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On the splitting type of an equivariant vector bundle over a toric manifold

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Abstract

From the work of Lian, Liu, and Yau on "Mirror Principle", in the explicit computation of the Euler data $Q = \{Q_0, Q_1, \cdots\}$ for an equivariant concavex bundle $\mathcal{E}$ over a toric manifold, there are two places the structure of the bundle comes into play: (1) the multiplicative characteric class Q_0 of V one starts with, and (2) the splitting type of $\mathcal{E}$. Equivariant bundles over a toric manifold has been classified by Kaneyama, using data related to the linearization of the toric action on the base toric manifold. In this article, we relate the splitting type of $\mathcal{E}$ to the classifying data of Kaneyama. From these relations, we compute the splitting type of a couple of nonsplittable equivariant vector bundles over toric manifolds that may be of interest to string theory and mirror symmetry. A code in Mathematica that carries out the computation of some of these examples is attached.

Key words: toric manifold, equivariant vector bundle, spliting numbers, splitting type.

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0. Introduction and outline.

Introduction.

From the work of Lian, Liu, and Yau on "Mirror Principle", in the explicit computation of the Euler data $Q = \{Q_0, Q_1, \cdots\}$ for an equivariant concavex bundle $\mathcal{E}$ over a toric manifold, there are two places the structure of the bundle comes into play: (1) the multiplicative characteric class Q_0 of V one starts with, and (2) the splitting type of $\mathcal{E}$. Equivariant bundles over a toric manifold has been classified by Kaneyama, using data related to the linearization of the toric action on the base toric manifold. The purpose of these notes is to relate the splitting type of $\mathcal{E}$ to these classification data of Kaneyama.

In Sec. 1, we provide some general backgrounds and notations for this article. Some basics of equivariant vector bundles over toric manifolds are provides in Sec. 2. In Sec. 3, we discuss how the splitting type of an equivariant vector bundle over a toric manifold, if exists, can be obtained from the bundle data. In Sec. 4, we give two classes of examples. In Example 4.1, we determine which non-decomposable equivariant rank 2 bundles over $\mathbb{C}\mathrm{P}^2$ admit a splitting type and work out their splitting type. In Example 4.2, we discuss the splitting type of tangent/cotangent bundles of toric manifolds. We start with the splitting type for the (co)tangent bundle of $\mathbb{C}\mathrm{P}^n$ and then turn to the case of toric surfaces. The isomorphism classes of the latter are coded in a weighted circular graph. From these weights, one can decide whose (co)tangent bundle admits a splitting type. Since all toric surfaces arise from consecutive equivariant blowups of either $\mathbb{C}\mathrm{P}^2$ or one of the the Hirzebruch surfaces $\mathbb{F}_a$ and how the weights on the weighted circular graph behavior under equivariant blowup is known, the task of deciding which (co)tangent bundle admits a splitting type and determining them for those that admit one can be done with the aid of computer. For the interest of string theory and mirror symmetry, from the toric surfaces arising from equivaraint blowups of $\mathbb{C}\mathrm{P}^2$ up to 9 points, we sort out those whose (co)tangent bundle admits a splitting type. The splitting type for their tangent bundle is also computed and listed. A package in Mathematica that carries out this computation is attached for reference.

Overall, this is part of the much bigger ambition of toric mirror symmetry computation via Euler data, as discussed in [L-L-Y1, L-L-Y2, L-L-Y3]. We leave the application of the current article to this goal for another work.

Outline.

1 Essential backgrounds and notations for physicists.

In this section, we collect some basic facts and notations that will be needed in the discussion. The part that is related to equivariant vector bundles over toric manifolds is singled out in Sec. 2. Readers are referred to the listed literatures for more details.

• **Toric geometry.** ([A-G-M], [C-K], [Da], [Ew], [Fu], [Gre], [G-K-Z], [Ke], and [Od1,Od2].) Physicists are referred particularly to [A-G-M] or [Gre] for a nice expository of toric geometry. Let us fix the terminology and notations here and refer the details to [Fu].

Notation:

$N \cong \mathbb{Z}^n$: a lattice;

$M = Hom(N, \mathbb{Z})$: the dual lattice of N;

$T_N = Hom(M, \mathbb{C}^*)$: the (complex) n-torus;

Σ: a fan in $N_{\mathbb{R}}$;

X_Σ: the toric variety associeted to Σ;

$\Sigma(i)$: the i-skeleton of Σ;

U_σ: the local affine chart of X_Σ associated to σ in Σ;

$x_\sigma \in U_\sigma$: the distinguished points associated to σ;

O_σ: the T_N-orbit of x_σ under the T_N-action on X_Σ;

$V(\sigma)$: the orbit closure of O_σ;

$M(\sigma) = \sigma^\perp \cap M$.

Recall that points v in the interior of $\sigma \cap N$ represent one-parameter subgroups λ_v in $\mathbb{T}_N$ such that $\lim_{z\to 0} \lambda_v(z) = x_\tau$. Recall also that the normal cones associated to a polyhedron Δ with vertices in M form a fan in $N_{\mathbb{R}}$, called the normal fan of Δ. This determines a projective toric variety.

• **Toric surface.** ([Od2].) Any complete nonsingular toric surface is obtained from consecutive equivariant blowups of either $\mathbb{CP}^2$ or one of the Hurzebruch surfaces $\mathbb{F}_a$ at T_N fixed points. Indeed, one has a complete classification of them as follows:

Fact 1.1 [toric surface]. ([Od2].) *The set of isomorphism classes of complete nonsingular toric surfaces X_Σ is in one-to-one corresponce with the set of equivalent classes of weighted circular graphs $w = (w_1, \cdots, w_s)$ (under rotation and reflection) of the following form:* (FIGURE 1-1.)

(1) The circular graph having 3 *vertices with weights* 1, 1, 1.

(2) The circular graph having 4 *vertices with weights in circular order* 0, a, 0, $-a$.

(3) The weighted circular graphs with $s \geq 5$ vertices that is obtained from one with $(s-1)$ vertices by adding a vertex of weight 1 and reducing the weight of each of its two adjacent vertices by 1.

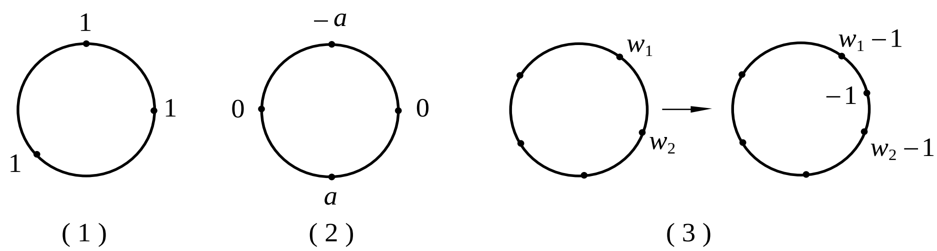

FIGURE 1-1. The weighted circular graphs that labels the isomorphism classes of complete nonsingular toric surfaces.

Let $\Sigma(1) = (v_1, \cdots, v_s)$ in, say, counterclockwise order in $N_{\mathbb{R}}$, then for each i, there exists a unique integer a_i such that $v_{i-1} + v_{i+1} + a_i v_i = 0$ (here, $s+1 \equiv 1$). The correspondence is then given by $X_\Sigma \mapsto (a_1, \cdots, a_s)$.

• **Line bundles with positive/negative c_1.** ([C-K], [Hi], and [Re].) Recall that the Kähler cone in $H^2(X_\Sigma, \mathbb{R})$ consists of all the Kähler classes of X_Σ and the Mori cone of X_Σ consists of all the classes in $H_2(X_\Sigma, \mathbb{R})$ representable by effective 2-cycles. From [Re], the Mori cone of X_Σ is generated by $V(\tau)$, $\tau \in \Sigma(n-1)$. We call a class $\omega \in H^2(X_\Sigma, \mathbb{R})$ *positive* (resp. *negative*), in notation $\omega > 0$ (resp. $\omega < 0$), if ω (resp. $-\omega$) lies in the Kähler cone of X_Σ. The fact that the Kähler cone and the Mori cone of a complete toric manifold are dual to each other gives us a criterion for a line bundle over X_Σ to have positive c_1:

Fact 1.2 [positive/negative line bundle]. *Given an n-dimensional toric manifold X_Σ. Let $L \in \operatorname{Pic}(X_\Sigma)$ be a line bundle over X_Σ and D be the associated divisor class as an element in $H_{2n-2}(X_\Sigma, \mathbb{Z})$. Then $c_1(L) > 0$ (resp. < 0) if and only if $D \cdot V(\tau) > 0$ (resp. < 0) for all $\tau \in \Sigma(n-1)$.*

• **The augmented intersection matrix.** ([Fu].) Given an n-dimensional complete nonsingular fan Σ. Let $\Sigma(1) = \{ v_1, \cdots, v_J \}$ and $\Sigma(n-1) = \{ \tau_1, \cdots, \tau_I \}$. Let A_1 and A_{n-1} be respectively the first and $(n-1)$-th Chow group of X_Σ. Then A_1 is generated by $V(\tau_i)$, $i = 1, \cdots, I$, and A_{n-1} is generated by $D(v_j)$, $j = 1, \cdots, J$. There is a nondegenerate pairing $A_1 \times A_{n-1} \to \mathbb{Z}$ by taking the intersection number. Let Q be the $I \times J$ matrix whose (i,j)-entry is the intersection number $V(\tau_i) \cdot D(v_j)$. Since the generators for A_1 and A_{n-1} used here may not be linearly independent, we shall call Q the *augmented intersection matrix* (with respect to the generators). Explicitly, Q can be determined as follows.

Let $\tau_i = [v_{j_1}, \cdots, v_{j_{n-1}}] \in \Sigma(n-1)$. Then τ_i is the intersection of two n-cones

$$\sigma_1 = [v_{j_1}, \cdots, v_{j_{n-1}}, v_{j_n}] \quad \text{and} \quad \sigma_2 = [v_{j_1}, \cdots, v_{j_{n-1}}, v_{j'_n}]$$

in Σ. These vertices in $\sigma_1 \cup \sigma_2$ satisfy a linear equation of the form

$$v_{j_n} + v_{j'_n} + a_1 v_{j_1} + \cdots + a_{n-1} v_{j_{n-1}} = 0,$$

for some unique integers $a_1, \cdots, a_n$ determined by $\sigma_1 \cup \sigma_2$. In terms of this, the i-th row of Q is simply the coefficient (row) vector of the above equation. I.e. $V(\tau_i) \cdot D(v_{j_k}) = a_k$ for $k = 1, \cdots, n-1$; $V(\tau_i) \cdot D(v_{j_n}) = V(\tau_i) \cdot D(v_{j'_n}) = 1$; and $V(\tau_i) \cdot D(v_j) = 0$ for all other j.

• **Cox homogeneous coordinate ring of a toric manifold.** ([Co] and [C-K], also [Au] and [Do].) Let Σ be a fan in $\mathbb{R}^n$ with $\Sigma(1)$ generated by $\{v_1, \cdots, v_a\}$ and $A_{n-1}(X_\Sigma)$ be the Chow group of X_Σ. Let $(z_1, \cdots, z_a)$ be the coordinates of $\mathbb{C}^a$. For $\sigma = [v_{j_1}, \cdots, v_{j_k}] \in \Sigma$, denote by $z^{\widehat{\sigma}}$ the monomial from $(z_1 \cdots z_a)/(z_{j_1} \cdots z_{j_k})$ after cancellation. Then X_Σ can be realized as the geometric quotient

$$X_\Sigma = (\mathbb{C}^{\Sigma(1)} - Z(\Sigma))/G,$$

where $Z(\Sigma)$ is the exceptional subset $\{(z_1, \cdots, z_a) \,|\, z^{\widehat{\sigma}} = 0 \text{ for all } \sigma \text{ in } \Sigma\}$ in $\mathbb{C}^a$ and G is the group $Hom_{\mathbb{Z}}(A_{n-1}(X_\Sigma), \mathbb{C}^*)$ that acts on $\mathbb{C}^a$ via the embedding in $(\mathbb{C}^*)^a$, obtained by taking $Hom(\,\cdot\,, \mathbb{C}^*)$ of the following exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & \mathbb{Z}^a & \longrightarrow & A_{n-1}(X_\Sigma) & \longrightarrow & 0 \\ & & m & \longmapsto & (m(v_1), \cdots, m(v_a)) & & & & . \end{array}$$

More facts will be recalled along the way when we need them. Their details can be found in Sec. 1-3 in [Co].

2 Basics of equivariant vector bundles over toric manifolds.

Equivariant vector bundles over a toric manifold have been classified by Kaneyama and Klyachko independently, using differently sets of data ([Ka1] and [Kl]). In this article, we use Kaneyama's data as the starting point to compute splitting types. Some necessary facts from [Ka1] and [Kl] are summarized below with possibly slight modification/rephrasing to make the geometric picture more transparent.

Equivariant vector bundles over a toric manifold.

A vector bundle $\mathcal{E}$ over a toric manifold X_Σ is equivariant if $g^*\mathcal{E} = \mathcal{E}$ for all $g \in \mathbb{T}_N$. An equivariant bundle is linearizable if the action on the base can be lifted to a fiberwise linear action on the total space of the bundle.

Fact 2.1 [linearizability]. *Every equivaraint vector bundle $\mathcal{E}$ over a toric manifold X_Σ is linearizable.*

In general, a bundle can be described by its local trivializations and the pasting maps. For an equivariant vector bundle $\mathcal{E}$ over a toric manifold X_Σ, these data can be integrated with the linearization of the toric action.

(1) *Local trivializations*: Over each invariant affine chart U_σ for $\sigma \in \Sigma$, the bundle splits:

$$\mathcal{E}|_{U_\sigma} = \oplus_\chi U_\sigma \times E_\sigma^\chi ,$$

where E_σ^χ is the representation of T_N associated to the weight $\chi \in M$.

(2) *The pasting maps*: Over each orbit $O_\tau \hookrightarrow U_{\sigma_1} \cap U_{\sigma_2}$, the pasting map $\varphi_{\sigma_2\sigma_1} : U_{\sigma_1} \cap U_{\sigma_2} \to GL(r, \mathbb{C})$ is determined by its restriction at a point, say, x_τ, in O_τ, due to the equivariant requirement. The pasting maps over different orbits in $U_{\sigma_1} \cap U_{\sigma_2}$ are related to each other by the *holomorphicity requirement* that $\varphi_{\sigma_2\sigma_1} : U_{\sigma_1} \cap U_{\sigma_2} \to GL(r, \mathbb{C})$ must be holomorphic for every pair of σ_1, σ_2 in Σ. This implies that indeed $\varphi_{\sigma_2\sigma_1}$ is completely determined by its restriction to a point, say, x_0, in the dense open orbit $U_0 \subset U_{\sigma_1} \cap U_{\sigma_2}$. Together with the local splitting propertities in (1) above, in fact $\varphi_{\sigma_2\sigma_1}$ is a regular matrix-valued function on $U_{\sigma_1} \cap U_{\sigma_2}$. Pasting maps also have to satisfy the *cocycle condition*: $\varphi_{\sigma_3\sigma_1} = \varphi_{\sigma_3\sigma_2} \circ \varphi_{\sigma_2\sigma_1}$ over $U_{\sigma_1} \cap U_{\sigma_2} \cap U_{\sigma_3}$ for every triple of σ_1, σ_2, σ_3 in Σ. This implies that the full set of pasting maps between affine charts of X_Σ is determined by the set of pasting maps $\varphi_{\sigma_2\sigma_1}$ with σ_1, $\sigma_2 \in \Sigma(n)$ that satisfy the cocycle condition.

These observations lead to Kaneyama's data for equivariant vector bundles over X_Σ.

The bundle data and the classification after Kaneyama.

(a) *The data of local trivialization*: *a collection of weight systems.*

For $\sigma \in \Sigma(n)$, x_σ is a fixed point of the toric action; thus $\mathcal{E}_{x_\sigma}$ is an invariant fiber of the lifted toric action. Associated to the representation of $\mathbb{T}^n$ on $\mathcal{E}_{x_\sigma}$ is the weight system $\mathcal{W}_\sigma \subset M$. $\mathcal{W}_\sigma$ determones the local trivialization of $\mathcal{E}|_{U_\sigma}$: $\mathcal{E}|_{U_\sigma} = \oplus_{\chi \in \mathcal{W}_\sigma} U_\sigma \times E_\sigma^\chi$.

(b) *The data of pasting*: *net of weight systems and pasting maps.*

These weight systems must satisfy a compatibility condition as follows. Let $\tau = \sigma_1 \cap \sigma_2 \in \Sigma(n-1)$ be the common codimension-1 wall of two maximal cones σ_1 and σ_2, then the stabilizer $\mathit{Stab}\,(x_\tau)$ of x_τ is an $(n-1)$-subtorus in $\mathbb{T}_N$ associated to the sublattice in N spanned by τ. Associated to the representation of $\mathit{Stab}\,(x_\tau)$ on the fiber $\mathcal{E}_{x_\tau}$ is a weight system $\mathcal{W}_\tau \subset M/M(\tau)$, where $M(\tau) = \tau^\perp \cap M$. The projection map $M \to M/M(\tau)$ induces the maps

$$\mathcal{W}_{\sigma_1} \xrightarrow{\pi_{\tau\sigma_1}} \mathcal{W}_\tau \xleftarrow{\pi_{\tau\sigma_2}} \mathcal{W}_{\sigma_2}$$

between weight systems. Since they correspond to the refinement of the $\mathit{Stab}\,(\tau)$-weight spaces to the $\mathbb{T}_N$-weight spaces, the holomorphicity requiremnent of equivariant pasting maps implied that both of these maps are surjective. Thus, $\{\mathcal{W}_\sigma \,|\, \sigma \in \Sigma(n)\}$ form a net of weight systems. FIGURE 2-1 indicates how the net of weight systems may look like for Σ that comes from the normal cone of a strong convex polyhedron Δ in M.

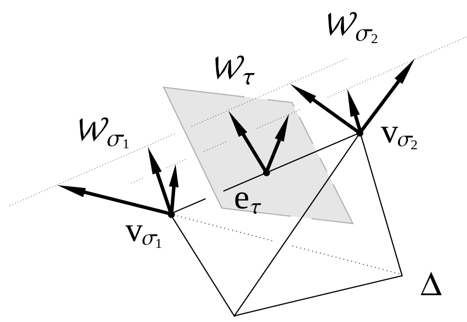

FIGURE 2-1. For Σ being a normal fan of a convex polyhedron in M, the weight system $\mathcal{W}$ can be realized as a collection of vectors (weights) at the vertices and the barycenter of the edges of Δ. The compatibility condition is translated into the condition that, for vertices v_{σ_1} and v_{σ_2} connected by an edge e_τ of Δ, the three sets $\mathcal{W}_{\sigma_1}$, $\mathcal{W}_\tau$, and $\mathcal{W}_{\sigma_2}$ have to match up under the projection along the direction parallel to e_τ.

The equivariant pasting maps are given by a map

$$P \;:\; \Sigma(n)\times\Sigma(n) \;\longrightarrow\; GL\,(r,\mathbb{C})$$

that satisfies

$$P(\sigma_3,\sigma_2)\,P(\sigma_2,\sigma_1) \;=\; P(\sigma_3,\sigma_1)\,.$$

P gives a set of compatible pasting maps for the fiber $\mathcal{E}_{x_0}$ in different $\mathcal{E}|_{U_\sigma}$ with respect to the bases given by the weight space decomposition. Holomorphicity condition for its equivariant extension to over $U_{\sigma_1}\cap U_{\sigma_2}$ requires that:

> For any $\tau = \sigma_1\cap\sigma_2 \in \Sigma(n-1)$, let $\mathcal{W}_{\sigma_i} = (\chi_{\sigma_i 1}, \cdots, \chi_{\sigma_i r})$, written with multiplicity given by the dimension of the corresponding weight space. Then $P(\sigma_2,\sigma_1)_{ij} = 0$ if $\chi_{\sigma_2 i} - \chi_{\sigma_1 j} \in M - \tau^\vee$.

(c) *Equivalence of the bundle data.*

Given Σ, two bundle data $(\mathcal{W},P)$ and $(\mathcal{W}',P')$ are said to be equivariant if $\mathcal{W}=\mathcal{W}'$ and there is a map $\rho : \Sigma(n) \to GL\,(r,\mathbb{C})$ such that $P'(\sigma_2,\sigma_1) = \rho(\sigma_2)P(\sigma_2,\sigma_1)\rho(\sigma_1)^{-1}$. Equivariant data determine isomorphic linearized equivariant vector bundles over X_Σ.

3 The splitting type of an equivariant vector bundle.

Recall first a theorem of Grothendieck ([Gro] Theorem 2.1), which says that any holomorphic vector bundle over $\mathbb{C}\mathrm{P}^1$ splits into the direct sum of a unique set of line bundles. The following definition follows from [L-L-Y2]:

Definition 3.1 [splitting type]. Let $\mathcal{E}$ be an equivariant vector bundle of rank r over a toric manifold X_Σ. Suppose that there exist nontrivial equivariant line bundles $L_1, \cdots, L_r$ over X_Σ such that each $c_1(L_i)$ is either ≥ 0 or < 0 and that the restriction $\mathcal{E}|_{V(\tau)}$ is isomorphic to the direct sum $(\oplus_{i=1}^r L_i)|_{V(\tau)}$ for any $\tau \in \Sigma(n-1)$. Then $\{L_1, \cdots, L_r\}$ is called a *splitting type* of $\mathcal{E}$.

Definition 3.2 [system of splitting numbers]. For each $\tau \in \Sigma(n-1)$, suppose that

$$\mathcal{E}|_{V(\tau)} \;=\; \oplus_{i=1}^r \mathcal{O}(d_i^\tau) \quad \text{with} \quad d_1^\tau \;\geq\; d_2^\tau \;\geq\; \cdots \;\geq\; d_r^\tau\,.$$

From Grothendieck's theorem, $(d_1^\tau, \cdots, d_r^\tau)$ is uniquely determined by $\mathcal{E}$. We shall call the set

$$\Xi(\mathcal{E}) \;=\; \{\,(d_1^\tau, \cdots, d_r^\tau)\,|\,\tau \in \Sigma(n-1)\,\}$$

the *system of splitting numbers associated to* $\mathcal{E}$.

To compute the splitting types of $\mathcal{E}$, we first extract the bundle data of $\mathcal{E}|_{V(\tau)}$ from that of $\mathcal{E}$ and then compute $\Xi(\mathcal{E})$ from the bundle data of $\mathcal{E}|_{V(\tau)}$ by weight bootstrapping. Using these numbers, one can then determine all the splitting types of $\mathcal{E}$ by the augmented intersection matrix Q associated to Σ. Let us now turn to the details.

We shall assume that the rank r of $\mathcal{E} \geq 2$.

The bundle data of $\mathcal{E}|_{V(\tau)}$.

Let $\tau = \sigma_1 \cap \sigma_2 \in \Sigma(n-1)$, $E = \mathcal{E}|_{V(\tau)}$, $U_1 = U_{\sigma_1} \cap V(\tau)$, and $U_2 = U_{\sigma_1} \cap V(\tau)$. Let v_τ be a lattice point in the interior of τ, then the pasting map $\varphi_{21}(x_\tau)$ for E from $E|_{U_1}$ to $E|_{U_2}$ over x_τ is given by $\lim_{z\to 0}(\lambda_{v_\tau}(z)P(\sigma_2,\sigma_1)\lambda_{v_\tau}(z)^{-1})$, where $z \in \mathbb{C}^*$. Since $\mathit{Stab}\,(V(\tau))$ acts on E via the T_N-action on E and the pasting map for E is T_N-equivariant and, hence, commutes with the $\mathit{Stab}\,(V(\tau))$-action, E can be decomposed into a direct sum of $\mathit{Stab}\,(V(\tau))$-weight subbundles:

$$E \;=\; \oplus_{\chi\in\mathcal{W}_\tau} E^\chi\,.$$

Indeed, for $\chi \in \mathcal{W}_\tau$, $E^\chi|_{U_1} = \oplus_{\chi' \in \pi_{\tau\sigma_1}^{-1}(\chi)} U_1 \times E_{\sigma_1}^{\chi'}$; and similarly for $E^\chi|_{U_2}$. Thus, up to a permutation of elements in the basis, we may assume that $\varphi_{21}(x_\tau)$ is in a block diagonal form with each block labelled by a distinct $\chi \in \mathcal{W}_\tau$. Let us now turn to the weight system for E at x_{σ_1} and x_{σ_2}.

Let $\tau_{\sigma_1}^{\perp}$ be the primitive lattice point of $\sigma_1^{\vee} \cap \tau^{\perp}$ in M and v_{σ_1} be a lattice point in N such that $\langle \tau_{\sigma_1}^{\perp}, v_{\sigma_1} \rangle = 1$. Let $\lambda_{v_{\sigma_1}}$ be the corresponding one-parameter subgroup in $\mathbb{T}_N$. Then $\lambda_{v_{\sigma_1}}$ acts on O_τ freely and transitively with $\lim_{z\to 0} \lambda_{v_{\sigma_1}}(z) \cdot x_\tau = x_{\sigma_1}$. Let

$$\mathcal{W}_{\sigma_1} \;=\; \{\, \chi_{\sigma_1 1},\, \chi_{\sigma_1 2},\, \cdots \} \quad \text{and} \quad \mathcal{W}_{\sigma_2} \;=\; \{\, \chi_{\sigma_2 1},\, \chi_{\sigma_2 2},\, \cdots \}$$

be the set of $\mathbb{T}_N$-weights at x_{σ_1} and x_{σ_2} respectively. Then, as a $\lambda_{v_{\sigma_1}}$-equivariant bundle with the induced linearization from the linearization of $\mathcal{E}$, the corresponding λ_{σ_1}-weights of $E_{x_{\sigma_1}}$ and $E_{x_{\sigma_2}}$ are given respectively by

$$\mathcal{W}_{\sigma_1}^{\mathbb{T}^1} \;=\; \{\, \langle \chi_{\sigma_1 1}, v_{\sigma_1} \rangle,\, \langle \chi_{\sigma_1 2}, v_{\sigma_1} \rangle,\, \cdots \} \quad \text{and} \quad \mathcal{W}_{\sigma_2}^{\mathbb{T}^1} \;=\; \{\, \langle \chi_{\sigma_2 1}, v_{\sigma_1} \rangle,\, \langle \chi_{\sigma_2 2}, v_{\sigma_1} \rangle,\, \cdots \}\,.$$

Notice that both sets depend on the choice of v_{σ_1}; however, different choices of v_{σ_1} will lead only to an overall shift of $\mathcal{W}_{\sigma_1}^{\mathbb{T}^1} \cup \mathcal{W}_{\sigma_2}^{\mathbb{T}^1}$ by an integer.

From bundle data to splitting numbers: weight bootstrapping.

Recall first the following fact by Grothendieck [Gro]:

Fact 3.3 [Grothendieck]. *Given a holomorphic vector bundle E of rank r over $\mathbb{C}\mathrm{P}^1$. Let $E_0 = \{0\} \subset E_1 \subset \cdots \subset E_r = E$ be a filtration of E such that E_i/E_{i-1} is a line bundle for $1 \le i \le r$ and that the degree d_i of E_i/E_{i-1} form a non-increasing sequence. Then E is isomorphic to the direct sum $\oplus_{i=1}^{r}(E_i/E_{i-1})$.*

Following previous discussions and notations, we only need to work out the splitting numbers for each E^{χ}. Thus, without loss of generality, we may assume that $\mathcal{W}_\tau = \chi$ in the following discussion.

Fix a v_{σ_1}. Note that in terms of the one-parameter subgroup $\lambda_{v_{\sigma_1}}$ acting on $V(\tau)$, x_{σ_1} has coordinate 0 while x_{σ_2} has coordinate ∞. Let

$$E|_{U_1} = \oplus_{i=1}^{a} U_1 \times E_1^{\chi_{1i}} \quad \text{and} \quad E|_{U_2} = \oplus_{j=1}^{b} U_2 \times E_2^{\chi_{2j}}$$

be the induced $\mathbb{T}^1$-weight space decomposition of $E|_{U_1}$ and $E|_{U_2}$ respectively, and $\varphi_{12}(x_\tau)$ be the pasting map at x_τ from $(E|_{U_1})|_{x_\tau}$ to $(E|_{U_2})|_{x_\tau}$. We assume that $\chi_{11} > \cdots > \chi_{1a}$ and $\chi_{21} < \cdots < \chi_{2b}$. Our goal is now to work out a filtration of E, using the given bundle data, that satisfies the property in the above fact.

Let v be a non-zero vector in the fiber E_{x_τ} over x_τ. Then associated to the $\mathbb{T}^1$-orbit $\mathbb{T}^1 \cdot v$ of v is a line bundle $\mathcal{L}_v$ that contains $\mathbb{T}^1 \cdot v$ as a meromorphic section s_v. Let $v_1^{\chi_{1i'}}$ be the lowest $\mathbb{T}^1$-weight component of v in $E|_{U_1}$ and $v_2^{\chi_{2j'}}$) be the highest $\mathbb{T}^1$-weight component of v in $E|_{U_2}$. Then $\mathcal{L}_v|_{x_{\sigma_1}}$ (resp. $\mathcal{L}_v|_{x_{\sigma_2}}$) lies in $E_1^{\chi_{1i'}}$ (resp. $E_2^{\chi_{2j'}}$) and the meromorphic section s_v is holomorphic over O_τ with a zero at x_{σ_1} (resp. x_{σ_2}) of order $\chi_{1i'}$ (resp. $-\chi_{2j'}$). (Here, a zero of order $-k$ means the same as a pole of order k.) This shows that

$$\mathcal{L}_v \;=\; \mathcal{O}(\chi_{1i'} - \chi_{2j'})\,.$$

Notice that, from the previous discussion, this degree is independent of the choices of v_{σ_1}. We shall now proceed to construct a $\mathbb{T}^1$-equivariant line subbundle in E that achieves the maximal degree.

Fix a basis for the $\mathbb{T}^1$-weight spaces in the local trivialization of E, then $\varphi_{21}(x_\tau)$ is expressed by a matrix A which admits a weight-block decomposition $A = [\, A_{\chi_{2j},\chi_{1i}}\,]_{ji}$. Consider the chain of submatrices B_{kl} in A that consists of weight blocks $A_{\chi_{2j},\chi_{1i}}$ with $j = k, \cdots, b$ and $i = 1, \cdots, l$, as indicated in FIGURE 3-1. Each B_{kl} gives the linear map

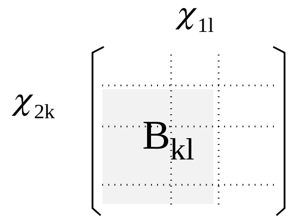

FIGURE 3-1. The submatrix B_{kl} of A, which consists of weight blocks, are indicated by the shaded part.

from $\oplus_{i=1}^{l} E_1^{\chi_{1i}}$ to $\oplus_{j=k}^{b} E_2^{\chi_{2j}}$ induced by A. Let N_{kl} be the kernel of B_{kl}, as a subspace in $(E|_{U_1})_{x_\tau}$. Define

$$\mathcal{E}_i \;=\; E_1^{\chi_{11}} \oplus \cdots \oplus E_1^{\chi_{1i}} \quad \text{at } x_\tau\,, \quad i = 1, \cdots, a\,.$$

Then one has the following sequence of filtrations:

$$\begin{array}{ccccccccc}
\{0\} & \subset & \mathcal{E}_1 & \subset & \cdots & \subset & \mathcal{E}_a & = & (E|_{U_1})_{x_\tau} \\
 & & \cup & & \cdots & & \cup & & \\
 & & N_{b1} & \subset & \cdots & \subset & N_{ba} & & \\
 & & \cup & & \cdots & & \cup & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \cup & & \cdots & & \cup & & \\
 & & N_{11} & \subset & \cdots & \subset & N_{1a} & & .
\end{array}$$

Let

$$\mathcal{E}_{ij} \;=\; \mathcal{E}_i - \mathcal{E}_{i-1} - N_{ji}\,, \quad \text{for } i = 1, \cdots, a\,, j = 1, \cdots, b\,;$$

then

$$(E|_{U_1})_{x_\tau} \;=\; \cup_{i,j}\, \mathcal{E}_{ij}\,.$$

By construction, if $\mathcal{E}_{ij}$ is non-empty, then, for any v in $\mathcal{E}_{ij}$, $deg\,(\mathcal{L}_v) = \chi_{1i} - \chi_{2j}$. Thus we may define the characteristic number $\mu(\mathcal{E}_{ij})$ for $\mathcal{E}_{ij}$ non-empty to be $\chi_{1i} + \chi_{2j}$. Now let

$$d_1 \;=\; max\{\, \mu(\mathcal{E}_{ij}) \,|\, \mathcal{E}_{ij} \text{ non-empty}, \, i = 1, \cdots, a, \, j = 1, \cdots, b\,\}$$

and $v \in \mathcal{E}_{ij}$ for some (ij) that realizes d_1; then by construction $E_1 = \mathcal{L}_v$ is a $\mathbb{T}^1$-equivariant line subbundle of E that achieves the maximal possible degree.

Suppose the basis for the weight spaces in the local trivialization of E are given by $(e_{11}, \cdots, e_{1r})$ and $(e_{21}, \cdots, e_{2r})$ respectively. Let $v = \sum_{i=1}^{k} c_i e_{1i}$ with $c_k \neq 0$ in $E|_{U_1}$ and $v = \sum_{i=1}^{k'} c'_i e_{2i}$ with $c'_{k'} \neq 0$ in $E|_{U_2}$. Then, by replacing e_{1k} and $e_{2k'}$ by v and putting it as the first element in the bases, one shows

Lemma 3.4. *There exist a new weight space decomposition of $E|_{U_1}$ and $E|_{U_2}$ respectively and a choice of the basis for the new weight spaces such that $E_1|_{x_\tau}$ is spanned by the first element in the basis.*

This renders the pasting map into the form:

$$A = \begin{bmatrix} 1 & * \\ 0 & A_1 \end{bmatrix},$$

where 0 is the $(r-1)$-dimensional zero vector and A_1 is a nondegenerate $(r-1)\times(r-1)$ matrix. With respect to the new trivialization, the $\mathbb{T}^1$-weight spaces and their basis descends then to the quotient E/E_1 with pasting map given by A_1.

Repeating the discussion r times, one obtains $\mathbb{T}^1$-equivariant line subbundles

$$E_1 \subset E\,, \quad E_2/E_1 \subset E/E_1\,, \quad \cdots, \quad E_{r-1}/E_{r-2} \subset E/E_{r-2}\,, \quad E_r/E_{r-1}\,.$$

of maximal degree in each pair and the associated filtration

$$E_0 = \{0\} \subset E_1 \subset \cdots \subset E_r = E\,.$$

Lemma 3.5. *The filtration of E obtained above satisfies the condition of Grothendieck in Fact 3.3.*

Proof. Since E_i/E_{i-1} is a $\mathbb{T}^1$-equivariant line subbundle of maximal degree in E/E_{i-1}, it must be so also in E_{i+1}/E_{i-1}. Together with the fact that $E_{i+1}/E_i = (E_{i+1}/E_{i-1})/(E_i/E_{i-1})$, we only need to justify the claim for the rank 2 case. Thus, assume that E is of rank 2 and the filtration is given by $\{0\} \subset E_1 \subset E$. By Lemma 3.4, we may choose a basis compatibe with the $\mathbb{T}^1$-weight space decomposition and the filtration such that the pasting map $\varphi_{\infty 0}$ and its inverse are given respectively by the following matrices with the $\mathbb{T}^1$-weight indicated:

$$A = \begin{array}{cc} \begin{array}{cc} \chi_{11} & \chi_{12} \end{array} & \\ \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} & \begin{array}{c} \chi_{21} \\ \chi_{22} \end{array} \end{array} \qquad \text{and} \qquad A^{-1} = \begin{array}{cc} \begin{array}{cc} \chi_{21} & \chi_{22} \end{array} & \\ \begin{bmatrix} 1 & -* \\ 0 & 1 \end{bmatrix} & \begin{array}{c} \chi_{11} \\ \chi_{12} \end{array} \end{array}.$$

This implies that $deg\, E_1 = \chi_{11} - \chi_{21}$ and $deg\,(E/E_1) = \chi_{12} - \chi_{22}$. The linear independency of the line bundles associated to the column vectors of A at $z = 0$ requires that $\chi_{22} \geq \chi_{21}$.

Similarly, the linear independency of the line bundles associated to the column vectors of A^{-1} at $z = \infty$ requires that $\chi_{12} \leq \chi_{11}$. Consequently, $deg\, E_1 \geq deg\,(E/E_1)$. This concludes the proof.

$\square$

Consequently, by Fact 3.3 one obtains the decomposition of E as the direct sum of line bundles and the splitting numbers.

Remark 3.6 [*weight matching*]. For $\tau = \sigma_1 \cap \sigma_2 \in \Sigma(n-1)$, if all the $Stab\,(x_\tau)$-weight spaces are 1-dimensional, then up to a permutation of the elements in bases, the pasting map $\varphi_{21}(x_\tau)$ becomes diagonal and the correpondences $\mathcal{W}_{\sigma_1} \to \mathcal{W}_\tau \leftarrow \mathcal{W}_{\sigma_2}$ are bijective. Let $\chi_{\sigma_1 i} \to \chi_{\tau i} \leftarrow \chi_{\sigma_2 i}$, $i = 1, \cdots, r$, be the correpondences of weights. Then, up to a permutation, the splitting number of $\mathcal{E}$ over $V(\tau)$ is given by

$$(\,\langle \chi_{\sigma_1 1} - \chi_{\sigma_2 1}\,,\, v_{\sigma_1} \rangle\,,\, \cdots\,,\, \langle \chi_{\sigma_1 r} - \chi_{\sigma_2 r}\,,\, v_{\sigma_1} \rangle\,)\ ,$$

where recall that $\tau^\perp_{\sigma_1} = \tau^\perp \cap \sigma_1^\vee$ and $\langle \tau^\perp_{\sigma_1}, v_{\sigma_1} \rangle = 1$.

From the system of splitting numbers to the splitting types of $\mathcal{E}$.

Following the notation in Sec. 1, let $\Sigma(n-1) = \{\,\tau_1, \cdots, \tau_I\,\}$ and

$$\Xi(\mathcal{E}) \;=\; \{\,(d_1^{\tau_i}, \cdots, d_r^{\tau_i})\,|\, i = 1, \cdots, r\,\}\,.$$

be the system of splitting numbers associated to $\mathcal{E}$. Let R be the $I \times r$ matrix whose i-th row is $(d_1^{\tau_i}, \cdots, d_r^{\tau_i})$. Recall the augmented intersection matrix Q from Sec. 1. Then the problem of finding splitting types of $\mathcal{E}$ is equivalent to finding out matrices R' obtained by row-wise permutations of R such that:

(1) Each column of R' has only all positive, all zero, or all negative entries.

(2) The following matrix linear equation has an integral solution:

$$Q\,X \;=\; R'\,,$$

where X is an $J \times r$ matrix.

Let X_{kl} be the (k,l)-entry of X. Then associated to the r-many column vectors of the solution matrix X are the line bundles L_l represented by $\sum_{k=1}^J X_{kl} D(v_k)$, for $l = 1, \cdots, r$. From Fact 1.2 in Sec.1, Condition (1) above for R' means that $c_1(L_l)$ is either ≥ 0 or < 0. Such set of line bundles gives then a splitting type of $\mathcal{E}$ by construction. If there exist no such (R', X), then $\mathcal{E}$ does not admit a splitting type. Finding all such R' and solving the matrix X can be achieved by using a computer.

4 The splitting type of some examples.

In this section, we compute the splitting types of some equivaraint vector bundles over toric manifolds to illustrate the ideas in previous sections and also for future use. The details of the toric manifolds used here can be found in [Fu] and [Od2].

Example 4.1 [equivariant vector bundles of rank 2 over $\mathbb{C}\mathrm{P}^2$]. Recall first ([Fu]) the toric data for $\mathbb{C}\mathrm{P}^2$, as illustrated in FIGURE 4-1(a). Let $\mathcal{E}$ be an indecomposable equivariant vector bundle of rank 2 over $\mathbb{C}\mathrm{P}^2 = \mathrm{Proj}\,(\mathbb{C}[u_0, u_1, u_2])$. From [Ka1], $\mathcal{E}$ is isomorphic to $\mathcal{E}_{a,b,c,n} = \mathcal{E}(a,b,c) \otimes \mathcal{O}(n)$ or its dual bundle for some positive integers a, b, c and integer n, where $\mathcal{E}(a,b,c)$ is the rank 2 bundle defined by the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{C}\mathrm{P}^2} & \longrightarrow & \mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c) & \longrightarrow & \mathcal{E}(a,b,c) & \longrightarrow & 0 \\ & & 1 & \longmapsto & (\, u_0^a,\, u_1^b,\, u_2^c\,) & & & & . \end{array}$$

From the bundle data of $\mathcal{E}(a,b,c)$ as worked out in [Ka1], the weight systems for $\mathcal{E}(a,b,c)$ at the distinguished points x_{σ_1}, x_{σ_1} and x_{σ_3} are given respectively by (cf. FIGURE 4-1(b))

$$W_1 \;=\; \{(a,0),\,(0,b)\}\,, \qquad W_2 \;=\; \{(-b,b),\,(-c,0)\}\,, \qquad W_3 \;=\; \{(a,-a),\,(0,-c)\}\,.$$

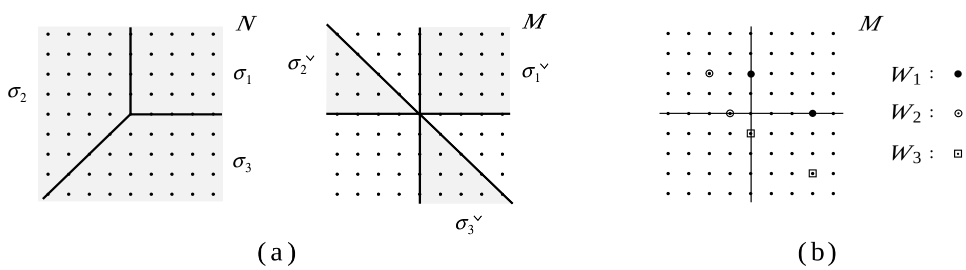

FIGURE 4-1. In (a), the fan and its dual cones for $\mathbb{C}\mathrm{P}^2$ are illustrated. In (b), the weight systems W_1, W_2, W_3 associated to $\mathcal{E}(a,b,c)$, a, b, c positive integers, are illustrated.

Comparing with the toric data for $\mathbb{C}\mathrm{P}^2$ and following the discussions in Sec. 3, in particular Remark 3.6, one has

$$\mathcal{E}(a,b,c)|_{\overline{x_{\sigma_1}x_{\sigma_2}}} \;=\; \mathcal{O}(a+c) \oplus \mathcal{O}(b)\,, \quad \mathcal{E}(a,b,c)|_{\overline{x_{\sigma_2}x_{\sigma_3}}} \;=\; \mathcal{O}(a+b) \oplus \mathcal{O}(c)\,, \quad \text{and}$$
$$\mathcal{E}(a,b,c)|_{\overline{x_{\sigma_1}x_{\sigma_3}}} \;=\; \mathcal{O}(b+c) \oplus \mathcal{O}(a)\,.$$

Consequently,

$$\mathcal{E}|_{\overline{x_{\sigma_1}x_{\sigma_2}}} \;=\; \mathcal{O}(a+c+n) \oplus \mathcal{O}(b+n)\,, \quad \mathcal{E}|_{\overline{x_{\sigma_2}x_{\sigma_3}}} \;=\; \mathcal{O}(a+b+n) \oplus \mathcal{O}(c+n)\,, \quad \text{and}$$
$$\mathcal{E}|_{\overline{x_{\sigma_1}x_{\sigma_3}}} \;=\; \mathcal{O}(b+c+n) \oplus \mathcal{O}(a+n)\,.$$

Thus, up to permutations, the system of splitting numbers associated to $\mathcal{E}$ is

$$\Xi(\mathcal{E}) \;=\; \{\,(a+c+n, b+n),\, (a+b+n, c+n),\, (b+c+n, a+n)\,\}$$

and

$$R \;=\; \begin{bmatrix} a+c+n & b+n \\ a+b+n & c+n \\ b+c+n & a+n \end{bmatrix}.$$

From FIGURE 4-1(a), the augmented intersection matrix Q for $\mathbb{C}\mathrm{P}^2$ is given by

$$Q \;=\; \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Performing row-wise permutations to R, one obtains four possible R', up to overall permutations of column vectors. Solving the matrix equation $QX = R'$, one concludes that

Corollary. *For the indecomposable equivaraint rank* 2 *bundle $\mathcal{E}_{a,b,c,n}$ over $\mathbb{C}\mathrm{P}^2$ to admit a splitting type, one must have $a = b = c$. In this case, the splitting type of $\mathcal{E}_{a,a,a,n}$ is unique and is given by $(\,\mathcal{O}(2a+n),\, \mathcal{O}(a+n)\,)$.*

This concludes the example.

$\square$

Example 4.2 [(co)tangent bundle of toric manifolds]. Notice that, since T_*X and T^*X are dual to each other, their splitting types are negative to each other. Thus, we only need to consider T_*X. Let us compute first the splitting numbers of T_*X_Σ. Let $\tau = \sigma_1 \cap \sigma_2 \in \Sigma(n-1)$ with

$$\sigma_1 = [v_1,\, \cdots,\, v_{n-1}, v_n] \quad \text{and} \quad \sigma_2 = [v_1,\, \cdots,\, v_{n-1}, v_{n'}]\,.$$

These vertices in $\sigma_1 \cup \sigma_2$ satisfy a linear equation of the form

$$v_{j_n} \,+\, v_{j'_n} \,+\, a_1 v_{j_1} \,+\, \cdots \,+\, a_{n-1} v_{j_{n-1}} \;=\; 0\,,$$

for some unique integers $a_1,\, \cdots,\, a_n$ determined by $\sigma_1 \cup \sigma_2$. Let $(e^1,\, \cdots,\, e^n)$ be the dual basis in M with respect to $(v_1,\, \cdots,\, v_n)$. Then

$$\sigma_1^\vee = [e^1,\, \cdots,\, e^{n-1}, e^n] \quad \text{and} \quad \sigma_2^\vee = [e^1 - a_1 e^n,\, \cdots,\, e^{n-1} - a_{n-1}e^n, -e^n]\,.$$

Consequently, $\chi_{\sigma_1 i} = e^i$ for $i = 1,\, \cdots,\, n$, $\chi_{\sigma_2 i} = e^i - a_i e^n$ for $i = 1,\, \cdots,\, n-1$; and $\chi_{\sigma_2 n} = -e^n$. Choosing $v_{\sigma_1} = v_n$ and by the discussion in Sec. 3, one concludes that the splitting number of T_*X_Σ over $V(\tau)$ is given by

$$(\,a_1,\, \cdots,\, a_{n-1},\, 2\,)\,,$$

up to a permutation. Thus, the system $\Xi(T_*X_\Sigma)$ of spliting numbers associated to T_*X_Σ is already coded in Σ, as it should be.

In the dual picture, if X_Σ is projective and, hence, Σ is realized as the normal fan of a strongly convex polyhedron Δ in M. Let m_σ be the vertex of Δ associated to $\sigma \in \Sigma(n)$. Then $\mathcal{W}_\sigma$ is the set of primitive vectors that generate the tangent cone of Δ at m_σ. If $\tau = \sigma_1 \cap \sigma_2 \in \Sigma(n-1)$, then m_{σ_1} and m_{σ_2} are connected by an edge $\overline{m_{\sigma_1}m_{\sigma_2}}$ of Δ that is parallel to $\tau^\perp$. Thus the projection $M \to M/M(\tau)$ is given by the projection along the $\overline{m_{\sigma_1}m_{\sigma_2}}$-direction. The strong convexity of Δ implies that $\mathcal{W}_{\sigma_1}$ and $\mathcal{W}_{\sigma_2}$ match up bijectively under this projection. Thus $\Xi(X_\Sigma)$ can be also read off directly from Δ.

We can now compute the splitting type of some concrete examples. The result shows that: *Not every tangent bundle of a toric manifold admits a splitting type.*

(a) *The projective space* $\mathbb{C}\mathrm{P}^n$. Let $(v_1, \cdots, v_n)$ be a basis of N and $v_{n+1} = -(v_1 \cdots v_n)$ and Σ be the fan whose maximal cones are generated by every independent n elements in $\{v_1, \cdots v_{n+1}\}$. Then $\mathbb{C}\mathrm{P}^n = X_\Sigma$. By construction, $\Sigma(1)$ is given by $\{v_1, \cdots, v_{n+1}\}$ with $v_1 + \cdots v_{n+1} = 0$. Thus the splitting number for $T_*\mathbb{C}\mathrm{P}^n$ over any invariant $\mathbb{C}\mathrm{P}^1$ is given by

$$(2, \underbrace{1, \cdots, 1}_{n-1}).$$

The augmented intersection matrix Q has all of its entries equal to 1. From this, one concludes that the splitting type of $\mathbb{C}\mathrm{P}^n$ is unique and is given by

$$(\mathcal{O}(2), \underbrace{\mathcal{O}(1), \cdots, \mathcal{O}(1)}_{n-1}).$$

(b) *The Hirzebruch surface* $\mathbb{F}_a$. The toric data for the Hurzebruch $\mathbb{F}_a$ and its weighted circular graph [Od2] is given in FIGURE 4-2.

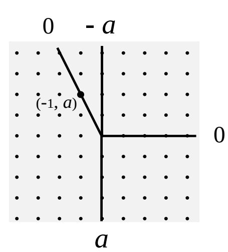

FIGURE 4-2. The fan for the Hirzebruch surface $\mathbb{F}_a$ and its weights.

Consequently,

$$\Xi(T_*\mathbb{F}_a) = \{(2,0), (2,a), (2,0), (2,-a)\}$$

and its augmented intersection matrix is given by

$$Q = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & a & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & -a \end{bmatrix}.$$

Thus, the only Hirzebruch surface whose tangent bundle can admit a splitting type is when $a = 0$ since the line bundle in a splitting type must be either positive or negative. For $a = 0$, $\mathbb{F}_0 = \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$. Let $H^2(\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ from the product structure. Then direct computations as in Example 4.1 concludes that $T_*(\mathbb{F}_0)$ admits a unique splitting type

$$(\, \mathcal{O}(2,2),\ \mathcal{O}(\mathbb{F}_0)\,)\,,$$

where $\mathcal{O}(2,2)$ is the line bundle associated to $(2,2)$ in $H^2(\mathbb{F}_0, \mathbb{Z})$ and $\mathcal{O}(\mathbb{F}_0)$ is the trivial line bundle.

(c) *The blowups of* $\mathbb{C}\mathrm{P}^2$ *or* $\mathbb{F}_a$. Recall from [Fu] and [Od2] that every complete nonsingular toric surface X is obtained from $\mathbb{C}\mathrm{P}^2$ or $\mathbb{F}_a$, $a > 0$, by a succession of blowups at the T_N-fixed points. Let $(a_1, \cdots, a_s)$ be the sequence of weights that appear in the weighted circular graph for X. Then

$$\Xi(T_*X) \;=\; \{\, (2, a_1), \cdots, (2, a_s)\,\}\,.$$

Consequently, a necessary condition for T_*X to admit a splitting type is that the weights that appear in the graph must be all positive, all zero, or all negative. The augmented intersection matrix is given by

$$Q = \begin{bmatrix} a_1 & 1 & & & & 1 \\ 1 & a_2 & 1 & & & \\ & 1 & a_3 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & a_{n-1} & 1 \\ 1 & & & & 1 & a_n \end{bmatrix},$$

where all the missing entries are 0. From these data, the splitting type of T_*X, if exists, can be worked out.

To provide more examples and also for the interest of string theory, equivariant blowups of $\mathbb{C}\mathrm{P}^2$ up to 9 points that admits a splitting type are searched out by computer. These include del Pezzo type and $\frac{1}{2}$K3 type surfaces. It turns out that there are only 8 of them, 4 of del Pezzo type and 4 of $\frac{1}{2}$K3 type. Their splitting types are listed in TABLE 4-1.

topology of X	k	$w = (a_1, \cdots, a_s)$	$H_2(X;\mathbb{Z})$	splitting type of T_*X	Remark
$\mathbb{C}\mathrm{P}^2 \sharp 3 \overline{\mathbb{C}\mathrm{P}^2}$	3	$(-1, -1, -1, -1, -1, -1)$	$\mathbb{Z}^4$	$((2,4,4,2),\ (-1,-2,-2,-1))$	del Pezzo type
$\mathbb{C}\mathrm{P}^2 \sharp 5 \overline{\mathbb{C}\mathrm{P}^2}$	5	$(-1, -2, -1, -2, -1, -2, -1, -2)$	$\mathbb{Z}^6$	$((2,4,8,6,6,2),\ (-2,-3,-6,-4,-4,-1))$	del Pezzo type
$\mathbb{C}\mathrm{P}^2 \sharp 6 \overline{\mathbb{C}\mathrm{P}^2}$	6	$(-1, -2, -2, -1, -2, -2, -1, -2, -2)$	$\mathbb{Z}^7$	$((2,4,8,14,8,4,2),\ (-2,-3,-6,-11,-6,-3,-2))$	del Pezzo type
$\mathbb{C}\mathrm{P}^2 \sharp 7 \overline{\mathbb{C}\mathrm{P}^2}$	7	$(-1, -2, -2, -1, -3, -1, -2, -2, -1, -3)$	$\mathbb{Z}^8$	$((2,4,8,14,8,12,6,2),\ (-3,-4,-7,-12,-6,-9,-4,-1))$	del Pezzo type
$\mathbb{C}\mathrm{P}^2 \sharp 9 \overline{\mathbb{C}\mathrm{P}^2}$	9	$(-1, -2, -2, -2, -1, -4, -1, -2, -2, -2, -1, -4)$	$\mathbb{Z}^{10}$	$((2,4,8,14,22,10,20,12,6,2),\ (-4,-5,-8,-13,-20,-8,-16,-9,-4,-1))$	$\frac{1}{2}$ K3 type
$\mathbb{C}\mathrm{P}^2 \sharp 9 \overline{\mathbb{C}\mathrm{P}^2}$	9	$(-1, -2, -2, -3, -1, -2, -2, -3, -1, -2, -2, -3)$	$\mathbb{Z}^{10}$	$((2,4,8,14,36,24,14,6,6,2),\ (-3,-4,-7,-12,-32,-21,-12,-5,-6,-2))$	$\frac{1}{2}$ K3 type
$\mathbb{C}\mathrm{P}^2 \sharp 9 \overline{\mathbb{C}\mathrm{P}^2}$	9	$(-1, -2, -3, -1, -2, -3, -1, -2, -3, -1, -2, -3)$	$\mathbb{Z}^{10}$	$((2,4,8,22,16,12,22,12,4,2),\ (-3,-4,-7,-20,-14,-10,-19,-10,-3,-2))$	$\frac{1}{2}$ K3 type
$\mathbb{C}\mathrm{P}^2 \sharp 9 \overline{\mathbb{C}\mathrm{P}^2}$	9	$(-1, -3, -1, -3, -1, -3, -1, -3, -1, -3, -1, -3)$	$\mathbb{Z}^{10}$	$((2,4,12,10,20,12,18,8,8,2),\ (-3,-4,-12,-9,-18,-10,-15,-6,-6,-1))$	$\frac{1}{2}$ K3 type

(1) k is the number of points blown up from $\mathbb{C}\mathrm{P}^2$.
(2) $H_2(X;\mathbb{Z})$ is generated by the first $(s-2)$ divisors in the list w.
(3) The line bundles in the splitting type are represented by divisors of X as elements in $H_2(X;\mathbb{Z})$.

TABLE 4-1. Complete list of T_*S and its splitting type for toric surfaces obtained from $\mathbb{C}\mathrm{P}^2$ via equivariant blowups up to 9 points.

The fan for these toric surfaces are indicated in FIGURE 4-3.

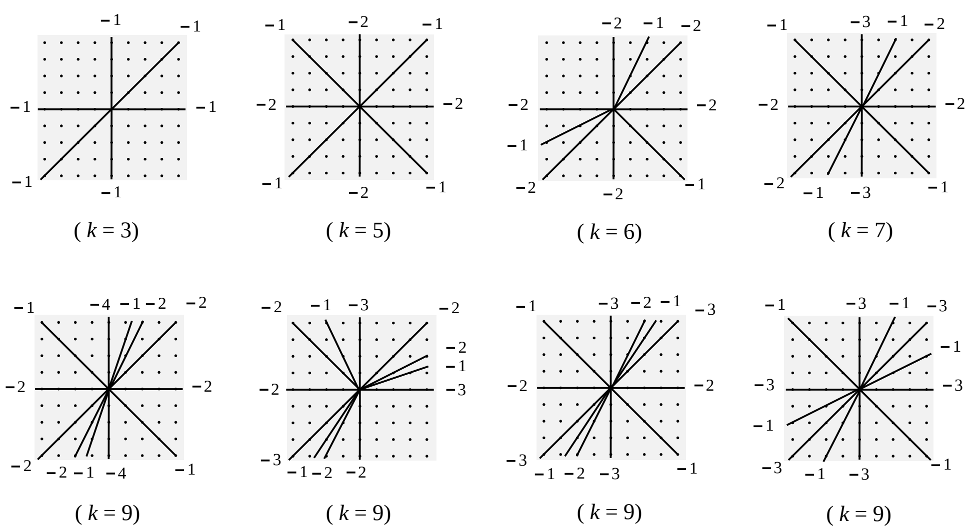

FIGURE 4-3. The fan for the blowups of $\mathbb{C}\mathrm{P}^2$ up to nine points that admit a splitting type, together with the weights, are indicated.

This concludes the example.

□

5 Remarks and issues for further study.

In the previous section, we have illustrated how the data of a linearized equivariant vector bundle $\mathcal{E}$ over a toric manifold X_Σ is used to determine its splitting type if it exists. It turns out that these examples are related to the following kind of exact sequence[1]:

$$0 \longrightarrow \mathcal{O}_{X_\Sigma} \xrightarrow{\eta} \oplus_{i=1}^{r+1} \mathcal{O}_{X_\Sigma}(D_i) \longrightarrow \mathcal{E} \longrightarrow 0\,,$$

where D_i are Cartier T-Weil divisors of X_Σ and η is a holomorphic bundle inclusion. For such $\mathcal{E}$, the system of splitting numbers $\Xi(\mathcal{E})$ may be obtained directly from this exact sequence.

Example 5.1 [equivariant vector bundles of rank n over $\mathbb{C}\mathrm{P}^n$]. Let $[z_0 : \cdots, z_n]$ be the homogeneous coordinates of the projective space $\mathbb{C}\mathrm{P}^n$. Recall from [Ka2] that an indecomposable equivariant vector bundle $\mathcal{E}$ of rank n over $\mathbb{C}\mathrm{P}^n$ is isomorphic to either $E \otimes \mathcal{O}_{\mathbb{C}\mathrm{P}^n}(d)$ or $E^* \otimes \mathcal{O}_{\mathbb{C}\mathrm{P}^n}(d)$ for some integer d where E is the equivariant vector bundle defined by the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{C}\mathrm{P}^n} \xrightarrow{\eta} \oplus_{i=0}^{n} \mathcal{O}_{\mathbb{C}\mathrm{P}^n}(m_i) \longrightarrow E \longrightarrow 0\,,$$

where m_i are positive integers and η sends 1 to $(z_0^{m_0}, \cdots, z_n^{m_n})$. Since the $(n+1)n/2$ many invariant $\mathbb{C}\mathrm{P}^1$ in $\mathbb{C}\mathrm{P}^n$ are given by

$$V_{ij} \;=\; \{[\,0, \cdots, 0, z_i, 0, \cdots, 0, z_j, 0, \cdots, 0\,]\,|\,(z_i, z_j) \in \mathbb{C}^2 - \{(0,0)\}\}\,,$$

$0 \le i < j \le n$, the above exact sequence, when restricted to V_{ij}, reduces to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{V_{ij}} & \longrightarrow & \oplus_{k=1}^{r+1}\mathcal{O}_{V_{ij}}(m_k) & \longrightarrow & E|_{V_{ij}} & \longrightarrow & 0 \\ & & 1 & \longmapsto & (\,0, \cdots, 0, z_i^{m_i}, 0, \cdots, 0, z_j^{m_j}, 0, \cdots, 0\,) & & & & \end{array}.$$

It follows from the multiplicativity of total Chern class that

$$E|_{V_{ij}} \;\simeq\; \mathcal{O}_{\mathbb{C}\mathrm{P}^1}(m_i{+}m_j) \oplus \mathcal{O}_{\mathbb{C}\mathrm{P}^1}(m_0) \oplus \cdots \oplus \widehat{\mathcal{O}_{\mathbb{C}\mathrm{P}^1}(m_i)} \oplus \cdots \oplus \widehat{\mathcal{O}_{\mathbb{C}\mathrm{P}^1}(m_j)} \oplus \cdots \oplus \mathcal{O}_{\mathbb{C}\mathrm{P}^1}(m_n)$$

and, hence, the system of splitting numbers of E is

$$\Xi(E) \;=\; \{\,(m_i + m_j, m_0, \cdots, \widehat{m_i}, \cdots, \widehat{m_j}, \cdots, m_n)\,|\,0 \le i < j \le n\,\}\,,$$

where terms with $\widehat{\ }$ are deleted. The augmented matrix in this case is

$$Q \;=\; \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \vdots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}.$$

Without loss of generality, one may assume that $0 < m_0 \le \cdots \le m_n$, then $m_i < m_n{+}m_{n-1}$ for all i. Consequently, with the notation from Sec. 3, for $QX = R'$ to have a solution,

[1]We thank Bong H. Lian for drawing our attention to this and the reference [Ja]

one must have $m_i + m_j = m_n + m_{n-1}$ for all $i < j$, which implies that $m_0 = \cdots = m_n$. One concludes therefore

Corollary. *For the indecomposable equivaraint rank n bundle $\mathcal{E} = E \otimes \mathcal{O}_{\mathbb{C}\mathrm{P}^n}(d)$ over $\mathbb{C}\mathrm{P}^n$ to admit a splitting type, one must have $m_0 = \cdots = m_n$. In this case, the splitting type of $\mathcal{E}$ is unique and is given by $(\,\mathcal{O}_{\mathbb{C}\mathrm{P}^n}(2m_0 + d),\, \cdots,\, \mathcal{O}_{\mathbb{C}\mathrm{P}^n}(m_0 + d)\,)$.*

This generalizes Example 4.1. Note also that the case $m_0 = \cdots = m_n = 1$ with $d = 0$ corresponds to $T_*\mathbb{C}\mathrm{P}^n$ and the above discussion double-checks part of Example 4.2.

$\square$

One can generalize this example slightly to toric manifolds as follows. First, let us state a lemma, whose proof is straightforward.

Lemma 5.2 *Given an exact sequence*

$$0 \longrightarrow \mathcal{O}_{\mathbb{C}\mathrm{P}^1} \xrightarrow{\eta} \mathcal{O}_{\mathbb{C}\mathrm{P}^1} \oplus (\oplus_{i=1}^r \mathcal{O}_{\mathbb{C}\mathrm{P}^1}(m_i)) \longrightarrow E \longrightarrow 0\,,$$

where $\eta(1) = (s_0, s_1, \cdots, s_r)$, such that s_0 is non-zero. Then $E \simeq \oplus_{i=1}^r \mathcal{O}_{\mathbb{C}\mathrm{P}^1}(m_i)$.

Given a toric n-fold X_Σ, consider now the exact sequence

$$0 \longrightarrow \mathcal{O}_{X_\Sigma} \xrightarrow{\eta} \oplus_{i=1}^{r+1} \mathcal{O}_{X_\Sigma}(D_i) \longrightarrow \mathcal{E} \longrightarrow 0\,.$$

The restriction of the sequence to $V(\tau)$, $\tau \in \Sigma(n-1)$, is given by

$$0 \longrightarrow \mathcal{O}_{\mathbb{C}\mathrm{P}^1} \xrightarrow{\eta} \oplus_{i=1}^{r+1} \mathcal{O}_{\mathbb{C}\mathrm{P}^1}(D_i \cdot V(\tau)) \longrightarrow E \longrightarrow 0\,.$$

If this exact sequence is of the kind in Example 5.1 or Lemma 5.2 for all $\tau \in \Sigma(n-1)$, then $\Xi(\mathcal{E})$ can be readily obtained and the splitting type of such $\mathcal{E}$, if exists, can then be determined. Inspired from Example 5.1, to realize this, recall the Cox homogeneous coordinates of X_Σ from [Co] (cf. Sec. 1): let $a = |\Sigma(1)|$, then X_Σ can be realized as a quotient $X_\Sigma = (\mathbb{C}^{\Sigma(1)} - Z(\Sigma))/G$. Let $(z_1, \cdots, z_a)$ be the standard coordinates of $\mathbb{C}^a$ and $\tau = [v_{j_1}, \cdots, v_{j_{n-1}}] \in \Sigma(n-1)$, then $V(\tau)$ can be realized as the quotient of the coordinate subspace: $V(\tau) = \{z_{j_1} = \cdots = z_{j_{n-1}}\}/G$. Furthermore, if $\tau = \sigma_1 \cap \sigma_2$, where

$$\sigma_1 = [v_{j_1}, \cdots, v_{j_{n-1}}, v_{j_n}] \quad \text{and} \quad \sigma_2 = [v_{j_1}, \cdots, v_{j_{n-1}}, v_{j'_n}]\,,$$

then $[z_{j_n} : z_{j'_n}]$ serves as a homogeneous coordinates for $V(\tau) \simeq \mathbb{C}\mathrm{P}^1$. For all other i, $\{z_i = 0\} \cap \{z_{j_1} = \cdots = z_{j_{n-1}}\}$ lies in the exceptional subset $Z(\Sigma)$ and, hence, z_i as an element in the homogeneous coordinate ring $\mathbb{C}[z_1, \cdots, z_a]$, graded by the Chow group $A_{n-1}(X_\Sigma)$, descends to a non-zero section in $\mathcal{O}_{X_\Sigma}(D(v_i))|_{V(\tau)} \simeq \mathcal{O}_{\mathbb{C}\mathrm{P}^1}$. In general, since $\mathcal{O}_{X_\Sigma}(D_1) \otimes \mathcal{O}_{X_\Sigma}(D_2) = \mathcal{O}_{X_\Sigma}(D_1 + D_2)$ for any Cartier T-Weil divisor D_1, D_2, for any monomial $\prod_k z_{j_k}^{\alpha_k}$ with $j_k \notin \{j_1, \cdots, j_{n-1}, j_n, j'_n\}$ and α_k positive integers, $\prod_k z_{j_k}^{\alpha_k}$ descends to a non-zero section in $\mathcal{O}_{X_\Sigma}(\sum_k \alpha_k D(v_{j_k}))|_{V(\tau)} \simeq \mathcal{O}_{\mathbb{C}\mathrm{P}^1}$. This fact provides us with a guideline for defining η so that exact sequences as in Lemma 5.2 can appear when restricted to invariant $\mathbb{C}\mathrm{P}^1$'s in X_Σ. Such examples can be constructed plenty. Let us give an example below to illustrate the idea.

Example 5.3 [simple rank 3 bundle over Hirzebruch surface]. Let $X_\Sigma = \mathbb{F}_a$ be a Hirzebruch surface (cf. Example 4.2 (b)). Consider the rank 3 bundle $\mathcal{E}(m_1, m_2, m_3, m_4)$ over $\mathbb{F}_a$ defined by the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{F}_a} & \longrightarrow & \oplus_{k=1}^4 \mathcal{O}_{\mathbb{F}_a}(m_k D_{v_k}) & \longrightarrow & \mathcal{E}(m_1, m_2, m_3, m_4) & \longrightarrow & 0 \\ & & 1 & \longmapsto & (\, z_1^{m_1},\, z_2^{m_2},\, z_3^{m_3},\, z_4^{m_4}) & & & & , \end{array}$$

where m_i are positive integers. From Lemma 5.2 and the discussions above, one concludes that the system of splitting numbers of $\mathcal{E}(m_1, m_2, m_3, m_4)$ is given by

$$\varXi(\mathcal{E}) \;=\; \{\, (m_1, m_2, m_4), (m_1, m_2, m_3), (m_2, m_3, m_4), (m_1, m_3, m_4) \,\}\,.$$

Recall the augmented intersection matrix Q for $\mathbb{F}_a$ from Example 4.2 (b). Through a tedious but straightforward algebra, one can show that the only case when $\mathcal{E}(m_1, m_2, m_3, m_4)$ admits a splitting type is when $a = 0$ (i.e. $X = \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$) with $m_1 = m_3$ and $m_2 = m_4$. In this case, the splitting type is unique and is given by

$$\mathcal{O}_{\mathbb{C}\mathrm{P}^1\times\mathbb{C}\mathrm{P}^1}(m_2, m_1) \,\oplus\, \mathcal{O}_{\mathbb{C}\mathrm{P}^1\times\mathbb{C}\mathrm{P}^1}(m_1, m_2) \,\oplus\, \mathcal{O}_{\mathbb{C}\mathrm{P}^1\times\mathbb{C}\mathrm{P}^1}(m_1, m_2)\,,$$

where we idetify $\mathrm{Pic}\,(X)$ with $H_2(X, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}$, with generators

$$\mathcal{O}_{\mathbb{C}\mathrm{P}^1\times\mathbb{C}\mathrm{P}^1}(1,0) \;\longmapsto\; [\mathbb{C}\mathrm{P}^1 \times *]\,, \qquad \mathcal{O}_{\mathbb{C}\mathrm{P}^1\times\mathbb{C}\mathrm{P}^1}(0,1) \;\longmapsto\; [* \times \mathbb{C}\mathrm{P}^1]\,.$$

$\square$

For more general η, the restriction of the exact sequence over X_Σ to each invariant $\mathbb{C}\mathrm{P}^1$ in X_Σ leads to an exact sequence of the form

$$0 \;\longrightarrow\; \mathcal{O}_{\mathbb{C}\mathrm{P}^1} \;\xrightarrow{\eta}\; \oplus_{i=1}^{r+1}\, \mathcal{O}_{\mathbb{C}\mathrm{P}^1}(m_i) \;\longrightarrow \mathcal{E}_{\mathbb{C}\mathrm{P}^1} \;\longrightarrow\; 0\,.$$

A complete study of how η determines the splitting of $\mathcal{E}_{\mathbb{C}\mathrm{P}^1}$ as a direct sum of line bundles requires more work[2].

We conclude the discussion of splitting types here and leave its further study and applications for another work.

[2] We thank Jason Starr for discussions on this and the references [Bri], [E-VV1], and [E-VV2].

Appendix. The computer code.

The computer code in Mathematica that carries out the computation in Example 4.2 (c) is attached below for reference,

```
(* This is a code in Mathematica. *)
(* The purpose of this code is to sort out and compute the splitting type of the tangent bundle
   of toric surfaces. The result of computation is written to the file 'ma-result.txt'. *)
(* Subroutines enclosed: BlowUp, BlowUpN, GenerateMatrix, (MAIN) SplittingType *)

(* Definition of the function 'BlowUp'. *)
(* 'BlowUp[weightlist]' generates the list of weights on the circular weighted graph obtained by
    equivariant blowup at a $T_N$-fixed point of a toric surface represented by 'weightlist'.
   Date of completion: 10/15/1999. Test: Tested correct. Date of last revision: 10/16/1999.
 *)

  BlowUp[ weightlist_ ] :=
    Module[ { a1, a2, b, b1, b2, list, list1, list2, list3, m1, m2, newlist },

            m1=Length[weightlist];

            list1[i_] := ReplacePart[ weightlist,
                           { weightlist[[i]]-1, -1, weightlist[[i+1]]-1 }, i+1 ];
            list2[i_] := Delete[ list1[i], i];
            list3[i_] := Flatten[ list2[i] ] ;

            list=ReplacePart[ weightlist, {-1, weightlist[[1]]-1 }, 1 ];
            list=ReplacePart[ list, list[[m1]]-1, m1];
            list={ Flatten[list] };

            newlist=Join[ list, Table[ list3[i] , {i, 1, m1-1}] ];
            newlist=Union[newlist];
            m2=Length[newlist];

            Do[
                a1=newlist[[i]];
                a2=Reverse[a1];
                b1=Table[ RotateRight[a1, i], {i, 1, m1+1} ];
                b2=Table[ RotateRight[a2, i], {i, 1, m1+1} ];
                b=Union[b1, b2];
                newlist=Union[{a1}, Complement[newlist, b] ];
                If[ m2>Length[newlist],
                    Return[newlist]
                  ],
               {i, 1, m2}
              ];

            Return[newlist]
          ]


(* Definition of the function 'BlowUpN'. *)
(* 'BlowUpN[weightlist, n]' generates the list of weights on the circular weighted graph obtained
    by consecutive equivariant blowup at a $T_N$-fixed points of a toric surface 'n' times, starting
    from the one represented by 'weightlist'.
   Date of completion: 10/15/1999. Test: Tested correct. Date of last revision: 10/16/1999.
 *)
```

```
 BlowUpN[ weight_, n_ ] :=
   Module[ { m, newlist, oldlist, totallist},

           totallist={weight};
           oldlist={weight};
           newlist={};

           Do[
               m=Length[oldlist];
               Do[
                   newlist=Union[ newlist, BlowUp[ oldlist[[j]] ] ],
                  {j, 1, m}
                 ];
               oldlist=newlist;
               totallist=Join[ totallist, newlist],
              {i, 1, n}
             ];
           totallist=Union[totallist];
           Return[totallist];
         ]


(* Definition of the function 'GenerateMatrix'. *)
(* 'GenerateMatrix[weightlist]' generates a matrix following the rule discussed in the paper on
    splitting types of equivariant vector bundle on toric manifolds .
   Date of completion: 10/15/1999. Test: Tested correct. Date of last revision: 10/15/1999.
 *)

 GenerateMatrix[ weightlist_ ] :=
   Module[ { listfirst, listlast, list1, list2, m, newlist, v  },

           m=Length[weightlist];
           v=Table[ 0, {i, 1, m-2} ];

           list1[i_]:=ReplacePart[ v, { 1, weightlist[[i]], 1 }, i-1 ];
           list2[i_]:=Flatten[ list1[i] ];

           listfirst=Flatten[ ReplacePart[ v, {1, weightlist[[1]], 1 }, 1 ] ];
           listfirst={ RotateLeft[listfirst, 1] };

           listlast=Flatten[ ReplacePart[ v, {1, weightlist[[m]], 1 }, 1 ] ];
           listlast={ RotateLeft[listlast, 2] };

           newlist=Join[ listfirst, Table[ list2[i], {i, 2, m-1} ], listlast ];

           Return[newlist]
         ]



(* MAIN ROUTINE *)

(* Definition of the function 'SplittingType'. *)
(* 'SplittingType[weight, n]' sorts out from all the toric surfaces that arise from equivariant
    blowups up to 'n' times of the toric surface whose associated weighted circular graph is given
    by 'weight' those that admit a splitting type and computes their splitting types.
   Date of completion: 10/15/1999. Test: Tested correct. Date of last revision: 10/17/1999.
 *)

 SplittingType[weight_, n_] :=
  Module[ { b, m, matrix, t, totallist, t1, x1, x2 },
```

```
          totallist=BlowUpN[weight, n];
          m=Length[totallist];

          Do[
              t=totallist[[i]];
              t1=Union[t];
              If[ Complement[t1,{0}]===t1,
                  matrix=GenerateMatrix[t];
                  b=Table[2, {j, 1, Length[t]}];
                  x1=LinearSolve[matrix, b1];
                  If[ Length[x1]>=3,
                      x2=LinearSolve[matrix, t];
                      PutAppend[i, "ma-result.txt"];
                      PutAppend[t, "ma-result.txt"];
                      PutAppend[x1, "ma-result.txt"];
                      PutAppend[x2, "ma-result.txt"]
                    ]
                ],
             {i, 1, m}
            ];
        ]


(* Case of study *)

  DeleteFile["ma-result.txt"];
  SplittingType[{1, 1, 1}, 9];
```

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