ERRATUM FOR RICCI-FLAT GRAPHS WITH GIRTH AT LEAST FIVE

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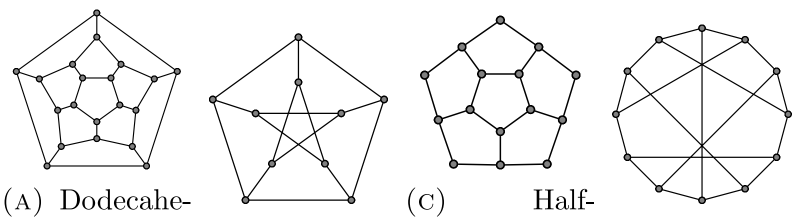
Abstract. This erratum will correct the classification of Theorem 1 in [1] that misses the Triplex graph.

In Theorem 1 of [1], the classification of Ricci-flat graph with girth $g(G) \geq 5$ missed one graph – the Triplex graph, as discovered by three authors: Cushing, Kangaslampi, and Liu. Here is the correct theorem.

Theorem 1. *Suppose that G is a Ricci-flat graph with girth $g(G) \geq 5$. Then G is one of the following graphs,*

(1) *the infinite path,*
(2) *cycle C_n with $n \geq 6$,*
(3) *the dodecahedral graph,*
(4) *the Petersen graph,*
(5) *the half-dodecahedral graph.*
(6) *the Triplex graph.*



(A) Dodecahedral (B) Petersen (C) Half-Dodecahedral (D) Triplex

Figure 1. The four Ricci-flat graphs with girth 5

This error was caused by an incorrect implicit statement (in [1]) that any 3-regular Ricci-flat graph G has a surface embedding whose faces are all pentagons. In this erratum, we analyze the case that G does not have a surface embedding whose faces are all pentagons. We will show that this case leads a unique missing graph — the Triplex graph. An alternative method to correct Theorem 1 in [1] is given in [2].

Recall that Lemma 3 item 2 in [1] states:

Lemma 1. *For any edge xy of a graph of girth at least 5, if $d_x = d_y = 3$ and $\kappa(x, y) = 0$, then xy belongs to two 5-cycles P_1 and P_2 such that $P_1 \cap P_2 = xy$.*

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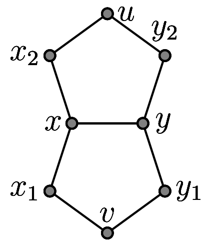


Since G contains no cycle of length 3 or 4, any C_5 containing the edge xy is uniquely determined by a 3-path passing through xy. Since $d_x = d_y = 3$, there are four 3-paths of form x_ixyy_j for $i, j = 1, 2$. Here x_1, x_2 are two neighbors of x other than y and y_1, y_2 are two neighbors of y other than x. We say two C_5's are *opposite* to each other at xy if one C_5 passes through x_ixyy_j and the other one passes through $x_{3-i}xyy_{3-j}$. The above lemma says that there is a pair of opposite C_5's sharing the edge xy. We say an edge xy is *irregular* if there are three or four C_5 passing through it.

From this lemma, we have the following corollary.

Corollary 1. *If G is a 3-regular Ricci-flat graph and contains no irregular edge, then G can be embedded into a surface so that all faces are pentagons.*

Proof. View G as 1-dimension skeleton and glue pentagons to G recursively. Starting with any C_5 and glue a pentagon to it as a face, call the two-dimensional region M.

Let xy be a boundary edge of M, that is, an edge belonging to only one pentagon f in M. This pentagon f determines an opposite C_5 of G with respect to the edge xy. We glue a pentagon face to the opposite C_5 at xy to enlarge M. Since G contains no irregular edge, every edge must be in exactly two pentagons. Therefore, the process will continue until M has no boundary edge. When this process ends, we get an embedding of G into some surface so that every face is a C_5. $\square$

We are ready to fix the proof of Theorem 1 in [1].

Proof of Theorem 1: In the original proof of Theorem 1 in [1], we have taken care of all the cases except that G is 3-regular and contains an irregular edge xy. The edge xy is either in three C_5's or four C_5's. We will show that the first case leads to the Triplex graph while the second case leads to the Petersen graph.

First assume the edge xy is contained in three C_5's: ux_2xyy_2u, vx_1xyy_1v, and wx_2xyy_1w. The path x_1xyy_2 is not in any C_5. Let w_1 be the third neighbor of x_1, and w_2 be the third neighbor of y_2. Then w_1, w_2 are two distinct vertices, and they cannot be coincident with any vertex on the three C_5's. This is our starting configuration (See Figure 2 with solid lines).

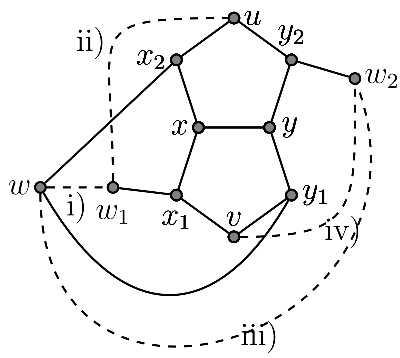


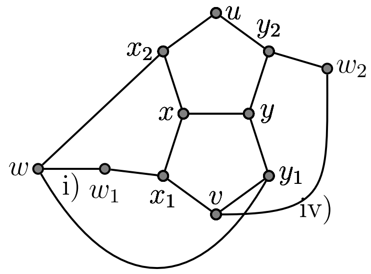
FIGURE 2. Starting configuration and possible extensions

Now consider the edge xx_1. Observe that the path w_1x_1xy is not on any C_5. Thus, the path $w_1x_1xx_2$ must be extended to a C_5. Either w_1u is an edge or w_1w is an edge. Similarly, by considering the edge yy_2, either w_2w or w_2v is an edge. These four possible edges are shown as dashed lines i), ii), iii), and iv) in Figure 2. There are four combinations: i)+iii), i)+iv), ii)+iii), ii)+iv). The combination i)+iii) is impossible since $d_w = 3$. The two cases i)+iv) and ii)+iii) are symmetric. Essentially we have two cases to consider:

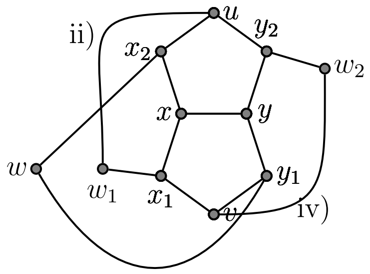
Case: i)+iv): Now consider the edge w_1x_1. By Lemma 1, there are a pair of opposite C_5 sharing the edge w_1x_1. Such a pair of opposite pentagons can be obtained only by adding a new vertex w_3 as the third neighbor of w_1 and connecting w_3 to w_2, since connecting w_1 to u would cause a C_4. Now $x_1w_1w_3w_2vx_1$ and $x_1w_1wx_2xx_1$ are the two opposite pentagons at x_1w_1. But, in order to have two opposite pentagons also at the new edge w_1w_3 we must have w_3u as an edge, which then creates a C_4: $w_3uy_2w_2w_3$. Contradiction!

Case: ii)+iv). Let w_3 be the third neighbor of w. (w_3 is distinct from w_1 and w_2 since the girth of G is at least 5.) Applying Lemma 1 on the edge wx_2, we must have a pair of opposite C_5's passing through wx_2. This will force w_3w_1 to be an edge. Similarly, by considering wy_1, we conclude that w_3w_2 must be an edge. This completes a 3-regular graph. It is easy to check this is the Triplex graph.

Now we assume xy is in four C_5's. For $i = 1, 2$ and $j = 1, 2$, write $k = 2(i-1) + j$ and let u_k be the vertex in the C_5 extending the path x_ixyy_j. Observe that connecting any pair u_1u_2, u_2u_4, u_4u_3, or u_1u_3



(A) Case 1



(B) Case 2

FIGURE 3. Two non-isomorphic ways to continue

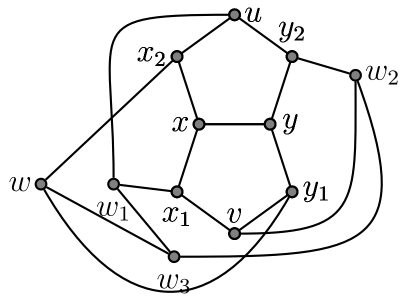


FIGURE 4. Unique way to complete into the Triplex graph.

results a triangle. So only u_2u_3 and u_1u_4 can be connected (See Figure 5).

Note that yy_1 are in two non-opposite C_5's: $xyy_1u_1x_1x$ and $xyy_1u_3x_2x$. So either u_2u_3 or u_1u_4 must be an edge.

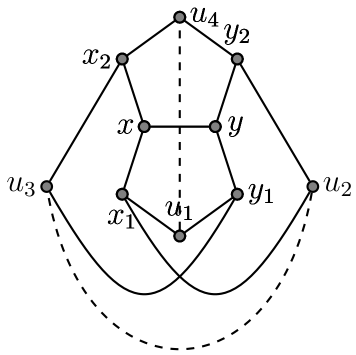


FIGURE 5. Starting configuration and possible extension when xy is in four C_5's.

If both u_1u_4 and u_2u_3 are edges, then the graph is completed and it is the Petersen graph.

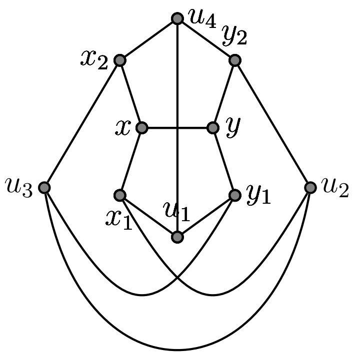


FIGURE 6. The Petersen Graph.

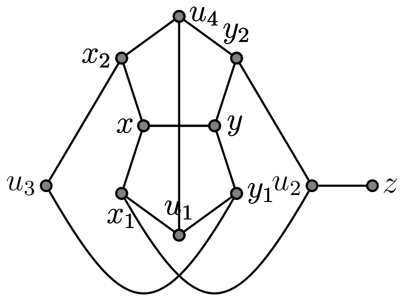


FIGURE 7. The edge zu_2 is not in any C_5. Contradiction!

If only one of them is an edge, by symmetry, we can assume u_1u_4 is an edge but u_2u_3 are not. Then u_2 must have an new neighbor, called z. Now the edge zu_2 can not be in any C_5. Otherwise, say zu_2XYZz is the C_5. We must have $X \in \{x_1, y_2\}$, $Y \in \{x, u_1, y, u_4\}$, and $Z \in \{x_2, y_1\}$. But now Zz is not an edge. Contradiction! □

References

[1] Y. Lin, L. Lu, and S.-T. Yau, *Ricci-flat graphs with girth at least five*, Comm. Anal. Geom. 22 (2014), no 4, 671-687.

[2] D. Cushing, R. Kangaslampi, Y. Lin, S. Liu, L. Lu, and S.-T. Yau, *Ricci-flat cubic graphs with girth five*, preprint.

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