CHAPTER IX

Group Actions on $\mathbf{R}^3$

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The Smith conjecture has many equivalent forms and each of these forms has various consequences and generalizations. Our point of view is that the Smith conjecture is a structure theorem about symmetries of the product of a compact surface with an interval. Here the interval may be closed or open. For example, the usual Smith conjecture is equivalent to proving the smooth Z_n actions on $S^2 \times [0, 1]$ are conjugate to actions that preserve the product structure. Thus this generalized Smith conjecture represents the belief that all the symmetries of the product of a compact surface with an interval actually arise from the symmetries of the surface extended trivially to the product structure.

In case the compact surface Ω has boundary, we make the additional assumption that symmetries of the three-dimensional manifold $M = \Omega \times [0, 1]$ preserve the ends $\Omega \times \{0, 1\}$. In this form the usual Smith conjecture can be restated as follows. Let $M = D \times [0, 1]$, where D is the unit disk.

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Then if $f: M \to M$, where $f^n = id$, $f(D \times \{0\}) = D \times \{0\}$ and $f(D \times \{1\}) = D \times \{1\}$, then f is conjugate to a diffeomorphism $g: M \to M$, where $g(D \times \{t\} = D \times \{t\}$ and the conjugating diffeomorphism $L: M \to M$ preserves the ends of M. More generally, let $M = \Omega \times [0, 1]$, where Ω is a compact surface (possibly with boundary). Suppose that $f: M \to M$ is a diffeomorphism that preserves the ends of M. We shall say that f preserves a product structure on M if there is a diffeomorphism $h: M \to M$ so that h preserves ends and hfh^{-1} preserves the original product structure.

In this chapter we shall show how to apply minimal surfaces to study the generalized Smith conjecture on $S^2 \times I$, where I is an open or closed interval. This analysis leads to a classification of compact groups that can act smoothly on $\mathbf{R}^3$. Furthermore, we shall prove that if the compact group is not algebraically isomorphic to the icosahedral group, then the action of the compact group on $\mathbf{R}^3$ is linear. John Morgan informed us that he and Michael Davis have similar results in the case S^3.

In [M–Y 3] the authors proved a geodesic version of the loop theorem and a new equivariant version of the loop theorem. The new equivariant loop theorem is easier to apply in the study of group actions, and it will be used to prove the theorems in this section. We now state these theorems.

THEOREM 1 (The Geodesic Loop Theorem). *Suppose that M is a compact three-dimensional manifold with boundary and let Z denote the collection of all loops on ∂M that are homotopically nontrivial in ∂M but homotopically trivial in M. Then with respect to any riemannian metric on ∂M*

(1) *There exists a collection of geodesics $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ contained in Z so that Z is the smallest normal subgroup of $\pi_1(\partial M)$ generated by the conjugacy classes represented by $\gamma_1, \ldots, \gamma_n$. Furthermore, γ_1 is a geodesic of least length in Z and γ_i is a geodesic of least length in the complement of the normal subgroup generated by the conjugacy classes represented by $\gamma_1, \ldots, \gamma_{i-1}$.*

(2) *Any such collection Γ is a pairwise disjoint collection of embedded geodesics.*

(3) *If $\Gamma' = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ is another such collection of geodesics and α_i intersects some γ_j, then $\alpha_i = \gamma_j$.*

We shall call a collection of geodesics Γ that arise in the above theorem a *short generating set* for Z. The disjointness property (3), coupled with the equivariant Dehn's lemma in [M–Y 3], proves the following equivariant version of the loop theorem.

THEOREM 2 (Equivariant Loop Theorem). *Let M and Z be as above and suppose that M is orientable. Suppose that G is a group of orientation-preserving isometries of a riemannian metric on M and let Γ be a short generating* *set for Z in this metric. Then* $G\Gamma = \{g(\gamma_i) | g \in G, \gamma \in \Gamma\}$ *is a pairwise disjoint collection of geodesics that bound a pairwise disjoint collection C of disks. Furthermore, if G acts freely on the union of* Γ, *then we may assume that G preserves the union of C.*

With the above theorems we are now in a position to examine the compact groups that act on $S^2 \times [0, 1]$. First, we state the following lemma of a relative version of the Smith conjecture. The proof, which uses the solution of the Smith conjecture, will be left to the reader. Throughout the remainder of this section all diffeomorphisms will be orientation-preserving unless otherwise stated.

LEMMA 1. *Let* $M = D \times [0, 1]$ *and F be a foliation of* $\partial D \times [0, 1]$ *by circles* S_t^1. *If* $f: M \to M$ *is a diffeomorphism of finite order so that* $f(S_t^1) = S_t^1$ *for all t, then there is a foliation of M by disks* D_t *for* $t \in [0, 1]$ *such that*

(1) $\partial D_t = S_t^1$ and
(2) $f(D_t) = D_t$.

DEFINITION. The term Ω_k denotes a compact planar domain with k boundary curves, $M_k = \Omega_k \times [0, 1]$.

LEMMA 2. *Suppose that* $f: M_2 \to M_2$ *is a diffeomorphism of order two which preserves ends and that f interchanges the two circles of* $\partial\Omega_2 \times \{0\}$. *Then f preserves a product structure on* M_2.

Proof. First, consider the usual linear action R of M_2 given in this lemma, which is induced by rotation around the Z-axis by 180° (see Fig. 1). The reader should note that in the product structure chosen for M_2, the ends $\Omega_2 \times \{0, 1\}$ consist of interior and exterior annuli of the solid cylinder. Note that there exists a disk D_1 such that D_1 and $D_2 = R(D_1)$ are disjoint and $D_1 \cup D_2$ disconnects the ends of M_2 into disks V_1, V_2 and W_1, W_2, respectively. We now prove the existence of similar disks for f rather than R.

It is not difficult to check that there is an f-invariant metric on ∂M_2, so that any curve γ of least length in the kernel K of $i_*: \pi_1(\partial M_2) \to \pi_1(M_2)$ is disjoint from the fixed point set of f and also that $\gamma \cap (\partial\Omega_2 \times [0, 1])$ are two embedded intervals. (In fact, there exists such an invariant metric that is a product metric on one component of $\partial\Omega_2 \times [0, 1]$ and is flat on the other boundary component. The metric on the second boundary component is induced by f from the first metric. We claim that if we choose the curves $\{x\} \times [0, 1]$ to be long enough in the product metric of the first component, the geodesic of least length in K can have only one interval component in

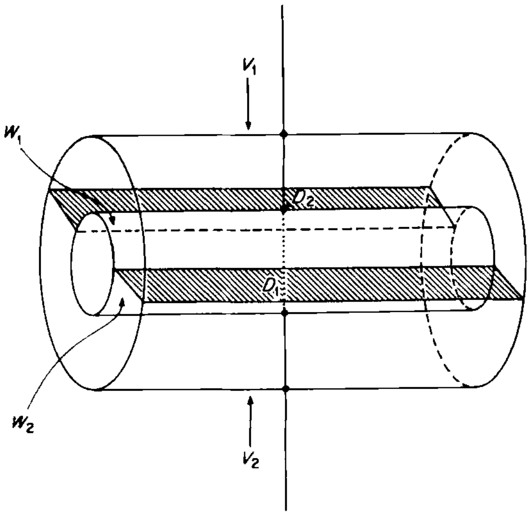

Figure 1

each component of $\partial\Omega_2 \times [0, 1]$. To see this, let σ be a closed curve in K whose intersection with the first component of $\partial\Omega_2 \times [0, 1]$ has the form $\{x\} \times [0, 1]$ and whose intersection with the second component of $\partial\Omega_2 \times [0, 1]$ is the arc induced from the first one by f. Let σ' be the part of σ that lies in the complement of $\partial\Omega_2 \times [0, 1]$ and let l be its length. If we pick the product metric on the first component so that $\{x\} \times [0, 1]$ has length greater than l, then any closed geodesic in K whose intersection with either the first component or the second component more than once has length not less than three times the length of $\{x\} \times [0, 1]$. By comparison with the length of σ, we obtain a contradiction and we have proved the claim. To guarantee that the geodesic avoids the fixed point set of f, we may just blow up the metric in a small neighborhood of each fixed point of f.)

By the geodesic loop theorem, γ and $f(\gamma)$ are equal or disjoint. By hypothesis, f preserves the ends of M_2 and so cannot act freely on γ. Because f has no fixed points on γ, f cannot leave γ invariant, which shows that γ and $f(\gamma)$ must be disjoint. Therefore, γ and $f(\gamma)$ disconnect $\Omega_2 \times \{0\}$ into two disks V_1 and V_2. The equivariant loop theorem stated in this section implies that there exists a disk D_1 such that $\partial D_1 = \gamma$ and $D_2 = f(D_1)$ is disjoint from D_1. The disks D_1 and D_2 disconnect M_2 into two balls B_1 and B_2, where $V_i \subset B_i$. By hypothesis, f interchanges the circles $\partial\Omega_2 \times \{0\}$. The Lefschetz fixed point theorem implies that $f|\Omega_3 \times \{0\}$ has a fixed point. Because the balls B_i have fixed points in their interiors, they are left invariant by f. Let W_i be the intersection of B_i with $\Omega \times \{1\}$ for $i = 1, 2$.

Now choose product foliations F_i of $\partial B_2 - (\mathring{V}_i \cup \mathring{W}_i)$ for $i = 1, 2$ that agree on the intersection $\partial B_1 \cap \partial B_2$ and that are preserved by f. Because B_1 and B_2 are invariant under the restriction of f, Lemma 1 implies that the foliations F_1 and F_2 are the boundary curves of a product foliation of M_2 preserved by f. Fitting together the associated invariant product structures along $\partial B_1 \cap \partial B_2$ gives rise to a product structure on M_2 that is preserved by f. This proves the lemma. ■

LEMMA 3. *If $f: M_3 \to M_3$ is a diffeomorphism of order two or three that preserves the ends, then f preserves a product structure on M_3.*

Proof. First consider the case in which f has order two. The usual linear action R on M_3 of order two given in this lemma is induced by rotating M_3 around the Z-axis by 180° (see Fig. 2).

Note that there exists a disk D_1 such that D_1 and $D_2 = R(D_1)$ are disjoint and $D_1 \cup D_2$ disconnects the ends of M_3 into disks C, E and annuli A_1, A_2, B_1, B_2. Furthermore, D_1 and D_2 each intersect $\partial\Omega_3 \times [0, 1]$ only along the invariant component and in two intervals. We now prove the existence of similar disks for f rather than R.

First, note that since f has order two and acts on $\partial\Omega_3 \times \{0\}$, f leaves invariant one or three of these circles. However, if f leaves invariant all three circles, then $f|\Omega_3 \times \{0\}$ would induce a diffeomorphism of order two on the disk with two or more fixed points. As this is impossible by the Lefschetz fixed point theorem, f leaves invariant exactly one circle. As in Lemma 2, there is an f-invariant metric on ∂M_3, so that any curve γ of least length in

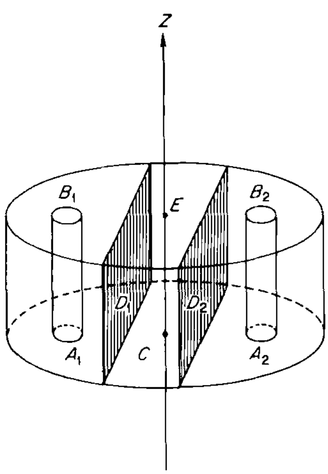

Figure 2

the kernel K of $i_*: \pi_1(\partial M_3) \to \pi_1(M_3)$ is disjoint from the fixed point set f and $\gamma \cap (\partial\Omega_3 \times [0, 1])$ is two intervals on the component of $\partial\Omega_3 \times [0, 1]$ invariant under f. (There is an invariant metric that is a product metric on two components X_1, X_2 of $\partial\Omega_3 \times [0, 1]$ and is flat on the other component $X_3 = f(X_2)$. In fact, choose a metric so that the length of $\{x\} \times [0, 1]$ on X_1 is sufficiently long and the length of $\{y\} \times [0, 1]$ on X_2 is much longer. The arguments in Lemma 2 now show that γ has the correct properties.)

By the geodesic loop theorem, γ and $f(\gamma)$ are equal or disjoint. The same argument as in the proof of Lemma 2 shows that γ and $f(\gamma)$ are disjoint. Therefore, γ and $f(\gamma)$ disconnect $\Omega_3 \times \{0\}$ into three parts A_1, A_2, and C, where $f(A_1) = A_2$ and C is an invariant disk. The equivariant loop theorem implies that there exists a disk D_1 such that $\partial D_1 = \gamma$ and $D_2 = f(D_1)$ is disjoint from D_1. The disks D_1 and D_2 disconnect M_3 into a ball B and two regions R_1, R_2 diffeomorphic to $A_1 \times [0, 1]$. Let $B_i = R_i \cap (\Omega_3 \times \{1\})$ and $E = B \cap (\Omega_3 \times \{1\})$.

Now choose a product foliation F_1 of R_1 with end leaves A_1, B_1 and such that $F_1 \cap D_1$ is a product foliation by intervals. Let F_2 be the foliation on R_2 induced from F_1 by f. The partial foliations on $\partial B \cap (D_1 \cup D_2)$ induced by $F_1 \cup F_2$ can be extended to a product foliation on $\partial B \cap (\partial\Omega_3 \times [0, 1])$ by circles S_t^1 with $f(S_t^1) = S_t^1$. Lemma 1 implies that there is a foliation F_3 of disks of B with $\partial D_t = S_t^1$ and $f(D_t) = D_t$. Fitting together the foliations F_1, F_2, and F_3 along ∂B gives rise to a product structure for M_3 that is preserved by f. This proves the first case in Lemma 3. The proof of the second case in Lemma 3 is similar and will be left to the reader. ∎

Remark. The technique used in the above lemmas generalizes to characterize finite group actions which preserve the ends of M_n as being conjugate to actions preserving the product structure, and the similar statement holds for an M that is a product of an interval with any compact surface with nonempty boundary.

THEOREM 3. *Suppose that G is a compact Lie group that acts smoothly and preserves orientation on the ball $B = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$ and that G is not isomorphic to the alternating group A_5. Then G is conjugate to a linear action on B.*

PROOF. Since the group G acts on the boundary sphere of B and compact group actions on S^2 are standard, G is isomorphic to a compact subgroup of SO(3). The nonfinite compact subgroups of SO(3) are SO(3), S^1, and a Z_2 extension of S^1. In the case of SO(3), the orbit space is an interval and in the other two cases the orbit space is a disk. In the nonfinite cases it is straightforward to check that the action of G on B is conjugate to the linear action.

Because A_5 is the only nonsolvable finite subgroup of SO(3), our hypotheses imply that the group G is solvable. Standard Smith theory implies that G has a global fixed point and the fixed point set fix(g) of any diffeomorphism $g \in G$ is an interval with boundary in the boundary sphere of B. The solution of the Smith conjecture shows that the interval fix(g) is unknotted and that $g: B \to B$ is conjugate to a rotation. Hence, for G cyclic, the theorem is known. The proof of the theorem will depend on the analysis of the remaining possibilities. The other possible groups that act on S^2 are listed in [W, pp. 83–85] and are the dihedral groups D_n, the tetrahedral group T, and the octahedral group.

Since the above solvable groups G have a global fixed point p and the action of G is linear near p, the action will be conjugate to the linear action if the restricted action on the complement of a small invariant open ball centered at p is standard. In other words, if a G action of $M = S^2 \times [0, 1]$ is conjugate to an action that preserves the product structure, then G acts on the compact three-dimensional ball standardly. Because product structures on $S^2 \times [0, 1]$ differ by diffeomorphisms, we need only find some product structure that is preserved by G. To do this we shall use the solvability of G.

The finite groups G that we are considering have a normal series $H_1 = \{e\} \subset H_2 \subset \cdots \subset H_n = G$, where H_k/H_{k-1} is cyclic and this cyclic group is equal to Z_2 or Z_3 when $k = 2$. In fact, for D_{2n}, $k = 3$, for the tetrahedral group, $k = 4$, and for the octahedral group, $k = 5$. This normal series of G gives rise to a family of cyclic branched covering spaces

$$X_1 \underset{p_2}{\to} X_2 \underset{p_3}{\to} \cdots \to X_{n-1} \underset{p_n}{\to} X_n,$$

where $X_k = M/H_k$ and H_{k+1}/H_k are the covering transformations for $p_{k+1}: X_k \to X_{k+1}$.

After averaging a riemannian metric on X_1, G acts as a group of isometries. By the solution of the Smith conjecture, X_2 is diffeomorphic to $S^2 \times [0, 1]$. Furthermore, the branch locus for $p_2: X_1 \to X_2$ consists of two intervals I_1 and I_2 in X_2 such that the small open ε-neighborhood N of $I_1 \cup I_2$ is invariant under H_3. Also, $X_2 - N$ is naturally diffeomorphic to $M_2 = \Omega_2 \times [0, 1]$. By solvability of the group G, M_2 is invariant under the generator f of the group of the covering transformations H_3/H_2 and f preserves the ends of M_2.

By suitable choice of the normal series, we may assume that f has order two at this stage and f interchanges the circles $\partial\Omega_2 \times \{0, 1\}$. Thus by Lemma 2, f preserves a product structure on M_2. This implies that the complement C of the branch set of the induced action of H_3/H_2 on M_2 has an invariant product structure. The quotient of this manifold C is naturally diffeomorphic to $M_3 = \Omega_3 \times [0, 1]$. Furthermore, H_4/H_3 induces an action on M_3 that preserves ends. Lemma 3 implies that H_4/H_3 preserves a product structure on M_3. Therefore, the complement of the ε-neighborhood of the branch set of H_4/H_3 has an invariant product structure.

Continuing the above arguments for the other groups H_k/H_{k-1}, we can show that the complement of the branch locus of $p_n \circ p_{n-1} \circ \cdots \circ p_2 : M_{k-1} \to M_k$ has a product structure. This implies that $G = H_n$ acting on M preserves a product structure. As was noted earlier, this proves Theorem 3. ■

LEMMA 4. *If G is a compact subgroup of* $\text{diff}^+(\mathbf{R}^3)$, *then there exists a sequence of balls $B_1 \subset B_2 \subset \cdots \subset B_n \subset \cdots \subset \mathbf{R}^3$ satisfying*

(1) $B_i \subset \text{int}(B_{i+1})$,
(2) $\bigcup_{i=1}^{\infty} B_i = \mathbf{R}^3$, *and*
(3) $g(B_i) = B_i$ *for all* $g \in G$.

Proof. The proof of the lemma for nonfinite groups is rather straightforward and will be left to the reader. Now suppose that G is finite. Smith's theory [B] for the powers g^k of elements g of G with prime order easily implies that the fixed point set fix(g) of g is diffeomorphic to $\mathbf{R}$ properly embedded in $\mathbf{R}^3$ and that the cyclic subgroup generated by g acts freely outside fix(g). The isotropy subgroups $I_p \subset G$ of any point $p \in \mathbf{R}^3$ has a natural representation from their differentials on $T_p\mathbf{R}^3$ in the group SO(3). From this representation, it is straightforward to put a differential structure on the orbit space $M = \mathbf{R}^3/G$ so that the projection map $p\colon \mathbf{R}^3 \to M$ is a smooth, branched immersion. Furthermore, if X is a smooth, compact, embedded surface in M that is in general position with respect to the branch locus of p, then $p^{-1}(X)$ is a smooth, embedded surface in $\mathbf{R}^3$. ■

ASSERTION 1. There exists a proper Morse function $H\colon M \to \mathbf{R}$ and an increasing sequence of numbers $r_i \in \mathbf{R}$ that satisfy the following properties.

(1) $N_i = (H \cdot p)^{-1}[r_i, r_{i+1}]$ is a smooth, compact three-dimensional submanifold of $\mathbf{R}^3$ that is invariant under f.

(2) There exists an embedded 2-sphere in N_i that intersects $\alpha[0, \infty]$ and $\alpha[-\infty, 0]$ transversely in an odd number of points, where $\alpha\colon \mathbf{R} \to \text{fix}(f)$ is a fixed diffeomorphism for some fixed, nontrivial $f \in G$.

(3) $H \circ p \circ \alpha(0) < r_1$ and $\lim_{i \to \infty} r_i = \infty$.

Proof. Let $H\colon M \to \mathbf{R}$ be a proper Morse function that is in general position with respect to $p \circ \alpha\colon \mathbf{R} \to M$. Then choose $r_1 \in \mathbf{R}$ large enough so that $\alpha(0) \in \text{fix}(f)$ is in the open submanifold $(H \circ p)^{-1}(-\infty, r_1)$. Pick D_1 to be a smooth ball in $\mathbf{R}^3$ that contains $(H \circ p)^{-1}((-\infty, r_1])$ and such that ∂D_1 is in general position with respect to the embedded curve $\alpha(\mathbf{R})$. Pick an $r_2 > \max(H \circ p(D_1)) + 1$. Now choose D_2 to be a smooth ball in $\mathbf{R}^3$ that contains $(H \circ p^{-1}((-\infty, r_2])$ and let $r_3 > \max(H \circ p(D_2)) + 1$. Continuing this process, one has an infinite set of r_i with $r_i < r_{i+1}$, $H \circ p \circ \alpha(0) < r_1$

and $\lim_{i\to\infty} r_i = \infty$. As H is a Morse function, we may choose r_i to be regular values of H and $(H \circ p \circ \alpha)$. By choice of r_i, $H^{-1}(r_i)$ is a compact surface, possibly disconnected, that is transversal to $(p \circ \alpha)(\mathbf{R})$ in a finite number of points. This implies that $N_i = (H \circ p)^{-1} \times ([r_i, r_i])$ is a smooth, compact submanifold of $\mathbf{R}^3$ that is invariant under f.

We now show that each sphere $S_i = \partial D_i$ intersects each of the intervals $\alpha([0, \infty))$ and $\alpha((-\infty])$ in an odd number of points. In fact, by the construction of D_i, $\alpha([0, \infty))$ intersects S_i transversally in a finite number of points. Since only a compact part of $\alpha[0, \infty)$ lies in D_i, there is an integer $m_i > 0$ so that $\alpha([m_i, \infty))$ is disjoint from D_i. As $\alpha(0)$ is contained in D_i and $\alpha(m_i)$ is contained in the complement of D_i, the curve $\alpha(0, m_i]$ must intersect S_i an odd number of times. Hence, $\alpha([0, \infty))$ must intersect S_i transversely an odd number of times. The same proof shows that $\alpha((-\infty, 0])$ intersects S_i in an odd number of points.

ASSERTION 2. There exists a sequence of closed, smooth balls

$$B_1 \subset B_2 \subset \cdots \subset B_n \subset \cdots \subset \mathbf{R}^3$$

that satisfy

(1) $B_i \subset \operatorname{int}(B_{i+1})$,
(2) $\bigcup_{i=1}^{\infty} B_i = \mathbf{R}^3$, and
(3) $f(B_i) = B_i$.

Proof. By Assertion 1 there exists an embedded sphere S_i in N_i such that $\alpha([0, \infty))$ intersects S_i in an odd number of points. By construction, $\alpha([0, \infty]) \cap N_i$ is a finite number of embedded intervals whose boundaries are contained in ∂N_i. We may consider $\alpha[0, \infty) \cap N_i$ as representing an element $\bar{\alpha}$ in $H_1(N_i, \partial N_i, Z_2)$. Considering the sphere S_i as representing an element $\bar{S}_i$ in $H_2(N_i, Z_2)$, the above geometric intersection shows that $\bar{S}_i \cap \bar{\alpha}_i$ is nonzero, where $\cap$ is the intersection pairing on homology. In particular, the class $\bar{S}_i$ is nonzero, which implies that S_i represents a nontrivial element in the second homotopy group $\pi_2(N_i)$ of N_i.

Since the group G_i of diffeomorphisms of N_i generated by the restrictions $g | N_i$ of elements of G to N_i is finite, we may assume G_i acts as a group of isometries with respect to a convex metric on N_i. Now apply the equivariant sphere theorem to get a finite generating set $g_1, g_2, \ldots, g_n: S^2 \to N_i$ of embedded spheres such that if $(f | N_i)(g_k(S^2))$ intersects $g_k(S^2)$, then f leaves this sphere invariant. Since $g_1, g_2, \ldots, g_n$ represent a $\pi_1(N_i)$-generating set for $\pi_2(N_i)$, the homology class $\bar{S}_i$ of the sphere S_i can be expressed as a linear combination of the associated homology classes $\bar{g}_1, \ldots, \bar{g}_n$ in $H_2(N_i, Z_2)$. As the intersection number $\bar{S}_i \cap \bar{\alpha}$ is nonzero, $\bar{\alpha} \cap \bar{g}_k$ is also nonzero for some k. Assume that $k = 1$.

Since $\bar{\alpha} \cap \bar{g}_1$ is nonzero, $\alpha \cap g_1(S^2)$ is nonempty. This implies that f has a fixed point on $g_1(S^2)$. Therefore, $f(g_1(S^2))$ intersects $g_1(S^2)$ and f must leave the sphere $g_1(S^2)$ invariant. By Alexander's theorem, $g_1(S^2)$ disconnects $\mathbf{R}^3$ into a ball B_i and a noncompact component. As f leaves invariant the boundary sphere of the closed ball B_i, f leaves B_i invariant. Since f is orientation-preserving, $f \mid B_i : B_i \to B_i$ is orientation-preserving. This shows $f \mid g_1(S^2)$ is orientation-preserving as a diffeomorphism of the sphere $g_1(S^2)$. Since finite groups act standardly on two spheres, f has two fixed points on $g_1(S^2)$. The local behavior of f near its fixed point set shows that $\text{fix}(f) = \alpha(\mathbf{R})$ is transversal to $g_1(S^2)$ at the two intersection points. As the intersection number of $\alpha([0, \infty))$ and $g_1(S^2)$ is odd, $\alpha([0, \infty))$ intersects $g_1(S^2)$ transversely in exactly one point. This shows that $\alpha(0)$ must be in the compact component of $\mathbf{R}^3$ bounded by the sphere S_i and hence $\alpha(0)$ is contained in the ball B_i. Since N_i and N_{i+2} are disjoint, the boundary sphere S_i of B_i is disjoint from the boundary sphere S_{i+2} of the ball B_{i+2}. This fact, together with the fact that $\alpha(0) \in B_i$ for all i, shows that $B_i \subset \text{int}(B_{i+2})$.

We now show that the union of $\bigcup_{i=1}^{\infty} B_{2i} = \mathbf{R}^3$. (Hence, after reindexing, $B_2, B_4, \ldots, B_{2n}, \ldots$ will provide the required balls for Assertion 2.) Consider the compact, possibly disconnected, manifolds $M_i = H^{-1}((-\infty, r_i])$ for each i. Let C_i be the component of M_i that contains the point $\alpha(0)$. If the union $\cup_i C_i$ is not all of $\mathbf{R}^3$, there is a boundary point x_0 of $\cup C_i$. As C_i is contained in the interior of C_{i+1}, there is a sequence $x_i \in \partial C_i$ that converges to x_0. However, $H(x_i) = r_i$ and hence $H(x) = \lim_{i\to\infty} H(x_i) = \lim_{i\to\infty} r_i = \infty$, which is impossible. This shows that $\cup_i C_i = \mathbf{R}^3$.

The sphere $\partial B_{2i} = \bar{S}_{2i} \subset N_{2i}$ is disjoint from the manifold C_{2i-2} and the ball B_{2i} intersects C_{2i-2} at the point $\alpha(0)$. Since C_{2i-2} is connected and disjoint from the boundary sphere of B_{2i}, C_{2i-2} is a subset of B_{2i}. Thus $\bigcup_i C_{2i} = \mathbf{R}^3 \subset \bigcup_i B_{2i} = \mathbf{R}^3$, and therefore $\mathbf{R}^3 = \bigcup_{i=1}^{\infty} B_{2i}$. This equation completes the proof of the assertion.

Assertion 3. There is an N such that for $i > N$ the ball B_i in Assertion 2 is invariant under all elements of G.

Proof. By construction of the ball B_i and the proof of the previous assertion, the ball B_i is invariant under a diffeomorphism $g \in G$ if $\text{fix}(g)$ intersects B_i nontrivially. However, as $\text{fix}(g)$ is nonempty and $\bigcup_{i=1}^{\infty} B_i = \mathbf{R}^3$, $\text{fix}(g)$ must intersect some ball B_i for i large. As there are only a finite number of elements in G, the assertion holds. The lemma follows immediately from Assertion 3. ∎

Lemma 5. *Suppose that M is a compact surface with boundary and let $X = M \times [0, 1]$. Let $Z_1 = M \times \{0\}$ and $Z_2 = \partial M \times [0, 1]$. If $f: Z_1 \cup Z_2 \to$*

$Z_1 \cup Z_2$ is a homeomorphism of finite order with $f(Z_i) = Z_i$ and $f|Z_i$ smooth for $i = 1, 2$, then f extends to a diffeomorphism of $M \times [0, 1]$.

Proof. First, suppose that $f(p, t) = (f(p, 0), t)$. In this case, f can be extended to X by defining $f(p, t) = (f(p, 0), t)$. This case is easy because f preserves the fixed-product structure on Z_2.

There is an elementary classical result that states there exists an isotopy between two diffeomorphisms of an annulus that preserve boundary curves and whose restriction to one of the boundary circles is preassigned. This classical result is equivalent to proving that the lemma holds when every component of M is an annulus. We now apply the fact that the lemma holds in this generality.

Let N be an invariant neighborhood of ∂M in M that is a collection of annular components. Let $Z_3 = (\partial N \backslash \partial M) \times [0, 1]$ and define $g\colon Z_1 \cup Z_2 \cup Z_3 \to X$ by

$$g(p, t) = \begin{cases} f(p, t) & \text{if } (p, t) \in Z_1 \cup c_2, \\ (f(p, 0), t) & \text{if } (p, t) \in Z_3. \end{cases}$$

By the extension property for M a collection of annular surfaces, it follows that g can be extended to $N \times [0, 1]$. As observed in the first case in the proof, g has natural extension as a homeomorphism to the rest of X. By making an appropriate choice of the extension of g to $N \times [0, 1]$, the required extension will be a diffeomorphism. This proves the lemma. ■

THEOREM 4. *If G is a compact subgroup of* $\operatorname{diff}^+(\mathbf{R}^3)$, *then G is isomorphic to a compact subgroup of* SO(3). *If G is not isomorphic to alternating group A_5 (also called the icosahedral group), then G is conjugate in* $\operatorname{diff}^+(\mathbf{R}^3)$ *to a subgroup of* SO(3).

Proof. That G is isomorphic to a compact subgroup of SO(3) follows immediately from Lemma 4 because G acts as a compact group of orientation-preserving diffeomorphism on the boundary sphere ∂B_i and all such actions are conjugate to linear actions. Now assume that G is not isomorphic to A_5.

Let $B_1 \subset B_2 \subset \cdots \subset B_n \subset \cdots \mathbf{R}^3$ be the equivariant balls given in Lemma 4. By the Schoenflies theorem for annular domains in $\mathbf{R}^3$ we may assume, after conjugation by some diffeomorphism, that the balls B_i are balls of radius i centered at the origin in $\mathbf{R}^3$. Suppose for the moment that there exist diffeomorphisms $\tilde{h}_n\colon B_n \to B_n$ so that $\tilde{h}_n G \tilde{h}_n^{-1}$ is a group of linear isometries of B_n, and the $\tilde{h}_n(B_{n-1})$ and $\tilde{h}_n|B_{n-1} = \tilde{h}_{n-1}$. Define $\tilde{h}\colon \mathbf{R}^3 \to \mathbf{R}^3$ by $\tilde{h}(x) = \tilde{h}_n(x)$, where $|x| \leq n$. Clearly, hGh^{-1} is contained in the subgroup SO(3) of $\operatorname{diff}^+(\mathbf{R}^3)$. Hence, the theorem will follow from the existence of the diffeomorphisms $\tilde{h}_n\colon B_n \to B_n$. We now give a proof by induction of the existence of $\tilde{h}_n$. By Theorem 3, any action of G on the closed three-dimensional ball of radius 1 is conjugate to the linear action. Translated into algebraic terms on the associated branched covering spaces, this conjugacy can be expressed by stating that the commutative diagram

$$\begin{array}{ccc} B_1 & \xrightarrow{\tilde{h}_1} & B \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p'_1} \\ B & \xrightarrow[h_1]{} & B' \end{array}$$

exists.

Here p_1 is the natural branched covering space arising from the action of G on B, p'_1 is the natural branched covering space arising from a fixed linear action of G on the ball B_1; h_1 is a diffeomorphism that maps the unknotted branch set of p, which is diffeomorphic to an unknotted interval to the branch set of p' in the ball B'; $\tilde{h}_1$ is a lift of h_1 to the branched covering spaces. Actually, once we have constructed a diffeomorphism h_1, a lifting $\tilde{h}_1$ exists if h_1 preserves the branch locus and the local monodromy. This fact can be proved using the theory of normal covering spaces in this case. If $\tilde{h}_1$ is some lifting, then clearly $\tilde{h}_1 g \tilde{h}_1^{-1}$ is a covering transformation for the branched covering space $p'_1 : B_1 \to B'$. As the covering transformations of this branched covering space is G, $\tilde{h}_1$ is a diffeomorphism that conjugates G to a linear action.

Note that in the above discussion the conjugating diffeomorphism $\tilde{h}_1$ depends on h_1 and the choice of the lifting of this h_1 to the total spaces of the branched covering spaces. Now suppose by induction that the commutative diagram

$$\begin{array}{ccc} B_{n-1} & \xrightarrow{\tilde{h}_{n-1}} & B_{n-1} \\ {\scriptstyle p_{n-1}}\downarrow & & \downarrow{\scriptstyle p'_{n-1}} \\ B & \xrightarrow[h_{n-1}]{} & B \end{array}$$

exists.

We now want to extend the diagram above to get the commutative diagram

$$\begin{array}{ccc} B_n & \xrightarrow{\tilde{h}_n} & B_n \\ {\scriptstyle p_n}\downarrow & & \downarrow{\scriptstyle p'_n} \\ B & \xrightarrow[h_n]{} & B' \end{array}$$

From the previous discussion this extension problem is equivalent to showing that we can extend the diffeomorphism h_{n-1} to a diffeomorphism $h_n: B \to B'$ so that $h_n | B_{n-1} = h_{n-1}$ and the image of the branch set of p is the branch set of p'.

Let Y_n denote the branch set of p_n and $\overline{Y}_n$ the branch set of p'_n. To construct h_n, we first extend h_{n-1} to a diffeomorphism h'_n of some small regular neighborhood N_n of Y_n to a small regular neighborhood $\overline{N}_n$ of $\overline{Y}_n$ in such a way that $h'_n(Y_n) = \overline{Y}_n$ and $h'_n(N_n \cup \partial B) = \overline{N}_n \cap \partial B'$. As Y_n and $\overline{Y}_n$ are unknotted in their respective balls, $B \backslash \mathrm{int}(N_n \cup B)$ and $B' \backslash \mathrm{int}(\overline{N}_n \cup B')$ are products of a planar domain with an interval. Lemma 5 implies h'_n extends to the required diffeomorphism.

It follows that the lift $\tilde{h}_n$ of h_n completes the proof of the existence of the above diagram. Thus, by induction, we have proved the existence of the required diffeomorphism $\tilde{h}_n$. As noted earlier, this completes the proof of the theorem. ∎

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