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Nonlinear Partial Differential Equations in Geometry and Physics

The 1995 Barrett Lectures

Garth Baker
Alexandre Freire
Editors

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Garth Baker and Alexandre Freire
Department of Mathematics
University of Tennessee
Knoxville, TN 37996-1300
USA

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Preface

This volume presents the proceedings of a series of lectures hosted by the Mathematics Department of The University of Tennessee, Knoxville, March 22–24, 1995, under the title "Nonlinear Partial Differential Equations in Geometry and Physics".

While the relevance of partial differential equations to problems in differential geometry has been recognized since the early days of the latter subject, the idea that differential equations of differential-geometric origin can be useful in the formulation of physical theories is a much more recent one. Perhaps the earliest emergence of systems of nonlinear partial differential equations having deep geometric and physical importance were the Einstein equations of general relativity (1915). Several basic aspects of the initial value problem for the Einstein equations, such as existence, regularity and stability of solutions remain prime research areas today, eighty years after Einstein's work.

An even more recent development is the realization that structures originally introduced in the context of models in theoretical physics may turn out to have important geometric or topological applications. Perhaps its emergence can be traced back to 1954, with the introduction of a non-abelian version of Maxwell's equations as a model in elementary-particle physics, by the physicists C.N. Yang and R. Mills. The rich geometric structure of the Yang-Mills equations was brought to the attention of mathematicians through work of M.F. Atiyah, N. Hitchin, I. Manin, I.M. Singer and others, dating back to the early 1970's. By the early 1980's, through the work of S.K. Donaldson, C.H. Taubes, K.K. Uhlenbeck and others, studying moduli spaces of anti-self-dual solutions of these systems had led to profound developments in understanding the differential topology of compact simply connected 4-manifolds.

Simultaneously, in the 1970's and 80's, several other researchers pursued the formulation of global geometric and topological questions on manifolds in terms of questions of solvability of systems of nonlinear partial differential equations involving special geometric differential operators, in both Riemannian and Hermitian geometry. Some of the key developments were S.-T. Yau's solution of the Calabi conjecture involving Monge-Ampere equations (1977), Y.-T. Siu's rigidity theorems using harmonic maps (1980), R. Hamilton's work on evolution of Riemannian metrics through curvature-driven flows (1982) and R. Schoen's analysis of the scalar curvature equation, which led to the crucial step in the resolution of Yamabe's conjecture (1984).

The present volume contains four sets of lectures on topics which represent some of the most active areas of research today in the field of partial differential equations originating from problems in geometry, topology or theoretical physics. The lectures are of an expository nature, and directed at graduate students and researchers in the interface of geometric analysis and mathematical physics.

In 1994 N. Seiberg and E. Witten introduced new topological invariants for smooth 4-manifolds, computed from moduli spaces of solutions to equations which have their origin in 4-dimensional "$N = 2$" supersymmetric Yang-Mills theories in quantum chromodynamics. Understanding the relationship between S. Donaldson's theory and these new invariants is a topic of intense current activity. In the first set of lectures, Ronald Fintushel begins with a brief overview of Donaldson's invariants and an introduction to the Seiberg-Witten equations, and describes some of their applications to the theory of differentiable 4-manifolds. His second and third lectures deal primarily with the recent proof of the "Immersed Thom conjecture" using the new invariants, and with very recent progress towards the "11/8 conjecture" – an attempt to characterize the indefinite integral quadratic forms which occur as intersection forms of smooth 4-manifolds.

Sergiu Klainerman's lectures deal with the nonlinear hyperbolic differential equations arising from classical field theories described by a Lagrangean variational principle in Minkowski spacetime. The key issues in the mathematical analysis are break-down of solutions in finite time from smooth initial data, global existence of solutions, and the minimal regularity of the data required for well-posedness of the Cauchy problem. The main examples are wave maps, the Yang-Mills equations and the Einstein field equations. In each case the natural scaling invariance of the equations characterizes the number of space dimensions for which the problem is "critical", "supercritical" or "subcritical". General conjectures regarding local well-posedness and asymptotic behavior are set forth in each case, and the currently available evidence is discussed. Klainerman's lectures then focus on the question of optimal conditions for local well-posedness, including a detailed survey of his recent and ongoing work (jointly with M. Machedon) in which a structural feature of many equations with physical origin ("null forms") plays a key role. This has enabled them to obtain results on local well-posedness for the wave maps and Yang-Mills systems that are much closer to the expected minimal Sobolev regularity class than those known classically.

The topic of Fang-Hua Lin's three survey lectures are the Ginzburg-Landau equations occurring in a classical model for superconductivity. The main mathematical issue discussed is the limiting behavior of vortex lines as the "Ginzburg-Landau parameter" approaches zero, in both static and time-dependent models. The starting point is the recent work of Bethuel, Brezis and Helein on the limiting distribution of vortices for energy minimizers in two-dimensional domains. In these results a key role is played by finite configurations of points which are critical for a "renormalized energy"; the problem of finding Ginzburg-Landau minimizers (or critical points) corresponding to given critical configurations is described in the second lecture. The third lecture deals with the time-dependent problem; specifically, the determination of the precise time scale in which motion of the vortices is observed, and of the finite-dimensional dynamical system describing the motion.

In the final set of lectures, Michael Struwe surveys "wave maps"-harmonic maps from Minkowski spacetime to a Riemannian manifold. Wave maps occur naturally for example in non-linear "sigma models" and in the Einstein vacuum equations in the presence of symmetry. Following a survey of local existence results for the quasilinear hyperbolic system defined by wave maps (obtained by energy methods), Struwe's lectures concentrate on the problems of blow-up, global existence and well-posedness in the energy class (in two space dimensions). The attempt to understand the latter problem has led to weak-compactness results for wave maps obtained by "concentration compactness" techniques, which parallel similar developments in the elliptic and parabolic cases; these are described in the last lecture in the volume.

The 1995 Barrett Lectures received financial support from the National Science Foundation (grant DMS-9404089) and from Science Alliance. The organizational assistance provided by the conference secretary, Ms. Kelly Nicely, was also essential for their success.

Garth A. Baker and Alexandre Freire
April 1996

John H. Barrett (1922–1969)

John H. Barrett was born in 1922 and grew up on a farm In Kansas. He received his A.B. in 1944 from Fort Hays Kansas State College. He did his graduate work at the University of Texas and was an instructor there from 1946–1951. His thesis was on differential equations of non-integer order and under the direction of H.J. Ettlinger. He was an assistant professor at Delaware from 1951–1956, an associate professor at Utah from 1956–1961, and professor at Tennessee from 1961–1969, where he was also head from 1964–1969. He also spent the year 1955–1956 visiting Yale and the year 1959–1960 visiting the Research Center at Wisconsin. In January, 1969, he died from complications following a kidney transplant. His wife, Lida, was also a faculty member at Tennessee, where she was head from 1973 to 1980. They had three children.

John was well known for his work on oscillation and disconjugacy theory of linear differential equations as well as the study of boundary value problems. One of his best known contributions was the extension of the classical Prüfer transformation for 2nd order scalar differential equations to systems of differential equations. He had several Ph.D. students. In a time when a common teaching load was three courses, he regularly ran a seminar on differential equations in addition. Typically it would meet very early on a Tuesday or Thursday morning with John as the most frequent lecturer. Through his efforts and influence, an active group in differential equations evolved at the University of Tennessee. This tradition has continued since that time. The Barrett Lecture series is a fitting reminder of his many contributions to the university and community.

(A tribute delivered by Don Hinton, March 22, 1995)

A List of Previous

John H. Barrett Memorial Lectures

The University of Tennessee, Knoxville
Department of Mathematics

Year	Principal Lecturers
1995	**Ronald Fintushel** Michigan State University **Sergiu Klainerman** Princeton University **Fang-Hua Lin** Courant Institute **Michael Struwe** ETH, Zürich
1994	**Robert Gilmer** Florida State University
1993	**Donald Dawson** Carleton University **Eugene Dynkin** Cornell University **Gopinath Kallianpur** University of North Carolina, Chapel Hill
1992	**Kyoshi Igusa** Brandeis University
1991	**John Ball** Herriot-Watt University
1989	**Sir Michael Atiyah** Oxford University **I. M. Singer** Massachusetts Institute of Technology **Clifford H. Taubes** Harvard University **Karen Uhlenbeck** University of Texas at Austin
1988	**Alan F. Beardon** Cambridge University

1987	**Joyce R. McLaughlin** Rensselaer Polytechnic Institute
1986	**Shreeram S. Abhyankar** Purdue University
1985	**Richard K. Miller** Iowa State University
1984	**Donald Ludwig** University of British Columbia
1983	**David Sarason** University of California, Berkeley
1982	**Jean Mawhin** Catholic University of Louvain
1981	**James H. Bramble** Cornell University
1980	**William T. Eaton** University of Texas at Austin
1979	**Fred Brauer** University of Wisconsin
1978	**Hyman Bass** Columbia University
1977	**W.T. Reid** University of Texas
1976	**Paul Waltman** University of Iowa
1975	**Jack E. Hale** Brown University
1974	**Zeev Nehari** Carnegie-Mellon University
1973	**W.N. Everett** University of Dundee
1972	**Garrett Birkhoff** Harvard University
1970	**Einar Hille,** (Emeritus Yale University) University of New Mexico

List of Participants

Alama, Stan	McMaster Univ.	alama@mcmail.cis.mcmaster.ca
Alexiades, Vasilios	Univ. of Tenn.	vasili@math.utk.edu
Alikakos, Nicholas	Univ. of Tenn.	alikakos@utkvx.utk.edu
Belchev, Evgeni	Michigan State Univ.	belchev@math.msu.edu
Bronsard, Lia	McMaster Univ.	bronsard@icarus.math.mcmaster.ca
Capogna, Luca	Purdue Univ.	capogna@math.purdue.edu
Chmaj, Adam	Univ. of Utah	chmaj@math.utah.edu
Demoulini, Sophia	Univ. of Cal., Davis	demoulin@ucdmath.ucdavis.edu
Deng, Qingping	Univ. of Tenn.	deng@math.utk.edu
Dydak, Jerzy	Univ. of Tenn.	dydak@novell.math.utk.edu
Fintushel, Ron	Michigan State Univ.	ronfint@math.msu.edu
Guidry, Michael	Univ. of Tenn.	guidry@utkvx.utk.edu
Han, Qing	Univ. of Notre Dame	qhan@yansu.math.nd.edu
Hinton, Don	Univ. of Tenn.	hinton@novell.math.utk.edu
Ianakiev, Krassimir	SUNY, Buffalo	ianakiev@newton.math.buffalo.edu
Ishigaki, Makoto	Univ. of Tenn.	
Kepka, Mariusz	Michigan State Univ.	kepka@math.msu.edu
Klainerman, Sergiu	Princeton Univ.	seri@math.princeton.edu
Kowalczyk, Michael	Univ. of Tenn.	
Kukavica, Igor	Univ. of Chicago	kukavica@cs.uchicago.edu
Liang, Min	Univ. of Tenn.	mliang@math.utk.edu
Lin, Fang-Hua	Courant Institute	linf@math.nyu.edu
Lu, Guozhen	Wright State Univ.	gzlu@discover.wright.edu
Mou, Libin	Univ. of Iowa	mou@math.uiowa.edu
del Pino, Manuel	Univ. of Chicago	delpino@math.uchicago.edu
Pinto, Joao	Georgia Tech.	pinto@math.gatech.edu
Plexisakis, Michael	Univ. of Tenn.	plex@math.utk.edu
Promislow, Keith	Penn State Univ.	ksp@math.psu.edu
Schlag, Wilhelm	Univ. of Cal., Berkeley	schlag@math.berkeley.edu
Simpson, Henry	Univ. of Tenn.	simpson@novell.math.utk.edu
Sinkala, Zachariah	Middle Tenn. State Univ.	zsinkala@mtsu.edu
Sowa, Artur	CUNY	sow@cunyvmsl.gc.cuny.edu
Struwe, Michael	E.T.H. Zurich	struwe@math.ethz.ch
Stuart, David	Univ. of Cal., Davis	dmstuart@aztec.ucdavis.edu
Tataru, Daniel	Northwestern Univ.	tataru@math.nwu.edu
Wang, Junping	Brigham Young Univ.	junw@math.byu.edu
Weinstein, Gilbert	Univ. of Alabama, Birmingham	weinstei@vorteb.math.uab.edu
Xiong, Jie	Univ. of Tenn.	xiong@novell.math.utk.edu

Organizers

Baker, Garth	Univ. of Tenn.	garth@math.utk.edu
Freire, Alex	Univ. of Tenn.	freire@math.utk.edu