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D0-brane realizations of the resolution of a reduced singular curve

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Abstract

Based on examples from superstring/D-brane theory since the work of Douglas and Moore on resolution of singularities of a superstring target-space Y via a D-brane probe, the richness and the complexity of the stack of punctual D0-branes on a variety, and as a guiding question, we lay down a conjecture that any resolution $Y' \to Y$ of a variety Y over $\mathbb{C}$ can be factored through an embedding of Y' into the stack $\mathfrak{M}_r^{0^{Az^f}_p}(Y)$ of punctual D0-branes of rank r on Y for $r \geq r_0$ in $\mathbb{N}$, where r_0 depends on the germ of singularities of Y. We prove that this conjecture holds for the resolution $\rho : C' \to C$ of a reduced singular curve C over $\mathbb{C}$. In string-theoretical language, this says that the resolution C' of a singular curve C always arises from an appropriate D0-brane aggregation on C and that the rank of the Chan-Paton module of the D0-branes involved can be chosen to be arbitrarily large.

Key words: D-brane, resolution, singularity; punctual D0-brane, stack; singular curve, normalization; embedding, separation of points, separation of tangents.

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0. Introduction and outline.

The work [D-M] of Michael Douglas and Gregory Moore on resolution of singularities of a superstring target-space Y via a D-brane probe (i.e., the realization of a resolution Y' of Y as a space of vacua – namely, a moduli space in quantum-field-theoretical sense – of the world-volume quantum field theory of the D-brane probe) has influenced many studies both on the mathematics and the string-theory side. (See also a related work [J-M] of Clifford Johnson and Robert Myers.) The attempt to understand the underlying geometry behind the setup of [D-M] is indeed part of the driving force that leads us to the current setting of D-branes in the project (cf. [L-Y1] and [L-Y2]). Based on examples[1] from superstring/D-brane theory since [D-M], the richness and the complexity of the stack $\mathfrak{M}^{0^{Az^f}_{\;p}}(Y)$ of punctual D0-branes on a variety Y, and as a guiding question, we lay down in this note[2] a conjecture that any resolution $Y' \to Y$ of a variety Y over $\mathbb{C}$ can be factored through an embedding of Y' into the stack $\mathfrak{M}_r^{0^{Az^f}_{\;p}}(Y)$ of punctual D0-branes of rank r on Y for $r \geq r_0$ in $\mathbb{N}$, where r_0 depends on the germ of singularities of Y; cf. Sec. 1. For the one-dimensional case, we prove that this conjecture holds for the resolution $\rho: C' \to C$ of a reduced singular curve C over $\mathbb{C}$; cf. Sec. 2. In string-theoretical language, this says that the resolution C' of a singular curve C always arises from an appropriate D0-brane aggregation on C and that the rank of the Chan-Paton module of the D0-branes involved can be chosen to be arbitrarily large.

Remark 0.1. [*another aspect*]. It should be noted that there is another direction of D-brane resolutions of singularities (e.g. [As], [Br], [Ch]), from the point of view of (hard/massive/solitonic) D-branes (or more precisely B-branes) as objects in the bounded derived category of coherent sheaves. Conceptually that aspect and ours (for which D-branes are soft in terms of string tension) are in different regimes of a refined Wilson's theory-space of $d = 2$ supersymmetric field theory-with-boundary on the open-string world-sheet.[3] Being so, there should be an interpolation between these two aspects. It would be very interesting to understand such details.

Convention. Standard notations, terminology, operations, facts in (1) algebraic geometry; (2) coherent sheaves; (3) resolution of singularities; (4) stacks can be found respectively in (1) [Ha]; (2) [H-L]; (3) [Hi], [Ko]; (4) [L-MB].

- All varieties, schemes and their products are over $\mathbb{C}$; a '*curve*' means a 1-dimensional proper scheme over $\mathbb{C}$; a '*stack*' means an *Artin stack*.

- The '*support*' *Supp* $(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ on a scheme Y means the *scheme-theoretical support* of $\mathcal{F}$; $\mathcal{I}_Z$ denotes the *ideal sheaf* of a subscheme of Z of a scheme Y.

- The current note continues the study in [L-Y1] (arXiv:0709.1515 [math.AG], D(1)), [L-Y2] (arXiv:0901.0342 [math.AG], D(3)), and [L-Y3] (arXiv:0907.0268 [math.AG], D(4)) with some background from [L-L-S-Y] (arXiv:0809.2121 [math.AG], (2)). A partial review of D-branes and Azumaya noncommutative geometry is given in [L-Y4] (arXiv:1003.1178 [math.SG], D(6)). Notations and conventions follow these early works when applicable.

[1] Unfamiliar readers are highly recommended to use keyword search to get a taste of the vast literature.

[2] In part, for a subsection of a talk under the title '*Azumaya noncommutative geometry and D-branes - an origin of the master nature of D-branes*' to be delivered in the workshop *Noncommutative algebraic geometry and D-branes*, December 12 – 16, 2011, organized by Charlie Beil, Michael Douglas, and Peng Gao, at Simons Center for Geometry and Physics, Stony Brook University, Stony Brook, NY.

[3] *For mathematicians*: See [W-K] for the origin of the notion of Wilson's theory-space and, for example, [H-I-V] and [H-H-P] for the case of $d = 2$ supersymmetric quantum field theories with boundary.

Outline.

1 The stack of punctual D0-branes on a variety and an abundance conjecture.

We collect a few most essential definitions and setups for this sub-line of the project. Readers are referred to [L-Y4] for a more thorough review of the first part of the project and stringy-theoretical remarks on how inputs from [Po1], [Po2], and [Wi] lead to such a setting.

D-branes as morphisms from Azumaya noncommutative spaces with a fundamental module.

Our starting point is the following prototypical definition of D-branes that comes from a mathematical understanding of [Po1], [Po2] from Joseph Polchinski and [Wi] from Edward Witten based on how Alexandre Grothendieck developed the theory of schemes in modern (commutative) algebraic geometry:

Definition 1.1. [D-brane]. Let Y be a variety (over $\mathbb{C}$). A *D-brane* on Y is a *morphism* φ from an Azumaya noncommutative space-with-a-fundamental-module $(X^{Az}, \mathcal{E}) := (X, \mathcal{O}_X^{Az}, \mathcal{E})$ to Y. Here, X is a scheme over $\mathbb{C}$, $\mathcal{E}$ a locally free $\mathcal{O}_X$-module, and $\mathcal{O}_X^{Az} = \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$; and φ is defined through *an equivalence class of gluing systems of ring homomorphisms* given by $\varphi^\sharp : \mathcal{O}_Y \to \mathcal{O}_X^{Az}$. The rank of $\mathcal{E}$ is called the *rank* of the D-brane.

Similar to the fact that the data of a morphism $f : X \to Y$ between schemes can be encoded completely by its graph Γ_f as a subscheme in $X \times Y$, the data of φ is also encoded completely by its graph Γ_φ :

Definition 1.2. [φ in terms of its graph Γ_φ]. The *graph* of a morphism in Definition 1.1 is given by an $\mathcal{O}_{X\times Y}$-module $\tilde{\mathcal{E}}$ that is flat over X and of relative dimension 0. In detail, let $pr_1 : X \times Y \to X$, $pr_2 : X \times Y \to Y$ be the projection map, and $f_\varphi : Supp\,(\tilde{\mathcal{E}}) \to Y$ be the restriction of pr_2. Then $\tilde{\mathcal{E}}$ defines a morphism φ in Definition 1.1 as follows:

- $\mathcal{E} = pr_{1*}\tilde{\mathcal{E}}$;
- note that $Supp\,(\tilde{\mathcal{E}})$ is affine over X; thus, the gluing system of ring homomorphisms $f_\varphi^\sharp : \mathcal{O}_Y \to \mathcal{O}_{Supp(\tilde{\mathcal{E}})}$ defines a gluing system of ring-homomorphisms $\varphi^\sharp : \mathcal{O}_Y \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) = \mathcal{O}_X^{Az}$, which defines φ.

It is worth emphasizing that, *unlike* the standard setting for a morphism between ringed topological spaces in commutative geometry, in general φ specifies only a correspondence from X to Y via the diagram

$$\begin{array}{ccc} X_\varphi := Supp\,(\tilde{\mathcal{E}}) & \xrightarrow{f_\varphi} & Y \\ {\scriptstyle \pi_\varphi}\downarrow & & \\ X & & \end{array},$$

not a morphism from X to Y.

Definition 1.2 suggests another equivalent description of φ.

Definition 1.3. [φ as morphism to stack of D0-branes]. Let $\mathfrak{M}^{0^{Az^f}}(Y)$ be the stack of 0-dimensional $\mathcal{O}_Y$-modules. It follows from Definition 1.2 that this is precisely the stack of D0-branes on Y in the sense of Definition 1.1 and, hence, the notation. Then, a morphism φ in Definition 1.1 is specified by a morphism $\hat{\varphi}: X \to \mathfrak{M}^{0^{Az^f}}(Y)$.

The stack of punctual D0-branes on a variety and an abundance conjecture.

Definition 1.4. [stack of punctual D0-branes]. Let Y be a variety. By a *punctual* 0-dimensional $\mathcal{O}_Y$-module, we mean a 0-dimensional $\mathcal{O}_Y$-module $\mathcal{F}$ whose $Supp\,(\mathcal{F})$ is a single point (with structure sheaf an Artin local ring). By Definition 1.2, $\mathcal{F}$ specifies a D0-brane on Y, which is called a *punctual D0-brane*. It is a morphism from an Azumaya point with a fundamental module to Y that takes the fundamental module to a punctual 0-dimensional $\mathcal{O}_Y$-module. Let $\mathfrak{M}_r^{0^{Az^f}_p}(Y)$ be the *stack of punctual D0-branes of rank r on a variety Y*. It has an Artin stack with atlas constructed from Quot-schemes. There is a morphism $\pi_Y: \mathfrak{M}^{0^{Az^f}_p}(Y) \to Y$ that takes $\mathcal{F}$ to $Supp\,(\mathcal{F})$ with the reduced scheme structure. π_Y is essentially the Hilbert-Chow/Quot-Chow morphism.

The following two conjectures are motivated by the various examples in string theory concerning D-brane resolution of singularities of a superstring target-space and the richness and the complexity of the stack $\mathfrak{M}^{0^{Az^f}_p}(Y)$:

Conjecture 1.5. [D0-brane resolution of singularity]. *Any resolution $Y' \to Y$ of a variety Y can be factored through an embedding of Y' into the stack $\mathfrak{M}_r^{0^{Az^f}_p}(Y)$ of punctual D0-branes of rank r on Y for any $r \geq r_0$ in $\mathbb{N}$, where r_0 depends only on the germ of singularities of Y.*

Conjecture 1.5 is a weaker form of the following stronger form of an abundance conjecture:

Conjecture 1.6. [abundance]. *Any birational morphism $Y' \to Y$ between varieties over $\mathbb{C}$ can be factored through an embedding of Y' into the stack $\mathfrak{M}_r^{0^{Az^f}_p}(Y)$ of punctual D0-branes of rank r on Y for any $r \geq r_0$ in $\mathbb{N}$, where r_0 depends only on the germ of singularities of Y and the germ of singularities of Y'.*

This says that all the birational models of and over Y are already contained in the stack $\mathfrak{M}^{0^{Az^f}_p}(Y)$ of punctual D0-branes on Y. All the birational transitions between birational models of and over Y happens as correspondences inside $\mathfrak{M}^{0^{Az^f}_p}(Y)$ (and hence the name of the conjecture) – an intrinsic stack over Y, locally of finite type, that is canonically associated to Y.

Remark 1.7. [*string-theoretical remark*]. A standard setting (cf. [D-M]) in D-brane resolution of singularities of a (complex) variety Y (which is a singular Calabi-Yau space in the context of string theory) is to consider a super-string target-space-time of the form $\mathbb{R}^{(9-2d)+1} \times Y$ and an (effective-space-time-filling) D$(9-2d)$-brane whose world-volume sits in the target space-time as a submanifold of the form $\mathbb{R}^{(9-2d)+1} \times \{p\}$. Here, d is the complex dimension of the variety Y and $p \in Y$ is an isolated singularity of Y. When considering only the geometry of the internal part of this setting, one sees only a D0-brane on Y. This explains the role of D0-branes in the statement of Conjecture 1.5 and Conjecture 1.6. In the physics side, the exact dimension of the D-brane (rather than just the internal part) matters since supersymmetries and their superfield representations in different dimensions are not the same and, hence, dimension does play a role in writing down a supersymmetric quantum-field-theory action for the world-volume of the D$(9-2d)$-brane probe. In the above mathematical abstraction, these data are now reflected into the richness, complexity, and a master nature of the stack $\mathfrak{M}_r^{0^{Az^f}_{p}}(Y)$ that is intrinsically associated to the internal geometry. The precise dimension of the D-brane as an object sitting in or mapped to the whole space-time becomes irrelevant.

2 Realizations of resolution of singular curves via D0-branes.

Let C be a reduced singular curve over $\mathbb{C}$ and

$$\rho \;:\; C' \longrightarrow C$$

be the resolution of singularities of C. In the current 1-dimensional case, the singularities of C are isolated and ρ is realized by the normalization of C. In particular, ρ is an affine morphism. The built-in $\mathcal{O}_C$-module homomorphism $\rho^\sharp : \mathcal{O}_C \to \rho_*\mathcal{O}_{C'}$ determines a subsheaf $\mathcal{A}_C \subset \mathcal{O}_{C'}$ of $\mathbb{C}$-subalgebras with the induced morphism $C' \to \textbf{Spec}\,\mathcal{A}_C$ identical to ρ. Let $p' \in C'$ be a closed point, $p := \rho(p')$, and $\mathfrak{m}_{p'} = (t)$ (resp. $\mathfrak{m}_p$) be the maximal ideal of $\mathcal{O}_{C',p'}$ (resp. $\mathcal{O}_{C,p}$). Then $\rho^\sharp(\mathfrak{m}_p)\cdot\mathcal{O}_{C',p'} = (t^{n_{p'}})$ for some $n_{p'} \in \mathbb{N}$. $n_{p'} > 1$ if and only if $p \in C_{sing}$:= the singular locus of C. We show in this section that:

Proposition 2.1. [one-dimensional case]. *Conjecture 1.5 holds for $\rho : C' \to C$. Namely, there exists an $r_0 \in \mathbb{N}$ depending only on the tuple $(n_{p'})_{\rho(p') \in C_{sing}}$ and a (possibly empty) set $\{b.i.i.(p) \,:\, p \in C_{sing}\,,\ C \text{ has multiple branches at } p\}$ (cf. Definition 2.6), both associated to the germ of C_{sing} in C, such that, for any $r \geq r_0$, there exists an embedding $\tilde{\rho} : C' \hookrightarrow \mathfrak{M}_r^{0^{Az^f}_{p}}(C)$ that makes the following diagram commute:*

$$\begin{tikzcd} & \mathfrak{M}_r^{0^{Az^f}_{p}}(C) \arrow[d, "\pi_C"] \\ C' \arrow[ur, hook, "\tilde{\rho}"] \arrow[r, "\rho"'] & C \end{tikzcd} .$$

Basic setup and a criterion for nontrivial extensions of modules.

Consider the induced affine morphism $id_{C'} \times \rho : C' \times C' \to C' \times C$. Let $pr_1' : C' \times C' \to C'$ (the first factor), $pr_2' : C' \times C' \to C'$ (the second factor), $pr_1 : C' \times C \to C'$, and $pr_2 : C' \times C \to C$ be the projection maps. Let $\tilde{\mathcal{E}}'$ be a coherent sheaf on $C' \times C'$ that is flat over C' under pr_1', with support in an infinitesimal neighborhood of the diagonal $\Delta_{C'} \subset C' \times C'$. Then $\tilde{\mathcal{E}} := (id_{C'} \times \rho)_*(\tilde{\mathcal{E}}')$ is a coherent sheaf on $C' \times C$ that is flat over C' under pr_1, with support in an infinitesimal neighborhood of the graph Γ_ρ of ρ in $C' \times C$.

Lemma 2.2. [commutativity of push-forward and restriction]. *Let $p' \in C'$ be a closed point. Then $(id_{C'} \times \rho)_*(\tilde{\mathcal{E}}'|_{\{p'\}\times C'}) = \tilde{\mathcal{E}}|_{\{p'\}\times C}$.*

Proof. As $\tilde{\mathcal{E}}'$ is flat over C' under pr_1', one has the exact sequence

$$0 \longrightarrow \mathcal{I}_{\{p'\}\times C'} \otimes_{\mathcal{O}_{C'}} \tilde{\mathcal{E}}' \longrightarrow \tilde{\mathcal{E}}' \longrightarrow \tilde{\mathcal{E}}'|_{\{p'\}\times C'} \longrightarrow 0\,.$$

Since $id_{C'} \times \rho$ is affine, $(id_{C'} \times \rho)_* : \mathcal{C}oh\,(C') \to \mathcal{C}oh\,(C)$ is exact and one has

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (id_{C'} \times \rho)_*(\mathcal{I}_{\{p'\}\times C'} \otimes_{\mathcal{O}_{C'}} \tilde{\mathcal{E}}') & \longrightarrow & \tilde{\mathcal{E}} & \longrightarrow & (id_{C'} \times \rho)_*(\tilde{\mathcal{E}}'|_{\{p'\}\times C'}) & \longrightarrow & 0 \\
 & & \| & & & & & & \\
 & & \mathcal{I}_{\{p'\}\times C} \otimes_{\mathcal{O}_{C'}} \tilde{\mathcal{E}} & & & & & & ,
\end{array}$$

where the top horizontal line is an exact sequence. This proves the lemma.

□

Remark/Notation 2.3. [*general restriction over a base*]. Lemma 2.2 holds more generally with p' replaced by a subscheme of C', by the same proof with the replacement. We'll denote the restriction of a coherent sheaf $\tilde{\mathcal{F}}'$ (resp. $\tilde{\mathcal{F}}$) on $C' \times C'$ (resp. $C' \times C$) over a subscheme Z' of the base C' by $\tilde{\mathcal{F}}'_{Z'}$ (resp. $\tilde{\mathcal{F}}_{Z'}$).

Let $v_{p'} \simeq Spec\,(\mathbb{C}[\varepsilon])$, where $\varepsilon^2 = 0$, be the subscheme of the base C' that corresponds to the $\mathbb{C}$-algebra quotient $\mathcal{O}_{C',p'} \to \mathbb{C}[\varepsilon]$ with $t \mapsto \varepsilon$. Then the restriction of $\tilde{\mathcal{E}}'$ over $v_{p'}$ determines an element $\alpha'_{p'} \in \mathrm{Ext}^{\,1}_{C'}(\tilde{\mathcal{E}}'_{p'}, \tilde{\mathcal{E}}'_{p'})$. Similarly, the restriction of $\mathcal{E}$ over $v_{p'}$ determines an element $\alpha_{p'} =: \rho_*\alpha'_{p'} \in \mathrm{Ext}^{\,1}_{C}(\tilde{\mathcal{E}}_{p'}, \tilde{\mathcal{E}}_{p'})$. Let $p := \rho(p')$ and recall $t \in \mathcal{O}_{C',p'}$ and $n_{p'} \in \mathbb{N}$ from the beginning of this section. Let us first state an elementary criterion for non-splitability of a short exact sequence, whose proof is immediate:

Lemma 2.4. [criterion of non-splitability]. *Let W be a scheme and*

$$0 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F}_1 \longrightarrow 0$$

be an exact sequence of $\mathcal{O}_W$-modules that represents a class $\beta \in \mathrm{Ext}^{\,1}_W(\mathcal{F}_1, \mathcal{F}_2)$. Suppose that there exist a point $w \in W$ and a local function $f \in \mathcal{O}_{W,w}$ such that, for the associated $\mathcal{O}_{W,w}$-modules (still denoted the same), $f \cdot \mathcal{F}_1 = f \cdot \mathcal{F}_2 = 0$ while $f \cdot \mathcal{G} \neq 0$. Then, $\beta \neq 0$; namely, the above sequence doesn't split.

Corollary 2.5. [push-forward of jet]. *Continuing the main-line discussions and notations. Let α' be given by the exact sequence*

$$0 \longrightarrow \tilde{\mathcal{E}}'_{p'} \longrightarrow \tilde{\mathcal{F}}' \stackrel{j}{\longrightarrow} \tilde{\mathcal{E}}'_{p'} \longrightarrow 0$$

of $\mathcal{O}_{C'}$-modules. Denote the same for the associated exact sequence of $\mathcal{O}_{C',p'}$-modules. As such, suppose that there is an $l \in \mathbb{N}$ such that $(t^{n_{p'}})^l \cdot \tilde{\mathcal{E}}' = 0$ while $(t^{n_{p'}})^{l+1} \cdot \tilde{\mathcal{F}}' \neq 0$. Then $\alpha \neq 0$ in $\mathrm{Ext}^{\,1}_C(\tilde{\mathcal{E}}_{p'}, \tilde{\mathcal{E}}_{p'})$.

Proof. Note that the multiplication of t by an invertible element in $\mathcal{O}_{C',p'}$ (i.e. by an element in $\mathcal{O}_{C',p'} - \mathfrak{m}_{p'}$) won't alter its nilpotency behavior on the modules in question. The corollary follows immediately from Lemma 2.4 and the observation that, up to a multiplication by an invertible element in $\mathcal{O}_{C',p'}$, one may assume that $t^{n_{p'}} \in \rho^\sharp(\mathcal{O}_{C,p})$.

□

Separation of points in $\rho^{-1}(p)$ via punctual D0-branes at p.

Let $p \in C_{sing}$ and $\hat{C}$ be the formal neighborhood (as an ind-scheme) of p in C. Then each irreducible component $\hat{C}_i$, $i = 1, \cdots, k$, of $\hat{C}$ corresponds to a branch of the germ of p in C. Assume that $k \geq 2$. Then the intersection of two distinct components $\hat{C}_i$ and $\hat{C}_j$ of $\hat{C}$ is represented by a punctual 0-dimensional subscheme $Z_{ij} = Z_{ji}$ of C at p of finite length $l_{ij} = l_{ji}$.

Definition 2.6. [branch intersection index]. For $k \geq 2$, define the *branch intersection index* $b.i.i.(p)$ at $p \in C_{sing}$ to be

$$b.i.i.(p) \;:=\; \max\{l_{ij} \;:\; 1 \leq i, j \leq k; i \neq j\}\,.$$

Let $p \in C_{sing}$, $\rho^{-1}(p) = \{p_1^\prime, \cdots p_k^\prime\}$, and $\hat{C}_i^\prime$ be the formal neighborhood of $p_i^\prime$ in $C^\prime$. Then $\rho : C^\prime \to C$ induces a morphism $\hat{\rho}_i : \hat{C}_i^\prime \to \hat{C}$ of ind-schemes, for $i = 1, \ldots, k$. The image $\hat{\rho}_i(\hat{C}_i^\prime)$ is a branch of $\hat{C}$, which we may assume to be $\hat{C}_i$, after relabeling, since different $\hat{C}_i^\prime$'s are mapped to different branches of $\hat{C}$ under $\hat{\rho}_i$. Let $\mathfrak{m}_{p_i^\prime} = (u_i)$ be the maximal ideal of $\mathcal{O}_{C^\prime, p_i^\prime}$;

- $\mathcal{F}_{i;l}^\prime$ be the 0-dimensional $\mathcal{O}_{C^\prime}$-module $\mathcal{O}_{C^\prime, p_i^\prime}/(u_i^{n_{p_i^\prime} l})$;
- $\hat{\mathcal{F}}_{i;l}^\prime$ be the $\mathcal{O}_{\hat{C}_i^\prime}$-module associated to $\mathcal{F}_{i;l}^\prime$;
- $\mathcal{F}_{i;l}$ be the $\mathcal{O}_C$-module $\rho_* \mathcal{F}_{i;l}^\prime$;
- $\hat{\mathcal{F}}_{i;l}$ be the $\mathcal{O}_{\hat{C}}$-module $\hat{\rho}_{i\,*} \hat{\mathcal{F}}_{i;l}^\prime = \widehat{\rho_* \mathcal{F}_{i;l}^\prime}$.

Then, one has the following lemma:

Lemma 2.7. [separation by punctual modules]. *length* $(Supp\,(\mathcal{F}_{i;l})) \geq l$ *and* $Supp\,(\hat{\mathcal{F}}_{i;l}) \subset \hat{C}_i$*. In particular, if* $l > b.i.i.(p)$*, then* $\mathcal{F}_{1;l}, \cdots, \mathcal{F}_{k;l}$ *are punctual* 0*-dimensional* $\mathcal{O}_C$*-modules at* p *that are non-isomorphic to each other.*

Proof. As in the previous theme, we may assume that $u_i^{n_{p_i^\prime}} = \rho^\sharp(f_i)$ for some $f_i \in \mathfrak{m}_p \subset \mathcal{O}_{C,p}$. Let $h \in \mathbb{C}[x]$ be a polynomial in one variable. Then, by construction, $h(u_i^{n_{p_i^\prime}}) \cdot \mathcal{F}_{i;l}^\prime = 0$ if and only if $h \in (x^l)$. In other words, $h(f_i) \cdot \mathcal{F}_{i;l} = 0$ if and only of $h \in (x^l)$. It follows that there exists a local section $m_{i;l}$ of $\mathcal{F}_{i;l}$ such that $f_i^{l-1} \cdot m_{i;l} \neq 0$. Consider the sub-$\mathcal{O}_C$-module $\mathcal{O}_C \cdot m_{i;l} \simeq \mathcal{O}_C / Ann\,(m_{i;l})$ of $\mathcal{F}_{i;l}$, where $Ann\,(m_{i;l})$ is the annihilator of $m_{i;l}$ in $\mathcal{O}_{C,p}$. Then,

$$m_{i;l}\,,\; f_i \cdot m_{i;l}\,,\; \cdots\,,\; f_i^{l-1} \cdot m_{i;l}$$

are $\mathbb{C}$-linearly independent in $\mathcal{F}_{i;l}$, which implies that

$$1\,,\; f_i\,,\; \cdots\,,\; f_i^{l-1}$$

are $\mathbb{C}$-linearly independent in $\mathcal{O}_{C,p}$. Since

$$Span_{\mathbb{C}}\{1\,,\, f_i\,,\, \cdots\,,\, f_i^{l-1}\} \cap Ann\,(m_{i;l}) \;=\; 0$$

as $\mathbb{C}$-vector subspaces in $\mathcal{O}_{C,p}$, one has that *length* $(Supp\,(\mathcal{O}_{C,p}/Ann\,(m_i))) \geq l$ and, hence, that *length* $(Supp\,(\mathcal{F}_{i;l})) \geq l$. The rest of the lemma are immediate.

$\square$

We say that $p_1^\prime, \cdots, p_k^\prime \in \rho^{-1}(p) \subset C^\prime$ *are separated by the punctual* $\mathcal{O}_C$*-modules* $\mathcal{F}_{1;l}, \cdots, \mathcal{F}_{k;l}$ *at* $p \in C$ when $\mathcal{F}_{1;l}, \cdots, \mathcal{F}_{k;l}$ as constructed above are non-isomorphic to each other.

Construction of embeddings $C' \hookrightarrow \mathfrak{M}^{0^{Az^f}_p}(C)$ that descend to ρ.

We now proceed to prove Proposition 2.1 in three steps.

Step (a): Examination of a local model.

Consider the local ring $\mathcal{O}_{C'\times C',(p',p')} = \mathcal{O}_{C',p'} \otimes_{\mathbb{C}} \mathcal{O}_{C',p'}$. (For simplicity of phrasing, here we use '=' to mean 'standard canonical isomorphism'.) Let $\mathfrak{m}_{p'} = (t_1) \subset \mathcal{O}_{C',p'}$ be the maximal ideal of the first factor and $\mathfrak{m}_{p'} = (t_2) \subset \mathcal{O}_{C',p'}$ be the maximal ideal of the second factor. Given $r \in \mathbb{N}$, compare the following two quotient $\mathcal{O}_{C'\times C',(p',p')}$-modules:

$$M_1 \;:=\; \frac{\mathcal{O}_{C',p'} \otimes_{\mathbb{C}} \mathcal{O}_{C',p'}}{((t_1\otimes 1 - 1\otimes t_2)^r\,,\, t_1\otimes 1)} \qquad \text{and} \qquad M_2 \;:=\; \frac{\mathcal{O}_{C',p'} \otimes_{\mathbb{C}} \mathcal{O}_{C',p'}}{\big((t_1\otimes 1 - 1\otimes t_2)^r\,,\, t_1^2\otimes 1\big)}\,.$$

M_1 corresponds to the restriction of the $\mathcal{O}_{C',p'}\otimes_{\mathbb{C}}\mathcal{O}_{C',p'}$-module $\mathcal{O}_{C',p'}\otimes_{\mathbb{C}}\mathcal{O}_{C',p'}/(t_1\otimes 1 - 1\otimes t_2)^r$, which is flat over C' (the first factor), to over $p' \in C'$ (the first factor) while M_2 corresponds to the restriction of the same $\mathcal{O}_{C',p'}\otimes_{\mathbb{C}}\mathcal{O}_{C',p'}$-module to over $v_{p'} \simeq \mathit{Spec}\,(\mathbb{C}[t_1]/(t_1^2)) \simeq \mathit{Spec}\,(\mathbb{C}[\varepsilon]) \subset C'$ (the first factor). They fit into an exact sequence, representing a class in $\mathrm{Ext}^1_{C'}(M_1, M_1)$ (here C' = the second factor),

$$0 \longrightarrow M_1 \stackrel{a}{\longrightarrow} M_2 \stackrel{b}{\longrightarrow} M_1 \longrightarrow 0$$

of $\mathbb{C}[\varepsilon]$-modules with

$$\begin{aligned} M_1 &= \mathit{Span}_{\mathbb{C}}\left\{1\otimes 1\,,\, 1\otimes t_2^2\,,\, \cdots\,,\, 1\otimes t_2^{r-1}\right\}\,;\\ M_2 &= \mathit{Span}_{\mathbb{C}[\varepsilon]}\left\{1\otimes 1\,,\, 1\otimes t_2^2\,,\, \cdots\,,\, 1\otimes t_2^{r-1}\right\}\\ &= \mathit{Span}_{\mathbb{C}}\left\{1\otimes 1\,,\, 1\otimes t_2^2\,,\, \cdots\,,\, 1\otimes t_2^{r-1}\,,\, \varepsilon\otimes 1\,,\, \varepsilon\otimes t_2^2\,,\, \cdots\,,\, \varepsilon\otimes t_2^{r-1}\right\}\,, \end{aligned}$$

where a = multiplication by ε, and b = quotient by εM_1. As $\mathbb{C}[\varepsilon]$-modules and with respect to the above bases (and with a vector identified as a column vector),

$$t_2 \;\text{on}\; M_1 \;=\; \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{bmatrix}_{r\times r} \qquad \text{and} \qquad t_2 \;\text{on}\; M_2 \;=\; \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & r\varepsilon \end{bmatrix}_{r\times r}\,.$$

Here all the missing entries in the $r\times r$-matrices are 0. It follows that, as $\mathcal{O}_{C',p'}$ (the second factor) -modules,

$$\cdot\;\; t_2^l\cdot M_1 = 0 \;\; \textit{if and only if} \;\; l \geq r \quad \textit{while} \quad t_2^l \cdot M_2 = 0 \;\; \textit{if and only if} \;\; l \geq r+1\,.$$

In particular, the above short exact sequence (of $\mathcal{O}_{C',p'}$-modules) doesn't split.

Step (b): Construction of a local embedding $C' \to \mathfrak{M}_{r_0}^{0^{Az^f}_p}(C)$ that descend to ρ, for some $r_0 \in \mathbb{N}$.

Let

$$l_0 \;:=\; 1 \,+\, \max\{\, b.i.i.(p) \,:\, p \in C_{sing}\,,\, \text{C has multiple branches at } p\,\}$$

(by convention, $l_0 = 1$ if C has only single branch at each $p \in C_{sing}$) and

$$r_0 \;:=\; l_0 \cdot l.c.m.\{n_{p'} : p' \in C'\} \;\in\; \mathbb{N}\,.$$

(Here, $l.c.m.$ = the '*least common multiple*' in $\mathbb{N}$.) Since $n_{p'} = 1$ except for $\rho(p')$ in the finite set C_{sing}, r_0 is well-defined. Furthermore, since $\{n_{p'}\}_{\rho(p')\in C_{sing}}$ and $\{b.i.i.(p) : p \in C_{sing}\}$ (possibly empty) depend only on the germ of C_{sing} in C, r_0 depends only on the germ of C_{sing} in C. Let $\tilde{\mathcal{E}}'$ be the $\mathcal{O}_{C'\times C'}$-module

$$\tilde{\mathcal{E}}' \;=\; \mathcal{O}_{C'\times C'}/\mathcal{I}_{\Delta_{C'}}^{r_0}$$

and $\tilde{\mathcal{E}} := (id_{C'}\times\rho)_*(\tilde{\mathcal{E}}')$ on $C'\times C$. Then, it follows from the construction and Lemma 2.7 that the induced morphism

$$\tilde{\rho}_0 \;:\; C' \;\longrightarrow\; \mathfrak{M}_{r_0}^{0^{Az^f}_{\ p}}(C)$$

descends to ρ and sends distinct closed points of C' to distinct geometric points on $\mathfrak{M}_{r_0}^{0^{Az^f}_{\ p}}(C)$ (i.e. $\tilde{\rho}$ separates points of C'). Furthermore, it follow from the local study in Step (a) and Corollary 2.5 that all the extension classes $\alpha_{p'}\in \mathrm{Ext}^1_C(\tilde{\mathcal{E}}_{p'},\tilde{\mathcal{E}}_{p'})$, $p'\in C'$, $\tilde{\mathcal{E}}$ specifies are non-zero. This shows that $\tilde{\rho}_0$ separates also tangents of C' and hence is an embedding.

Step (c): Embeddings $C'\hookrightarrow \mathfrak{M}_r^{0^{Az^f}_{\ p}}(C)$ that descend to ρ, for all $r>r_0$.

Finally, to obtain an embedding $\tilde{\rho}: C'\rightarrow \mathfrak{M}_r^{0^{Az^f}_{\ p}}(C)$ for $r>r_0$ that descends to ρ, observe that the $\mathcal{O}_{C'\times C}$-module $\mathcal{O}_{\Gamma_\rho}$ has the following properties:

- The corresponding extension class $\bar{\alpha}_{p'}$ in $\mathrm{Ext}^1_C(\mathcal{O}_p,\mathcal{O}_p)$, where $p:=\rho(p')$, vanishes if and only if $p\in C_{sing}$.

This implies that all the extension classes $\hat{\alpha}_{p'}\in\mathrm{Ext}^1_C(\hat{\mathcal{E}}_{p'},\hat{\mathcal{E}}_{p'})$, $p'\in C'$, as specified by the direct sum

$$\hat{\mathcal{E}} \;:=\; \tilde{\mathcal{E}}\oplus\mathcal{O}_{\Gamma_\rho}^{\oplus(r-r_0)}$$

of $\mathcal{O}_{C'\times C}$-modules, remain non-zero. Furthermore,

$$Supp\,((\tilde{\mathcal{E}}\oplus\mathcal{O}_{\Gamma_\rho}^{\oplus(r-r_0)})|_{p'\times C}) \;=\; Supp\,(\tilde{\mathcal{E}}_{p'}) \qquad \text{for all }\; p'\in C'\,.$$

It follows that the morphism $\tilde{\rho}:C'\rightarrow\mathfrak{M}_r^{0^{Az^f}_{\ p}}(C)$ specified by $\hat{\mathcal{E}}$ on $C'\times C$ separates both points and tangents of C' and, hence, is an embedding that descend to ρ.

This concludes the proof of Proposition 2.1.

Remark 2.8. [*non-uniqueness*]. In general there can be other embeddings of C' into $\mathfrak{M}_r^{0^{Az^f}_{\ p}}(C)$ that descend also to ρ. Hence, the one constructed in the proof above is by no means unique.

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