Quasilocal energy-momentum at null infinity

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Abstract

[1] We study the limit of quasilocal energy defined in [7] and [8] for a family of spacelike 2-surfaces approaching null infinity of an asymptotically flat spacetime. It is shown that Lorentzian symmetry is recovered and an energy-momentum 4-vector is obtained. In particular, the result is consistent with the Bondi–Sachs energy-momentum at a retarded time. The quasilocal mass in [7] and [8] is defined by minimizing quasilocal energy among admissible isometric embeddings and observers. The solvability of the Euler-Lagrange equation for this variational problem is also discussed in both the asymptotically flat and asymptotically null cases.

1 Introduction

This is a continuation of [9] in which the spatial limit of the new quasilocal energy defined in [7] and [8] is analyzed. In the present article, we address the question of the null limit in Bondi–Sachs coordinates for an asymptotically flat spacetime. Let N be a spacetime with metric $g_{\alpha\beta}$ in Bondi–Sachs coordinates given by

$$-UVdw^2 - 2Udwdr + \sigma_{ab}(dx^a + W^a dw)(dx^b + W^b dw) \;\; a, b = 2, 3$$

where

$$W^a = O(r^{-2}),$$

$$U = 1 - \frac{X^2 + Y^2}{2r^2} + o(r^{-2}),$$

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$$V = 1 - \frac{2m}{r} + o(r^{-1})$$

and the metric σ_{ab} is given by

$$\begin{pmatrix} r^2 + 2Xr + 2(X^2+Y^2) & -2Yr\sin\theta \\ -2Yr\sin\theta & \sin^2\theta[r^2 - 2Xr + 2(X^2+Y^2)] \end{pmatrix}$$

with

$$\det \sigma_{ab} = r^4 \sin^2\theta.$$

The inverse of the metric is

$$g^{ww} = g^{wa} = 0, g^{wr} = -U^{-1}, g^{rr} = U^{-1}V, g^{ra} = U^{-1}W^a, \text{ and } g^{ab} = \sigma^{ab}.$$

Throughout the paper, coordinates are labeled by $x^0 = w, x^1 = r, x^2 = \theta, x^3 = \phi$ and the indexes are for $\alpha, \beta, \gamma, \cdots = 0, 1, 2, 3$, $i, j, k, \cdots = 1, 2, 3$, $a, b, c, \cdots = 2, 3$.

At a retarded time $w = c$, the Bondi–Sachs energy-momentum vector ([1] [6]) is defined as

$$(E, P_1, P_2, P_3) = \frac{1}{8\pi}\left(\int_{S^2} 2m dS^2, \int_{S^2} 2m\tilde{X}_1 dS^2, \int_{S^2} 2m\tilde{X}_2 dS^2, \int_{S^2} 2m\tilde{X}_3 dS^2\right) \quad (1.1)$$

where $m = m(c, \theta, \phi)$ is the mass aspect function in the expansion of V and $\tilde{X}_i$, $i = 1, 2, 3$ are the three eigenfunctions $\sin\theta\sin\phi$, $\sin\theta\cos\phi$ and $\cos\theta$ of the Laplace operator $\widetilde{\Delta}$ on S^2 with eigenvalue -2.

We recall that given a 2-surface Σ in a spacetime, a quasilocal energy $E(\Sigma, X, T_0)$ is defined in [7], [8] with respect to an isometric embedding $X : \Sigma \to \mathbb{R}^{3,1}$ and a constant future timelike vector $T_0 \in \mathbb{R}^{3,1}$. For a family of surface Σ_r and a family of isometric embeddings X_r of Σ_r into $\mathbb{R}^{3,1}$, the limit of $E(\Sigma_r, X_r, T_0)$ is evaluated in [9, Theorem 2.1] under the assumption that

$$\lim_{r\to\infty} \frac{|H_0|}{|H|} = 1 \quad (1.2)$$

where H and H_0 are spacelike mean curvature vectors of Σ_r in N and the image of X_r in $\mathbb{R}^{3,1}$, respectively. In fact, the limit of $E(\Sigma_r, X_r, T_0)$ with respect to a constant future timelike vector $T_0 \in \mathbb{R}^{3,1}$ is given by

$$\lim_{r\to\infty} \frac{1}{8\pi} \int_{\Sigma_r} \left[-\langle T_0, \frac{J_0}{|H_0|}\rangle (|H_0| - |H|) - \langle \nabla^{\mathbb{R}^{3,1}}_{\nabla\tau} \frac{J_0}{|H_0|}, \frac{H_0}{|H_0|}\rangle + \langle \nabla^N_{\nabla\tau} \frac{J}{|H|}, \frac{H}{|H|}\rangle \right] d\Sigma_r \quad (1.3)$$

where $\tau = -\langle T_0, X_r\rangle$ is the time function with respect to T_0. This expression is linear in T_0 and defines an energy-momentum 4-vector at infinity.

In this article, we consider a family of 2-surface Σ_r on a null cone $w = c$ as r goes to infinity in Bondi–Sachs coordinates. The limit of the quasilocal energy is first computed with respect to isometric embedding X_r into $\mathbb{R}^3$ which are essentially unique and satisfy (1.2). We show in particular,

$$\lim_{r\to\infty} \frac{1}{8\pi}\int_{\Sigma_r} (|H_0| - |H|) d\Sigma_r = E, \text{ and } \lim_{r\to\infty} \frac{1}{8\pi}\int_{\Sigma_r} \langle \nabla^N_{-\nabla X_i} \frac{J}{|H|}, \frac{H}{|H|}\rangle d\Sigma_r = P_i \quad (1.4)$$

where (X_1, X_2, X_3) are the coordinate functions of the isometric embedding X_r into $\mathbb{R}^3$. We remark that exactly the same limit expression on coordinate spheres of asymptotically flat hypersurface gives the ADM energy-momentum in [9].

This computation is stable with respect to any $O(1)$ perturbation of X_r in $\mathbb{R}^{3,1}$ and is equivariant with respect to Lorentzian transformation acting on X_r. In particular, the reference hypersurface spanned by X_r can be either asymptotically flat or asymptotically null.

In [7] and [8], the quasilocal mass of a 2-surface Σ is defined to be the minimum of $E(\Sigma, X, T_0)$ among all admissible isometric embeddings X into $\mathbb{R}^{3,1}$ and the Euler-Lagrange equation is derived for an optimal isometric embedding. In the last section, we show that the isometric embedding can be solved as an $O(1)$ perturbation of embeddings into a boosted totally geodesic slice in $\mathbb{R}^{3,1}$ and it locally minimizes the quasilocal energy.

Brown–Lau–York [2] and Lau [4] compute the null limit of the Brown–York energy and we compare our calculation with theirs in the following:

1) Brown–York mass is gauge dependent. After fixing a reference isometric embedding (either to flat $\mathbb{R}^3$ [2] or to the null cone in $\mathbb{R}^{3,1}$[4]), a gauge is chosen arbitrarily so that the limit of the mass coincide with the Bondi mass. In contrast, in our case, once a reference isometric embedding is picked, the quasilocal energy is determined by the canonical gauge condition (Eq (1.1) in [9]). Our calculation is robust with respect to the choice of reference isometric embedding. In particular, the reference family can be arranged to be asymptotically flat or asymptotically null in $\mathbb{R}^{3,1}$.

2) In [2], the momentum part came from the smear energy while in our case, the momentum part came from the connection one-form associated with the mean curvature gauge. This one form gives the right momentum contribution in the asymptotically flat case as well (see [9]).

3) In [2], the energy and momentum are defined separately. In our case, the Lorentzian symmetric is recovered at infinity and the energy-momentum form a co-

variant 4-(co)vector. We show that this (co)vector is equivariant with respect to the reference isometric embeddings into $\mathbb{R}^{3,1}$.

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2 The geometry of 2-surface Σ_r in Bondi coordinates

Let N be an asymptotically flat spacetime with Bondi–Sachs coordinates. Let Σ_r be the 2-surface defined by $w = c$ and a fixed r. In this section, we compute the mean curvature vector H of Σ_r in N and the connection one-form of the normal bundle of Σ_r in the mean curvature gauge. Denote $W_a = \sigma_{ab} W^b$ and let $\delta^a W_a$ be the divergence of the 1-form W_a on Σ_r with respect to the induced metric σ_{ab}.

Lemma 1. *Let Σ_r be the 2-surface defined by $w = c$ and a fixed r. The mean curvature vector H of Σ_r in N is given by*

$$H = \frac{1}{U}\left[\frac{2}{r}(\frac{\partial}{\partial w} - W^a \frac{\partial}{\partial x^a}) - (\frac{2V}{r} + \delta^a W_a)\frac{\partial}{\partial r}\right]. \tag{2.1}$$

In particular, H is spacelike when r is large enough with

$$|H|^2 = \frac{4}{Ur}(\frac{V}{r} + \delta^a W_a). \tag{2.2}$$

Suppose J is the future timelike normal vector dual to H, then

$$\langle \nabla_{\frac{\partial}{\partial x^b}} J, H\rangle = \partial_b[\frac{2}{rU}(\frac{V}{r} + \delta^a W_a)] + \frac{2}{rU^2}(\frac{V}{r} + \delta^a W_a)\sigma_{cb}\partial_r W^c. \tag{2.3}$$

Proof. By definition, we have

$$\begin{aligned} H &= \sigma^{ab}(\nabla_{\frac{\partial}{\partial x^a}} \frac{\partial}{\partial x^b} - (\nabla_{\frac{\partial}{\partial x^a}} \frac{\partial}{\partial x^b})^T) \\ &= \sigma^{ab}(\Gamma^r_{ab}\frac{\partial}{\partial r} + \Gamma^w_{ab}\frac{\partial}{\partial w} + (\Gamma^c_{ab} - \langle \nabla_{\frac{\partial}{\partial x^a}} \frac{\partial}{\partial x^b}, \frac{\partial}{\partial x^d}\rangle \sigma^{dc})\frac{\partial}{\partial x^c}). \end{aligned}$$

The last coefficient can be computed explicitly as

$$\begin{aligned} &\Gamma^c_{ab} - \langle \nabla_{\frac{\partial}{\partial x^a}} \frac{\partial}{\partial x^b}, \frac{\partial}{\partial x^d}\rangle \sigma^{dc} \\ =& \Gamma^c_{ab} - (\Gamma^r_{ab} g_{rd} + \Gamma^w_{ab} g_{wd} + \Gamma^e_{ab}\sigma_{ed})\sigma^{dc} \\ =& -\Gamma^r_{ab} g_{rd}\sigma^{dc} - \Gamma^w_{ab} g_{wd} g^{dc} \\ =& \Gamma^w_{ab} g_{wr}\sigma^{rc}. \end{aligned}$$

Thus

$$H = \sigma^{ab}(\Gamma^r_{ab}\frac{\partial}{\partial r} + \Gamma^w_{ab}\frac{\partial}{\partial w} + \Gamma^w_{ab} g_{wr}\sigma^{rc}\frac{\partial}{\partial x^c}).$$

The relevant Christoffel symbols of $g_{\mu\nu}$ are given by

$$\Gamma^w_{ab} = \frac{1}{2}U^{-1}\partial_r \sigma_{ab},$$

and

$$\begin{aligned}\Gamma^r_{ab} &= -\frac{1}{2}U^{-1}[\partial_b(W^c)\sigma_{ac} + \partial_a(W^c)\sigma_{bc} + \partial_w\sigma_{ab} + V\partial_r\sigma_{ab} + W^c\partial_c\sigma_{ab}] \\ &= -\frac{1}{2}U^{-1}[\partial_b W_a + \partial_a W_b + \partial_w \sigma_{ab} + V\partial_r\sigma_{ab} - 2\gamma^d_{ab}W_d]\end{aligned}$$

where γ^d_{ab} is the Christoffel symbol of the metric σ_{ab}. When tracing with σ^{ab}, we notice that

$$\sigma^{ab}\partial_\alpha\sigma_{ab} = \partial_\alpha(\ln\det\sigma_{ab}) = \partial_\alpha \ln(r^4\sin^2\theta). \tag{2.4}$$

Thus, we obtain equations (2.1) and (2.2). To compute the connection one-form, we rewrite equation (2.1) as

$$\begin{aligned}UH =& -(\frac{2V}{r} + \delta^a W_a)\frac{\partial}{\partial r} + \frac{2}{r}(\frac{\partial}{\partial w} - W^c\frac{\partial}{\partial x^c}) \\ =& -(\frac{V}{r} + \delta^a W_a)\frac{\partial}{\partial r} + \frac{2}{r}(\frac{\partial}{\partial w} - W^c\frac{\partial}{\partial x^c} - \frac{V}{2}\frac{\partial}{\partial r})\end{aligned}$$

where $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial w} - W^c\frac{\partial}{\partial x^c} - \frac{V}{2}\frac{\partial}{\partial r}$ are null vectors.

Thus we have

$$UJ = -(\frac{V}{r} + \delta^a W_a)\frac{\partial}{\partial r} - \frac{2}{r}(\frac{\partial}{\partial w} - W^c\frac{\partial}{\partial x^c} - \frac{V}{2}\frac{\partial}{\partial r}).$$

For simplicity, let's denote $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial w} - W^c\frac{\partial}{\partial x^c} - \frac{V}{2}\frac{\partial}{\partial r}$ by $\vec{n}_1$ and $\vec{n}_2$ and the coefficients $\frac{V}{r} + \delta^a W_a$ and $\frac{2}{r}$ by x and y in the following computation. Then,

$$\begin{aligned}\langle\nabla_{\frac{\partial}{\partial x^b}} J, H\rangle &= U^{-2}\langle\nabla_{\frac{\partial}{\partial x^b}} x\vec{n}_1 + y\vec{n}_2, x\vec{n}_1 - y\vec{n}_2\rangle \\ &= U^{-2}\{[(\partial_b x)(-y) + (\partial_b y)(x)]\langle\vec{n}_1, \vec{n}_2\rangle - xy(\langle\nabla_{\frac{\partial}{\partial x^b}}\vec{n}_1, \vec{n}_2\rangle - \langle\nabla_{\frac{\partial}{\partial x^b}}\vec{n}_2, \vec{n}_1\rangle)\} \\ &= U^{-2}[(\partial_b x)(-y)\langle\vec{n}_1, \vec{n}_2\rangle - 2xy\langle\nabla_{\frac{\partial}{\partial x^b}}\vec{n}_1, \vec{n}_2\rangle + xy\partial_b\langle\vec{n}_2, \vec{n}_1\rangle] \\ &= \partial_b[\frac{2}{Ur}(\frac{V}{r} + \delta^a W_a)] - \frac{4}{rU^2}(\frac{V}{r} + \delta^a W_a)\langle\Gamma^r_{br}\frac{\partial}{\partial r} + \Gamma^c_{br}\frac{\partial}{\partial x^c}, \frac{\partial}{\partial w} - W^d\frac{\partial}{\partial x^d} - \frac{V}{2}\frac{\partial}{\partial r}\rangle.\end{aligned}$$

Since $\frac{\partial}{\partial x^c}$ is a tangent vector field, while $\frac{\partial}{\partial w} - W^d \frac{\partial}{\partial x^d} - \frac{V}{2}\frac{\partial}{\partial r}$ is normal, we have

$$\langle \nabla_{\frac{\partial}{\partial x^b}} J, H\rangle = \partial_b \left[\frac{2}{rU} \left(\frac{V}{r} + \delta^a W_a\right)\right] - \frac{4}{rU^2}\left(\frac{V}{r} + \delta^a W_a\right)\Gamma^r_{br}$$

Substitute $\Gamma^r_{br} = \frac{1}{2}U^{-1}\partial_b U - \frac{1}{2}U^{-1}\sigma_{bc}\partial_r W^c$ and rearrange terms, we obtain (2.3). $\square$

3 Limit of quasi-local energy

In this section, we compute the limit of quasi-local energy with respect to isometric embeddings X_r of Σ_r into $O(1)$ perturbations of a boosted totally geodesic slice in $\mathbb{R}^{3,1}$. First we quote the following lemma whose proof can be found in [3]:

Lemma 2. *Let σ^r_{ab} be a family of metric on $\Sigma_r \simeq S^2$ with $\sigma^r_{ab} = r^2\tilde{\sigma}_{ab} + O(r)$ in which $\tilde{\sigma}_{ab}$ is the standard round metric on S^2. Let $X_r = (X_1, X_2, X_3)$ be the isometric embedding into $\mathbb{R}^3$ for r large and H_0 be the mean curvature of X_r. Then*

$$|H_0| = \frac{2}{r} + O(r^{-2}) \text{ and } \int_{\Sigma_r} |H_0| d\Sigma_r = 4\pi r + \frac{Area(\Sigma_r)}{r} + O(r^{-1}).$$

We note that up to an isometry of $\mathbb{R}^3$, X_r can be arranged so that the coordinate functions satisfy $X_i = r\tilde{X}_i + O(1)$.

Theorem 1. *Let Σ_r be the 2-surface defined by $w = c$ and a fixed r in an asymptotically flat spacetime with Bondi–Sachs coordinates. Suppose X_r is the (unique) family of isometric embeddings of Σ_r into $\mathbb{R}^3$ for r large, the limit of quasi-local energy with respect to $T_0 = (\sqrt{1+|a|^2}, a^1, a^2, a^3)$ is*

$$\lim_{r\to\infty} E(\Sigma_r, X_r, T_0) = \frac{1}{8\pi}\int_{S^2} 2m(\sqrt{1+|a|^2} + a^i\tilde{X}_i)dS^2. \tag{3.1}$$

Proof. Let $(0, X_1, X_2, X_3)$ be the isometric embedding X_r of Σ_r into $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$. In this case, $\frac{J_0}{|H_0|}$ is simply the vector $(1, 0, 0, 0)$. By the assumption on σ_{ab} we can apply Lemma 2 and

$$\int_{\Sigma_r} |H_0| d\Sigma_r = 8\pi r + O(r^{-1}).$$

On the other hand, from equation (2.2) and the expansion for V, we obtain

$$|H| = \frac{2}{r} - \frac{2m}{r^2} + \delta^a W_a + O(r^{-3}) \tag{3.2}$$

and thus

$$\int_{\Sigma_r} |H| d\Sigma_r = 8\pi r - \int_{S^2} 2m dS^2 + O(r^{-1}).$$

Next we compute the physical hamiltonian

$$\frac{1}{8\pi}\int_{\Sigma_r} \langle \nabla^N_{\nabla\tau} \frac{J}{|H|}, \frac{H}{|H|}\rangle d\Sigma_r = -a^i \frac{1}{8\pi}\int_{\Sigma_r} \langle \nabla^N_{\nabla X_i} \frac{J}{|H|}, \frac{H}{|H|}\rangle d\Sigma_r.$$

From equation (2.3) and the asymptotic expansions of V and W^a , we derive

$$\langle \nabla_{\frac{\partial}{\partial x^b}} J, H\rangle = \frac{2}{r}[\partial_b(\delta^a W_a - \frac{2m}{r^2})] + \frac{4}{r^3} W_b + O(r^{-4}).$$

Let Z be the vector on Σ dual to the connection one-from $\langle \nabla^N \frac{J}{|H|}, \frac{H}{|H|}\rangle$. From the above computation,

$$div_{\Sigma_r} Z = \frac{1}{r}[\frac{1}{2}(\widetilde{\Delta}+2)(\delta^a W_a - \frac{2m}{r^2}) + \frac{2m}{r^2}] + O(r^{-4}).$$

The limit of $\int_{\Sigma_r} \langle \nabla^N_{\nabla X_i} \frac{J}{|H|}, \frac{H}{|H|}\rangle d\Sigma_r$ as $r \to \infty$ is thus the same as

$$\lim_{r\to\infty} \int_{\Sigma_r} X_i div_{\Sigma_r} Z d\Sigma_r = \int_{S^2} \tilde{X}_i [\frac{1}{2}(\widetilde{\Delta}+2)(\widetilde{\delta^a}\widetilde{W_a} - 2m) + 2m] dS^2 = \int_{S^2} \tilde{X}_i 2m dS^2$$

In this case, the reference Hamiltonian term is zero as $\frac{J_0}{|H_0|}$ is a constant vector. In view of expression (1.3), the theorem is proved.

$\square$

Next we show that the limit of the quasilocal energy is invariant under $O(1)$ perturbations of embeddings into totally geodesic $\mathbb{R}^3$ and that it is Lorentzian equivariant.

Corollary 1. *Suppose $X_r = (\tau_0, X_1, X_2, X_3)$ is a family of isometric embeddings of Σ_r into $\mathbb{R}^{3,1}$ with $\lim_{r\to\infty} r^2 \nabla \tau_0 = \widetilde{\nabla}\tilde{\tau}_0$ for some function $\tilde{\tau}_0$ on S^2. Then we still have*

$$\lim_{r\to\infty} E(\Sigma_r, X_r, T_0) = \frac{1}{8\pi}\int_{S^2} 2m(\sqrt{1+|a|^2} + a^i \tilde{X}_i) dS^2.$$

Proof. Let $\hat{X}_r$ be the embedding of Σ_r by projecting X_r onto $\mathbb{R}^3$ which is given by, $(0, X_1, X_2, X_3)$. It is not hard to check that the induced metric by the embedding $\hat{X}_r$ agrees with the standard round metric of radius r up to the top order term and its area agrees with that of the standard round metric of radius r up to the second order term. The mean curvature of the embedding $\hat{X}_r$ is then $(0, \widehat{\Delta} X_1, \widehat{\Delta} X_2, \widehat{\Delta} X_3)$. By Lemma 2, the mean curvature $\hat{H}_0$ satisfies

$$|\hat{H}_0| = \frac{2}{r} + O(r^{-2}), \text{ and } \int_{\Sigma_r} |\hat{H}_0| d\Sigma_r = 8\pi r + O(r^{-1}). \tag{3.3}$$

The mean curvature H_0 of X_r is given by

$$(\Delta\tau_0, \Delta X_1, \Delta X_2, \Delta X_3).$$

The difference between σ_{ab} and $\hat{\sigma}_{ab}$ is of order

$$\sigma_{ab} - \hat{\sigma}_{ab} = O(1) \text{ and } \sigma^{ab} - \hat{\sigma}^{ab} = O(r^{-4}).$$

As a result, the difference between the two laplace operators is of order

$$\widehat{\Delta} X_i - \Delta X_i = O(r^{-3}).$$

Hence $|\hat{H}_0|^2 - |H_0|^2 = O(r^{-4})$, and thus $|\hat{H}_0| - |H_0| = O(r^{-3})$.

By equation (1.3), the limit of quasi-local energy with respect to the embedding (τ_0, X_1, X_2, X_3) is thus

$$\frac{1}{8\pi} \lim_{r\to\infty} \int_{\Sigma_r} (|H_0| - |H|) d\Sigma_r = \frac{1}{8\pi} \lim_{r\to\infty} \int_{\Sigma_r} (|\hat{H}_0| - |H|) d\Sigma_r.$$

Unlike the previous case, $\frac{J_0}{|H_0|}$ is no longer a constant vector for such an isometric embedding X_r. However, the asymptotic expansion

$$-\langle T_0, \frac{J_0}{|H_0|} \rangle = \sqrt{1 + |a|^2} + O(r^{-1})$$

is valid and the energy component is the same as the limit of quasi-local energy of the isometric embedding into $\mathbb{R}^3$ in view of (3.3).

Next we compute the physical hamiltonian. Since the induced metric on the projection still agrees with the standard one up to lower order term, up to an isometry of $\mathbb{R}^3$, $X_i = r\tilde{X}_i + O(1)$. The corresponding time function τ is

$$\tau = -(\sum_i a_i X_i) + \tau_0 \sqrt{1 + \sum_i a_i^2} = -(\sum_i a_i \tilde{X}_i) r + O(1).$$

Thus the physical hamiltonian remains the same.

Lastly, we claim that the reference hamiltonian goes to 0 as r goes to infinity.

Lemma 3. *Let Σ_r be a family of surfaces with metric $\sigma^r_{ab} = r^2\tilde{\sigma}_{ab} + O(r)$. Given an $O(1)$ time function τ_0, let $X_r(\Sigma_r)$ be the images of Σ_r under the isometric embedding into $\mathbb{R}^{3,1}$ determined by τ_0. Let Z_0 be the vector dual to the one form $\langle \nabla_{(\cdot)} \frac{J_0}{|H_0|}, \frac{H_0}{|H_0|}\rangle$ on $X_r(\Sigma_r)$ then*

$$div_{\Sigma_r} Z_0 = \frac{1}{2r^3}\widetilde{\Delta}(\widetilde{\Delta} + 2)\tau_0 + O(r^{-4}).$$

Proof. We need to compute $\frac{J_0}{|H_0|}$ up to $O(r^{-2})$. For this purpose, it is enough to assume that the embedding is $(\tau_0, r\tilde{X}_1, r\tilde{X}_2, r\tilde{X}_3)$. Using $\sum_i \partial_c \tilde{X}_i \partial_b \tilde{X}_i = \tilde{\sigma}_{bc}$, we derive that a normal vector is

$$(1, \frac{1}{r}\partial_a\tau_0\partial_b\tilde{X}_i\tilde{\sigma}^{ab})$$

where $\tilde{\sigma}$ is the standard metric on S^2. As $\sum_i \tilde{X}_i\partial_a\tilde{X}_i = 0$, we check that

$$(1, \frac{1}{r}\partial_a\tau_0\partial_b\tilde{X}_i\tilde{\sigma}^{ab}) + \frac{\Delta\tau_0}{|H_0|^2}H_0$$

is a normal vector perpendicular to H_0. Thus, $\frac{J_0}{|H_0|}$, which is the unit normal perpendicular to H_0 is, up to lower order, given by the same expression. As a result, we compute

$$\langle \nabla_{\frac{\partial}{\partial x^a}} \frac{J_0}{|H_0|}, H_0\rangle = \sum_i \partial_a \left[\frac{1}{r}(\partial_b\tau_0\partial_c\tilde{X}_i)\tilde{\sigma}^{bc} + \frac{\Delta\tau_0}{|H_0|^2}\Delta X_i\right]\Delta X_i.$$

The right hand side equals to

$$\left[\sum_i \frac{1}{r}(\partial_b\tau_0)(\partial_a\partial_c\tilde{X}_i)\tilde{\sigma}^{bc}\Delta X_i\right] + \partial_a\Delta\tau_0 + O(r^{-3}).$$

Here one uses again that $\sum_i \tilde{X}_i\partial_a\tilde{X}_i = 0$ and thus from the first term, one has nonzero contribution only when the derivative ∂_a falls on $\partial_b\tilde{X}_i$. For the second term, the leading term of $|H_0|$ is independent of θ and ϕ and $\sum_i(\Delta X_i)^2 = |H_0|^2$ up to lower order terms. Thus one only has contribution when the derivative hits $\Delta\tau_0$. Direct computation using $\sum_i \tilde{X}_i\partial_a\partial_c\tilde{X}_i = -\tilde{\sigma}_{ac}$ shows that

$$\sum_i \frac{1}{r}(\partial_b\tau_0)(\partial_a\partial_c\tilde{X}_i)\tilde{\sigma}^{bc}\Delta X_i = \frac{2}{r^2}\partial_a\tau_0.$$

As a result,

$$div_{\Sigma_r} Z_0 = \frac{1}{2r^3}\widetilde{\Delta}(\widetilde{\Delta} + 2)\tau_0 + O(r^{-4})$$

$\square$

Using the above lemma, the reference hamiltonian at infinity is

$$\lim_{r\to\infty}\int_{\Sigma_r} X_i div_{\Sigma_r} Z_0 d\Sigma = \int_{S^2}\frac{1}{2}\tilde{X}_i\widetilde{\Delta}(\widetilde{\Delta}+2)\tau_0 dS^2 = 0$$

□

Corollary 2. *Suppose X'_r is another family of isometric embeddings into $\mathbb{R}^{3,1}$ such that $X'_r = \tilde{L}(r)X_r$ for some X_r in the previous Corollary, and a family of Lorentzian transformation $\tilde{L}(r)$ such that the limit of the $SO(3,1)$ part of $\tilde{L}(r)$ converges to an L_∞, then the energy-momentum 4-vector also transform by L_∞.*

Proof. Both $|H_0|$ and the connection one form $\langle\nabla^{\mathbb{R}^{3,1}}_{(\cdot)}\frac{J_0}{|H_0|},\frac{H_0}{|H_0|}\rangle$ are invariant under Lorentzian transformation, while $\langle T_0,\frac{J_0}{|H_0|}\rangle$ and $\nabla\tau = -\nabla\langle T_0, X_r\rangle$ are Lorentzian equivariant. □

For example, if we take a family of isometric embedding X_r into $\mathbb{R}^3$ and define $X'_r = X_r + r$, it is not hard to see that the hypersurface spanned by X'_r is asymptotically null.

4 Optimal embedding equation

In the previous sections, we compute the null limit of quasi-local mass with respect to $O(1)$ perturbations of embeddings into a boosted totally geodesic slice in $\mathbb{R}^{3,1}$. In [8], the quasilocal mass is defined to be the minimum of quasilocal energy among admissible isometric embeddings into $\mathbb{R}^{3,1}$. We address the following problem in this section : is there an $O(1)$ perturbation of embeddings into a boosted totally geodesic slice that is a critical point of this variational problem?

The optimal embedding equation for minimizing the quasi-local energy is derived in [8, Proposition 6.2]. The equation reads

$$-(\widehat{H}\hat{\sigma}^{ab}-\hat{\sigma}^{ac}\hat{\sigma}^{bd}\hat{h}_{cd})\frac{\nabla_b\nabla_a\tau}{\sqrt{1+|\nabla\tau|^2}}+div_\Sigma(\frac{\nabla\tau}{\sqrt{1+|\nabla\tau|^2}}\cosh\theta|H|-\nabla\theta-V)=0 \quad (4.1)$$

where $\sinh\theta = \frac{-\Delta\tau}{|H|\sqrt{1+|\nabla\tau|^2}}$ and V is the tangent vector on Σ that is dual to the connection one-form $\langle\nabla^N_{(\cdot)}\frac{J}{|H|},\frac{H}{|H|}\rangle$. To solve for this equation, we start with data on the 2-surface Σ given by $(\sigma_{ab},|H|,V)$. Take a function τ on Σ, we consider the isometric embedding $\hat{X}:(\Sigma,\hat{\sigma})\to\mathbb{R}^3$ with the metric $\hat{\sigma}_{ab}=\sigma_{ab}+\tau_a\tau_b$. $\widehat{H}$ and $\hat{h}_{ab}$

are the mean curvature and the second fundamental form of the image of $\hat{X}$ in $\mathbb{R}^3$ respectively.

We first observe that momentum become an obstruction to solving the optimal embedding equation near $\mathbb{R}^3$ and then discuss how this can be resolved by boosting the embedding. The discussion covers the spatial infinity case discussed in [9] as well. In the last subsection, we show the solution obtained is locally energy-minimizing up to lower order terms in r.

4.1 Embedding near $\mathbb{R}^3$

In this subsection, we investigate the situation where the embedding is near the flat $\mathbb{R}^3$ in $\mathbb{R}^{3,1}$. Namely. we assume that there exists a function $\tilde{\tau}$ on S^2 such that

$$\nabla\tau = \frac{\widetilde{\nabla}\tilde{\tau}}{r^2} + o(r^{-2}).$$

We have the following asymptotic expansion:

$$\widehat{H}\hat{\sigma}^{ab} - \hat{\sigma}^{ac}\hat{\sigma}^{bd}\hat{h}_{cd} = \frac{\hat{\sigma}^{ab}}{r} + O(r^{-4}),$$

$$\sinh\theta = \frac{-\Delta\tau}{|H|\sqrt{1+|\nabla\tau|^2}} = \frac{-\widetilde{\Delta}\tilde{\tau}}{2r} + O(r^{-2}),$$

and

$$|H| = \frac{2}{r} + O(r^{-2}) \text{ and } 1+|\nabla\tau|^2 = 1 + O(r^{-2}).$$

Further, let's assume that $div_{\Sigma_r}V$ has the following asymptotic expansion

$$div_{\Sigma_r}V = \frac{v(\theta,\phi)}{r^3} + O(r^{-4}).$$

This assumption is true on the null cone $w = c$ as r tends to infinity and at spatial infinity, see [9]. We derive

$$div_{\Sigma_r}\left(\frac{\nabla\tau}{\sqrt{1+|\nabla\tau|^2}}\cosh\theta|H|\right) = \frac{2\widetilde{\Delta}\tilde{\tau}}{r^3} + O(r^{-4}),$$

$$\Delta\theta = \Delta\left(\frac{-\Delta\tau}{|H|\sqrt{1+|\nabla\tau|^2}} + O(r^{-2})\right) = \frac{-1}{2r^3}\widetilde{\Delta}(\widetilde{\Delta}\tilde{\tau}) + O(r^{-4}),$$

and

$$-(\widehat{H}\hat{\sigma}^{ab} - \hat{\sigma}^{ac}\hat{\sigma}^{bd}\hat{h}_{cd})\frac{\nabla_b\nabla_a\tau}{\sqrt{1+|\nabla\tau|^2}} = -\frac{\widetilde{\Delta}\tilde{\tau}}{r^3} + O(r^{-4}).$$

Hence, the leading order term of equation (4.1) is $O(r^{-3})$ which reads

$$\widetilde{\Delta}\tilde{\tau} + \frac{1}{2}\widetilde{\Delta}(\widetilde{\Delta}\tilde{\tau}) = v. \tag{4.2}$$

Equation (4.2) can be solved if and only if v is perpendicular to the kernel of the operators $L = \widetilde{\Delta} + 2$ and $\widetilde{\Delta}$. The second condition always holds. For the first condition, the kernel of L is spanned by $\tilde{X}_i$ and hence momentum becomes an obstruction to solving optimal embedding equation near $\mathbb{R}^3$. In the next subsection, we boost the embedding inside $\mathbb{R}^{3,1}$ by an Lorentzian isometry to resolve this problem.

4.2 Embedding near a boosted slice in $\mathbb{R}^{3,1}$

Suppose Σ_r is a family of spacelike 2-surfaces in spacetime such that
(1) The induced metric satisfies $\sigma^r_{ab} = r^2\tilde{\sigma}_{ab} + O(r)$.
(2) The norm of the mean curvature vector satisfies $\frac{2}{r} - |H_r| = \frac{2h(\theta,\phi)}{r^2} + o(r^{-2})$.
(3) The connection one-form in mean curvature gauge V satisfies $div_{\Sigma_r}V = \frac{v(\theta,\phi)}{r^3} + o(r^{-3})$.

These assumptions hold on coordinate spheres of an asymptotically flat hypersurface as well as the r level surfaces at a retarded time in Bondi-Sachs coordinates. Altogether they guarantee the limit of the quasi-local energy-momentum (e, p_1, p_2, p_3) with respect to isometric embeddings of σ^r_{ab} into $\mathbb{R}^3$ is well-defined.

Theorem 2. *Suppose Σ_r satisfies (1), (2), and (3) and the limit of the quasi-local energy-momentum (e, p_1, p_2, p_3) is timelike. Take (b_1, b_2, b_3) such that $\frac{b_i}{\sqrt{1+\sum_i b_i^2}} = \frac{p_i}{e}$, then there exists a function τ' on S^2 such that an isometric embedding $X_r : \Sigma \to \mathbb{R}^{3,1}$ with time function*

$$\tau = (\sum_i b_i\tilde{X}_i)r + \sqrt{1+b_i^2}\tau'$$

solves equation (4.1) up to $O(r^{-4})$.

Remark 1. *It is not hard to see that $X_r = LX'_r$ for an isometric embedding $X'_r = (\tau', \tilde{X}_1 r, \tilde{X}_2 r, \tilde{X}_3 r) + O(1)$ and $L \in O(3,1)$.*

Proof. Under the assumption, the energy momentum vector (e, p_1, p_2, p_3) and the functions v and h are related by

$$\int_{S^2} h dS^2 = 8\pi e \text{ and } \int_{S^2} v\tilde{X}_i dS^2 = 8\pi p_i. \tag{4.3}$$

Let Σ_r^0 be the image of the isometric embedding determined by τ in $\mathbb{R}^{3,1}$.

The optimal embedding equation for the function τ on Σ_r is

$$-(\widehat{H}\hat{\sigma}^{ab} - \hat{\sigma}^{ac}\hat{\sigma}^{bd}\hat{h}_{cd})\frac{\nabla_b\nabla_a\tau}{\sqrt{1+|\nabla\tau|^2}} + div_\Sigma(\frac{\nabla\tau}{\sqrt{1+|\nabla\tau|^2}}\cosh\theta_r|H_r| - \nabla\theta_r - V_r) = 0 \tag{4.4}$$

We compare the above equation with the optimal embedding equation for τ over Σ_r^0.

$$-(\widehat{H}\hat{\sigma}^{ab} - \hat{\sigma}^{ac}\hat{\sigma}^{bd}\hat{h}_{cd})\frac{\nabla_b\nabla_a\tau}{\sqrt{1+|\nabla\tau|^2}} + div_\Sigma(\frac{\nabla\tau}{\sqrt{1+|\nabla\tau|^2}}\cosh\theta_0|H_0| - \nabla\theta_0 - V_0) = 0 \tag{4.5}$$

which is automatically true since the surface Σ_r^0 is in $\mathbb{R}^{3,1}$.

Subtracting equation (4.4) from equation (4.5), equation (4.4) is equivalent to

$$div[\frac{\nabla\tau}{\sqrt{1+|\nabla\tau|^2}}(\cosh\theta_r|H_r| - \cosh\theta_0|H_0|) - \nabla(\theta_r - \theta_0) - V_r + V_0] = 0.$$

where

$$\cosh\theta_r|H_r| - \cosh\theta_0|H_0| = \sqrt{|H_r|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}} - \sqrt{|H_0|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}}$$

$$\sinh(\theta_r - \theta_0) = \frac{\Delta\tau}{|H_r||H_0|\sqrt{1+|\nabla\tau|^2}}(\sqrt{|H_r|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}} - \sqrt{|H_0|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}}).$$

Set

$$f = \frac{\sqrt{|H_r|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}} - \sqrt{|H_0|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}}}{\sqrt{1+|\nabla\tau|^2}} = O(r^{-2}).$$

Equation (4.4) is equivalent to

$$div(f\nabla\tau) - \Delta[\sinh^{-1}(\frac{\Delta\tau f}{|H_r||H_0|})] = divV_r - divV_0.$$

Using the explicit form of τ and the given asymptotic expansion of $|H_0| - |H_r|$, we can compute directly that

$$f = \frac{f_2}{r^2} + O(r^{-3}) \text{ where } f_2 = \frac{-h}{\sqrt{1 + \sum_i b_i^2}}.$$

Denote $\sum_i b_i \tilde{X}_i$ by τ_1, we derive,

$$\theta_r - \theta_0 = \frac{1}{4}(r\Delta\tau_1)f_2 + O(r^{-2}).$$

On the other hand, by Lemma 3,

$$divV_0 = \frac{1}{2r^3}\widetilde{\Delta}(\widetilde{\Delta} + 2)\tau' + O(r^{-4}).$$

The leading term of equation (4.4) is equivalent to

$$\frac{3}{2}\widetilde{\Delta}(\tau_1)f_2 + 2\widetilde{\nabla}\tau_1 \cdot \widetilde{\nabla} f_2 + \frac{1}{2}\tau_1\widetilde{\Delta} f_2 = -\frac{1}{2}\widetilde{\Delta}(\widetilde{\Delta} + 2)\tau' + v. \tag{4.6}$$

The left hand side of the equation is

$$(\widetilde{\Delta} + 2)(f_2\tau_1) - \tau_1(3f_2 + \frac{1}{2}\widetilde{\Delta} f_2).$$

Thus, equation (4.6) can be solved if and only if

$$v + \tau_1(3f_2 + \frac{1}{2}\widetilde{\Delta} f_2)$$

is perpendicular to the kernel of $\widetilde{\Delta}+2$, i.e. the vector space spanned by $\tilde{X}_j$, $j = 1, 2, 3$.

For each j, we compute

$$\int_{S^2} v\tilde{X}_j + \tau_1(3f_2 + \frac{1}{2}\widetilde{\Delta} f_2)\tilde{X}_j dS^2 = \int_{S^2}[v\tilde{X}_j - b_j f_2]dS^2 = \int_{S^2}[v\tilde{X}_j + \frac{b_j}{\sqrt{1 + |b|^2}}h]dS^2$$

where we integrate by parts and use $\nabla\tilde{X}_i \cdot \nabla\tilde{X}_j = \delta_{ij} - \tilde{X}_i\tilde{X}_j$ and $\nabla\tau_1 \cdot \nabla\tilde{X}_j = b_j - \tau_1\tilde{X}_j$. Thus $\int_{S^2}[v\tilde{X}_j + \frac{b_j}{\sqrt{1+|b|^2}}h]dS^2$ is zero if one picks

$$\frac{b_i}{\sqrt{1 + |b|^2}} = \frac{p_i}{e}.$$

$\square$

4.3 Locally energy minimizing

In this subsection, we show the solution obtained in Theorem 2 is locally energy-minimizing up to lower order of r. We consider isometric embeddings into $\mathbb{R}^{3,1}$ with time functions

$$\tau_s = (\sum_i b_i \tilde{X}_i) r + \sqrt{1+b_i^2}\tau_s' + O(r^{-1})$$

where τ_s' is a family of $O(1)$ function such that τ_0' gives the solution in Theorem 2. That is, we take the metric σ^r on Σ_r and consider isometric embedding $\hat{X}_r$ of the metric $\hat{\sigma}^r = \sigma^r + (d\tau_s)^2$ into $\mathbb{R}^3$. Let Σ_r^0 be the graph of τ_s over the image of $\hat{X}_r$ in $\mathbb{R}^{3,1}$. The induced metric on Σ_r^0 is thus the same as σ^r. From Proposition 6.2 in [8], the first variation of the quasilocal energy is

$$\int_{\Sigma_r} \left[-(\widehat{H}\hat{\sigma}^{ab} - \hat{\sigma}^{ac}\hat{\sigma}^{bd}\hat{h}_{cd}) \frac{\nabla_b \nabla_a \tau_s}{\sqrt{1+|\nabla\tau_s|^2}} + div_{\Sigma_r}\left(\frac{\nabla\tau_s}{\sqrt{1+|\nabla\tau_s|^2}}\cosh\theta|H| - \nabla\theta - V_r\right)\right] \delta\tau_s d\Sigma_r.$$

Using (4.5), we can rewrite this as

$$\int_{\Sigma_r} div_{\Sigma_r}\left[\frac{\nabla\tau_s}{\sqrt{1+|\nabla\tau_s|^2}}(\cosh\theta_r|H_r| - \cosh\theta_0|H_0|) - \nabla(\theta_r - \theta_0) - V_r + V_0\right]\delta\tau_s d\Sigma_r$$

where $|H_0|$, θ_0 and V_0 are data associated with Σ_r^0.

Using the expansions computed in the previous subsection, the first variation has the following expansion:

$$r^{-1}\int_{S^2} [f_2\tilde{\Delta}\tau_1 + \tilde{\nabla} f_2 \cdot \tilde{\nabla}\tau_1 + \frac{1}{2}\tilde{\Delta}(\tau_1 f_2) - v + \frac{1}{2}\tilde{\Delta}(\tilde{\Delta}+2)\tau_s']\delta\tau_s' dS^2 + O(r^{-2}).$$

We derive that the second variation of the quasilocal energy at $s=0$ is

$$\frac{1}{2r}\int_{S^2}[\tilde{\Delta}(\tilde{\Delta}+2)\delta\tau']\delta\tau' dS^2 + O(r^{-2}).$$

We may assume $\int_{S^2}\delta\tau' dS^2 = 0$ by normalization. By decomposing $\delta\tau'$ into sum of eigenfunctions of S^2 and noting that the first non-zero eigenvalue of S^2 is -2, this is always non-negative.

References

[1] H. Bondi, M. G. van der Burg and A. W. K. Metzner, *Gravitational waves in general relativity. VII. Waves from axi-symmetric isolated systems.* Proc. Roy. Soc. Ser. A, **269** (1962) 21–52.

[2] J. D. Brown, S. R. Lau, and J. W. York, Jr.,*Energy of isolated systems at retarded times as the null limit of quasilocal energy.* Phys. Rev. D (3) **55** (1997), no. 4, 1977–1984.

[3] X.-Q. Fan, Y. Shi, and L.-F. Tam, *Large-sphere and small-sphere limits of the Brown-York mass.* Comm. Anal. Geom. **17** (2009), no. 1, 37-72.

[4] S. R. Lau, *Light-cone reference for total gravitational energy.* Phys. Rev. D (3) **60** (1999), no. 10, 104034, 4 pp.

[5] L. Nirenberg, *The Weyl and Minkowski problems in differential geometry in the large.* Comm. Pure Appl. Math. **6**, (1953), 337-394.

[6] R. K. Sachs, *Gravitational waves in general relativity. VIII. Waves in asymptotically flat space-time. Proc.* Roy. Soc. Ser. A **270** (1962) 103–126.

[7] M.-T. Wang and S.-T. Yau, *Quasilocal mass in general relativity.* Phys. Rev. Lett. **102** (2009), no. 2, no. 021101.

[8] M.-T. Wang and S.-T. Yau, *Isometric embeddings into the Minkowski space and new quasi-local mass.* Comm. Math. Phys. **288** (2009), no. 3, 919-942.

[9] M.-T. Wang and S.-T. Yau, *Limit of quasilocal mass at spatial infinity.* arXiv:0906.0200, to appear in Comm. Math. Phys.