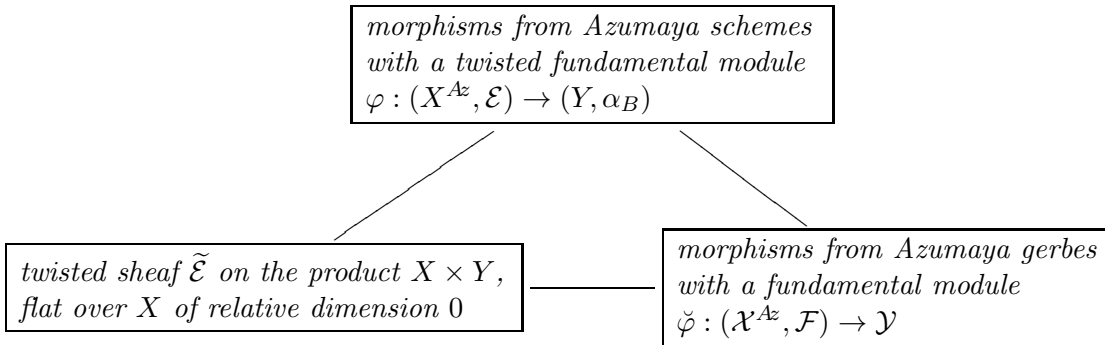
**Lemma 2.2.8. [presentation as morphism from Azumaya gerbe].**  $f_\varphi : X_\varphi \rightarrow Y$  induces a morphism  $\check{f}_\varphi : \mathcal{X}_\varphi \rightarrow \mathcal{Y}$  of  $\mathcal{O}_\bullet^*$ -gerbes via the induced scheme-morphism  $\hat{f}_\varphi$  on their atlases;  $\check{f}_\varphi$  is independent of the choices in the above discussion.

It follows that the morphism  $\varphi : (X^{Az}, \mathcal{E}) \rightarrow (Y, \alpha_B)$  can be presented also as a morphism  $\check{\varphi} : (\mathcal{X}^{Az}, \mathcal{F}) \rightarrow \mathcal{Y}$  from an Azumaya  $\mathcal{O}_X^*$ -gerbe with a fundamental module, specified by a pair

$$(\mathcal{O}_X \subset \check{\mathcal{A}}_\varphi \subset \mathcal{O}_X^{Az} := \text{End}_{\mathcal{O}_X}(\mathcal{F}), \check{f}_\varphi : \mathcal{X}_\varphi \rightarrow \mathcal{Y}).$$

The  $\alpha_B$ -twisted sheaf  $\varphi_*\mathcal{E}$  on  $Y$  presented on the  $\alpha_B$ -admissible étale cover  $q : V \rightarrow Y$  of  $Y$  is given then by  $\check{q}^*\check{f}_{\varphi,*}(\mathcal{O}_{X_\varphi}\mathcal{F})$ . Define  $\check{\varphi}_*\mathcal{F} := \check{f}_{\varphi,*}(\mathcal{O}_{X_\varphi}\mathcal{F})$  and let  $Z = \text{Im } \varphi \subset Y$ . Then,  $\check{\varphi}_*\mathcal{F}$  is supported on a substack of  $\mathcal{Y}$  that is a  $\mathcal{O}_Z^*$ -gerbe over  $Z$ .

In summary:



Cf. [L-L-S-Y: FIGURE 2-2-1].

### 3 The moduli stack of morphisms.

#### 3.1 Family of D-branes in a $B$ -field background, twisted Hilbert polynomials, and boundedness.

Some preparations toward the moduli problem of D-branes in a  $B$ -field background are given in this subsection.

##### Family of D-branes in a $B$ -field background.

The discussion in Sec. 2.2 applies also to a family.

**Definition 3.1.1. [family of D-branes in  $B$ -field background].** Let  $S$  be a base scheme/ $\mathbb{C}$ . An  $S$ -family of morphisms from Azumaya schemes with a fundamental module to  $(Y, \alpha_B)$  consists of the following data:

- a flat family  $X_S/S$  of schemes over  $S$ ;
- a twisted coherent locally-free  $\mathcal{O}_{X_S}$ -module  $\mathcal{E}_S$  on  $X_S/S$  of class  $\alpha_S \in \check{C}_{\acute{e}t}^2(X_S, \mathcal{O}_{X_S}^*)$ ;
- a morphism  $\varphi_S : (X_S^{\text{Az}}, \mathcal{E}_S) := (X_S, \mathcal{O}_{X_S}^{\text{Az}} := \text{End}_{\mathcal{O}_{X_S}}(\mathcal{E}_S), \mathcal{E}_S) \longrightarrow (Y, \alpha_B)$  as defined in Definition 2.2.4.

Let  $(\mathcal{O}_{X_S} \subset \mathcal{A}_{\varphi_S} \subset \mathcal{O}_{X_S}^{\text{Az}}, f_{\varphi_S} : X_{\varphi_S} := \mathbf{Spec} \mathcal{A}_{\varphi_S} \rightarrow Y)$  be the pair underlying  $\varphi_S$  and  $\pi_{\varphi_S} : X_{\varphi_S} \rightarrow X$  be the built-in morphism.

Let  $h : T \rightarrow S$  be a morphism of  $\mathbb{C}$ -schemes,  $X_T = h^*X_S := T \times_S X_S$  with the built-in  $\hat{h} : X_T \rightarrow X_S$  that lifts  $h$ ,  $\mathcal{E}_T = \hat{h}^*\mathcal{E}_S$  the pull-back  $\hat{h}^*\alpha_S (= \alpha_T)$ -twisted coherent locally-free  $\mathcal{O}_{X_T}$ -module, and  $(X_T^{\text{Az}}, \mathcal{E}_T) = (X_T, \mathcal{O}_{X_T}^{\text{Az}} := \text{End}_{\mathcal{O}_{X_T}}(\mathcal{E}_T), \mathcal{E}_T)$ . Then, the *pull-back*  $h^*\varphi_S$  of  $\varphi_S$  to  $T$  is the morphism  $\varphi_T : (X_T^{\text{Az}}, \mathcal{E}_T) \rightarrow (Y, \alpha_B)$  with the underlying pair

$$(\mathcal{O}_{X_T} \subset \mathcal{A}_{\varphi_T} \subset \mathcal{O}_{X_T}^{\text{Az}}, f_{\varphi_T} : X_{\varphi_T} := \mathbf{Spec} \mathcal{A}_{\varphi_T} \rightarrow Y),$$

where

- $\mathcal{A}_{\varphi_T}$  is the image  $\mathcal{O}_{X_T}$ -subalgebra of  $\hat{h}^*\mathcal{A}_{\varphi_S} \rightarrow \hat{h}^*\mathcal{O}_{X_S}^{\text{Az}} = \mathcal{O}_{X_T}^{\text{Az}}$ ,
- $f_{\varphi_T}$  is the composition of the morphisms  $\mathbf{Spec} \mathcal{A}_{\varphi_T} \hookrightarrow h^*X_{\varphi_S} = \mathbf{Spec}(\hat{h}^*\mathcal{A}_S) \xrightarrow{h^*f_{\varphi_S}} Y$ .

Note that the minimal property of  $X_{\varphi_T}$  is automatic and the matching  $\pi_{\varphi_T}^*\alpha_T = f_{\varphi_T}^*\alpha_B$  follows from the matching  $\pi_{\varphi_S}^*\alpha_S = f_{\varphi_S}^*\alpha_B$  on  $X_{\varphi_S}$  and the built-in inclusion  $X_{\varphi_T} \hookrightarrow h^*X_{\varphi_S}$ . In particular, let  $\iota_s : s \rightarrow S$  be a closed point of  $S$ . Then the *fiber*  $\varphi_s$  of  $\varphi_S$  over  $s$  is defined to be the morphism  $\iota_s^*\varphi_S : (X_s^{\text{Az}}, \mathcal{E}_s) \rightarrow (Y, \alpha_B)$ .

*Remark 3.1.2. [surrogates in family].* Note that in general  $X_{\varphi_s}$ ,  $s \in S$ , do not form a flat family of schemes over  $S$ . See [L-L-S-Y: Remark 2.1.16] for more comments.

It follows from Lemma 2.2.7 that:

**Lemma 3.1.3. [equivalent description via Azumaya-w/o-Azumaya-'n'-morphisms-w/o-morphisms].** *An  $S$ -family of morphisms from Azumaya schemes with a fundamental module to  $(Y, \alpha_B)$  can be described equivalently by the following data:*

- a flat family  $X_S/S$  of schemes over  $S$ ;
- a  $pr_1^*\alpha_S$ -twisted coherent  $\mathcal{O}_{X_S \times Y}$ -module  $\tilde{\mathcal{E}}_S$  on  $X_S \times Y$  that satisfies:
  - (a)  $\tilde{\mathcal{E}}_S$  on  $(X_S \times Y)/X_S$  is flat over  $X_S$  of relative dimension 0 and of fixed relative length;
  - (b)  $pr_1^*\alpha_S = pr_2^*\alpha_B$  on  $\text{Supp } \tilde{\mathcal{E}}_S$ .

Here,  $pr_1 : X_S \times Y \rightarrow X_S$  and  $pr_2 : X_S \times Y \rightarrow Y$  are the projection maps.

### Twisted Hilbert polynomials of a morphism.

**Definition/Lemma 3.1.4. [twisted Hilbert polynomial of sheaf].** (Cf. [Yo: Sec. 2.1].) *Let  $(W, \mathcal{O}_W(1))$  be a projective scheme with a fixed  $\alpha$ -twisted coherent locally-free  $\mathcal{O}_W$ -module  $\mathcal{G}$  in the class  $\alpha \in \check{C}_{\acute{e}t}(W, \mathcal{O}_W^*)$  with  $[\alpha] \in Br(W)$ . Let  $\mathcal{F}$  be an  $\alpha$ -twisted coherent  $\mathcal{O}_W$ -module. Then the function*

$$P_{\mathcal{G}, \mathcal{F}} : m \longmapsto \chi((\mathcal{F} \otimes_{\mathcal{O}_W} \mathcal{G}^\vee)(m)), \quad m \in \mathbb{Z},$$

*is a polynomial in  $m$  of degree  $\dim \mathcal{F}$ . We shall call it the  $\mathcal{G}$ -twisted Hilbert polynomial of  $\mathcal{F}$  on  $(W, \mathcal{O}_W(1))$ .*

As tensoring by a twisted locally free sheaf leaves the flatness property of a twisted sheaf intact, one has the following proposition:

**Proposition 3.1.5. [invariance under flat deformation].** (Cf. [H-L] and [Ha].) *Let  $S$  be a base scheme,  $W_S \rightarrow S$  be a projective morphism with a relative ample line bundle  $\mathcal{O}_{W_S/S}(1)$  on  $W_S$ ,  $\mathcal{G}_S$  be an  $\alpha_S$ -twisted coherent locally-free  $\mathcal{O}_{W_S}$ -module in the class  $\alpha_S \in \check{C}_{\acute{e}t}(W_S, \mathcal{O}_{W_S}^*)$  with  $[\alpha_S] \in Br(W_S)$ . Let  $\mathcal{F}_S$  be an  $\alpha_S$ -twisted coherent  $\mathcal{O}_{W_S}$ -module. Denote by  $\mathcal{G}_s$  (resp.  $\mathcal{F}_s$ ) the restriction of  $\mathcal{G}_S$  (resp.  $\mathcal{F}_S$ ) to the fiber  $W_s$  of  $W_S/S$  at a closed point  $s \in S$ . Then, if  $\mathcal{F}_S$  is flat over  $S$ , the twisted Hilbert polynomial  $P_{\mathcal{G}_s, \mathcal{F}_s}$  is locally constant as a function of  $s \in S$ . When  $S$  is reduced, the converse also holds.*

Lemma 3.1.3 motivates then the following definition:

**Definition 3.1.6. [twisted Hilbert polynomial of morphism].** Assume that  $Y$  is projective with a very ample line bundle  $\mathcal{O}_Y(1)$ . Fix an  $\alpha$ -twisted locally free coherent  $\mathcal{O}_X$ -module  $\mathcal{G}$  on  $(X, \mathcal{O}_X(1))$  in the class  $\alpha$  with  $[\alpha] \in Br(X)$ . Let  $\varphi : (X^{\text{Az}}, \mathcal{E}) \rightarrow (Y, \alpha_B)$  be a morphism in the class  $\alpha$  and  $\tilde{\mathcal{E}}$  be the  $pr_1^*\alpha$ -twisted sheaf on  $X \times Y$  that represents  $\varphi$ . (Here,  $pr_1 : X \times Y \rightarrow X$  and  $pr_2 : X \times Y \rightarrow Y$  are the projection maps.) Then, the  $\mathcal{G}$ -twisted Hilbert polynomial  $P_{\mathcal{G}, \varphi}$  of  $\varphi$  is defined to be  $P_{pr_1^*\mathcal{G}, \tilde{\mathcal{E}}}$ , where  $\mathcal{O}_{X \times Y}(1)$  is taken to be  $\mathcal{O}_X(1) \boxtimes \mathcal{O}_Y(1) := pr_1^*\mathcal{O}_X(1) \otimes pr_2^*\mathcal{O}_Y(1)$ .

**Lemma 3.1.7. [invariance in a family].** *Fix an  $\alpha_S$ -twisted locally free coherent  $\mathcal{O}_{X_S}$ -module  $\mathcal{G}_S$  on  $(X_S, \mathcal{O}_{X_S/S}(1))$  in the class  $\alpha_S$  with  $[\alpha_S] \in Br(X_S)$ . Let  $\varphi_S : (X_S^{\text{Az}}, \mathcal{E}_S) \rightarrow (Y, \alpha_B)$  be an  $S$ -family of morphisms from Azumaya schemes with a fundamental module to a projective  $(Y, \alpha_B)$  in the class  $\alpha_S$ . Then, the  $\mathcal{G}_S$ -twisted Hilbert polynomial  $P_{\mathcal{G}_S, \varphi_S}$  of  $\varphi_S$  is locally constant as a function of  $s \in S$ .*

This is a consequence of Proposition 3.1.5. Thus, a twisted Hilbert polynomial of a morphism gives the notation of *combinatorial type* of a morphism  $\varphi : (X^{\text{Az}}, \mathcal{E}) \rightarrow (Y, \alpha_B)$ .

### Boundedness.

Recall first the following theorem from Căldăraru [Că]:

**Theorem 3.1.8. [equivalence of  $\text{Mod}(W, \alpha)$  and  $\text{Mod-}\mathcal{A}$ ].** ([Că: Theorem 1.3.7].) *Let  $\alpha \in \check{C}_{\text{ét}}^2(W, \mathcal{O}_W^*)$  with  $[\alpha] \in \text{Br}(W)$ ,  $\mathcal{A}$  be an Azumaya algebra on  $W$  with  $[\mathcal{A}] = [\alpha]$ ,  $\text{Mod}(W, \alpha)$  be the category of  $\alpha$ -twisted  $\mathcal{O}_W$ -modules, and  $\text{Mod-}\mathcal{A}$  be the category of right  $\mathcal{A}$ -modules on  $W$ . Let  $\mathcal{G}$  be an  $\alpha$ -twisted coherent locally-free  $\mathcal{O}_W$ -module on  $W$  such that  $\mathcal{A} \simeq \text{End}_{\mathcal{O}_W}(\mathcal{G})$  ( $\simeq \mathcal{G} \otimes_{\mathcal{O}_W} \mathcal{G}^\vee$  canonically) as  $\mathcal{O}_W$ -algebras. Note that  $\mathcal{G}$  is naturally a left  $\mathcal{A}$ -module. Then, the following pair of functors defines an equivalence of categories:*

$$\begin{array}{ccc} \text{Mod}(W, \alpha) & \xrightleftharpoons{\quad} & \text{Mod-}\mathcal{A} \\ \bullet & \xrightarrow{\quad} & \bullet \otimes_{\mathcal{O}_W} \mathcal{G}^\vee \\ \bullet \otimes_{\mathcal{A}} \mathcal{G} & \xleftarrow{\quad} & \bullet \end{array} .$$

The following proposition generalizes [L-L-S-Y: Proposition 2.3.1]:

**Proposition 3.1.9. [boundedness of morphisms].** *Assume that  $Y$  is projective with a very ample line bundle  $\mathcal{O}_Y(1)$ . Let  $(X_S/S, \mathcal{O}_{X_S/S}(1))$  be a flat family of projective schemes,  $\mathcal{G}_S$  be an  $\alpha$ -twisted locally free coherent  $\mathcal{O}_X$ -module on  $X_S$  in the class  $\alpha_S$  with  $[\alpha_S] \in \text{Br}(X_S)$ ,  $(X_S^{\text{Az}}, \mathcal{E}_S)/S$  be a flat family of Azumaya schemes with a fundamental module over  $X_S/S$  in the class  $\alpha_S$ , and  $P$  be a polynomial in one variable. Then the set  $\{\varphi_\bullet\}_\bullet$  of morphisms from fibers  $(X_s^{\text{Az}}, \mathcal{E}_s)$  of  $(X_S^{\text{Az}}, \mathcal{E}_S)/S$  to  $(Y, \alpha_B)$  with  $\mathcal{G}_s$ -twisted Hilbert polynomial  $P_{\mathcal{G}_s, \varphi_s} = P$  is bounded.*

*Proof.* Let  $pr_1$  be the projection map  $X_s \times Y \rightarrow X_s$  for  $s \in S$ . Observe that for the  $pr_1^* \alpha_s$ -twisted  $\mathcal{O}_{X_s \times Y}$ -module  $\tilde{\mathcal{E}}_s$  that represents a morphism  $\varphi_s : (X_s^{\text{Az}}, \mathcal{E}_s) \rightarrow (Y, \alpha_B)$ , there is a surjective homomorphism  $pr_1^* \mathcal{E}_s \rightarrow \tilde{\mathcal{E}}_s$  of  $pr_1^* \alpha_s$ -twisted modules. As  $pr_1^* \mathcal{G}_s^\vee$  is locally free, one has the following exact sequence of the underlying  $\mathcal{O}_{X_s \times Y}$ -modules of the right  $\text{End}_{\mathcal{O}_{X_s \times Y}}(pr_1^* \mathcal{G}_s)$ -modules in question:

$$pr_1^* \mathcal{E}_s \otimes_{\mathcal{O}_{X_s \times Y}} pr_1^* \mathcal{G}_s^\vee \longrightarrow \tilde{\mathcal{E}}_s \otimes_{\mathcal{O}_{X_s \times Y}} pr_1^* \mathcal{G}_s^\vee \longrightarrow 0.$$

The proposition follows now from Theorem 3.1.8 and [H-L: Lemma 1.7.6], which says that a family  $\{F_i\}_{i \in I}$  of (ordinary) coherent sheaves on a projective scheme is bounded if and only if the set of Hilbert polynomials  $\{P(F_i)\}_{i \in I}$  is finite and there is a coherent sheaf  $F$  such that all  $F_i$  admits surjective homomorphisms  $F \rightarrow F_i$ . □

### 3.2 $\mathfrak{M}_{\text{Az}(X_S/S, \alpha_S)^f}(Y, \alpha_B)$ is algebraic.

Let  $X_S/S$  be a (fixed) flat family of projective schemes over  $S$ ,  $\alpha_S \in \check{C}_{\text{ét}}^2(X_S, \mathcal{O}_{X_S}^*)$  with  $[\alpha] \in \text{Br}(X_S)$ , and  $\mathfrak{M}_{\text{Az}(X_S/S, \alpha_S)^f}(Y, \alpha_B)$  be the moduli stack of morphisms from (non-fixed) Azumaya schemes with a fundamental module on (fixed) fibers  $X_s$  of  $X_S/S$  to a (fixed) projective  $(Y, \alpha_B)$

in the class  $\alpha_s$ . As a sheaf of groupoids on the category  $\mathit{Scheme}_S$  of schemes over  $S$  with the fppf topology,

$$\mathfrak{M}_{\mathcal{A}_Z(X_S/S, \alpha_S)^f}(Y, \alpha_B)(T) = \{\varphi_T : (X_T^{\mathcal{A}_Z}, \mathcal{E}_T) \rightarrow (Y, \alpha_B)\},$$

for  $T \in \mathit{Scheme}_S$ . Here,  $X_T = T \times_S X_S$  with a built-in  $X_T \rightarrow X_S$ ,  $\alpha_T \in \check{C}_{\acute{e}t}^2(X_T)$  the pull-back of  $\alpha_S$ , and  $\mathcal{E}_T$  is a  $\alpha_T$ -twisted coherent locally free  $\mathcal{O}_{X_T}$ -module. A morphism  $\varphi_{T,1} \rightarrow \varphi_{T,2}$  for

$$(\varphi_{T,1} : (X_{T,1}^{\mathcal{A}_Z}, \mathcal{E}_{T,1}) \rightarrow (Y, \alpha_B)), (\varphi_{T,2} : (X_{T,2}^{\mathcal{A}_Z}, \mathcal{E}_{T,2}) \rightarrow (Y, \alpha_B)) \in \mathfrak{M}_{\mathcal{A}_Z(X_S/S, \alpha_S)^f}(Y, \alpha_B)(T)$$

is an isomorphism  $\hat{h} : \mathcal{E}_{T,1} \xrightarrow{\sim} \mathcal{E}_{T,2}$  of  $\alpha_T$ -twisted  $\mathcal{O}_{X_T}$ -modules that satisfies  $\hat{h}(\mathcal{A}_{\varphi_{T,1}}) = \mathcal{A}_{\varphi_{T,2}}$  and the diagram

$$\begin{array}{ccc} X_{\varphi_{T,1}} & \xrightarrow{f_{\varphi_{T,1}}} & Y \\ \hat{h} \uparrow & \nearrow f_{\varphi_{T,2}} & \\ X_{\varphi_{T,2}} & & \end{array}$$

commutes. Here,  $\tilde{h} : \mathcal{O}_{X_{T,1}}^{\mathcal{A}_Z} \xrightarrow{\sim} \mathcal{O}_{X_{T,2}}^{\mathcal{A}_Z}$  and  $\hat{h} : X_{\varphi_{T,2}} \xrightarrow{\sim} X_{\varphi_{T,1}}$  are  $h$ -induced isomorphisms of  $\mathcal{O}_{X_T}$ -algebras and  $X_T$ -schemes respectively. The goal of this subsection is to prove that  $\mathfrak{M}_{\mathcal{A}_Z(X_S/S, \alpha_S)^f}(Y, \alpha_B)$  is an algebraic stack, locally of finite type.

Denote by  $\mathfrak{T}((X_S \times Y)/S, pr_2^* \alpha_B)^{0/X_S/S}$  the moduli stack of  $pr_2^* \alpha_B$ -twisted coherent  $\mathcal{O}_{X_S \times Y}$ -modules on  $X_S \times Y$ ,  $s \in S$ , that are flat over  $X_s$  of relative dimension 0.

Recall first the following proposition of Lieblich [Liel]: (in the special case of schemes and with the sheaf  $\mu_r$  of groups of  $r$ -th roots of unity replaced by  $\mathcal{O}_\bullet^*$ )

**Proposition 3.2.1. [stack of twisted coherent sheaves algebraic].** ([Liel: Proposition 4.1.1.1]). *Let  $W$  be a scheme/ $\mathbb{C}$ ,  $\alpha \in \check{C}_{\acute{e}t}^2(W)$ , and  $\mathfrak{T}(W, \alpha)$  be the stack of  $\alpha$ -twisted coherent  $\mathcal{O}_W$ -modules. Then,  $\mathfrak{T}(W, \alpha)$  is algebraic, locally of finite type.*

The proof follows from Artin's criteria for algebraic stacks, [Art] and [Schl].

**Proposition 3.2.2. [ $\mathfrak{T}((X_S \times Y)/S, pr_2^* \alpha_B)^{0/X_S/S}$  algebraic].**  $\mathfrak{T}((X_S \times Y)/S, pr_2^* \alpha_B)^{0/X_S/S}$  is an algebraic stack, locally of finite type.

*Proof.* This follows from Proposition 3.2.1 and the observation that  $\mathfrak{T}((X_S \times Y)/S, pr_2^* \alpha_B)^{0/X_S/S}$  can be identified with the stack of morphisms from the fibers of the fixed  $X_S/S$  to the connected components of  $\mathfrak{T}(Y, \alpha_B)$  that parameterizes 0-dimensional  $\alpha_B$ -twisted  $\mathcal{O}_Y$ -modules. □

**Corollary 3.2.3. [ $\mathfrak{M}_{\mathcal{A}_Z(X_S/S, \alpha_S)^f}(Y, \alpha_B)$  algebraic].**  $\mathfrak{M}_{\mathcal{A}_Z(X_S/S, \alpha_S)^f}(Y, \alpha_B)$  is an algebraic stack, locally of finite type.

*Proof.* Let  $pr_1 : X_S \times Y \rightarrow X_S$  and  $pr_2 : X_S \times Y \rightarrow Y$  be projective maps and  $U \rightarrow X_S \times Y$  be an étale cover of  $X_S \times Y$  that is both  $pr_1^* \alpha_S$ - and  $pr_2^* \alpha_B$ -admissible. Then the pair  $(pr_1^* \alpha_S, pr_2^* \alpha_B) \in \check{C}_{\acute{e}t}^2(X_S \times Y, \mathcal{O}_{X_S \times Y}^*) \times \check{C}_{\acute{e}t}^2(X_S \times Y, \mathcal{O}_{X_S \times Y}^*)$  of 2-cocycles with value in  $\mathcal{O}_{X_S \times Y}^*$  determines the 2-cochain  $pr_1^* \alpha_S - pr_2^* \alpha_B$  of ideal sheaves<sup>13</sup> of  $\mathcal{O}_{X_S \times Y}$ , and hence a 2-cochain  $(Z_{ijk})_{ijk}$  of closed subscheme, presented via  $U$ . The matching condition  $\pi_{\varphi_s}^* \alpha_S = f_{\varphi_s}^* \alpha_B$  of twists on  $X_{s, \varphi_s}$  for a  $\varphi_s : (X_s^{\mathcal{A}_Z}, \mathcal{E}_s) \rightarrow (Y, \alpha_B)$  is equivalent to the condition that

<sup>13</sup>In this language, we have to allow the term “ideal sheaf” to include also the nonproper one, i.e.  $\mathcal{O}_\bullet$  itself.

(†) *the lift of the closed subscheme  $\text{Supp } \tilde{\mathcal{E}}_s$  of  $X_S \times Y$  to  $U \times_{X_S \times Y} U \times_{X_S \times Y} U$  is contained in the 2-cochain  $(Z_{ijk})_{ijk}$  of closed subschemes*

on the corresponding  $pr_1^* \alpha_s$ -twisted sheaf  $\tilde{\mathcal{E}}_s$  on  $X_S \times Y$  that represents  $\varphi_s$ . Note that each  $Z_{ijk}$  projects to a constructible subset of  $X_S \times Y$  under the étale morphism  $U \times_{X_S \times Y} U \times_{X_S \times Y} U \rightarrow X_S \times Y$ . Note also that, given a  $\tilde{\mathcal{E}}_s$  that represents a  $\varphi_s$ , it follows from the projectivity of  $X_S/S$  and  $Y$  and a generalization of the construction in [H-L: Sec. 2.2 and Chap. 4] to twisted sheaves, e.g. [H-S: Sec. 2], [Lie1: Sec. 4.1], [Yo: Sec. 2], that a small enough local chart  $T$  of  $\mathfrak{T}((X_S \times Y)/S, pr_2^* \alpha_B)^{0/X_S/S}$  around  $[\tilde{\mathcal{E}}_s]$  comes from a Quot-scheme construction on a pair of  $pr_2^* \alpha_B$ -twisted locally free  $\mathcal{O}_{X_S \times Y}$ -modules on  $X_S \times Y$  for a local chart of  $[\tilde{\mathcal{E}}_s]$  in  $\mathfrak{T}(X_S \times Y, pr_2^* \alpha_B)$ . In other words,  $T$  is realized as the subscheme of the variety of homomorphisms  $\mathcal{F}_1 \xrightarrow{h} \mathcal{F}_0$  for two fixed  $pr_2^* \alpha_B$ -twisted locally free  $\mathcal{O}_{X_S \times Y}$ -modules, whose *coker*  $h$  correspond to objects of  $\mathfrak{T}(X_S \times Y, pr_2^* \alpha_B)$ . Condition (†) imposes now a system of determinantal-type constructible-subset conditions on  $T$ . Thus, it selects a local chart  $T'$  of  $\mathfrak{M}_{\mathcal{A}_z(X_S/S, \alpha_S)^f}(Y, \alpha_B)$  around  $[\tilde{\mathcal{E}}_s]$  as the intersection of a system of determinantal-type constructible subset of  $T$ . Functoriality of the construction realizes  $\mathfrak{M}_{\mathcal{A}_z(X_S/S, \alpha_S)^f}(Y, \alpha_B)$  then as a constructible substack of  $\mathfrak{T}((X_S \times Y)/S, pr_2^* \alpha_B)^{0/X_S/S}$ . The proposition now follows from Proposition 3.2.2.  $\square$

The following discussion shows that the twist-matching condition is a closed condition from the nature of the basic lemma below, which is immediate:

**Lemma 3.2.4. [basic].** *Let  $T$  be a discrete valuation ring with the field of fraction  $K$ ,  $R$  be a  $T$ -algebra,  $M$  be an  $R$ -module that is flat over  $T$ , and  $M_K := M \otimes_T K$ . Let  $r \in R$  such that  $r \cdot M_K = 0$ . Then,  $r \cdot M = 0$ .*

**Lemma 3.2.5. [matching of twists as closed condition].** *Let  $T = \text{Spec } R$ ,  $R$  a discrete valuation ring, with the generic point  $\eta$  and the closed point  $\mathbf{0} : T \rightarrow S$ ,  $X_T = h^* X_S = T \times_S X_S$  with the built-in map  $\hat{h} : X_T \rightarrow X_S$ ,  $\alpha_T = \hat{h}^* \alpha_S$ ;  $pr_1 : X_T \times Y \rightarrow X_T$  and  $pr_2 : X_T \times Y \rightarrow Y$  be the projection maps, and  $\pi : X_T \times Y \rightarrow T$  be the built-in morphism. Let  $\mathcal{F}_T$  be a  $pr_2^* \alpha_B$ -twisted coherent  $\mathcal{O}_{X_T \times Y}$ -module on  $X_T \times Y$  such that*

- $\mathcal{F}_T$  is flat over  $T$ ,
- $pr_1^* \alpha_\eta = pr_2^* \alpha_B$  on  $\text{Supp}(\mathcal{F}_T|_\eta)$ .

*Then,  $pr_1^* \alpha_T = pr_2^* \alpha_B$  holds on  $\text{Supp } \mathcal{F}_T$  over  $T$ . The same statement holds also for  $\mathcal{F}_T$  being a  $pr_1^* \alpha_T$ -twisted coherent  $\mathcal{O}_{X_T \times Y}$ -module on  $X_T \times Y$ .*

*Proof.* Let  $p : U \rightarrow X_T \times Y$  be an étale cover of  $X_T \times Y$  that is admissible to both  $pr_1^* \alpha_T$  and  $pr_2^* \alpha_B$  and  $p^{(2)} : U^{(2)} := U \times_{X_T \times Y} U \times_{X_T \times Y} U \rightarrow X_T \times Y$  be the built-in morphism from the fibered product. Then, the pullback family  $p^{(2),*} \mathcal{F}_T$  on  $U^{(2)}$  is flat over  $\mathbf{0} \in T$ . The pair  $(pr_1^* \alpha_T, pr_2^* \alpha_B)$  on  $U^{(2)}$ , each of which takes values in  $\mathcal{O}_{U^{(2)}}^*$ , determines the principal ideal sheaf<sup>14</sup>  $pr_1^* \alpha_T - pr_2^* \alpha_B$  of  $\mathcal{O}_{U^{(2)}}$ . The lemma now follows from Lemma 3.2.4.  $\square$

<sup>14</sup>Here, we allow a local generator of a principal ideal sheaf to be invertible, cf. footnote 13.

**Corollary 3.2.6. [closed substack].**  $\mathfrak{M}_{\text{Az}(X_S/S, \alpha_S)^f}(Y, \alpha_B)$  is a closed substack of  $\mathfrak{T}((X_S \times Y)/S, pr_2^* \alpha_B)^{0/X_S/S}$ .

We remark that in general  $\mathfrak{T}((X_S \times Y)/S, pr_2^* \alpha_B)^{0/X_S/S}$  and, hence,  $\mathfrak{M}_{\text{Az}(X_S/S, \alpha_S)^f}(Y, \alpha_B)$  are not closed in  $\mathfrak{T}(X_S \times Y, pr_2^* \alpha_B)$ .

## 4 The case of holomorphic D-strings.

In this section<sup>15</sup>, we consider the case when  $X$  is a nodal/prestable curve and  $Y$  is a smooth projective variety.

### 4.1 The moduli stack $\mathfrak{M}_{\text{Az}(g, r, \chi)^f}(Y, \alpha_B; \beta)$ of morphisms from Azumaya prestable curves to $(Y, \alpha_B)$ .

#### The big vs. the small moduli problem.

Recall the following lemma, which follows from the normalization sequence and Tsen's Theorem: (see [Liel: Sec. 5.1.1] for more general discussions.)

**Lemma 4.1.1. [Br(C) vanishes].** (Cf. [Liel: Lemma 5.1.1.1].) *Let  $C$  be a nodal curve. Then  $Br(C) = 0$ .*

Thus, for  $C$  a prestable curve and  $\alpha \in \check{C}_{\text{ét}}(C, \mathcal{O}_C^*)$  with  $[\alpha] \in Br(C)$ ,  $[\alpha]$  indeed vanishes and

- there is an  $\alpha$ -twisted line bundle  $\mathcal{L}$  on  $C$  and the correspondence

$$\begin{array}{ccc} \text{Mod}(C, \alpha) & \longrightarrow & \text{Mod}(C) \\ \mathcal{F} & \longmapsto & \mathcal{F} \otimes_{\mathcal{O}_C} \mathcal{L}^\vee \end{array}$$

is an equivalence of categories,

- any Azumaya algebra  $\mathcal{A}$  over  $C$  is isomorphic to  $\text{End}_{\mathcal{O}_C}(\mathcal{E})$  for some (ordinary) locally free  $\mathcal{O}_C$ -module  $\mathcal{E}$ .

Despite the fact that these special features reduce the study of Azumaya schemes and modules on prestable curve to the case as in [L-L-S-Y], the moduli stacks  $\mathfrak{M}_\alpha(Y, \alpha_B)$  of morphisms from Azumaya prestable curves with an  $\alpha$ -twisted fundamental module to  $(Y, \alpha_B)$  in general are *not* isomorphic for different choices of  $\alpha$ 's. The image of such morphisms for different  $\alpha$  are in general distinct due to the effect of  $\alpha_B$  via the twist-matching condition. Thus, one has two moduli problems:

- (1) *The big moduli problem:* moduli of morphisms from Azumaya prestable curves with a possibly twisted fundamental module to  $(Y, \alpha_B)$ .
- (2) *The small moduli problem:* moduli of morphisms from Azumaya prestable curves with an ordinary/untwisted fundamental module to  $(Y, \alpha_B)$ .

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<sup>15</sup>The current section continues the previous work [L-L-S-Y] (D(2)) with Si Li and Ruifang Song, fall 2008. C.-H.L. thank them for the participation of the biweekly Saturday D-brane working seminar, spring 2008, and Liang Kong and S.L. for a communication/conversation on further issues while editing the current manuscript.

While Problem (1) is a final goal, in this work we address only Problem (2), a sub-problem to Problem (1).

### The small moduli problem.<sup>16</sup>

Let  $\mathfrak{M}_{Az(g,r,\chi)^f}(Y, \alpha_B; \beta)$  be the moduli stack of morphisms  $\varphi : (C^{Az}, \mathcal{E}) \rightarrow (Y, \alpha_B)$  from (unfixed) Azumaya prestable curves with a fundamental module to  $(Y, \alpha)$  (cf. Definition 2.2.4) of type  $(g, r, \chi, |\beta)$  in the sense that:

- $C$  has (arithmetic) genus  $g$ ,
- $\mathcal{E}$  has rank  $r$  and Euler characteristic  $\chi = \text{deg } \mathcal{E} + r(1 - g)$ ,
- the image curve class  $\varphi_*[C] = \beta \in N_1(Y)$ .

([L-L-S-Y: Definition 2.1.11, Definition 2.1.12, Definition 2.1.13, Remark 2.1.14, and Lemma 2.2.4].)  
Then

**Proposition 4.1.2.** [ $\mathfrak{M}_{Az(g,r,\chi)^f}(Y, \alpha_B; \beta)$  algebraic].  $\mathfrak{M}_{Az(g,r,\chi)^f}(Y, \alpha_B; \beta)$  is an algebraic stack, locally of finite type.

This follows from [L-L-S-Y: Sec. 3.2] and Corollary 3.2.3.

## 4.2 Fillability/valuation-criterion property of $\mathfrak{M}_{Az(g,r,\chi)^f}(Y, \alpha_B; \beta)$ .

We prove in this subsection the following fillability/valuation-criterion property of the moduli stack  $\mathfrak{M}_{Az(g,r,\chi)^f}(Y, \alpha_B; \beta)$ . This indicates that  $\mathfrak{M}_{Az(g,r,\chi)^f}(Y, \alpha_B; \beta)$  is a sufficiently large moduli space for the study of more restrictive moduli problems for curves.

**Proposition 4.2.1.** [valuation-criterion property]. Let  $S = \text{Spec } R$ ,  $R$  a discrete valuation ring, with the generic point  $\eta$  and the closed point  $\mathbf{0}$  and  $f : \eta \rightarrow \mathfrak{M}_{Az(g,r,\chi)^f}(Y, \alpha_B; \beta)$  be a morphism. Then, after a base change on  $S$  if necessary,  $f$  extends to a morphism  $\hat{f} : S \rightarrow \mathfrak{M}_{Az(g,r,\chi)^f}(Y, \alpha_B; \beta)$ .

(However, the extension in general is not unique.)

*Proof.* Let  $C_\eta$  be a flat family of prestable curves (of genus  $g$ ) over  $\eta$  and  $\tilde{\mathcal{E}}_\eta$  be the coherent  $\mathcal{O}_{C_\eta \times Y}$ -module on  $C_\eta \times Y$ , flat over  $C_\eta$  with relative dimension 0 (and of relative length  $r$ ), that corresponds to the morphism  $f : \eta \rightarrow \mathfrak{M}_{Az(g,r,\chi)^f}(Y, \alpha_B; \beta)$ . Let  $pr_1 : C_\eta \times Y \rightarrow C_S$  and  $pr_2 : C_\eta \times Y \rightarrow Y$  be the projection maps. From the projection formula,  $pr_{1,*}(\mathcal{F} \otimes_{\mathcal{O}_{C_\eta \times Y}} pr_1^* \mathcal{L}) \simeq (pr_{1,*} \mathcal{F}) \otimes_{\mathcal{O}_{C_\eta}} \mathcal{L}$  for an  $\mathcal{O}_{C_\eta \times Y}$ -module  $\mathcal{F}$  and a coherent locally free  $\mathcal{O}_{C_\eta}$ -module  $\mathcal{L}$ , we may assume, after tensoring  $pr_1^* \mathcal{L}$  for an appropriate relative ample line bundle  $\mathcal{L}$  on  $C_\eta/\eta$ , that  $\mathcal{E}_\eta := pr_{1,*} \tilde{\mathcal{E}}_\eta$  fits into the exact sequence

$$\mathcal{O}_{C_\eta}^{\oplus N} \longrightarrow \mathcal{E}_\eta \longrightarrow 0.$$

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<sup>16</sup>This theme is taken/adapted from [L-L-S-Y]. Readers are referred ibidem for more details.



for some  $N \gg 0$ . It follows from the built-in exact sequence  $pr_1^* \mathcal{E}_\eta \rightarrow \tilde{\mathcal{E}}_\eta \rightarrow 0$  that

$$\mathcal{O}_{C_\eta \times Y}^{\oplus N} \longrightarrow \tilde{\mathcal{E}}_\eta \longrightarrow 0.$$

Regarding this as an exact sequence of  $\mathcal{O}_{C_\eta \times Y}$ -modules on  $(C_\eta \times Y)/C_\eta/S$ , one obtains a morphism

$$\tilde{f}_\eta : C_\eta/\eta \longrightarrow \text{Quot}_Y(\mathcal{O}_Y^{\oplus N}, r),$$

where  $\text{Quot}_Y(\mathcal{O}_Y^{\oplus N}, r)$  is the Quot-scheme that parameterizes the 0-dimensional quotient sheaves of  $\mathcal{O}_Y^{\oplus N}$  with length  $r$ . Rigidifying  $\tilde{f}_\eta$  as a prestable map from a curve over  $\eta$  if necessary, the properness property of the moduli stack  $\overline{\mathcal{M}}_\bullet(\text{Quot}_Y(\mathcal{O}_Y^{\oplus N}, r), \bullet)$  of stable maps to the projective scheme  $\text{Quot}_Y(\mathcal{O}_Y^{\oplus N}, r)$  implies that, subject to a base change<sup>17</sup> on  $S$ ,  $\tilde{f}_\eta$  extends to a morphism  $\tilde{f}_S : C_S/S \rightarrow \text{Quot}_Y(\mathcal{O}_Y^{\oplus N}, r)$ , where  $C_S/S$  is a flat family of prestable curves that extends  $C_\eta/\eta$  (after the above base change). The associated quotient sheaf  $\tilde{\mathcal{E}}_S$  on  $C_S \times Y$  extends  $\tilde{\mathcal{E}}_\eta$  on  $C_\eta \times Y$  and has the property that  $\tilde{\mathcal{E}}_S$  is flat over  $C_S$  of relative dimension 0 and relative length  $r$ . This defines thus  $\hat{f} : S \rightarrow \mathfrak{M}_{\text{Az}(g,r,\chi)^f}(Y, \beta)$ . Corollary 3.2.6 implies then that  $\hat{f}$  indeed has the image in  $\mathfrak{M}_{\text{Az}(g,r,\chi)^f}(Y, \alpha_B; \beta)$  and, hence, is an extension of  $f : S - \{0\} \rightarrow \mathfrak{M}_{\text{Az}(g,r,\chi)}^f(Y, \alpha_B; \beta)$ , after a base change. This proves the proposition.  $\square$

## 5 The extension by the sheaf $\mathcal{D}$ of differential operators.

In this section, the second effect - namely, the deformation quantization of both the target space(-time) and D-brane world-volumes - of the background  $B$ -field to a smooth D-brane world-volume  $X$  along the line of the Polchinski-Grothendieck Ansatz is brought into consideration as well. We focus on the case when the deformation quantizations that occur are modelled directly on that for the phase space in quantum mechanics and when the study in Sec. 2.2 can be extended/applied immediately. The special case of morphisms from  $X$  with the new structure to a target-space  $Y$  being the total space  $\Omega_W$  of the cotangent bundle  $\Omega_W$  of a smooth variety  $W$  is considered. An application of this gives the notion of deformation quantizations of the spectral curves that appear in Hitchin's integrable systems. For language simplicity, we use the analytic topology for smooth varieties in the discussion below whenever it is more convenient.

### 5.1 Azumaya schemes with a fundamental module with a flat connection.

The discussion in Sec. 2.2 has a direct generalization to incorporate the sheaf  $\mathcal{D}$  of algebras of differential operators and  $\mathcal{D}$ -modules.

#### Weyl algebras, the sheaf $\mathcal{D}$ of differential operators, and $\mathcal{D}$ -modules.

Let  $X$  be a smooth variety over  $\mathbb{C}$ ,  $\Theta_X = \text{Der}_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$  be the sheaf of  $\mathbb{C}$ -derivations on  $\mathcal{O}_X$ , and  $\Omega_X$  be the sheaf of Kähler differentials on  $X$ . We recall a few necessary objects and facts for our study. Their details are referred to [Be3], [Bj], and [B-E-G-H-K-M]:<sup>18</sup>

<sup>17</sup>For the simplicity of notation and expressions, a base change on  $S$  and the new family over  $S$  will still be denoted by  $S$  and  $C_S/S$  respectively.

<sup>18</sup>See Bernstein [Be3: §0. Introduction], Björk [Bj: Introduction], and Borel [B-E-G-H-K-M: Introduction] for a list of people who contribute to the early development of the subject.

(1) the *Weyl algebra*

$$A_n(\mathbb{C}) := \mathbb{C}\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle / ([x_i, x_j], [\partial_i, \partial_j], [\partial_i, x_j] - \delta_{ij} : 1 \leq i, j \leq n),$$

which is the algebra of differential operators acting on  $\mathbb{C}[x_1, \dots, x_n]$  by formal differentiation; here,  $\mathbb{C}\langle \dots \rangle$  is the unital associative  $\mathbb{C}$ -algebra generated by elements  $\dots$  indicated,  $[ , ]$  is the commutator,  $\delta_{ij}$  is the Kronecker delta, and  $(\dots)$  is the 2-sided ideal generated by  $\dots$  indicated;

(2) the *sheaf  $\mathcal{D}_X$  of (linear algebraic) differential operators* on  $X$ , which is the sheaf of unital associative algebras that extends  $\mathcal{O}_X$  by new generators from the sheaf  $\Theta_X$ ;

(3)  *$\mathcal{D}_X$ -modules* (or directly  *$\mathcal{D}$ -modules* when  $X$  is understood), which are sheaves on  $X$  on which  $\mathcal{D}_X$  acts from the left.

**Lemma 5.1.1.** [ *$A_n(\mathbb{C})$  simple*].  *$A_n(\mathbb{C})$  is a simple algebra: the only 2-sided ideal therein is the zero ideal (0).*

**Proposition 5.1.2.** [ *$\mathcal{O}$ -coherent  $\mathcal{D}$ -module*]. *Let  $\mathcal{M}$  be a  $\mathcal{D}_X$ -module that is coherent as an  $\mathcal{O}_X$ -module. Then,  $\mathcal{M}$  is  $\mathcal{O}_X$ -locally-free. Furthermore, in this case, the action of  $\mathcal{D}_X$  on  $\mathcal{M}$  defines a flat connection  $\nabla : \mathcal{M} \rightarrow \mathcal{M} \otimes \Omega_X$  on  $\mathcal{M}$  by assigning  $\nabla_\xi s = \xi \cdot s$  for  $s \in \mathcal{M}$  and  $\xi \in \Theta_X$ ; the converse also holds. This gives an equivalence of categories:*

$$\left\{ \mathcal{O}_X\text{-coherent } \mathcal{D}_X\text{-modules} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{coherent locally free } \mathcal{O}_X\text{-modules} \\ \text{with a flat connection} \end{array} \right\}.$$

**$\mathcal{D}$  as the structure sheaf of the deformation quantization of the cotangent bundle.**

From the presentation of the Weyl algebra  $A_n(\mathbb{C})$ , which resembles the quantization of a classical phase space with the position variable  $(x_1, \dots, x_n)$  and the dual momentum variable  $(p_1, \dots, p_n) = (\partial_1, \dots, \partial_n)$ , and the fact that  $\mathcal{D}_X$  is locally modelled on the pull-back of  $A_n(\mathbb{C})$  over  $\mathbb{A}^n$  under an étale morphism to  $\mathbb{A}^n$ , the sheaf  $\mathcal{D}_X$  of algebras with the built-in inclusion  $\mathcal{O}_X \subset \mathcal{D}_X$  can be thought of as the structure sheaf of a noncommutative space from the quantization<sup>19</sup> of the cotangent bundle, i.e. the total space  $\Omega_X$  of the sheaf  $\Omega_X$ , of  $X$ .

**Definition 5.1.3.** [*canonical deformation quantization of cotangent bundle*]. We will formally denote this noncommutative space by  $Space \mathcal{D}_X =: Q\Omega_X$  and call it the *canonical deformation quantization* of  $\Omega_X$ .

A special class of *morphisms* from or to  $Space \mathcal{D}_X$  can be defined contravariantly as homomorphisms of sheaves of  $\mathbb{C}$ -algebras.

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<sup>19</sup>The word “quantization” has received various meanings in mathematics. Here, we mean solely the one associated to quantum mechanics. This particular quantization is also called *deformation quantization*.

**Example 5.1.4.**  $[A_n(\mathbb{C})]$ . The noncommutative space  $Space(A_n(\mathbb{C}))$  defines a deformation quantization of  $\Omega_{\mathbb{A}^n}$ . Recall the presentation of  $A_n(\mathbb{C})$ . The  $\mathbb{C}$ -algebra homomorphism

$$\begin{aligned} f_{(k)}^\# &: \mathbb{C}[y_1, \dots, y_n] \longrightarrow A_n(\mathbb{C}) \\ y_i &\longmapsto x_i, \quad i = 1, \dots, k, \\ y_j &\longmapsto \partial_j, \quad j = k+1, \dots, n, \end{aligned}$$

defines a dominant morphism  $f_{(k)} : Space(A_n(\mathbb{C})) \rightarrow \mathbb{A}^n$ ,  $k = 0, \dots, n$ . The  $\mathbb{C}$ -algebra automorphism  $A_n(\mathbb{C}) \rightarrow A_n(\mathbb{C})$  with  $x_i \mapsto \partial_i$  and  $\partial_i \mapsto -x_i$  defines the *Fourier transform* on  $Space(A_n(\mathbb{C}))$ . Note that, since  $A_n(\mathbb{C})$  is simple, any morphisms to  $Space(A_n(\mathbb{C}))$  is dominant (i.e. the related  $\mathbb{C}$ -algebra homomorphism from  $A_n(\mathbb{C})$  is injective).

**$\alpha$ -twisted  $\mathcal{O}_X$ -coherent  $\mathcal{D}_X$ -modules and enlargements of  $\mathcal{O}_X^{Az}$  by  $\mathcal{D}_X$ .**

Let  $\alpha \in \check{C}\acute{e}t(X, \mathcal{O}_X^*)$  and  $\mathcal{F} = (\{U_i\}_{i \in I}, \{\mathcal{F}_i\}_{i \in I}, \{\phi_{ij}\}_{i,j \in I})$  be an  $\alpha$ -twisted  $\mathcal{O}_X$ -module.

**Definition 5.1.5. [connection on  $\mathcal{F}$ ].** A *connection*  $\nabla$  on  $\mathcal{F}$  is a set  $\{\nabla_i\}_{i \in I}$  where  $\nabla_i : \mathcal{F}_i \rightarrow \mathcal{F}_i \otimes_{\mathcal{O}_{U_i}} \Omega_{U_i}$  is a connection on  $\mathcal{F}_i$ , that satisfies  $\phi_{ij} \circ (\nabla_i|_{U_{ij}}) = (\nabla_j|_{U_{ij}}) \circ \phi_{ij}$ .  $\nabla$  is said to be *flat* if  $\nabla_i$  is flat for all  $i \in I$ .

Note that the existence of an  $\alpha$ -twisted  $\mathcal{O}_X$ -module with a connection imposes a condition on  $\alpha$  that  $\alpha$  has a presentation  $(\alpha_{ijk})_{ijk}$  with  $d\alpha := (d\alpha_{ijk})_{ijk} = (0)_{ijk}$ ; i.e.  $\alpha_{ijk} \in \mathbb{C}^*$  for all  $i, j, k$ .

As the proof of Proposition 5.1.2 is local, it generalizes to  $\alpha$ -twisted  $\mathcal{O}_X$ -coherent  $\mathcal{D}_X$ -modules:

**Proposition 5.1.6. [ $\alpha$ -twisted  $\mathcal{O}$ -coherent  $\mathcal{D}$ -module].** *Let  $\mathcal{M}$  be a  $\mathcal{D}_X$ -module that is  $\alpha$ -twisted  $\mathcal{O}_X$ -coherent. Then,  $\mathcal{M}$  is an  $\alpha$ -twisted  $\mathcal{O}_X$ -locally-free. Furthermore, in this case, the action of  $\mathcal{D}_X$  on  $\mathcal{M}$  defines a flat connection  $\nabla : \mathcal{M} \rightarrow \mathcal{M} \otimes \Omega_X$  on  $\mathcal{M}$  by assigning  $\nabla_\xi s = \xi \cdot s$  for  $s \in \mathcal{M}$  and  $\xi \in \Theta_X$ ; the converse also holds. This gives an equivalence of categories:*

$$\left\{ \alpha\text{-twisted } \mathcal{O}_X\text{-coherent } \mathcal{D}_X\text{-modules} \right\} \longleftrightarrow \left\{ \begin{array}{l} \alpha\text{-twisted coherent locally free } \mathcal{O}_X\text{-} \\ \text{modules with a flat connection} \end{array} \right\}.$$

Let  $\mathcal{E}$  be an  $\alpha$ -twisted  $\mathcal{O}_X$ -coherent  $\mathcal{D}_X$ -module. Then the  $\mathcal{D}_X$ -module structure on  $\mathcal{E}$  induces a natural  $\mathcal{D}_X$ -module structure on the (ordinary)  $\mathcal{O}_X$ -module  $\mathcal{O}_X^{Az} := \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$ . We will denote both the connection on  $\mathcal{E}$  and on  $\mathcal{O}_X^{Az}$  by  $\nabla$ . As both  $\mathcal{O}_X^{Az} := \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$  and  $\mathcal{D}_X$  act now on  $\mathcal{E}$  and  $\mathcal{D}_X$  acts also on  $\mathcal{O}_X^{Az}$ , one can define a sheaf  $\mathcal{O}_X^{Az, \mathcal{D}}$  of unital associative algebras generated by  $\mathcal{O}_X^{Az}$  and  $\mathcal{D}_X$  as follows:

- Over a (Zariski) open subset  $U$  of  $X$ ,  $\mathcal{O}_X^{Az, \mathcal{D}}(U)$  is the unital associative  $\mathbb{C}$ -algebra generated by  $\mathcal{O}_X^{Az}(U) \cup \mathcal{D}_X(U)$  subject to the following rules:

- (1) for  $\phi_1, \phi_2 \in \mathcal{O}_X^{Az}(U)$ ,  $\phi_1 \cdot \phi_2 \in \mathcal{O}_X^{Az, \mathcal{D}}(U)$  coincides with the existing  $\phi_1 \phi_2 \in \mathcal{O}_X^{Az}(U)$ ;
- (2) for  $\eta_1, \eta_2 \in \mathcal{D}_X(U)$ ,  $\eta_1 \cdot \eta_2 \in \mathcal{O}_X^{Az, \mathcal{D}}(U)$  coincides with the existing  $\eta_1 \eta_2 \in \mathcal{D}_X(U)$ ;
- (3) (*Leibniz rule*) for  $\phi \in \mathcal{O}_X^{Az}(U)$  and  $\xi \in \Theta_X(U) \subset \mathcal{D}_X(U)$ ,

$$\xi \cdot \phi = (\nabla_\xi \phi) + \phi \cdot \xi.$$

In notation,  $\mathcal{O}_X^{Az, \mathcal{D}} := \mathbb{C}\langle \mathcal{O}_X^{Az}, \mathcal{D}_X \rangle^\nabla$ .

**Definition 5.1.7. [Azumaya quantum scheme with fundamental module].** The non-commutative space

$$(X^{Az, \mathcal{D}}, \mathcal{E}^\nabla) := (X, \mathcal{O}_X^{Az, \mathcal{D}} = \mathbb{C}\langle \text{End}_{\mathcal{O}_X}(\mathcal{E}), \mathcal{D}_X \rangle^\nabla, (\mathcal{E}, \nabla))$$

will be called an *Azumaya quantum scheme with a fundamental module in the class  $\alpha$* .

Caution that  $\mathcal{O}_X \subset \mathcal{O}_X^{Az, \mathcal{D}}$  in general does not lie in the center of  $\mathcal{O}_X^{Az, \mathcal{D}}$ .

*Remark 5.1.8. [ $\mathcal{E}^\nabla$  as a module over  $\text{Space}(\mathcal{O}_X^{Az, \mathcal{D}})$ ].* The full notation for  $X^{Az, \mathcal{D}}$  in Definition 5.1.7 is meant to make two things manifest:

- (1) There is a built-in diagram of dominant morphisms of  $X$ -spaces :

$$\begin{array}{ccc} & X^{Az, \mathcal{D}} := \text{Space } \mathcal{O}_X^{Az, \mathcal{D}} & \\ & \swarrow \quad \downarrow \quad \searrow & \\ X^{Az} := \text{Space } \mathcal{O}_X^{Az} & & Q\Omega_X := \text{Space } \mathcal{D}_X \\ & \searrow \quad \downarrow \quad \swarrow & \\ & X & \end{array}$$

$\text{Space } \mathcal{O}_X^{Az, \mathcal{D}}$  is the major space one should focus on. The other three spaces -  $\text{Space } \mathcal{O}_X^{Az}$ ,  $\text{Space } \mathcal{D}_X$ , and  $X$  - should be treated as auxiliary spaces that are built into the construction to encode a special treatment that takes care of the issue of localizations of noncommutative rings in the current situation; cf. the next item.

- (2) Despite the fact that  $\mathcal{O}_X$  is in general not in the center of  $\mathcal{O}_X^{Az, \mathcal{D}}$ , there is a notion of localization and open sets on  $\text{Space } \mathcal{O}_X^{Az, \mathcal{D}}$  induced by those on  $X$ . I.e.  $\text{Space } \mathcal{O}_X^{Az, \mathcal{D}}$  has a built-in topology induced from the (Zariski) topology of  $X$ . Thus, one can still have the notion of *gluing systems of morphisms* and *sheaves* with respect to this topology.

In particular,  $\mathcal{E}^\nabla$  is a sheaf of  $\mathcal{O}_X^{Az, \mathcal{D}}$ -modules supported on the whole  $\text{Space } \mathcal{O}_X^{Az, \mathcal{D}}$  with this topology.

*Remark 5.1.9. [Azumaya algebra over  $\mathcal{D}_X$ ].* Note that  $\mathcal{O}_X^{Az, \mathcal{D}}$  can also be thought of as an *Azumaya algebra over  $\mathcal{D}_X$*  in the sense that it is a sheaf of algebras on  $X$ , locally modelled on the matrix ring  $M_r(\mathcal{D}_U)$  over  $\mathcal{D}_U$  for  $U$  an affine étale-open subset of  $X$ .

*Remark 5.1.10. [partially deformation-quantized target].* From the fact that Weyl algebras are simple, it is anticipated that a morphism to a totally deformation-quantized space  $Y = \Omega_W$  is a dominant morphism. In general, one may take  $Y$  to be a partial deformation quantization of a space along a foliation. E.g. a deformation quantization of  $\Omega_{W/B}$  along the fibers of a fibration  $W/B$ . For compact  $Y$ , one may consider the deformation quantization along torus fibers of a space fibered by even-dimensional tori.<sup>20</sup> (Cf. Example 5.1.11.)

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<sup>20</sup>Though we do not touch this here, readers should be aware that this is discussed in numerous literatures.

## Higgsing and un-Higgsing of D-branes via deformations of morphisms.

Same as the situation studied in [L-Y1], [L-L-S-Y], [L-Y2], and [L-Y3], the Higgsing and un-Higgsing of D-branes can occur when we deform morphisms in the current situation.

**Example 5.1.11. [Higgsing/un-Higgsing of D-brane].** Let  $(X^{Az, \mathcal{D}}, \mathcal{E}^\nabla)$  be the affine Azumaya quantum scheme with a fundamental module associated to the ring  $R := \mathbb{C}\langle M_2(\mathbb{C}[z]), \partial_z \rangle$  (with the implicit relation  $[\partial_z, z] = 1$  and the identification of  $\mathbb{C}[z]$  with the center of  $M_2(\mathbb{C}[z])$ ) with the  $R$ -module  $N := \mathbb{C}[z] \oplus \mathbb{C}[z]$ , on which  $M_2(\mathbb{C}[z])$  acts by multiplication and  $\partial_z$  acts by formal differentiation, and  $Y$  be the partially deformation-quantized space  $Q_\lambda \Omega_{\mathbb{A}^2/\mathbb{A}^1}$  associated to the ring  $S_\lambda := \mathbb{C}\langle u, v, w \rangle / ([v, w], [u, v], [u, w] - \lambda)$ , where  $\lambda \in \mathbb{C}$ . Note<sup>21</sup> that the action of  $\partial_z$  on  $N$  induces an action of  $\partial_z$  on  $M_2(\mathbb{C}[z])$  by the entry-wise formal differentiation and the  $\mathbb{A}^2/\mathbb{A}^1$  corresponds to  $\mathbb{C}[v] \hookrightarrow \mathbb{C}[v, w]$ . Consider the following special class of morphisms:

$$\begin{array}{ccc} X & \xrightarrow{\varphi_{(A,B)}} & Y \\ R & \xleftarrow{\varphi_{(A,B)}^\sharp} & S_\lambda \\ \lambda\partial_z + A & \xleftarrow{\quad} & u \\ B & \xleftarrow{\quad} & v \\ z & \xleftarrow{\quad} & w \end{array}, \quad A, B \in M_2(\mathbb{C}[z]),$$

subject to  $[\lambda\partial_z + A, B] = 0$ . (The other two constraints,  $[B, z] = 0$  and  $[\lambda\partial_z + A, z] - \lambda = 0$ , are automatic.) Let

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix},$$

where  $a_i, b_j \in \mathbb{C}[z]$  and assume that  $\lambda \neq 0$ . Then, the associated system  $\lambda\partial_z B + [A, B] = 0$  of homogeneous linear ordinary differential equations on  $B$  has a solution if and only if  $A$  satisfies

$$(a_1 - a_4)^2 + 4a_2a_3 = 0.$$

Under this condition on  $A$ , the system has four fundamental solutions:

$$\begin{aligned} B_1 &= \begin{bmatrix} 1 + \lambda^{-2}a_2a_3z^2 & \lambda^{-1}a_2z - \frac{1}{2}\lambda^{-2}(a_1 - a_4)a_2z^2 \\ -\lambda^{-1}a_3z - \frac{1}{2}\lambda^{-2}(a_1 - a_4)a_3z^2 & -\lambda^{-2}a_2a_3z^2 \end{bmatrix}, \\ B_2 &= \begin{bmatrix} \lambda^{-1}a_3z - \frac{1}{2}\lambda^{-2}(a_1 - a_4)a_3z^2 & 1 - \lambda^{-1}(a_1 - a_4)z - \lambda^{-2}a_2a_3z^2 \\ -\lambda^{-2}a_3^2z^2 & -\lambda^{-1}a_3z + \frac{1}{2}\lambda^{-2}(a_1 - a_4)a_3z^2 \end{bmatrix}, \\ B_3 &= \begin{bmatrix} -\lambda^{-1}a_2z - \frac{1}{2}\lambda^{-2}(a_1 - a_4)a_2z^2 & -\lambda^{-2}a_2^2z^2 \\ 1 + \lambda^{-1}(a_1 - a_4)z - \lambda^{-2}a_2a_3z^2 & \lambda^{-1}a_2z + \frac{1}{2}\lambda^{-2}(a_1 - a_4)a_2z^2 \end{bmatrix}, \\ B_4 &= \begin{bmatrix} -\lambda^{-2}a_2a_3z^2 & -\lambda^{-1}a_2z + \frac{1}{2}\lambda^{-2}(a_1 - a_4)a_2z^2 \\ \lambda^{-1}a_3z + \frac{1}{2}\lambda^{-2}(a_1 - a_4)a_3z^2 & 1 + \lambda^{-2}a_2a_3z^2 \end{bmatrix}. \end{aligned}$$

<sup>21</sup>Also, we take the convention that  $\partial_z \cdot m$  means the product in  $\mathbb{C}\langle M_2(\mathbb{C}[z]), \partial_z \rangle$  and  $\partial_z m$  means entry-wise formal differentiation of  $m$ , for  $m \in M_2(\mathbb{C}[z])$ .

Denote this solution space by  $\mathbb{C}_A^4$  with coordinates  $(\hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_4)$  and the correspondence

$$(\hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_4) \longleftrightarrow \hat{b}_1 B_1 + \hat{b}_2 B_2 + \hat{b}_3 B_3 + \hat{b}_4 B_4 =: B_{(\hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_4)}.$$

Then,

- the degree-0 term  $B_{(0)}$  of  $B = B_{(\hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_4)}$  (in  $z$ -powers) is given by  $\begin{bmatrix} \hat{b}_1 & \hat{b}_2 \\ \hat{b}_3 & \hat{b}_4 \end{bmatrix}$ ,
- the characteristic polynomial of  $B$  is identical to that of  $B_{(0)}$ .

It follows that the image  $Im \varphi_{(A,B)}$  of  $\varphi_{(A,B)}$  is a (complex-)codimension-1 sub-quantum scheme in  $Y$  whose associated ideal in  $S_\lambda$  contains the ideal

$$(v^2 - tr B_{(0)} v + det B_{(0)}) .$$

Let  $\mu_-$  and  $\mu_+$  be the eigen-values of  $B_{(0)}$ .

*Case (a) :  $\nu_- \neq \nu_+$ .* In this case, the above ideal  $((v - \nu_-)(v - \nu_+))$  coincides with  $Ker \varphi_{(A,B)}^\sharp$  and, hence, describes precisely  $Im \varphi_{(A,B)} \subset Y$ . Since  $\varphi_{(A,B)}^\sharp(v) = B$ , let  $N_- := Ker(B - \nu_-) \subset N$ . This is a rank-1  $\mathbb{C}[z]$ -submodule of  $\mathbb{C}[z] \oplus \mathbb{C}[z]$  that is invariant also under  $\varphi_{(A,B)}^\sharp(S_\lambda)$ . This gives  $N_-$  a  $S_\lambda/(v - \nu_-)$ -module structure that has rank-1 as  $\mathbb{C}[w]$ -module. Similarly,  $N_+ := Ker(B - \nu_+) \subset N$  is invariant under  $\varphi_{(A,B)}^\sharp(S_\lambda)$  and has a  $\varphi_{(A,B)}^\sharp$ -induced  $S_\lambda/(v - \nu_+)$ -module structure that is of rank-1 as  $\mathbb{C}[w]$ -module. Let

$$\begin{aligned} Z &:= Im \varphi_{(A,B)} = Space(S_\lambda / ((v - \nu_-)(v - \nu_+))) \\ &= Space(S_\lambda / (v - \nu_-)) \cup Space(S_\lambda / (v - \nu_+)) =: Z_- \cup Z_+ \end{aligned}$$

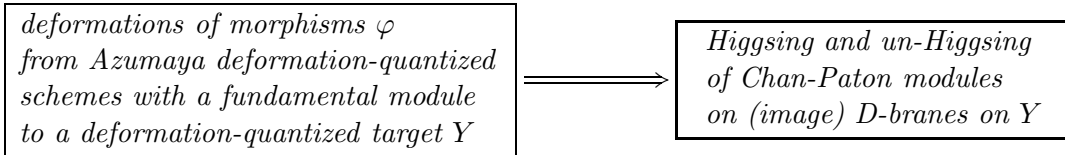
be the two connected components of the quantum subscheme  $Im \varphi_{(A,B)} \subset Y$  and denote the  $\mathcal{O}_{Z_-}$ -modules associated to  $N_-$  and  $N_+$  by  $(s_\lambda N_-)^\sim$  and  $(s_\lambda N_+)^\sim$  respectively. Then

$$\varphi_{(A,B),*} \mathcal{E} = (s_\lambda N_-)^\sim \oplus (s_\lambda N_+)^\sim \quad \text{with } (s_\lambda N_-)^\sim \text{ supported on } Z_- \text{ and } (s_\lambda N_+)^\sim \text{ on } Z_+ .$$

*Case (b) :  $\nu_- = \nu_+ = \nu$ .* In this case,  $Ker \varphi_{(A,B)}^\sharp$  can be either  $(v - \nu)$  or  $((v - \nu)^2)$  and both situations happen.

- When  $Ker \varphi_{(A,B)}^\sharp = (v - \nu)$ ,  $N = \mathbb{C}[z] \oplus \mathbb{C}[z]$  has a  $\varphi_{(A,B)}^\sharp$ -induced  $S_\lambda/(v - \nu)$ -module structure and  $\varphi_{(A,B),*} \mathcal{E}$  has support  $Im \varphi_{(A,B)} = Space(S_\lambda / (v - \nu)) \subset Y$ .
- When  $Ker \varphi_{(A,B)}^\sharp = ((v - \nu)^2)$ ,  $N = \mathbb{C}[z] \oplus \mathbb{C}[z]$  has a  $\varphi_{(A,B)}^\sharp$ -induced  $S_\lambda/((v - \nu)^2)$ -module structure and  $\varphi_{(A,B),*} \mathcal{E}$  has support  $Z := Im \varphi_{(A,B)} = Space(S_\lambda / ((v - \nu)^2)) \subset Y$ . It contains an  $\mathcal{O}_Z$ -submodule  $(s_\lambda N_0)^\sim$ , associated to  $N_0 := Ker(v - \nu) \subset N$ , that is supported on  $Z_0 := Space(S_\lambda / (v - \nu)) \subset Z$ . In other words, in the current situation,  $\varphi_{(A,B),*} \mathcal{E}$  not only is of rank-2 as a  $\mathbb{C}[w]$ -module but also has a built-in  $\varphi_{(A,B)}$ -induced filtration  $(s_\lambda N_0)^\sim \subset \varphi_{(A,B),*} \mathcal{E}$ .

Thus, by varying  $(A, B)$  in the solution space of  $\lambda\partial_z B + [A, B] = 0$  so that the eigen-values of  $B_{(0)}$  change from being distinct to being identical and vice versa, one realizes the Higgsing and un-Higgsing phenomena of D-branes in superstring theory for the current situation as deformations of morphisms from Azumaya quantum schemes to the open-string quantum target-space  $Y$ :



Cf. [L-L-S-Y: FIGURE 2-1-1] for a similar phenomenon.

This concludes the example.

## 5.2 Deformation quantizations of spectral covers in a cotangent bundle.

We employ the notions from the previous subsection to discuss the notion of “quantum spectral covers”<sup>22</sup> from the viewpoint of Azumaya geometry and the Polchinski-Grothendieck Ansatz. A special case of this gives the notion of deformation quantizations of spectral curves in Hitchin’s integrable systems.

### A 1-parameter family of deformation quantizations of the cotangent bundle $\Omega_W$ .

Let  $W$  be a smooth variety of dimension  $n$  over  $\mathbb{C}$ ,  $\Omega_W$  be its sheaf of Kähler differentials, and  $\mathbf{\Omega}_W$  be the total space  $\mathbf{Spec}(Sym^\bullet \Omega_W^\vee) = \mathbf{Spec}(Sym^\bullet \Theta_W)$  of  $\Omega_W$ . One may construct a 1-parameter family of deformation quantizations of  $\mathbf{\Omega}_W$  as follows.

Let  $p \in W$  be a geometric point on  $W$ . Then there exists a Zariski open neighborhood  $U$  of  $p$  in  $W$  such that  $\Omega_U$  is a free  $\mathcal{O}_U$ -module and that there admits an étale morphism  $\pi : U \rightarrow \mathbb{A}^n = \text{Spec}(\mathbb{C}[w_1, \dots, w_n])$ . Denote the lifting of  $w_i$  and  $\partial_{w_i}$  on  $\mathbb{A}^n$  to  $U$  under  $\pi$  also by  $w_i$  and  $\partial_{w_i}$  respectively, for  $i = 1, \dots, n$ . Then as both  $U$  and  $\pi$  are smooth,  $\mathcal{D}_U$  as an  $\mathcal{O}_W(U)$ -algebra given by  $\mathbb{C}\langle \mathcal{O}_W(U), \partial_{w_1}, \dots, \partial_{w_n} \rangle$ . This is abstractly the algebra  $\mathbb{C}\langle \mathcal{O}_W(U), p_1, \dots, p_n \rangle / I$  with  $I$  the two-sided ideal  $([p_i, p_j], [p_i, w_j] - \delta_{ij} : 1 \leq i, j \leq n)$ . Here, we think of  $p_i$  as a local section of the tangent sheaf  $\mathcal{T}_W$  of  $W$  without a pre-assigned action on  $\mathcal{O}_W$ . Note that, as  $\mathcal{O}_W(U)$  is integral over an open subset of  $\mathbb{A}^n$  under  $\pi$  and both  $U$  and  $\pi$  are smooth, the set of equations  $[p_i, w_j] = \delta_{ij}$ ,  $1 \leq i, j \leq n$ , determine the commutator  $[p_i, f] \in \mathcal{O}_W(U)$ , which is  $\partial_{w_i} f$ , for all  $f \in \mathcal{O}_W(U)$  and  $i = 1, \dots, n$ .

<sup>22</sup>The current subsection is written with the particular works [D-H-S-V] and [D-H-S] of Dijkgraaf, Hollands, Sulkowski, and Vafa in mind. We thank Cumrun Vafa for the illuminations of [D-H-S-V]. These works involve several mathematical themes. Here we focus on a particular one: the notion of *quantum spectral curves from the viewpoint of D-branes*. For that reason, it is not very appropriate to attach a sub-title like *Dijkgraaf-Holland-Sulkowski-Vafa vs. Polchinski-Grothendieck* to this subsection though this is indeed what this subsection is meant to be for the relevant part of [D-H-S-V] and [D-H-S]. Readers are referred *ibidem* and references therein for related string contents/pictures.

**Notation 5.2.1. [unital associative algebra generated by module].** (1) Let  $S$  be a commutative ring and  $R$  be a commutative  $S$ -algebra with a built-in  $S \subset R$ , and  $M$  be a finitely generated  $R$ -module. Denote by  $S\langle M \rangle$  the unital associative  $S$ -algebra generated by elements of  $M$  with the requirement that  $S$  be in the center, and by  $S\langle R, M \rangle$  be the unital associative  $S$ -algebra generated by  $R \cup S\langle M \rangle$  with a built-in  $S$ -algebra inclusions,  $R \subset S\langle R, M \rangle$  and  $S\langle M \rangle \subset S\langle R, M \rangle$ . Note that  $S$  is in the center of  $S\langle R, M \rangle$  while  $R$  in general is not.

(2) Let  $Z$  be a scheme over a base  $\mathbb{C}$ -scheme  $B$  and  $\mathcal{F}$  be a coherent  $\mathcal{O}_Z$ -module. Denote by  $\mathcal{O}_B\langle \mathcal{O}_Z, \mathcal{F} \rangle$  the sheaf of unital associative  $\mathcal{O}_Z$ -algebras from the enlargement of  $\mathcal{O}_Z$  by elements of  $\mathcal{F}$  with the requirement that the built-in  $\mathcal{O}_B \subset \mathcal{O}_B\langle \mathcal{O}_Z, \mathcal{F} \rangle$  be in the center. Over an affine open subset  $U$  of  $Z$  that sits over an affine open subset  $V$  of  $B$ ,  $\mathcal{O}_B\langle \mathcal{O}_Z, \mathcal{F} \rangle(U)$  is the unital associative algebra  $\mathcal{O}_B(V)\langle \mathcal{O}_Z(U), \mathcal{F}(U) \rangle$ . Note that the image of the built-in inclusion  $\mathcal{O}_Z \subset \mathcal{O}_B\langle \mathcal{O}_Z, \mathcal{F} \rangle$  in general does not lie in the center.

(3) Let  $\Theta_{Z/B}$  be the sheaf of  $\mathcal{O}_B$ -derivations on  $Z/B$  and  $\mathcal{T}_{Z/B}$  be the relative tangent sheaf of  $Z/B$ . They are canonically isomorphic  $\mathcal{O}_Z$ -modules. However, for convenience, we take the convention that  $\mathcal{O}_B\langle \mathcal{O}_Z, \Theta_{Z/B} \rangle$  is the sheaf  $\mathcal{D}_{Z/B}$  of algebras of differential operators on  $\mathcal{O}_{Z/B}$  (i.e. the  $\Theta_{Z/B}$ -action on  $\mathcal{O}_Z$  via derivations is already included into its construction by setting  $[p_i, f] = \partial_{w_i} f$ ) and that  $\mathcal{O}_B\langle \mathcal{O}_Z, \mathcal{T}_{Z/B} \rangle$  is constructed as Item (2) above, with  $\mathcal{T}_{Z/B}$  treated only as an abstract coherent  $\mathcal{O}_Z$ -module.

Let  $\mathcal{T}_{(\mathbb{A}^1 \times W)/\mathbb{A}^1}$  be the relative tangent sheaf of  $(\mathbb{A}^1 \times W)/\mathbb{A}^1$ . Consider the  $\mathcal{O}_{\mathbb{A}^1 \times W}$ -algebra  $\mathcal{O}_{\mathbb{A}^1}\langle \mathcal{O}_{\mathbb{A}^1 \times W}, \mathcal{T}_{(\mathbb{A}^1 \times W)/\mathbb{A}^1} \rangle$ . Here, we take  $\mathbb{A}^1$  as  $\text{Spec}(\mathbb{C}[\lambda])$ . Let  $I$  be the two-sided ideal sheaf of  $\mathcal{O}_{\mathbb{A}^1}\langle \mathcal{O}_{\mathbb{A}^1 \times W}, \mathcal{T}_{(\mathbb{A}^1 \times W)/\mathbb{A}^1} \rangle$  whose value over  $\mathbb{A}^1 \times U$ , for  $U$  being an affine open subset of  $W$  over which  $\mathcal{T}_W$  is trivialized by  $(p_1, \dots, p_n)$  corresponding to  $(\partial_{w_1}, \dots, \partial_{w_n})$ , is given by

$$([p_i, p_j], [p_i, w_j] - \lambda : 1 \leq i, j \leq n).$$

Note that, by definition,  $[\lambda, p_i] = 0$ , for  $i = 1, \dots, n$ , and that  $[p_i, f] = \lambda \partial_{w_i} f$ , where  $f \in \mathcal{O}_{\mathbb{A}^1 \times W}(\mathbb{A}^1 \times U)$ , in the quotient  $\mathcal{O}_{\mathbb{A}^1 \times W}$ -algebra  $\mathcal{O}_{\mathbb{A}^1}\langle \mathcal{O}_{\mathbb{A}^1 \times W}, \mathcal{T}_{(\mathbb{A}^1 \times W)/\mathbb{A}^1} \rangle/I$ . This gives a noncommutative space

$$Q_{\mathbb{A}^1} \Omega_W := \text{Space}(\mathcal{O}_{\mathbb{A}^1}\langle \mathcal{O}_{\mathbb{A}^1 \times W}, \mathcal{T}_{(\mathbb{A}^1 \times W)/\mathbb{A}^1} \rangle/I)$$

over  $\mathbb{A}^1$ . It has the following properties:

- (1) The fiber  $Q_\lambda \Omega_W$  over  $\lambda \neq 0$  is isomorphic to the noncommutative space  $\text{Space } \mathcal{D}_W$ .
- (2) The fiber over  $\lambda = 0$  is the commutative scheme  $\Omega_W$ .
- (3) There is no local section of  $\mathcal{O}_{\mathbb{A}^1}\langle \mathcal{O}_{\mathbb{A}^1 \times W}, \mathcal{T}_{(\mathbb{A}^1 \times W)/\mathbb{A}^1} \rangle$  that is annihilated, either from the left or from the right, by a non-zero element of  $\mathbb{C}[\lambda]$ . Thus, we may think of  $Q_{\mathbb{A}^1} \Omega_W/\mathbb{A}^1$  as a flat family of generically noncommutative spaces over  $\mathbb{A}^1$ , parameterized by  $\lambda$ .

**Definition 5.2.2. [canonical family of deformation quantization].** We shall call the noncommutative space  $Q_{\mathbb{A}^1} \Omega_W$  over  $\mathbb{A}^1$  the *canonical family of deformation quantizations* of  $\Omega_W$ .



Let  $W$  be as above and  $\mathcal{E}$  and  $\mathcal{N}$  be coherent locally free  $\mathcal{O}_W$ -modules. Then one has the canonical isomorphisms

$$\mathrm{Hom}_{\mathcal{O}_W}(\mathcal{E}, \mathcal{E} \otimes \mathcal{N}) \simeq \Gamma(\mathcal{E}^\vee \otimes \mathcal{E} \otimes \mathcal{N}) \simeq \mathrm{Hom}_{\mathcal{O}_W}(\mathcal{N}^\vee, \mathrm{End}_{\mathcal{O}_W}(\mathcal{E})).$$

Here,  $(\bullet)^\vee$  denotes the dual  $\mathcal{O}_W$ -module of  $(\bullet)$ .

**Definition 5.2.3.** [commutativity-admissible  $\phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{N}$ ]. An  $\mathcal{O}_W$ -module homomorphism  $\phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{N}$  is said to be *commutativity-admissible* if its corresponding  $\hat{\phi} : \mathcal{N}^\vee \rightarrow \mathrm{End}_{\mathcal{O}_W}(\mathcal{E})$  has image contained in a commutative  $\mathcal{O}_W$ -subalgebra of  $\mathrm{End}_{\mathcal{O}_W}(\mathcal{E})$ .

A commutativity-admissible  $\phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{N}$  induces an  $\mathcal{O}_W$ -algebra homomorphism

$$\varphi^\# : \mathrm{Sym}^\bullet(\mathcal{N}^\vee) \longrightarrow \mathrm{End}_{\mathcal{O}_W}(\mathcal{E}),$$

which defines a morphism  $\varphi$  from the Azumaya scheme with a fundamental module  $(W^{\mathrm{Az}}, \mathcal{E}) := (W, \mathcal{O}_W^{\mathrm{Az}} := \mathrm{End}_{\mathcal{O}_W}(\mathcal{E}), \mathcal{E})$  to the total space  $\mathcal{N} = \mathbf{Spec}(\mathrm{Sym}^\bullet(\mathcal{N}^\vee))$  of the  $\mathcal{O}_W$ -module  $\mathcal{N}$ . Note that, in this case, both  $W^{\mathrm{Az}}$  and  $\mathcal{N}$  are spaces over  $W$  and  $\varphi : (W^{\mathrm{Az}}, \mathcal{E}) \rightarrow \mathcal{N}$  is a morphism between spaces over  $W$ . Let  $\pi : \mathcal{N} \rightarrow W$  be the built-in morphism. Then, by construction,  $\pi_*\varphi_*\mathcal{E} \simeq \mathcal{E}$  canonically.

**Lemma 5.2.4.** [ $\phi$  vs.  $\varphi$ ]. *Given coherent locally free  $\mathcal{O}_W$ -modules  $\mathcal{E}$  and  $\mathcal{N}$ , there is a canonical one-to-one correspondence*

$$\left\{ \begin{array}{l} \text{commutativity-admissible} \\ \mathcal{O}_W\text{-module homomorphisms} \\ \phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{N} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{morphisms } \varphi : (W^{\mathrm{Az}}, \mathcal{E}) \rightarrow \mathcal{N} \\ \text{as spaces over } W \end{array} \right\}.$$

**Lemma 5.2.5.** [generalized spectral cover from morphisms from Azumaya scheme].

*When  $\mathcal{N}$  is a line bundle  $\mathcal{L}$  on  $W$ , any  $\phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{L}$  is commutativity-admissible and the image scheme  $\mathrm{Im} \varphi \subset \mathcal{N}$  of the corresponding  $\varphi$  lies in the spectral cover  $\Sigma_{(\mathcal{E}, \phi)}$  in  $\mathcal{N}$  associated to the pair  $(\mathcal{E}, \phi)$ . If furthermore  $\Sigma_{(\mathcal{E}, \phi)}$  is smooth, then  $\mathrm{Im} \varphi = \Sigma_{(\mathcal{E}, \phi)}$ .*

<sup>23</sup>Readers are referred to [Hi1], [B-N-R], [Ox], [Ni], and [Ma], [Don], [Do-M] for the classical study of *Higgs pairs* and their associated *spectral curves/covers*. The current theme continues the discussion of the theme “*Comparison with the spectral cover construction and the Hitchin system*” in [L-Y1: Sec. 4.1]. Here, we see one more example of the ubiquity of Azumaya geometry in mathematics and its recovering of D-brane phenomena. As illustrated in the precedent D(1) - D(4), Azumaya geometry is a very fundamental nature and geometry a D-brane world-volume carries. It gives the common origin of many of the D-brany phenomena. Furthermore, like what happens here, such a structure is actually hidden in many mathematical problems as well. Despite the introduction of the notion of “maximally central algebra” by Prof. Goro Azumaya in [Az] in year 1951, which later came to be called “Azumaya algebra”, and the study of it from the viewpoint of algebras and representation theory, the investigation of it as a geometric object started only much later, cf. related reference in [L-Y1], [L-Y2], and [L-Y3]. The full richness of Azumaya geometry remains to be explored.

<sup>24</sup>From C.-H.L. : The setting here rewrites and generalizes some discussions with *Mihnea Popa* on the connection between the D-brane Higgsing/un-Higgsing phenomenon and spectral covers in spring 2002 ([Liu] and [Popa]). During years 2001 - 2005, Mihnea was giving lectures on a wide span of topics in algebraic geometry: from Grothendieck’s foundation to geometric topics at the frontier, while *Shiraz Minwalla* was giving lectures on an equally wide span of topics in theoretical high energy physics: from quantum field theory and supersymmetry foundation to stringy topics at the frontier. Their systematic lectures in these four years were filled with insight and enthusiasm and played a definite role for the revival of the project in early 2007.

This is consistent with the fact that, in the case of the Lemma, the morphism  $\varphi$  carries the full information of the pair  $(\mathcal{E}, \phi)$  while the spectral cover  $\Sigma_{(\mathcal{E}, \phi)}$  may not. When  $\mathcal{N}$  is a general coherent locally-free  $\mathcal{O}_W$ -module, then  $\varphi$  can be thought of interchangeably as a *generalized Higgs pair*  $(\mathcal{E}, \phi)$  for  $\mathcal{N}$  with  $\phi$  commutativity-admissible.

*Remark 5.2.6.* [*general  $\phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{N}$* ]. Let  $\mathcal{E}$  and  $\mathcal{N}$  be as above. Let  $\mathcal{O}_W\langle \mathcal{N}^\vee \rangle$  be the unital associative  $\mathcal{O}_W$ -algebra generated by  $\mathcal{N}^\vee$  with the requirement that  $\mathcal{O}_W$  be in the center. Its associative “space”  $Space(\mathcal{O}_W\langle \mathcal{N}^\vee \rangle)$  can be thought of as a noncommutative-affine-space bundle over  $W$ . A general  $\mathcal{O}_W$ -module homomorphism  $\phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \mathcal{N}$  corresponds then to an  $\mathcal{O}_W$ -algebra homomorphism  $\varphi^\sharp : \mathcal{O}_W\langle \mathcal{N}^\vee \rangle \rightarrow \mathcal{E}nd_{\mathcal{O}_W}(\mathcal{E})$ . This can be thought of as defining a morphism  $\varphi : (W^{Az}, \mathcal{E}) \rightarrow Space(\mathcal{O}_W\langle \mathcal{N}^\vee \rangle)$  over  $W$ .

### Deformation quantization of spectral covers via morphisms from Azumaya schemes with a fundamental module with a flat connection.

Let  $W$  be a smooth variety over  $\mathbb{C}$ ,  $\mathcal{E}$  and  $\mathcal{N}$  be coherent locally free  $\mathcal{O}_W$ -modules,  $\phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_W$  be a commutativity-admissible  $\mathcal{O}_W$ -module homomorphism, and  $\varphi : (W^{Az}, \mathcal{E}) \rightarrow \Omega_W$  be the associated morphism between spaces over  $W$ . Let  $\mathcal{E}_{\mathbb{A}^1}$  be the locally-free  $\mathcal{O}_{\mathbb{A}^1 \times W}$ -module on  $\mathbb{A}^1 \times W$  from the pull-back of  $\mathcal{E}$  under the projection map  $\mathbb{A}^1 \times W \rightarrow W$ . Here we take  $\mathbb{A}^1$  as  $Spec(\mathbb{C}[\lambda])$  and  $\mathcal{E}_{\mathbb{A}^1}$  as a constant family of  $\mathcal{O}_W$ -modules over  $\mathbb{A}^1$ . Denote  $Spec(\mathbb{C}[\lambda, \lambda^{-1}])$  by  $\mathbb{A}^1 - \{0\}$  and the restriction of  $\mathcal{E}_{\mathbb{A}^1}$  to over  $\mathbb{A}^1 - \{0\}$  by  $\mathcal{E}_{\mathbb{A}^1 - \{0\}}$ . For convenience, we will denote a number in  $\mathbb{C}$  also by  $\lambda$ .

**Definition 5.2.7.** [ **$\lambda$ -connection**]. ([Ari: Definition 2.1 and Example 2.2].) For  $\lambda \in \mathbb{C}$ , a  $\lambda$ -connection on  $\mathcal{E}$  is a  $\mathbb{C}$ -linear map  $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_W$  which satisfies the  $\lambda$ -Leibniz rule:

$$\nabla(fs) = \lambda \cdot s \otimes df + f\nabla s$$

for any  $f \in \mathcal{O}_W$ ,  $s \in \mathcal{E}$ . Note that a 0-connection on  $\mathcal{E}$  is an  $\mathcal{O}_W$ -module homomorphism and, for  $\lambda \neq 0$ ,  $\nabla$  is a  $\lambda$ -connection if and only if  $\lambda^{-1}\nabla$  is an (ordinary) connection. A  $\lambda$ -connection  $\nabla$  on  $\mathcal{E}$  is said to be *flat* if the connection  $\lambda^{-1}\nabla$  is flat.

The notion of  $\lambda$ -connection was introduced by Deligne; it gives an interpolation between a Higgs field and a connection on  $\mathcal{E}$ , cf. [Sim].

**Definition 5.2.8.** [ **$\lambda$ -connection deformation of  $\phi$** ]. An  $\mathcal{O}_{\mathbb{A}^1 - \{0\}}$ -module homomorphism

$$\nabla : \mathcal{E}_{\mathbb{A}^1 - \{0\}} \longrightarrow \mathcal{E}_{\mathbb{A}^1 - \{0\}} \otimes \Omega_{((\mathbb{A}^1 - \{0\}) \times W)/(\mathbb{A}^1 - \{0\})}$$

that satisfies:

- (1) on each  $\mathcal{E}_\lambda := \mathcal{E}_{\mathbb{A}^1}|_\lambda$  over a closed point, parameterized by  $\lambda$ , of  $\mathbb{A}^1 - \{0\}$ ,  $\nabla^\lambda := \nabla|_\lambda : \mathcal{E}_\lambda \rightarrow \mathcal{E}_\lambda \otimes \Omega_W$  is a  $\lambda$ -connection on  $\mathcal{E}_\lambda$ ,
- (2)  $\nabla|_{\lambda=0} = \phi$ ,

is called a  $\lambda$ -connection deformation of  $\phi$ . If furthermore each  $\nabla^\lambda$  is flat, then  $\nabla$  is called a *flat  $\lambda$ -connection deformation* of  $\phi$ .

Let  $X_{\mathbb{A}^1} = \text{Space}(\mathcal{D}_{(\mathbb{A}^1 \times W)/\mathbb{A}^1})$ ,  $(\mathcal{E}_{\mathbb{A}^1}, \nabla \text{ on } \mathcal{E}_{\mathbb{A}^1 - \{0\}})$  be as above, and  $Y_{\mathbb{A}^1} = Q_{\mathbb{A}^1} \Omega_W$  be the canonical family of deformation quantizations of  $\Omega_W$ . Let

$$\begin{array}{ccc}
X_{\mathbb{A}^1} & \xrightarrow{\varphi_{\mathbb{A}^1}} & Y_{\mathbb{A}^1} \\
\mathcal{O}_{\mathbb{A}^1} \langle \mathcal{O}_{\mathbb{A}^1 \times W}, \Theta_{(\mathbb{A}^1 \times W)/\mathbb{A}^1} \rangle & \xleftarrow{\varphi_{\mathbb{A}^1}^\#} & \mathcal{R} \\
\lambda & \xleftarrow{\quad} & \lambda \\
w_i & \xleftarrow{\quad} & w_i \\
\lambda \partial_{w_i} & \xleftarrow{\quad} & p_i \quad .
\end{array}$$

Here we adopt the notation in the construction of  $Q_{\mathbb{A}^1} \Omega_W$ . Then,  $\varphi_{\mathbb{A}^1}$  is a morphism of spaces over  $\mathbb{A}^1$  with the following the properties:

- $\varphi_0 := \varphi_{\mathbb{A}^1}|_{\lambda=0}$  is the composition  $(W^{Az, \mathcal{D}}, \mathcal{E}^{\nabla'}) \longrightarrow (W^{Az}, \mathcal{E}) \xrightarrow{\varphi} \Omega_W$ ; here, as  $\lambda^{-1} \nabla$  does not extend over  $\lambda = 0$ , we take  $\nabla'$  to be an arbitrary auxiliary flat connection on  $\mathcal{E}$  to render  $\mathcal{E}$  a  $\mathcal{D}_W$ -module and let  $(W^{Az, \mathcal{D}}, \mathcal{E}^{\nabla'}) \longrightarrow (W^{Az}, \mathcal{E})$  be the built-in dominant morphism;
- $\varphi_\lambda := \varphi_{\mathbb{A}^1}|_\lambda : (W^{Az, \mathcal{D}}, \mathcal{E}^{\lambda^{-1} \nabla_\lambda}) \longrightarrow Q_\lambda \Omega_W$ , for  $\lambda \in \mathbb{A}^1 - \{0\}$ .

Note that  $\varphi_{0*} \mathcal{E} = \varphi_* \mathcal{E}$  is a  $\mathcal{O}_{\Omega_W}$ -module flat over  $W$  with relative dimension 0 and relative length =  $\text{rank } \mathcal{E}$ .  $\text{Supp}(\mathcal{E}) = \text{Im } \varphi$ , which is identical to the spectral curves  $\Sigma_{(\mathcal{E}, \phi)}$  when  $W$  is a smooth curve  $C$  and  $\Sigma_{(\mathcal{E}, \phi)}$  is smooth. On the other hand, for  $\lambda \neq 0$ ,  $\varphi_{\lambda*} \mathcal{E}$  has support  $\text{Im } \varphi_\lambda$ , which is the whole  $Q_\lambda \Omega_W$ . The characteristic variety for the  $\mathcal{O}_{Q_\lambda \Omega_W}$ -module  $\varphi_{\lambda*} \mathcal{E}$  is the zero-section of  $\Omega_W/W$ .

*Remark 5.2.9. [existence/interpretation of quantum spectral covers].* Due to the fact that the Weyl algebras are simple algebras, the spectral curve  $\Sigma_{(\mathcal{E}, \phi)}$  in  $\Omega_C$  in general do not have a direct deformation quantization into  $Q_\lambda \Omega_C$  by the ideal sheaf of  $\Sigma_{(\mathcal{E}, \phi)}$  in  $\mathcal{O}_{\Omega_C}$  since this will only give  $\mathcal{O}_{Q_\lambda \Omega_C}$ , which corresponds to the empty subspace of  $Q_\lambda \Omega_C$ . The setting above replaces the notion of quantum spectral curves by quantum deformation  $\varphi_\lambda$  of the morphism  $\varphi$  associated to the Higgs/spectral pair  $(\mathcal{E}, \phi)$ .

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