Commun. Math. Phys. 335, 1215–1233 (2015)
Digital Object Identifier (DOI) 10.1007/s00220-015-2337-5

Communications in
**Mathematical
Physics**

Complete cscK Metrics on the Local Models of the Conifold Transition

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Received: 7 January 2014 / Accepted: 30 December 2014
Published online: 27 February 2015 –

Abstract: In this paper, we construct complete constant scalar curvature Kähler (cscK) metrics on the complement of the zero section in the total space of $\mathcal{O}(-1)^{\oplus 2}$ over $\mathbb{P}^1$, which is biholomorphic to the smooth part of the cone C_0 in $\mathbb{C}^4$ defined by equation $\Sigma_{i=1}^4 w_i^2 = 0$. On its small resolution and its deformation, we also consider complete cscK metrics and find that if the cscK metrics are homogeneous, then they must be Ricci-flat.

1. Introduction

An important problem in Kähler geometry is to find on a Kähler manifold a canonical metric in each Kähler class, such as Kähler–Einstein (KE) metrics, constant scalar curvature Kähler (cscK) metrics or even extremely Kähler metrics. When the Kähler manifold is compact or is the complement of divisors in a compact Kähler manifold, there are lots of references on this problem.

Here, we are concerned with the question of how to look for a canonical metric in each Kähler class on the complement of varieties with higher codimension in a compact Kähler manifold. The cscK metric on such a manifold may be a good candidate. For example, let X be a three dimensional Calabi–Yau manifold with $(-1,-1)$-curves E_1, …, E_l. These curves are rational curves $\mathbb{P}^1$ and their neighborhoods are isomorphic to $\mathcal{O}(-1)^{\oplus 2}$ over $\mathbb{P}^1$. Let $M = X \setminus \cup_i E_i$. For simplicity, assume that all E_i are disjoint. So the question is whether there exists a complete cscK metric on M.

Let us first recall the geometry of X. It is well-known that if we blow down these curves on X, we get a singular Calabi–Yau threefold X_0 with singular double points $p_1, \cdots, p_l$, and have a biholomorphic map [4]

$$X \setminus \cup_i E_i \to X_0 \setminus \cup_i p_i.$$

X is called the small resolution of X_0. Moreover, under a very not strictly topological condition on $E_1, \ldots, E_l$, X_0 can be smoothed out to get a family X_t of smooth complex manifolds [8–10]. Sometimes, these complex manifolds may be non-Kähler. The progress

$$X \to X_0 \rightsquigarrow X_t$$

is called the *conifold transition*, which is very important in the superstring theory (c.f.[1, 2,7]). It is in this sense that Candelas and de la Ossa studied the existence of Ricci-flat Kähler metrics on the local models of the transition above

$$E \to C_0 \rightsquigarrow C_t.$$

Here E is the total space of the holomorphic vector bundle $\mathcal{O}(-1)^{\oplus 2}$ over $\mathbb{P}^1$, C_t is the (complex) hypersurface in $\mathbb{C}^4$ defined by equation

$$\sum_{i=1}^{4} w_i^2 = t.$$

When $t = 0$, C_0 is the cone. So zero is the singular point in C_0. Candelas and de la Ossa [3] wrote down homogeneous Ricci-flat metrics on these local models, which are used to construct balanced metrics on X_t by J. Li and the first two named authors [5] when X_t are non-Kähler. In this note, we use the adjective "homogeneous" as in [3]. That is, when the local model is C_0 or C_t for $t \neq 0$, the homogeneous Kähler potential is a function of r^2, where r is the distance function induced from $\mathbb{C}^4$; when the local model is E, the homogeneous Kähler potential is a function as (1).

Candelas-de la Ossa's metric on $C_0\backslash\{0\}$ is not complete near zero. Its Kähler potential is $\frac{3}{2}(r^2)^{\frac{2}{3}}$. Since $C_0\backslash\{0\}$ is biholomorphic to $E\backslash\mathbb{P}^1$, this metric can be viewed as the one defined on $E\backslash\mathbb{P}^1$, which is also not complete near $\mathbb{P}^1$. Here $\mathbb{P}^1$ is viewed as the zero section of $\mathcal{O}(-1)^{\oplus 2}$. In fact, we cannot find any complete KE metrics on $E\backslash\mathbb{P}^1$. In this note, as the first step of the problem presented above, we use the natural distance function, which is also denoted by r (see the next section), to study complete cscK metrics on $E\backslash\mathbb{P}^1$. We will look for cscK metrics g_a on $E \setminus \mathbb{P}^1$ such that its associated Kähler potential is

$$a \log(1 + |z|^2) + f(r^2), \tag{1}$$

where z is the local coordinate of $\mathbb{P}^1$ and $a \geq 0$. We will determine the function $f(r^2)$, which is called the *partial Kähler potential* in the following. In this note, the constant scalar curvature will be denoted by c.

Theorem 1.1. *When $c = 0$ and $a \geq 0$, there exists on $E\backslash\mathbb{P}^1$ a unique complete cscK metric g_a with associated Kähler potential (1). The partial Kähler potential $f(r^2)$ has the following properties:*

(i) *When $r \to +\infty$, $f(r^2)$ approximates to $\frac{3}{2}(r^2)^{\frac{2}{3}}$, i.e., Candelas-de la Ossa's metric;*

(ii) *When $r \to 0$, $f(r^2)$ approximates to $\log r^2 - \frac{a+1}{a+2}\log(-\log r^2)$.*

Theorem 1.2. (i) *When $c < 0$ and $a \geq 0$, on the open set $\mathbf{U} = \{0 < r < 1\}$ of $E \setminus \mathbb{P}^1$ there exists a unique complete cscK metric g_a with associated Kähler potential (1);*

(ii) *When $c > 0$ and $a \geq 0$ with the relation $4+2a-(a+1)c > 0$, there exists on $E \setminus \mathbb{P}^1$ a unique cscK metric g_a with associated Kähler potential (1) which is complete near $\mathbb{P}^1$ but not complete when r goes to infinity;*

(iii) *In the above two cases, the partial Kähler potentials near* $\mathbb{P}^1$

$$f(r^2) = \log r^2 - \frac{2(a+1)}{4+2a-(a+1)c}\log(-\log r^2) + O((\log r^2)^{-1}).$$

The first example of a complete Kähler metric on $D^n\backslash\{0\}$, the punctured unit ball of $\mathbb{C}^n$, is given by Mok and the second named author [6]:

$$\omega = \frac{i}{2}\partial\bar{\partial}\big(\log r^2 - \log(-\log r^2)\big),$$

where r^2 is the standard distance function on $\mathbb{C}^n$. Its Ricci curvature is bounded and its volume is finite. How to further use and generalize the metric above is our second motivation. Inspired by this, we give the following definition.

Definition 1.1. *Let X be a Kähler manifold and N a smooth subvariety of X with higher codimension. Let $M = X \setminus N$. A Kähler metric ω on M is called the* Poincaré–Mok–Yau type metric *if near the subvariety N,*

$$\omega = \frac{i}{2}\partial\bar{\partial}\big(a\log r^2 - b\log(-\log r^2) + O((\log r^2)^{-1})\big),$$

where r is a distance function to the subvariety N, and a and b are two positive constants.

It is easy to see that near N the Poincaré–Mok–Yau type metric is complete and has finite volume. Under this definition, our metrics near the zero section for all three cases are the Poincare–Mok–Yau type metrics.

We have mentioned above that Candelas and de la Ossa [3] also constructed homogeneous Kähler Ricci-flat metrics on the local models E and C_t. Since our interest is enlarged to cscK metrics, it is natural to ask whether there are cscK metrics that are not Ricci-flat. As a result, we have the following rigidity theorem.

Theorem 1.3. *Any complete homogeneous cscK metric on E or C_t is Ricci-flat.*

The paper will be organized as follows. We first consider complete cscK metrics on $E\backslash\mathbb{P}^1$. In Sect. 2, we derive an ODE on the partial Kähler potential and solve this ODE. In Sect. 3, we will use the standard method to give a necessary condition of the completeness of metrics. Then in Sect. 4, we will use this condition to discuss the completeness and the asymptotic behaviors of all three cases for c. Thus we finish the proof of the first two theorems above. In Sect. 5, we prove the rigidity theorem of cscK metrics on E and C_t.

2. Equations

Let E be the total space of $\mathcal{O}(-1)^{\oplus 2}$ bundle over $\mathbb{P}^1$ with the fibre $\mathbb{C}^2$. The base manifold $\mathbb{P}^1$ can be viewed as the zero section or a complex curve of E. Then, $E\backslash\mathbb{P}^1$ is biholomorphic to $C_0\backslash\{0\}$. In fact, let (z, u, v) be the coordinate of fibre bundle E, where z is a coordinate of $\mathbb{P}^1$, and u, v are the coordinates of fibres of $\mathcal{O}(-1)^{\oplus 2}$. The biholomorphic map between $E\backslash\mathbb{P}^1$ and $C_0\backslash\{0\}$ is given by

$$w_1 = v - zu, \quad w_2 = -i(v+zu), \quad w_3 = -i(u - zv), \quad w_4 = u + zv.$$

Under this map, the distance function

$$r^2 = \sum_{i=1}^{4} |w_i|^2$$

when restricted to C_0 can be transformed to the function on E

$$r^2 = (1 + |z|^2)(|u|^2 + |v|^2).$$

Hence, the above r^2 is a globally defined function on whole E. Actually, this can also be checked by the transition functions. Let (w, x, y) be the another coordinates of E, where w is the coordinate of $\mathbb{P}^1$ such that $w = \frac{1}{z}$ when $z \neq 0$. Then the transition functions of u, v to x, y are

$$x = zu, \quad y = zv.$$

Hence, it is easy to see that r^2 is well defined on E.

Now we begin to study the cscK metrics g_a on $E\backslash\mathbb{P}^1$. Let $f(r^2) + a \log(1 + |z|^2)$ for $a \geq 0$ be its Käher potential. Then

$$\begin{aligned} g_{\mu\bar{\nu}} &= \partial_\mu \partial_{\bar{\nu}} \left(f(r^2) + a \log(1 + |z|^2) \right) \\ &= f'(r^2)(\partial_\mu \partial_{\bar{\nu}} r^2) + f''(r^2)\partial_\mu r^2 \cdot \partial_{\bar{\nu}} r^2 + a \partial_\mu \partial_{\bar{\nu}} \log(1 + |z|^2), \end{aligned}$$

where μ, ν are any two coordinates of (z, u, v). By a direct calculation, we have

$$\det g \triangleq \det(g_{\mu\bar{\nu}}) = f'(r^2)\big(a + r^2 f'(r^2)\big)\big(f'(r^2) + r^2 f''(r^2)\big), \tag{2}$$

and the fact that the metric g is positive if and only if f satisfies the following conditions:

$$f'(r^2) > 0, \quad \text{and} \quad f'(r^2) + r^2 f''(r^2) > 0.$$

The Ricci tensor is given by

$$R_{\mu\bar{\nu}} = -\partial_\mu \partial_{\bar{\nu}} \log \det g,$$

and the scalar curvature by

$$c = g^{\mu\bar{\nu}} R_{\mu\bar{\nu}},$$

where $(g^{\mu\bar{\nu}})$ denotes the inverse matrix of $(g_{\mu\bar{\nu}})$. Explicitly, all elements of $(g^{\mu\bar{\nu}})$ are

$$\begin{aligned} g^{z\bar{z}} &= \frac{(1+|z|^2)^2}{a + r^2 f'}, \quad g^{z\bar{u}} = \overline{g^{u\bar{z}}} = -\frac{(1+|z|^2) z\bar{u}}{a + r^2 f'}, \quad g^{z\bar{v}} = \overline{g^{v\bar{z}}} = -\frac{(1+|z|^2) z\bar{v}}{a + r^2 f'}, \\ g^{u\bar{u}} &= \frac{f'^2 \cdot (r^2 - |z|^2|v|^2) + r^2 f' f'' \cdot (|v|^2 + |u|^2|z|^2) + a(f'/(1+|z|^2) + f'' \cdot |v|^2)}{f' \cdot (a + r^2 f')(f' + r^2 f'')}, \\ g^{v\bar{v}} &= \frac{f'^2 \cdot (r^2 - |z|^2|u|^2) + r^2 f' f'' \cdot (|u|^2 + |v|^2|z|^2) + a(f'/(1+|z|^2) + f'' \cdot |u|^2)}{f' \cdot (a + r^2 f')(f' + r^2 f'')}, \\ g^{u\bar{v}} &= \overline{g^{v\bar{u}}} = -\frac{f' f'' \cdot r^2(1 - |z|^2) - f'^2 \cdot |z|^2 + a f''}{f' \cdot (a + r^2 f')(f' + r^2 f'')} u\bar{v}, \end{aligned}$$

where $f' = f'(r^2)$ and $f'' = f''(r^2)$.

Let $h(r^2) = \log\det g$ and for brevity, denote

$$\phi(r^2) = r^2 f'(r^2), \qquad v(r^2) = r^2 h'(r^2).$$

Then through a standard but tedious calculation, we get the scalar curvature

$$c = -\frac{v'(r^2)}{\phi'(r^2)} - \frac{(a+2\phi)v}{(a+\phi)\phi}. \tag{3}$$

Integrating (3) with the integrating factor $\phi^2 v + a\phi v$, we obtain a relation between $\phi(r^2)$ and $v(r^2)$

$$-\frac{c}{3}\phi^3 - \frac{ac}{2}\phi^2 = a\phi v + \phi^2 v + c_1, \tag{4}$$

where c_1 is a constant to be determined later. Using the definition of ϕ, we can rewrite (2) as

$$\det g = \frac{\phi(a+\phi)\phi'}{r^2}.$$

Hence,

$$v = r^2 h'(r^2) = r^2\left(\frac{\phi'}{\phi} + \frac{\phi'}{a+\phi} + \frac{\phi''}{\phi'} - \frac{1}{r^2}\right).$$

Therefore, (4) is equivalent to

$$-(\frac{c}{3}\phi^3 + \frac{ac}{2}\phi^2)\phi' = r^2(a+\phi)\phi'^2 + r^2\phi\phi'^2 + r^2(a+\phi)\phi\phi'' - (a+\phi)\phi\phi' + c_1\phi'.$$

Integrating the equation with the factor $r^2(a+\phi)\phi\phi'$, we have

$$\phi'(r^2)r^2 = \frac{-\frac{c}{12}\phi^4 + (\frac{2}{3} - \frac{ac}{6})\phi^3 + a\phi^2 - c_1\phi - c_2}{\phi(a+\phi)}, \tag{5}$$

where c_2 is a constant also to be determined later. In the following, for brevity, we denote by $F(\phi)$ the numerator of the fraction of the above equation

$$F(\phi) = -\frac{c}{12}\phi^4 + (\frac{2}{3} - \frac{ac}{6})\phi^3 + a\phi^2 - c_1\phi - c_2. \tag{6}$$

Using the definition of ϕ, $f(r^2)$ is the partial Kähler potential if and only if

$$\phi(r^2) > 0, \quad \text{and} \quad \phi'(r^2)r^2 = \frac{F(\phi)}{\phi(a+\phi)} > 0.$$

Then Eq. (5) can be rewritten as

$$\frac{dr^2}{r^2} = \frac{\phi(a+\phi)d\phi}{F(\phi)}. \tag{7}$$

The integration of both sides establishes an implicit function $\phi(r^2)$ since $\phi(r^2)$ increases monotonically. Thus from $\phi(r^2) = r^2 f'(r^2)$ we determine the partial Kähler potential

$$f(r^2) = \int \frac{\phi(r^2)}{r^2} dr^2.$$

3. Completeness

Now we discuss the completeness of the Kähler metric g_a defined in the last section.

Lemma 3.1. *The metric g_a is complete near $r = 0$ (near $r = \infty$, respectively) if and only if the integration $\int_{s_1}^{s_2} \sqrt{f'(r^2) + r^2 f''(r^2)}dr$ is infinite as $s_1 \to 0$ ($s_2 \to \infty$, respectively). Moreover, g_a is complete near zero section is equivalent to that $F(\phi)$ defined in (6) has a factor $(\phi - b)^2$ and $\lim_{r\to 0} \phi(r^2) = b$, where b is any given positive constant.*

Proof. By a direct calculation (please see the "Appendix"), the gradient

$$\nabla r = \frac{u\partial_u + v\partial_v + \bar{u}\partial_{\bar{u}} + \bar{v}\partial_{\bar{v}}}{(f'(r^2) + r^2 f''(r^2))r}$$

and

$$\nabla_{\frac{\nabla r}{|\nabla r|}} \frac{\nabla r}{|\nabla r|} = 0.$$

The equation above implies that the gradient ∇r is the geodesic direction. Hence, for any point $(z_0, u_0, v_0) \in E\backslash\mathbb{P}^1$, the curve $\chi(s) = (z_0, su_0, sv_0)$ is a geodesic. Clearly

$$|\chi'(s)| = \sqrt{f'(z_0, su_0, sv_0)r_0^2 + s^2 f''(z_0, su_0, sv_0)r_0^4}.$$

The length l of $\chi(s)$ from $s = \frac{s_1}{r_0}$ to $s = \frac{s_2}{r_0}$ is $l = \int_{s_1/r_0}^{s_2/r_0} |\chi'(s)|ds$. Taking the coordinate transformation $s = \frac{r}{r_0}$, we have

$$\int_{s_1/r_0}^{s_2/r_0} |\chi'(s)|ds = \int_{s_1}^{s_2} \sqrt{f'(r^2) + r^2 f''(r^2)}dr = \int_{s_1}^{s_2} \sqrt{\frac{d\phi}{dr^2}}dr.$$

Now by (7), the above integration

$$\int_{s_1/r_0}^{s_2/r_0} |\chi'(s)|ds = \int_{\phi(s_1)}^{\phi(s_2)} \frac{1}{2} \frac{\sqrt{\phi(a+\phi)}}{\sqrt{F(\phi)}}d\phi.$$

Hence, the metric g_a is complete if and only if the integration of polynomial $(\phi^2 + a\phi)^{1/2}F^{-1/2}(\phi)$ is infinite when $s_1 \to 0$ as s_2 is fixed and when $s_2 \to +\infty$ as s_1 is fixed.

Since ϕ is the positive and monotonically increasing function, the function ϕ tends to a nonnegative constant b as $r \to 0$, the completeness near the zero section $\mathbb{P}^1$ forces the polynomial $F(\phi)$ including a factor $(\phi - b)^2$. Therefore, the constants c_1 and c_2 are determined and we have

$$F(\phi) = (\phi - b)^2 F_1(\phi), \tag{8}$$

where

$$F_1(\phi) = -\frac{c}{12}\phi^2 + \left(\frac{2}{3} - \frac{(a+b)c}{6}\right)\phi + a + \frac{4b}{3} - \frac{abc}{3} - \frac{b^2c}{4}. \tag{9}$$

We should prove $b > 0$. If $b = 0$, then

$$F(\phi) = \phi^2\big(-\frac{c}{12}\phi^2 + (\frac{2}{3} - \frac{ac}{6})\phi + a\big)$$

and hence,

$$\frac{dr^2}{r^2} = \frac{(a+\phi)d\phi}{-\frac{c}{12}\phi^3 + (\frac{2}{3} - \frac{ac}{6})\phi^2 + a\phi}. \tag{10}$$

We discuss the following two cases. The first case is $a > 0$. In this case, if $c = 0$, then we recover Candelas and de la Ossa's Ricci-flat metric on the small resolution E and it is complete on entire E; If $c < 0$, this metric is the complete cscK metric on the open set $\{(z, u, v) \in E \mid r < R\}$ for some positive constant R, which will be seen from the remark 4.1 in the next section; If $c > 0$, it follows that ϕ is defined on the entire E and not complete as $r^2 \to +\infty$. So for the completeness near the zero section $\mathbb{P}^1$ of $E\backslash\mathbb{P}^1$, the constant b should be positive in this case.

Next we discuss the second case $a = 0$. In this case, Eq. (10) is reduced to

$$\frac{dr^2}{r^2} = \frac{d\phi}{-\frac{c}{12}\phi^2 + \frac{2}{3}\phi}.$$

Now, if c = 0, then we recover Candelas and de la Ossa's Ricci-flat metric on $E\backslash\mathbb{P}^1$ and it is not complete; If $c < 0$, the metric is defined on an open set $\{(z, u, v) \in E \mid 0 < r < R\}$ for some positive constant R (see Remark 4.1 in the next section), by the above discussions, which is not complete near the zero section $\mathbb{P}^1$ but complete as $r \to R$; If $c > 0$, it follows $0 < \phi < \frac{8}{c}$ and the metric is complete neither near the zero section $\mathbb{P}^1$ nor $r \to \infty$. So for the completeness near the zero section $\mathbb{P}^1$, the constant b should also be positive in this case. □

4. Proof of Theorem 1.1 and 1.2

Without loss of generality, set $b = 1$.

4.1. Case c = 0. In this case, (8) becomes

$$F(\phi) = \frac{2}{3}(\phi - 1)^2(\phi + 2 + \frac{3a}{2}) > 0.$$

Hence,

$$\begin{aligned}\frac{dr^2}{r^2} &= \frac{\phi(a+\phi)d\phi}{F(\phi)} = \frac{\phi(a+\phi)d\phi}{\frac{2}{3}(\phi-1)^2(\phi+2+\frac{3}{2}a)} \\ &= d\phi\big(\frac{A}{(\phi-1)^2} + \frac{B}{(\phi-1)} + \frac{C}{\phi+2+\frac{3}{2}a}\big),\end{aligned} \tag{11}$$

where

$$A = \frac{a+1}{a+2}, \qquad B = \frac{3a^2+10a+10}{3(a+2)^2}, \qquad \text{and} \qquad C = \frac{\frac{3}{2}a^2+8a+8}{3(a+2)^2}.$$

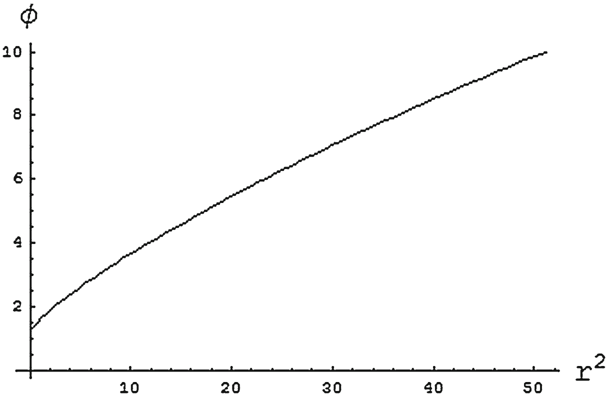

Fig. 1. The graph of $\phi(r^2)$ with $c = 0$ and $a = 1$

Therefore,

$$\log r^2 = \frac{A}{1-\phi} + B\log(\phi - 1) + C\log(\phi + 2 + \frac{3}{2}a).$$

Since when $r \to 0$, $\phi \to 1$, $\log r^2 \sim \frac{A}{1-\phi}$. Also since when $r \to +\infty$, $\phi \to +\infty$, $\log r^2 \sim \frac{3}{2}\log\phi$. It turns out from $\phi = r^2 f'(r^2)$ that the partial Kähler potential $f(r^2)$ has the asymptotic behavior

$$f(r^2) \sim \begin{cases} \log r^2 - \frac{a+1}{a+2}\log(-\log r^2), & r \to 0, \\ \frac{3}{2}(r^2)^{\frac{2}{3}}, & r \to +\infty. \end{cases}$$

Hence, the metric approximates to the Poincaré–Mok–Yau type metric near the zero section $\mathbb{P}^1$ (i.e. as $r \to 0$) and approximate to Candelas-de la Ossa's Ricci flat metric on $E\backslash\mathbb{P}^1$ as $r \to \infty$. Thus, we finish the proof of Theorem 1.1.

The picture of $\phi(r^2)$ in the case $a = 1$ is Fig. 1.

4.2. Case c < 0. In this case, since $a \geq 0$ and $c < 0$, $F_1(\phi)$ defined in (1) is automatically positive. Hence $F(\phi) > 0$. The equation (7) implies a function $\phi(r^2)$ or $r^2(\phi)$. Integrating both sides of (7), we have

$$\log r^2 + c_3 = \int \frac{\phi(a+\phi)}{F(\phi)}d\phi$$

for some constant c_3.

Since

$$\int_{a_0}^{\infty} \frac{\phi(a+\phi)}{F(\phi)}d\phi < +\infty$$

for any $a_0 > 1$, then any primitive function of $\frac{\phi(a+\phi)}{F(\phi)}$ is limited as $\phi \to +\infty$. We assume that $G(\phi)$ is such a primitive function. Then,

$$\log r^2 + c_3 = G(\phi)$$

and $\phi(r^2)$ is implied in this equation. Since $G(\phi)$ is tend to a constant as ϕ to $+\infty$, $\log r^2 + c_3$ is finite as ϕ to $+\infty$. Hence, we can choose some $G(\phi)$ and c_3 such that $\lim_{\phi\to+\infty} G(\phi) - c_3 = R^2$ for some positive constant R. Right now, $\phi \to +\infty$ as $r \to R^-$. Thus, we have a function $\phi(r^2)$ defined on open set $\mathbf{U}(R) = \{(z, u, v) \in E \mid 0 < r < R\}$ with the properties that $\lim_{r^2\to 0} \phi(r^2) = 1$ and $\lim_{r^2\to R^-} \phi(r^2) = +\infty$. In summary, for any $a \geq 0$ and $c < 0$, we can choose some constant $c_3 = c_3(a, c)$ such that $\phi(r^2)$ is only defined on $\mathbf{U}(R) = \{(z, u, v) \in E \mid 0 < r < R\}$. Without loss of generality, let $R = 1$ and $\mathbf{U} \triangleq \mathbf{U}(1) = \{(z, u, v) \in E \mid 0 < r^2 < 1\}$.

Remark 4.1. From the above discussion, for $c < 0$, the boundedness of the distance function r^2 is only dependent on the highest order of $F(\phi)$ but not the decomposition of $F(\phi)$. That's to say, any homogeneous negative cscK metric from Eq. (5) is only defined on some open set $\{(z, u, v) \in E \mid r < R\}$ for some positive constant R.

Next, we decompose

$$\frac{\phi(a+\phi)}{F(\phi)} = \frac{A}{(\phi-1)^2} + \frac{B}{\phi-1} + \frac{C}{F_1(\phi)}$$

where $A = A(a, c)$ and $B = B(a, c)$ are constants, C is a polynomial $C(a, c, \phi)$. As a record, we write down A, B and C here in detail

$$\begin{aligned}
A &= \frac{2(a+1)}{4+2a-(a+1)c},\\
B &= \frac{4(10+10a+3a^2-2c-3ac-a^2c)}{3(-4-2a+c+ac)^2},\\
C &= \frac{8(128+128a+24a^2-40c-50ac+8a^2c+12a^3c+3c^2}{(-4-2a+c+ac)^2}\\
&\quad+\frac{8(4ac^2-3a^2c^2-4a^3c^2+12a^3c+3c^2+4ac^2-3a^2c^2-4a^3c^2)}{(-4-2a+c+ac)^2}\\
&\quad+\frac{16(10c+10ac+3a^2c-2c^2-3ac^2+a^2c^2)\phi}{(-4-2a+c+ac)^2}.
\end{aligned}$$

Hence, near the zero section,

$$\frac{dr^2}{r^2} = \frac{\phi(a+\phi)}{F(\phi)}d\phi \sim \frac{A}{(\phi-1)^2}d\phi,$$

and the partial Kähler potential $f(r^2)$ has the following asymptotic behavior near the zero section

$$f(r^2) = \log r^2 - A\log(-\log r^2) + O((\log r^2)^{-1}).$$

On the other hand, when $r \to 1^-$, $\phi \to +\infty$. Then $\frac{dr^2}{r^2} = \frac{\phi(a+\phi)}{F(\phi)}d\phi \sim \frac{-12}{c\phi^2}d\phi$. Hence, when $r \to 1^-$

$$f(r^2) \sim \frac{12}{c}\log(-\log r^2) \sim \frac{12}{c}\log(1-r^2).$$

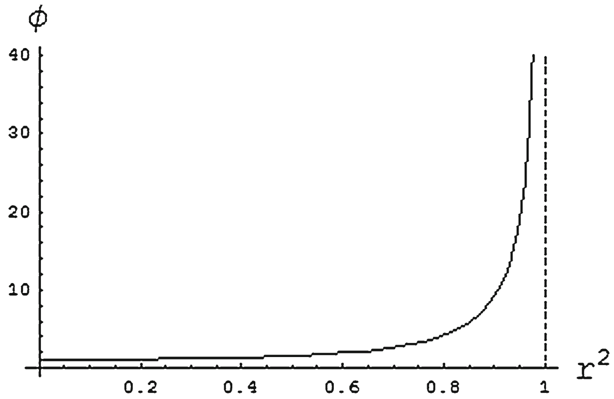

Fig. 2. The graph of $\phi(r^2)$ with $a = 0$ and $c = -12$

The Kähler metric on **U** approximates to the metric

$$\frac{i}{2}\partial\bar{\partial}\left(a\log(1+|z|^2)+\frac{12}{c}\log(1-r^2)\right).$$

Clearly, this metric is also complete when $r \to 1$. Actually, from Lemma 3.1, the curve $\chi(s) = (z_0, su_0, sv_0)$ is a geodesic through any point $(z_0, u_0, v_0) \in \mathbf{U}$. Since $r_0 < 1$ and $sr_0 < 1$, $s < \frac{1}{r_0}$. As $s \to \frac{1}{r_0}$, $\phi(s^2r_0^2) \to +\infty$ and

$$\int_1^s |\chi'(s)|ds = \int_{\phi(r_0^2)}^{\phi(s^2r_0^2)} \frac{\phi}{2\sqrt{F(\phi)}}d\phi \to +\infty,$$

which also implies the completeness.

We give a picture of $\phi(r^2)$ in the case of $a = 0$, $c = -12$. In this case,

$$c_3 = \frac{133\sqrt{23}\pi - 391\log 3}{4416}$$

and

$$G(\phi) = -\frac{1}{8(\phi-1)} + \frac{133}{96\sqrt{23}}\arctan\frac{3\phi+4}{\sqrt{23}} + \frac{17}{96}\log(\phi-1) - \frac{17}{192}\log(3\phi^2+8\phi+13).$$

4.3. $c > 0$. In this case, $F_1(\phi)$ defined in (1) is not automatically positive. We have assumed that $\lim_{r\to 0}\phi = 1$. Since $F_1(1) = 2 + a - 1/2(a+1)c$, in order to $F_1(\phi) > 0$, we must assume that (Fig. 2)

$$4 + 2a - (a+1)c > 0.$$

Under this assumption, $F_1(\phi) = 0$ has two real solutions

$$b_1 = \frac{4-(1+a)c-\sqrt{16+8c+4ac-2c^2-2ac^2+a^2c^2}}{c},$$

$$b_2 = \frac{4-(1+a)c+\sqrt{16+8c+4ac-2c^2-2ac^2+a^2c^2}}{c}.$$

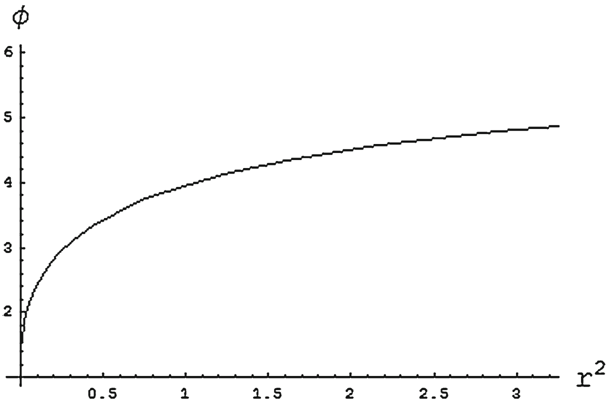

Fig. 3. The graph of $\phi(r^2)$ with $a = 1$ and $c = 1$

It is easy to see that $b_1 < 1 < b_2$. Clearly when $1 < \phi < b_2$, $F_1(\phi) > 0$, and hence $\frac{d\phi}{dr^2} > 0$. So we should prove that $1 < \phi < b_2$ for each $r \in (0, \infty)$. Write

$$F_1(\phi) = \frac{c}{12}(\phi - b_1)(b_2 - \phi)$$

and hence,

$$\frac{dr^2}{r^2} = \frac{\phi(a+\phi)d\phi}{\frac{c}{12}(\phi-1)^2(\phi-b_1)(b_2-\phi)}. \tag{12}$$

Now from the above equation, it is clear that there does not exist a r_0 such that $\phi(r_0^2) = b_2$.

Hence, we get a cscK metric with $0 < c < \frac{4+2a}{a+1}$ on entire manifold $E\backslash\mathbb{P}^1$ which is complete near $r = 0$. From Lemma 3.1, we also know that the metric is not complete as $r \to \infty$. This metric approximates to the Poincaré–Mok–Yau type metric when $r \to 0$. In fact, near the zero section $\mathbb{P}^1$, $f(r^2)$ also has the approximate value

$$f(r^2) \sim \log r^2 - \frac{2(a+1)}{4+2a-(a+1)c}\log(-\log r^2).$$

Whereas $r \to \infty$,

$$f(r^2) \sim b_2 \log r^2 + br^{-2c_4},$$

for

$$c_4 = \frac{c(b_2-1)^2(b_2-b_1)}{12b_2(a+b_2)} \text{ and some constant } b > 0.$$

This fact is, from the Eq. (12), as $r^2 \to +\infty$,

$$\frac{dr^2}{r^2} \sim \frac{b_2(a+b_2)}{\frac{c}{12}(b_2-1)^2(b_2-b_1)(b_2-\phi)}d\phi.$$

The picture of $\phi(r^2)$ in the case of $a = 1$ and $c = 1$ is Figure 3. In this case, $b_1 = -3,\ b_2 = 7$.

Together with the case of $c < 0$, we finish the proof of Theorem 1.2.

5. Rigidity Theorem of cscK Metrics

In this section, we prove Theorem 1.3 by proving the following two propositions.

Proposition 5.1. *The Kähler metric on E given by the Kähler potential $f(r^2)+a\log(1+|z|^2)$ for $a>0$ is a complete cscK metric if and only if it is Ricci-flat.*

Proof. Assuming the cscK metric g is given by the Kähler potential $f(r^2)+a\log(1+|z|^2)$. Since g is smooth, we have $f(r^2)$ is smooth at $r^2=0$. It follows that

$$\lim_{r^2\to 0}\phi(r^2)=\lim_{r^2\to 0} r^2 f'(r^2)=0,$$

and

$$\lim_{r^2\to 0}\phi'(r^2)=\lim_{r^2\to 0} f'(r^2)+r^2 f''(r^2)<+\infty.$$

From Sect. 2, the metric g is a cscK metric if and only if $f(r^2)$ satisfies Eq. (5). It requires $c_1=c_2=0$ form $\lim_{r^2\to 0}\phi(r^2)=0$ and the existence of $\lim_{r^2\to 0}\phi'(r^2)$. Then (5) becomes

$$\frac{dr^2}{r^2}=\frac{(a+\phi)d\phi}{-\frac{c}{12}\phi^3+(\frac{2}{3}-\frac{ac}{6})\phi^2+a\phi}. \tag{13}$$

From the discussion at the last part of the Sect. 3, the existence of complete cscK metric from Eq. (13) requires $c=0$. This implies g is Ricci flat. In fact, from (4) and $c_1=0$, we have $v'(r^2)=(r^2\log\det g)'(r^2)=0$. It is easy to see the metric is Ricci-flat. □

Proposition 5.2. *Any complete homogeneous cscK metric on the entire deformation manifold C_t is Ricci-flat.*

Proof. Let $r^2=\Sigma_{i=1}^4|w_i|^2$. Assume that the homogeneous metric $g(t)$ is given by the Kähler potential $f(r^2)$. If take (w_1,w_2,w_3) as the local coordinates of C_t, then for $1\le i,j\le 3$,

$$\begin{aligned}g(t)_{i\bar{j}}&=\partial_{w_i}\partial_{\bar{w}_j}f(r^2)=f'(r^2)\partial_{w_i}\partial_{\bar{w}_j}r^2+f''(r^2)\partial_{w_i}r^2\partial_{\bar{w}_j}r^2\\&=f'(r^2)\big(\delta_{ij}+\frac{w_i\bar{w}_j}{|w_4|^2}\big)+f''\big(r^2)(w_i\bar{w}_j+w_j\bar{w}_i-\frac{\bar{w}_4^2w_iw_j+w_4^2\bar{w}_i\bar{w}_j}{|w_4|^2}\big).\end{aligned}$$

By a direct calculation, we have

$$\det g(t)=\det(g(t)_{i\bar{j}})=\frac{(f')^2\left(r^2f'+(r^4-|t|^2)f''\right)}{|w_4|^2}.$$

The Ricci curvature

$$R_{i\bar{j}}=-\partial_{w_i}\partial_{\bar{w}_j}\log\det g(t)=-\partial_{w_i}\partial_{\bar{w}_j}\log((f')^2(r^2f'+(r^4-|t|^2)f'')), \tag{14}$$

since

$$\partial_i\bar{\partial}_j\log|w_4|^2=0.$$

Let

$$\tilde{h}(r^2) = \log((f')^2(r^2 f' + (r^4 - |t|^2) f'')).$$

Then the scalar curvature

$$c = g(t)^{i\bar{j}} R_{i\bar{j}} = -g(t)^{i\bar{j}} \partial_{w_i} \partial_{\bar{w}_j} \tilde{h},$$

where $g(t)^{i\bar{j}}$ is the inverse matrix of $g(t)_{i\bar{j}}$. We write $g(t)^{i\bar{j}}$ in detail as follows

$$\begin{aligned} g(t)^{i\bar{j}} = & \frac{f' \cdot \bar{w}_i w_j + (|w_4|^2 + |w_k|^2)(w_i \bar{w}_j + w_j \bar{w}_i) f''}{f'(r^2 f' + (r^4 - |t|^2) f'')} \\ & - \frac{((w_4^2 + w_k^2)\bar{w}_i \bar{w}_j + (\bar{w}_4^2 + \bar{w}_k^2) w_i w_j) f''}{f'(r^2 f' + (r^4 - |t|^2) f'')} \end{aligned}$$

for $i \neq j$, where $k \neq i,\ k \neq j$; and

$$g(t)^{i\bar{i}} = \frac{(r^2 - |w_i|^2) f' + ((r^2 - |w_i|^2)^2 - |t - w_i^2|^2) f''}{f'(r^2 f' + (r^4 - |t|^2) f'')}.$$

Based on these, we find its scalar curvature

$$c = -2\frac{\tilde{h}'}{f'} - \frac{r^2 \tilde{h}' + (r^4 - |t|^2)\tilde{h}''}{r^2 f' + (r^4 - |t|^2) f''}. \tag{15}$$

It is obvious that the completeness near $r \to +\infty$ of $g(t)$ is the same of $g_0(0)$. From Theorem 1.1 and 1.2, we know that the only case $c = 0$ can give us a complete metric near $r \to +\infty$. Thus we just need to consider the case $c = 0$. Hence, the above equation is reduced to

$$2\frac{\tilde{h}'}{f'} + \frac{r^2 \tilde{h}' + (r^4 - |t|^2)\tilde{h}''}{r^2 f' + (r^4 - |t|^2) f''} = 0,$$

that is

$$3\tilde{h}' f'^2 \cdot r^2 + (\tilde{h}' f'^2)'(r^4 - |t|^2) = 0.$$

The general solutions of the above equation are

$$f'^2 \tilde{h}' = c_0 (r^4 - |t|^2)^{-3/2} \tag{16}$$

where c_0 is any constant.

But the metric we need should be regular at $r^2 = |t|$, then c_0 should be zero since the left side of (16) is limited as $r^2 \to |t|$. That's to say

$$\tilde{h}' \equiv 0$$

which implies that the metric is Ricci flat from (14). □

In fact, by setting $r^2 = |t| \cosh \tau$, the Kähler potential $f(r^2)$ of Ricci flat in [3] satisfies

$$r^2 f' = \frac{2^{-1/3} |t|^{2/3}}{\tanh \tau} (\sinh 2\tau - 2\tau)^{1/3}.$$

Acknowledgements. Fu is supported in part by NSFC grants 11025103 and 11421061. Yau is supported in part by NSF grants.

6. Appendix

Lemma 6.1. *The gradient of r is*

$$\nabla r = \frac{u\partial_u + v\partial_v + \bar{u}\partial_{\bar{u}} + \bar{v}\partial_{\bar{v}}}{(f'(r^2) + r^2 f''(r^2))r}$$

and satisfies

$$\nabla_{\frac{\nabla r}{|\nabla r|}} \frac{\nabla r}{|\nabla r|} = 0.$$

Proof. Let $f(r^2) + a\log(1+|z|^2)$ be the Käher potential and let

$$g_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} f(r^2) + a\log(1+|z|^2)$$

where μ, ν are any two coordinates of (z, u, v) and $r^2 = (1+|z|^2)(|u|^2+|v|^2)$. The complex gradient $\nabla^{\mathbb{C}} r^2$ is computed as follows:

$$\begin{aligned}
\nabla^{\mathbb{C}} r^2 &= g^{\mu\bar{\nu}} \bar{\partial}_\nu r^2 \partial_\mu \\
&= \{g^{z\bar{z}} \frac{\partial r^2}{\partial \bar{z}} + g^{z\bar{u}} \frac{\partial r^2}{\partial \bar{u}} + g^{z\bar{v}} \frac{\partial r^2}{\partial \bar{v}}\} \frac{\partial}{\partial z} \\
&\quad + \{g^{u\bar{z}} \frac{\partial r^2}{\partial \bar{z}} + g^{u\bar{u}} \frac{\partial r^2}{\partial \bar{u}} + g^{u\bar{v}} \frac{\partial r^2}{\partial \bar{v}}\} \frac{\partial}{\partial u} \\
&\quad + \{g^{v\bar{z}} \frac{\partial r^2}{\partial \bar{z}} + g^{v\bar{u}} \frac{\partial r^2}{\partial \bar{u}} + g^{v\bar{v}} \frac{\partial r^2}{\partial \bar{v}}\} \frac{\partial}{\partial v}
\end{aligned}$$

We use the explicit expression of $\{g^{\mu\nu}\}$ in page 4 to compute the coefficients of $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ as follows.We assume $a = 0$ for simplicity since the case of $a \neq 0$ is the same and the expressions of its calculations are more tedious to write. First,

$$\begin{aligned}
&g^{z\bar{z}} \frac{\partial r^2}{\partial \bar{z}} + g^{z\bar{u}} \frac{\partial r^2}{\partial \bar{u}} + g^{z\bar{v}} \frac{\partial r^2}{\partial \bar{v}} \\
&= \frac{(1+|z|^2)^2}{f' \cdot r^2}(|u|^2+|v|^2)z - \frac{(1+|z|^2)^2}{f' \cdot r^2}|u|^2 z - \frac{(1+|z|^2)^2}{f' \cdot r^2}|v|^2 z = 0
\end{aligned}$$

Secondly,

$$\begin{aligned}
&g^{u\bar{z}} \frac{\partial r^2}{\partial \bar{z}} + g^{u\bar{u}} \frac{\partial r^2}{\partial \bar{u}} + g^{u\bar{v}} \frac{\partial r^2}{\partial \bar{v}} \\
&= -\frac{(1+|z|^2)|z|^2(|u|^2+|v|^2)}{f' \cdot r^2} u \\
&\quad + \frac{f'(r^2 - |z|^2|v|^2) + r^2 f''(|v|^2 + |u|^2|z|^2)}{f'(f' + r^2 f'')r^2}(1+|z|^2)u \\
&\quad - \frac{f'' r^2(1-|z|^2) - f' \cdot |z|^2}{f'(f' + r^2 f'')r^2}(1+|z|^2)|v|^2 u \\
&= \frac{-(f' + r^2 f'')r^2|z^2|u}{f'(f' + r^2 f'')r^2} + \frac{(f' + f''(|u|^2+|v|^2)|z|^2)r^2(1+|z|^2)u}{f'(f' + r^2 f'')r^2} \\
&= \frac{u}{f' + r^2 f''}
\end{aligned}$$

Similarly,

$$g^{v\bar{z}}\frac{\partial r^2}{\partial \bar{z}} + g^{v\bar{u}}\frac{\partial r^2}{\partial \bar{u}} + g^{v\bar{v}}\frac{\partial r^2}{\partial \bar{v}} = \frac{v}{f' + r^2 f''}.$$

Hence, we get

$$\nabla^{\mathbb{C}} r^2 = \frac{u\partial_u + v\partial_v}{f' + r^2 f''}$$

or

$$\nabla^{\mathbb{C}} r = \frac{1}{2}\frac{u\partial_u + v\partial_v}{(f' + r^2 f'')r}$$

Then the real gradient ∇r of r is

$$\nabla r = \frac{u\partial_u + v\partial_v + \bar{u}\partial_{\bar{u}} + \bar{v}\partial_{\bar{v}}}{(f' + r^2 f'')r}.$$

Next we prove

$$\nabla_{\frac{\nabla r}{|\nabla r|}}\frac{\nabla r}{|\nabla r|} = 0$$

By the direct calculation , we have

$$|\nabla r|^2 = \frac{1}{f' + r^2 f''}$$

and

$$\frac{\nabla r}{|\nabla r|} = \frac{u\partial_u + v\partial_v + \bar{u}\partial_{\bar{u}} + \bar{v}\partial_{\bar{v}}}{(f' + r^2 f'')^{1/2} r}$$

Hence, we should check

$$\nabla_{\frac{u\partial_u + v\partial_v + \bar{u}\partial_{\bar{u}} + \bar{v}\partial_{\bar{v}}}{(f' + r^2 f'')^{1/2} r}}\frac{u\partial_u + v\partial_v + \bar{u}\partial_{\bar{u}} + \bar{v}\partial_{\bar{v}}}{(f' + r^2 f'')^{1/2} r} = 0$$

This covariant derivative has two terms:

$$\frac{1}{(f' + r^2 f'')r^2}\nabla_{(u\partial_u + v\partial_v + \bar{u}\partial_{\bar{u}} + \bar{v}\partial_{\bar{v}})}(u\partial_u + v\partial_v + \bar{u}\partial_{\bar{u}} + \bar{v}\partial_{\bar{v}}) \tag{17}$$

and

$$\frac{(u\partial_u + v\partial_v + \bar{u}\partial_{\bar{u}} + \bar{v}\partial_{\bar{v}})}{(f' + r^2 f'')^{1/2} r}\nabla_{(u\partial_u + v\partial_v + \bar{u}\partial_{\bar{u}} + \bar{v}\partial_{\bar{v}})}((f' + r^2 f'')^{1/2} r)^{-1}$$

The second term is easily dealt:

$$\begin{aligned}&\frac{(u\partial_u + v\partial_v + \bar{u}\partial_{\bar{u}} + \bar{v}\partial_{\bar{v}})}{(f' + r^2 f'')^{1/2} r}\nabla_{(u\partial_u + v\partial_v + \bar{u}\partial_{\bar{u}} + \bar{v}\partial_{\bar{v}})}((f' + r^2 f'')^{1/2} r)^{-1}\\ &= -\frac{f' + 3f'' \cdot r^2 + f''' \cdot r^4}{(f' + r^2 f'')^2 r^2}(u\partial_u + v\partial_v + \bar{u}\partial_{\bar{u}} + \bar{v}\partial_{\bar{v}})\end{aligned} \tag{18}$$

Now we deal with the first term. We have

$$
\begin{aligned}
&\nabla_{(u\partial_u+v\partial_v+\bar{u}\partial_{\bar{u}}+\bar{v}\partial_{\bar{v}})}(u\partial_u+v\partial_v+\bar{u}\partial_{\bar{u}}+\bar{v}\partial_{\bar{v}})\\
&\quad=(u^2\nabla_{\partial_u}\partial_u+v^2\nabla_{\partial_v}\partial_v+uv\nabla_{\partial_u}\partial_v+uv\nabla_{\partial_v}\partial_u+u\partial_u+v\partial_v)\\
&\qquad+(\bar{u}^2\nabla_{\partial_{\bar{u}}}\partial_{\bar{u}}+\bar{v}^2\nabla_{\partial_{\bar{v}}}\partial_{\bar{v}}+\bar{u}\bar{v}\nabla_{\partial_{\bar{u}}}\partial_{\bar{v}}+\bar{u}\bar{v}\nabla_{\partial_{\bar{v}}}\partial_{\bar{u}}+\bar{u}\partial_{\bar{u}}+\bar{v}\partial_{\bar{v}})
\end{aligned}
$$

We only need to compute

$$
u^2\nabla_{\partial_u}\partial_u+v^2\nabla_{\partial_v}\partial_v+uv\nabla_{\partial_u}\partial_v+uv\nabla_{\partial_v}\partial u+u\partial_u+v\partial_v
$$

since the other part is its conjugation.

First we compute $\nabla_{\partial u}\partial u$. Let $\nabla_{\partial_\mu}\partial\nu=\Gamma^{\omega}_{\mu\nu}\partial_\omega$, then $\Gamma^{\omega}_{\mu\nu}=g^{\omega\bar{\lambda}}\frac{\partial g_{\mu\bar{\lambda}}}{\partial\nu}$. Hence,

$$
\begin{aligned}
\nabla_{\partial u}\partial u&=\Gamma^{\nu}_{uu}\partial_\nu=g^{\nu\bar{\mu}}\frac{\partial g_{u\bar{\mu}}}{\partial u}\partial_\nu\\
&=\{g^{z\bar{z}}\frac{\partial g_{u\bar{z}}}{\partial u}+g^{z\bar{u}}\frac{\partial g_{u\bar{u}}}{\partial u}+g^{z\bar{v}}\frac{\partial g_{u\bar{v}}}{\partial u}\}\frac{\partial}{\partial z}\\
&\quad+\{g^{u\bar{z}}\frac{\partial g_{u\bar{z}}}{\partial u}+g^{u\bar{u}}\frac{\partial g_{u\bar{u}}}{\partial u}+g^{u\bar{v}}\frac{\partial g_{u\bar{v}}}{\partial u}\}\frac{\partial}{\partial u}\\
&\quad+\{g^{v\bar{z}}\frac{\partial g_{u\bar{z}}}{\partial u}+g^{v\bar{u}}\frac{\partial g_{u\bar{u}}}{\partial u}+g^{v\bar{v}}\frac{\partial g_{u\bar{v}}}{\partial u}\}\frac{\partial}{\partial v}
\end{aligned}
$$

The above three terms are computed as follows:

$$
\begin{aligned}
&\big(g^{z\bar{z}}\frac{\partial g_{u\bar{z}}}{\partial u}+g^{z\bar{u}}\frac{\partial g_{u\bar{u}}}{\partial u}+g^{z\bar{v}}\frac{\partial g_{u\bar{v}}}{\partial u}\big)\frac{\partial}{\partial z}\\
&\quad=\Bigg(\frac{(1+|z|^2)^2}{f'\cdot r^2}(f'+r^2f'')'(1+|z|^2)\bar{u}^2z\\
&\qquad-\frac{(1+|z|^2)^2}{f'\cdot r^2}[2f''+f'''(1+|z|^2)|u|^2]\bar{u}^2z\\
&\qquad-\frac{(1+|z|^2)^2}{f'\cdot r^2}f'''(1+|z|^2)|v|^2\bar{u}^2z\Bigg)\frac{\partial}{\partial z}=0;\\
&\left(g^{u\bar{z}}\frac{\partial g_{u\bar{z}}}{\partial u}+g^{u\bar{u}}\frac{\partial g_{u\bar{u}}}{\partial u}+g^{u\bar{v}}\frac{\partial g_{u\bar{v}}}{\partial u}\right)\frac{\partial}{\partial u}\\
&\quad=\Bigg(-\frac{(1+|z|^2)^2}{f'\cdot r^2}(f'+r^2f'')'|z|^2|u|^2\\
&\qquad+\frac{f'\cdot(r^2-|z|^2|v|^2)+r^2f''\cdot(|v|^2+|u|^2|z|^2)}{f'(f'+r^2f'')r^2}\big(2f''(1+|z|^2)^2+sf'''(1+|z|^2)^3|u|^2\big)\\
&\qquad-\frac{f''\cdot r^2(1-|z|^2)-f'\cdot|z|^2}{f'(f'+r^2f'')r^2}f'''(1+|z|^2)^3|v|^2|u|^2\Bigg)\bar{u}\partial_u\\
&\quad=\frac{(1+|z|^2)^2\left(2f'f''(|u|^2+|v|^2)+2r^2(f'')^2|v|^2+f'f'''r^2|u|^2\right)\bar{u}\partial_u}{f'(f'+r^2f'')r^2};
\end{aligned}
$$

and

$$\begin{aligned}
&\Big(g^{v\bar z}\frac{\partial g_{u\bar z}}{\partial u}+g^{v\bar u}\frac{\partial g_{u\bar u}}{\partial u}+g^{v\bar v}\frac{\partial g_{u\bar v}}{\partial u}\Big)\frac{\partial}{\partial v}\\
&=\Bigg(-\frac{(1+|z|^2)^2}{f'\cdot r^2}(f'+r^2f'')'|z|^2\\
&\quad-\frac{f''\cdot r^2(1-|z|^2)-f'\cdot|z|^2}{f'(f'+r^2f'')r^2}\left(2f''(1+|z|^2)^2+f'''(1+|z|^2)^3|u|^2\right)\\
&\quad\frac{f'\cdot(r^2-|z|^2|u|^2)+r^2f''\cdot(|u|^2+|v|^2|z|^2)}{f'(f'+r^2f'')r^2}f'''(1+|z|^2)^3|v|^2|u|^2\Bigg)\bar u^2v\partial v\\
&=\frac{(1+|z|^2)^2(-2(f'')^2r^2+f'f'''\cdot r^2)\bar u^2v\partial v}{f'(f'+r^2f'')r^2}
\end{aligned}$$

Thus

$$\begin{aligned}
u^2\nabla_{\partial_u}\partial_u=&\frac{(1+|z|^2)^2\left(2f'f''(|u|^2+|v|^2)|u|^2+2r^2(f'')^2|u|^2|v|^2+r^2f'f'''|u|^4\right)u\partial_u}{f'(f'+r^2f'')r^2}\\
&+\frac{(1+|z|^2)^2\left(-2(f'')^2r^2+r^2f'f'''\right)|u|^4v\partial_v}{f'(f'+r^2f'')r^2}
\end{aligned}\tag{19}$$

Similarly,

$$\begin{aligned}
v^2\nabla_{\partial_v}\partial_v=&\frac{(1+|z|^2)^2\left(-2(f'')^2r^2+r^2f'f'''\right)|v|^4u\partial_u}{f'(f'+r^2f'')r^2}\\
&+\frac{(1+|z|^2)^2\left(2f'f''(|u|^2+|v|^2)|v|^2+2r^2(f'')^2|u|^2|v|^2+r^2f'f'''|v|^4\right)v\partial_v}{f'(f'+r^2f'')r^2}
\end{aligned}\tag{20}$$

Now we compute $\nabla_{\partial_u}\partial_v$

$$\begin{aligned}
\nabla_{\partial u}\partial v=\Gamma^{\nu}_{uv}\partial_\nu=g^{\nu\bar\mu}\frac{\partial g_{u\bar\mu}}{\partial v}\partial_\nu\\
=\{g^{z\bar z}\frac{\partial g_{u\bar z}}{\partial v}+g^{z\bar u}\frac{\partial g_{u\bar u}}{\partial v}+g^{z\bar v}\frac{\partial g_{u\bar v}}{\partial v}\}\frac{\partial}{\partial z}\\
+\{g^{u\bar z}\frac{\partial g_{u\bar z}}{\partial v}+g^{u\bar u}\frac{\partial g_{u\bar u}}{\partial v}+g^{u\bar v}\frac{\partial g_{u\bar v}}{\partial v}\}\frac{\partial}{\partial u}\\
+\{g^{v\bar z}\frac{\partial g_{u\bar z}}{\partial v}+g^{v\bar u}\frac{\partial g_{u\bar u}}{\partial v}+g^{v\bar v}\frac{\partial g_{u\bar v}}{\partial v}\}\frac{\partial}{\partial v}
\end{aligned}$$

We compute coefficients of ∂_z, ∂_u and ∂_v. First,

$$\begin{aligned}
g^{z\bar z}\frac{\partial g_{u\bar z}}{\partial v}+g^{z\bar u}\frac{\partial g_{u\bar u}}{\partial v}+g^{z\bar v}\frac{\partial g_{u\bar v}}{\partial v}=&\frac{(1+|z|^2)^2}{f'\cdot r^2}(f+r^2f'')'(1+|z|^2)z\bar u\bar v\\
&-\frac{(1+|z|^2)^2}{f'\cdot r^2}z\bar u\left(f''(1+|z|^2)^2\bar v+f'''(1+|z|^2)^3|u|^2\bar v\right)\\
&-\frac{(1+|z|^2)^2}{f'\cdot r^2}z\bar v\left(f''(1+|z|^2)^2\bar u+f'''(1+|z|^2)^3|v|^2\bar u\right)
\end{aligned}$$

$$= \frac{z\bar{u}\bar{v}(1+|z|^2)^3\left\{2f''+r^2f''' - f'' - f'''(1+|z|^2)|u|^2 - f'''(1+|z|^2)|v|^2 - f''\right\}}{f'\cdot r^2}$$
$$= 0.$$

Secondly,

$$\begin{aligned}
& g^{u\bar{z}}\frac{\partial g_{u\bar{z}}}{\partial v} + g^{u\bar{u}}\frac{\partial g_{u\bar{u}}}{\partial v} + g^{u\bar{v}}\frac{\partial g_{u\bar{v}}}{\partial v} \\
&= -\frac{(1+|z|^2)^2}{f'\cdot r^2}(f'+r^2f'')'|z|^2|u|^2\bar{v} \\
&\quad + \frac{f'\cdot(r^2-|z|^2|v|^2)+r^2f''(|v|^2+|u|^2|z|^2)}{f'(f'+r^2f'')r^2}(f''\cdot(1+|z|^2)^2+f'''(1+|z|^2)^3|u|^2)\bar{v} \\
&\quad - \frac{f''\cdot r^2(1-|z|^2)-f'\cdot|z|^2}{f'(f'+r^2f'')r^2}\left(f''(1+|z|^2)^2+f'''(1+|z|^2)^3|v|^2\right)|u|^2\bar{v} \\
&= \frac{(1+|z|^2)^2\left(f'f''(|u|^2+|v|^2)+r^2(f'')^2(|v|^2-|u|^2)+r^2f'f'''|u|^2\right)\bar{v}}{f'(f'+r^2f'')r^2}
\end{aligned}$$

Similarly,

$$\begin{aligned}
& g^{v\bar{z}}\frac{\partial g_{u\bar{z}}}{\partial v} + g^{v\bar{u}}\frac{\partial g_{u\bar{u}}}{\partial v} + g^{v\bar{v}}\frac{\partial g_{u\bar{v}}}{\partial v} \\
&\quad = \frac{(1+|z|^2)^2\left(f'f''(|u|^2+|v|^2)+r^2(f'')^2(|u|^2-|v|^2)+r^2f'f'''|v|^2\right)\bar{u}}{f'(f'+r^2f'')r^2}
\end{aligned}$$

Thus,

$$\begin{aligned}
uv\nabla_{\partial_u}\partial_v &= \frac{(1+|z|^2)^2\left(f'f''(|u|^2+|v|^2)+r^2(f'')^2(|v|^2-|u|^2)+r^2f'f'''|u|^2\right)|v|^2u\partial_u}{f'(f'+r^2f'')r^2} \\
&\quad + \frac{(1+|z|^2)^2\left(f'f''(|u|^2+|v|^2)+r^2(f'')^2(|u|^2-|v|^2)+r^2f'f'''|v|^2\right)|u|^2v\partial_v}{f'(f'+r^2f'')r^2}
\end{aligned}$$

We also have the similar formula of $uv\nabla_{\partial v}\partial u$. Combined above Eqs. (19–21) and the similar formula of $uv\nabla_{\partial v}\partial u$, we obtain

$$\begin{aligned}
& u^2\nabla_{\partial_u}\partial_u + v^2\nabla_{\partial_v}\partial_v + uv\nabla_{\partial_u}\partial_v + uv\nabla_{\partial_v}\partial u + u\partial_u + v\partial_v \\
&\quad = \frac{f'+3r^2f''+r^4f'''}{f'+r^2f''}(u\partial_u+v\partial_v)
\end{aligned}$$

So the first term (17) of $\nabla_{\frac{\nabla r}{|\nabla r|}}\frac{\nabla r}{|\nabla r|}$ is

$$\frac{f'+3f''\cdot r^2+f'''\cdot r^4}{(f'+r^2f'')^2r^2}(u\partial_u+v\partial_v+\bar{u}\partial_{\bar{u}}+\bar{v}\partial_{\bar{v}}),$$

which is the negative second term (18) of $\nabla_{\frac{\nabla r}{|\nabla r|}}\frac{\nabla r}{|\nabla r|}$. Thus,

$$\nabla_{\frac{\nabla r}{|\nabla r|}}\frac{\nabla r}{|\nabla r|} = 0.$$

□

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Communicated by H. Ooguri