

HAUSDORFF DIMENSION OF WEIGHTED SINGULAR VECTORS IN \mathbb{R}^2

LINGMIN LIAO, RONGGANG SHI, OMRI N. SOLAN, AND NATTALIE TAMAM

ABSTRACT. Let $w = (w_1, w_2)$ be a pair of positive real numbers with $w_1 + w_2 = 1$ and $w_1 \geq w_2$. We show that the set of w -weighted singular vectors in \mathbb{R}^2 has Hausdorff dimension $2 - \frac{1}{1+w_1}$. This extends the previous work of Yitwah Cheung on the Hausdorff dimension of the usual (unweighted) singular vectors in \mathbb{R}^2 .

1. INTRODUCTION

Let $w = (w_1, w_2)$ be a pair of positive real numbers such that $w_1 + w_2 = 1$. Dirichlet's theorem with weight w (see [12, Chapter II])¹ states that for all $x = (x_1, x_2) \in \mathbb{R}^2$ and $T > 1$ there is $(p, q) = (p_1, p_2, q) \in \mathbb{Z}^2 \times \mathbb{Z}$ such that

$$\begin{cases} |qx_1 - p_1| < T^{-w_1} \\ |qx_2 - p_2| < T^{-w_2} \\ 0 < q \leq T \end{cases} .$$

There are different classes of vectors in \mathbb{R}^2 with more elaborate Diophantine properties, e.g. badly approximable vectors, Dirichlet's improvable vectors and singular vectors. Usually these sets have zero Lebesgue measure. The estimation of the size of them has a long history and is still fast developing in recent years, see e.g. [1, 2, 3, 4, 6, 7].

A vector $x = (x_1, x_2) \in \mathbb{R}^2$ is said to be w -singular if for every $\varepsilon > 0$ there exists $T_0 > 1$ such that for all $T > T_0$ the system of inequalities

$$(1.1) \quad \begin{cases} |qx_1 - p_1| < \varepsilon^{w_1} T^{-w_1} \\ |qx_2 - p_2| < \varepsilon^{w_2} T^{-w_2} \\ 0 < q < T \end{cases}$$

admits an integer solution $(p, q) \in \mathbb{Z}^2 \times \mathbb{Z}$. The set of w -singular vectors is denoted by $\text{Sing}(w)$. A vector $x \in \mathbb{R}^2$ is said to be singular if it is w -singular in the case where w is unweighted, i.e. when $w_1 = w_2 = \frac{1}{2}$.

It is proved by Cheung [6] that the Hausdorff dimension of the set of singular vectors in \mathbb{R}^2 is $\frac{4}{3}$. Here and hereafter the Hausdorff dimension of a subset of \mathbb{R}^d ($d \in \mathbb{N} := \{1, 2, \dots\}$) is with respect to the usual Euclidean metric. Recently, Cheung and Chevallier [7] extended this result to \mathbb{R}^d ($d \geq 2$) and proved that the set of singular vectors in \mathbb{R}^d has Hausdorff dimension $\frac{d^2}{d+1}$. Recall that in \mathbb{R} only

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¹The form we state below is not explicitly stated in [12], but it is a special case of [12, Theorem II.2C].

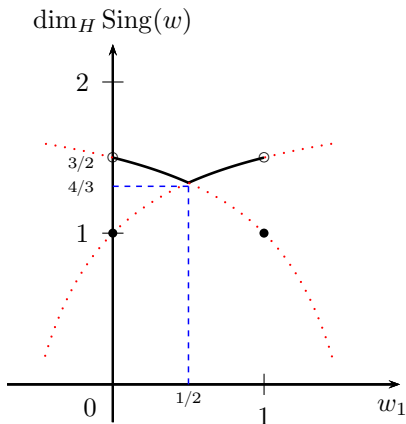


FIGURE 1. The Hausdorff dimension of $\text{Sing}(w)$.

rational numbers are singular, so we understand the Hausdorff dimension of the set of singular vectors in all Euclidean spaces.

The aim of this paper is to calculate the Hausdorff dimension of the set of w -singular vectors in \mathbb{R}^2 .

Theorem 1.1. *Suppose $w = (w_1, w_2)$ where $w_1 \geq w_2 > 0$ and $w_1 + w_2 = 1$. Then the Hausdorff dimension of $\text{Sing}(w)$ is $2 - \frac{1}{1+w_1}$.*

Remark 1.2. In the case where $w = (1, 0)$ one can also define w -singular vectors in a similar way. The above formula does not hold in this degenerate case where the Hausdorff dimension is 1. By symmetry, we can draw the whole picture of the Hausdorff dimension of $\text{Sing}(w)$ when w_1 goes from 0 to 1. We point out that the dimension graph has a non differential point $1/2$ and has jumps at 0 and 1 (see Figure 1).

Here and hereafter we always assume that the weight vector w satisfies the assumption of Theorem 1.1. It is observed by Dani [8] that w -singular vectors correspond to certain divergent trajectories in the space \mathcal{L}_3 of unimodular lattices in \mathbb{R}^3 with respect to the one-parameter semi-group

$$(1.2) \quad \mathcal{A}^+ = \{a_t = \text{diag}(e^{w_1 t}, e^{w_2 t}, e^{-t}) : t \geq 0\}.$$

More precisely, $x \in \mathbb{R}^2$ is w -singular if and only if $\mathcal{A}^+ h(x) \mathbb{Z}^3$ is divergent where

$$(1.3) \quad h(x) = \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using this dynamical interpretation it is not hard to see that given $y \in \mathbb{Q}^2$, a vector $x \in \mathbb{R}^2$ is w -singular if and only if $x + y$ is w -singular. Therefore the conclusion of Theorem 1.1 holds for any $U \cap \text{Sing}(w)$ where U is a nonempty open subset of \mathbb{R}^2 . Since $\mathbb{Q}^2 \subset \text{Sing}(w)$, the Minkowski dimension of $\text{Sing}(w)$ is 2 which is different from the Hausdorff dimension.

The lower bound of Theorem 1.1 is proved by constructing a subset of $\text{Sing}(w)$ with certain well-separated self-affine structure using this dynamical interpretation.

In fact, the denominator $1 + w_1$ in the dimension formula is the top Lyapunov exponent for the adjoint action of \mathcal{A}^+ on the group

$$(1.4) \quad \{h(x) : x \in \mathbb{R}^2\}.$$

This is reasonable since the top Lyapunov exponent corresponds to the shorter length of the rectangle in the self-affine structure and the shorter length is the length of the square after chopping a rectangle into squares. Since our construction has inductive nature it suffices to look at the first step to explain the ideas. We fix $t \gg 1$ and $\varepsilon < 1$ with $\varepsilon \gg e^{-t}$ which means ε^{-1} is negligible comparing to e^t . The lattice $a_{t'}\mathbb{Z}^3$ moves to the cusp in \mathcal{L}_3 as t' goes from 0 to t , since the Euclidean norm of $a_{t'}\mathbf{e}_3$ where $\mathbf{e}_3 = (0, 0, 1) \in \mathbb{Z}^3$ decays exponentially. If we want $x \in \mathbb{R}^2$ satisfy $\|a_{t'}h(x)\mathbf{e}_3\| \ll \varepsilon$ ($\|\cdot\|$ is the Euclidean norm) for t' away from 0, a reasonable condition is $x \in U_0$ where

$$U_0 = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \leq \varepsilon e^{-w_1 t}, |x_2| \leq \varepsilon e^{-w_2 t}\}.$$

To play this game again we need $a_t h(x)\mathbb{Z}^3 \in \mathcal{L}'_3$ where

$$(1.5) \quad \mathcal{L}'_3 = \{\Lambda \in \mathcal{L}_3 : \Lambda \cap \mathbb{R}\mathbf{e}_3 = r\mathbb{Z}\mathbf{e}_3 \text{ for some } r \text{ with } 1/2 < r \leq 1\}.$$

The cardinality of $\{x \in U_0 : a_t h(x)\mathbb{Z}^3 \in \mathcal{L}'_3\}$ is up to some constants the cardinality of $a_t \mathbb{Z}^3 \cap M$ where

$$M = \{(z_1, z_2, z_3) \in \mathbb{R}^3 : |z_1| \leq \varepsilon e^t |z_3|, |z_2| \leq \varepsilon e^t |z_3|, 1/2 < |z_3| \leq 1\}.$$

It will follow from lattice points counting that this cardinality is approximately the area of M which is $\approx \varepsilon^2 e^{2t} \approx e^{-t} \cdot e^{3t}$. Here e^{3t} is more or less the cardinality of the next subdivision of U_0 by rectangles of the size $2\varepsilon e^{-(2w_1+1)t} \times 2\varepsilon e^{-(2w_2+1)t}$, so it corresponds to the full dimension 2. In fact the -1 in the numerator of $\frac{-1}{1+w_1}$ comes from the factor e^{-t} . To make the self-affine structure well-separated we need more conditions than just $a_t h(x)\mathbb{Z}^3 \in \mathcal{L}'_3$ and the difficulty is to prove that the cardinality of those x is comparable to $\varepsilon^2 e^{2t}$ using geometry of numbers. In this part, different arguments are needed depending on whether $w_1 > w_2$ or $w_1 = w_2$. The lower bound is proved only in the genuine weighted case $w_1 > w_2$, since in the unweighted case the Hausdorff dimension is known.

Our proof of the upper bound follows the similar ideas of [6] and [7]. We use best approximation vectors with weight w to encode $\text{Sing}(w)$ to get a self-affine covering of the essential part of $\text{Sing}(w)$. The main difference comparing to the unweighted case is that our covering is self-affine instead of self-similar and this difference makes the calculation more subtle. In the unweighted case Einsiedler and Kadyrov [9] have an estimate of the upper bound using entropy. The method in [9] is further developed by Kadyrov, Kleinbock, Lindenstrauss and Margulis [11] to estimate the upper bound of the Hausdorff dimension of general (unweighted) singular systems of linear forms. The new input of this development is the use of the height function in Eskin–Margulis–Mozes [10] and its contracting property. Inspired by [11], it seems that the number 1 in the numerator of $\frac{1}{1+w_1}$ might also be interpreted as certain average contracting rate of the height function in [10] with respect to \mathcal{A}^+ and the group (1.4).

Based on our interpretation of Theorem 1.1 and [11, Corollary 1.2] it seems likely that the Hausdorff dimension of weighted singular vectors in \mathbb{R}^d ($d \geq 2$) can be formulated in a similar way. Namely, if we normalize the weights so that the

sum of positive weights is equal to 1, the Hausdorff dimension of weighted singular vectors in \mathbb{R}^d is

$$d - \frac{1}{\lambda_1}$$

where λ_1 is the top Lyapunov exponent for the adjoint action of the corresponding one-parameter semi-group on the corresponding unipotent group.

Now we turn to the Hausdorff dimension of vectors in \mathbb{R}^2 for which Dirichlet's theorem can be improved. For a positive real number $\varepsilon < 1$, we say w -weighted Dirichlet's theorem is ε -improvable for $x \in \mathbb{R}^2$ if (1.1) admits integer solutions $(p, q) \in \mathbb{Z}^2 \times \mathbb{Z}$ for T sufficiently large. Let $\text{DI}(w, \varepsilon)$ be the set of vectors $x \in \mathbb{R}^2$ for which w -weighted Dirichlet's theorem is ε -improvable. It follows directly from the definition that

$$\text{Sing}(w) = \bigcap_{0 < \varepsilon < 1} \text{DI}(w, \varepsilon).$$

We remark here that in the unweighted case our set $\text{DI}(w, \varepsilon)$ is $\text{DI}_{\sqrt{\varepsilon}}(2)$ defined in [6] and [7].

Denote by \dim_H the Hausdorff dimension. We have the following theorem.

Theorem 1.3. *Let $w = (w_1, w_2)$ where $w_1 \geq w_2 > 0$ and $w_1 + w_2 = 1$. There exists $C > 0$ such that for all $0 < \varepsilon \leq 2^{-5/w_2}$ one has*

$$(1.6) \quad 2 - \frac{1}{1 + w_1} \leq \dim_H \text{DI}(w, \varepsilon) \leq 2 - \frac{1}{1 + w_1} + C\sqrt{\varepsilon}.$$

Remark 1.4. The constant C in Theorem 1.3 is computable. In the unweighted case, the upper bound in (1.6) is the same as [6, Theorem 1.6]; while our method does not give good lower bound as in [7, Theorem 1.4].

Finally we discuss the divergent trajectories of \mathcal{A}^+ in \mathcal{L}_3 . We want to estimate the Hausdorff dimension of the set

$$\mathcal{D}(\mathcal{L}_3, \mathcal{A}^+) := \{\Lambda \in \mathcal{L}_3 : \mathcal{A}^+ \Lambda \text{ is divergent}\}.$$

Here the Hausdorff dimension is with respect to any Riemannian metric on the manifold $\mathcal{L}_3 \cong SL_3(\mathbb{R})/SL_3(\mathbb{Z})$. In the unweighted case the group (1.4) is the unstable horospherical subgroup of a_1 . Therefore as a corollary of the Hausdorff dimension of $\text{Sing}(\frac{1}{2}, \frac{1}{2})$ it is proved in [6, Corollary 1.2] that the Hausdorff dimension of $\mathcal{D}(\mathcal{L}_3, \mathcal{A}^+)$ is $7\frac{1}{3}$. In authentic weighted case where $w_1 > w_2$, the unstable horospherical subgroup of a_1 is the upper triangular unipotent group in $SL_3(\mathbb{R})$ and the group (1.4) is a proper subgroup of it. So we can not get the Hausdorff dimension of divergent trajectories from Theorem 1.1. On the other hand, our method for proving the lower bound of the Hausdorff dimension of $\text{Sing}(w)$ can also be used to prove the following result.

Theorem 1.5. *Suppose $w = (w_1, w_2)$ where $w_1 + w_2 = 1$ and $w_1 > w_2$. For any $\Lambda \in \mathcal{L}_3$ and any nonempty open subset U of \mathbb{R}^2 , the Hausdorff dimension of*

$$\{x \in U : \mathcal{A}^+ h(x)\Lambda \text{ is divergent}\}$$

is at least $2 - \frac{1}{1+w_1}$.

Theorem 1.5 immediately implies the following corollary.

Corollary 1.6. *Let $w = (w_1, w_2)$ where $w_1 > w_2 > 0$ and $w_1 + w_2 = 1$. Then the Hausdorff dimension of $\mathcal{D}(\mathcal{L}_3, \mathcal{A}^+)$ is at least $8 - \frac{1}{1+w_1}$.*

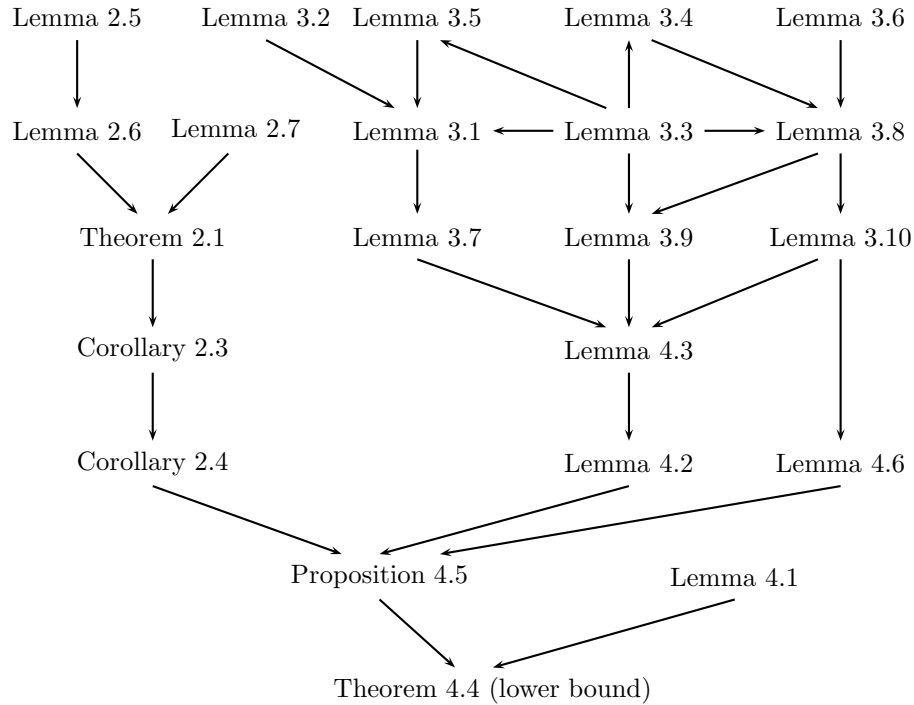


FIGURE 2. The relations between theorems for the lower bound.

We organize the paper as follows. In §2, we describe a fractal structure and develop some techniques for estimation of Hausdorff dimension. §3 is devoted to counting lattice points in a convex subset of Euclidean space. In §4, we give the proof of the lower bound of the Hausdorff dimension of $\text{Sing}(w)$ and the proof of Theorem 1.5. The proof of the upper bound of the Hausdorff dimension of $\text{Sing}(w)$ and the proof of Theorem 1.3 are given in the last section.

To make our presentation easier to follow, we give in Figures 2 and 3 the relations between the theorems for the lower and upper bounds of $\dim_H \text{Sing}(w)$ respectively.

2. FRACTAL STRUCTURE AND HAUSDORFF DIMENSION

In this section we first review the description of a fractal structure using a rooted tree and develop some techniques for estimating the lower bound of the Hausdorff dimension of a fractal set from this structure. Then we prove an upper bound estimate theorem for a fractal set given by certain relations.

2.1. Fractal structure. Recall that a rooted tree is a connected graph \mathcal{T} without cycles and with a distinguished vertex τ_0 , called the root of \mathcal{T} . In this paper we identify \mathcal{T} with the set of vertices of the tree \mathcal{T} . Any vertex $\tau \in \mathcal{T}$ is connected to τ_0 by a unique path whose length is called the height of τ . The set of vertices of height n is denoted by \mathcal{T}_n . Therefore $\mathcal{T}_0 = \{\tau_0\}$. A vertex $\tau \in \mathcal{T}_n$ is connected with a unique $\tau_{n-1} \in \mathcal{T}_{n-1}$ and we say τ is a son of τ_{n-1} . The set of sons of $\tau \in \mathcal{T}$ is denoted by $\mathcal{T}(\tau)$. A boundary point of \mathcal{T} is a sequence of vertices $\{\tau_n\}_{n \in \mathbb{N}}$ where

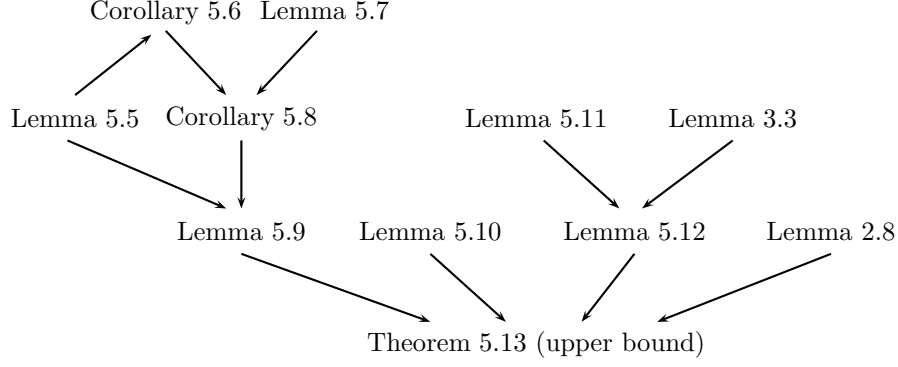


FIGURE 3. The relations between theorems for the upper bound.

τ_n is a son of τ_{n-1} . The boundary of \mathcal{T} consists of all the boundary points and is denoted by $\partial\mathcal{T}$. For a vertex $\tau \in \mathcal{T}_n$ the set of ancestors of τ is

$$\mathcal{A}(\tau) := \{\tau_i : 0 \leq i \leq n-1, \tau_{i+1} \in \mathcal{T}(\tau_i) \text{ where } \tau_n = \tau\}.$$

Let Y be a Polish space, i.e. a separable completely metrizable topological space. A fractal structure on Y is a pair (\mathcal{T}, β) where \mathcal{T} is a rooted tree and β is a map from \mathcal{T} to the set of nonempty compact subsets of Y . A fractal structure gives a set

$$\mathcal{F}(\mathcal{T}, \beta) := \{y \in Y : y \in \bigcap_{n=0}^{\infty} \beta(\tau_n) \text{ for some } \{\tau_n\} \in \partial\mathcal{T}\},$$

which is said to be the fractal with the structure (\mathcal{T}, β) . We remark that although each point of $\mathcal{F}(\mathcal{T}, \beta)$ should correspond to an infinite path, we do not assume each vertex of \mathcal{T} has a son. In particular if \mathcal{T} has only finitely many vertices, then the fractal set $\mathcal{F}(\mathcal{T}, \beta)$ has to be empty according to our definition.

We say that (\mathcal{T}, β) is a regular fractal structure if moreover the following properties hold:

- each vertex of \mathcal{T} has a nonempty set of sons;
- if τ is a son of τ' then $\beta(\tau) \subset \beta(\tau')$;
- for any $\{\tau_n\} \in \partial\mathcal{T}$ the diameter of $\beta(\tau_n)$ goes to zero as n tends to infinity.

We end up this section with several notations that will be used in the rest of the paper. For a set \mathcal{S} we use $\#\mathcal{S}$ to denote its cardinality. Let A, B be two subsets of a metric space (Y, d_Y) , then

$$\text{dist}(A, B) := \inf_{x \in A, y \in B} d_Y(x, y) \quad \text{and} \quad \text{diam}(A) := \sup_{x, y \in A} d_Y(x, y).$$

We will only use the above notation for the natural Euclidean metric on \mathbb{R}^d . For a real number s we take

$$\lceil s \rceil := \inf \{n \in \mathbb{Z} : n \geq s\} \quad \text{and} \quad \lfloor s \rfloor := \sup \{n \in \mathbb{Z} : n \leq s\}.$$

For two nonnegative real numbers s and t the notation $s \ll_{\mathcal{S}} t$ means that there is a constant $C \geq 1$ possibly depending on elements of the set \mathcal{S} such that $s \leq Ct$. We call C an implied constant for $s \ll_{\mathcal{S}} t$. The notation $s \gg_{\mathcal{S}} t$ means $t \ll_{\mathcal{S}} s$ and the notation $s \asymp_{\mathcal{S}} t$ means both $s \ll_{\mathcal{S}} t$ and $s \gg_{\mathcal{S}} t$.

2.2. Self-affine structure and lower bound. In this paper a rectangle means a rectangle in \mathbb{R}^2 with sides parallel to the axes. In particular, a rectangle with size $l_1 \times l_2$ and center $x \in \mathbb{R}^2$ refers to the set

$$\{y \in \mathbb{R}^2 : |y_1 - x_1| \leq l_1/2, |y_2 - x_2| \leq l_2/2\}.$$

A self-affine structure on \mathbb{R}^2 is a fractal structure (\mathcal{T}, β) on \mathbb{R}^2 such that for every $\tau \in \mathcal{T}$ the set $\beta(\tau)$ is a rectangle with size $W(\tau) \times L(\tau)$. A self-affine structure is said to be regular if the corresponding fractal structure is regular.

The main result of this section is Theorem 2.1. What we are going to use in the lower bound calculation are Corollaries 2.3 and 2.4 which are simplified versions of the theorem. We first prove these corollaries by assuming the theorem and then give the proof of the theorem.

Theorem 2.1. *Let (\mathcal{T}, β) be a regular self-affine structure on \mathbb{R}^2 . Suppose there are sequences of positive real numbers $\{W_n\}, \{L_n\}, \{\rho_n\}, \{C_n\}$ indexed by $\mathbb{N} \cup \{0\}$ with the following properties:*

- (1) $W(\tau) = W_n, L(\tau) = L_n$ and $W_n \leq L_n$ for all n and $\tau \in \mathcal{T}_n$;
- (2) $C_0 = 1$ and $\sharp\mathcal{T}(\tau) \geq C_n$ for all $n \in \mathbb{N}$ and $\tau \in \mathcal{T}_{n-1}$;
- (3) $\rho_n \leq 1$ for all $n \in \mathbb{N}$. Moreover, for all $\tau_n \in \mathcal{T}_n$ and different $\tau, \kappa \in \mathcal{T}(\tau_n)$

$$\text{dist}(\beta(\tau), \beta(\kappa)) \geq \rho_{n+1}W_n.$$

Let

$$P_n = \prod_{k=0}^n C_k,$$

$$D_n = \max\{k \geq n : L_k \geq W_n\},$$

$$s = \sup \left\{ t > 0 : \lim_{n \rightarrow \infty} \frac{\log \left(P_n W_n^t \rho_{n+1}^t \cdot \prod_{i=n+1}^{D_n} \rho_i C_i \right)}{\max\{D_n - n, 1\}} = \infty \right\}.$$

If $s > 1$, then $\dim_H \mathcal{F}(\mathcal{T}, \beta) \geq s$.

Remark 2.2. If $D_n = n$ we interpret $\prod_{i=n+1}^{D_n} \rho_i C_i = 1$. Since (\mathcal{T}, β) is regular, one has $W_n \rightarrow 0$ and hence $\rho_{n+1}W_n \rightarrow 0$. It follows that if $t < s$ then

$$\lim_{n \rightarrow \infty} \frac{\log \left(P_n W_n^t \rho_{n+1}^t \cdot \prod_{i=n+1}^{D_n} \rho_i C_i \right)}{\max\{D_n - n, 1\}} = \infty.$$

The main difference between Theorem 2.1 and lower bound theorems used in [6] and [7] is the factor $\prod_{i=n+1}^{D_n} \rho_i C_i$ which is trivial for the usual self-similar structures (here self-similar refers to $W_n = L_n$).

The formula of the lower bound s in Theorem 2.1 is much simpler in many interesting self-similar fractal structures where $L_n = W_n$ and $D_n = n$ for all $n \in \mathbb{N}$. The following two corollaries on refinement of the lower bound formula are only interesting in authentic self-affine cases where the following assumption (iv) of Corollary 2.3 is needed.

Corollary 2.3. *Let the notation be as in Theorem 2.1. We moreover assume that there exists $k \in \mathbb{N}$ such that for all sufficiently large $n \in \mathbb{N}$ the following conditions*

hold: (i) $D_n \leq kn$; (ii) $e^{n/k} \leq C_n \leq e^{kn}$; (iii) $\rho_n \geq e^{-nk}$; (iv) $\rho_n C_n L_n / L_{n-1} \geq n^{-k}$. Suppose that

$$s = \sup \left\{ t > 0 : \lim_{n \rightarrow \infty} \frac{L_n}{W_n} P_n \cdot W_n^t = \infty \right\} = \liminf_{n \rightarrow \infty} \frac{\log(L_n P_n)}{-\log W_n} + 1$$

is strictly bigger than 1, then $\dim_H \mathcal{F}(\mathcal{T}, \beta) \geq s$.

The conclusion of this corollary implies that the Hausdorff dimension of $\mathcal{F}(\mathcal{T}, \beta)$ is equal to the lower Minkowski dimension. Before the proof we explain the additional assumptions of the corollary. The assumptions (ii) and (iii) are standard and they are satisfied by the usual fractal structures of Cantor sets. The assumption (i) is a regularity condition which means that D_n grows at most linearly in n . If we fix $\tau \in \mathcal{T}_{n-1}$ and enlarge all the $\beta(\kappa)$ for $\kappa \in \mathcal{T}(\tau)$ to rectangles with the same centers and size $(W_n + \frac{1}{4}W_{n-1}\rho_n) \times (L_n + \frac{1}{4}W_{n-1}\rho_n)$, then they are mutually disjoint. Therefore, by assumption (2) in Theorem 2.1

$$C_n \leq \#\mathcal{T}(\tau) \leq \frac{W_{n-1}L_{n-1}}{(W_n + \frac{1}{4}W_{n-1}\rho_n)(L_n + \frac{1}{4}W_{n-1}\rho_n)}.$$

The assumption (iv) is satisfied if $L_n \gg \rho_n W_{n-1} \gg W_n$ and the above inequality is almost an equality up to sub-exponential factors. So (iv) means that the separation of $\beta(\kappa)$ ($\kappa \in \mathcal{T}(\tau)$) is almost optimal (see the remark after Lemma 4.6 for more explanations).

Proof of Corollary 2.3. It follows from the assumptions (i)-(iv) that given $t > 0$ and $\varepsilon > 0$ one has

$$(2.1) \quad \min \left\{ (\rho_{D_{n+1}} C_{D_{n+1}})^{-1}, \rho_{n+1}^t \prod_{i=n+1}^{D_n} \frac{\rho_i C_i L_i}{L_{i-1}} \right\} \geq P_n^{-\varepsilon}$$

for all n sufficiently large (depending on t and ε). We fix a real number t with $1 < t < s$. Let $\varepsilon > 0$ be sufficiently small so that $1 < \frac{t}{1-3\varepsilon} < s$. It follows from the definition of s that

$$(2.2) \quad \frac{L_n}{W_n} P_n \cdot W_n^{\frac{t}{1-3\varepsilon}} \geq 1$$

for n sufficiently large. Let $n_0 \in \mathbb{N}$ such that (2.1) and (2.2) hold for $n \geq n_0$. Then for all $n \geq n_0$ we have

$$\begin{aligned}
P_n W_n^t \rho_{n+1}^t \prod_{i=n+1}^{D_n} \rho_i C_i &\geq P_n^{1-\varepsilon} W_n^t \rho_{n+1}^t \prod_{i=n+1}^{D_{n+1}} \rho_i C_i && \text{by (2.1)} \\
&= P_n^{1-\varepsilon} W_n^t \cdot \rho_{n+1}^t \frac{L_n}{L_{D_{n+1}}} \cdot \prod_{i=n+1}^{D_{n+1}} \frac{\rho_i C_i L_i}{L_{i-1}} \\
&\geq P_n^{1-\varepsilon} W_n^t \cdot \frac{L_n}{W_n} \cdot \rho_{n+1}^t \prod_{i=n+1}^{D_{n+1}} \frac{\rho_i C_i L_i}{L_{i-1}} \\
&\geq P_n^{1-\varepsilon} W_n^t \cdot \frac{L_n}{W_n} \cdot P_n^{-\varepsilon} && \text{by (2.1)} \\
&\geq P_n^{1-\varepsilon} W_n^t \cdot \left(\frac{L_n}{W_n} \right)^{1-3\varepsilon} \cdot P_n^{-2\varepsilon} \cdot P_n^\varepsilon \\
&= \left[P_n W_n^{\frac{t}{1-3\varepsilon}} \frac{L_n}{W_n} \right]^{1-3\varepsilon} P_n^\varepsilon \\
&\geq P_n^\varepsilon && \text{by (2.2)}.
\end{aligned}$$

Therefore, the assumptions (i) (ii) and Theorem 2.1 imply

$$\dim_H \mathcal{F}(\mathcal{T}, \beta) \geq t.$$

The conclusion follows by considering an arbitrary real number t with $1 < t < s$. \square

The assumptions (ii)-(iv) in Corollary 2.3 are local, that is, they only depend on the data from height $n-1$ to n . The Hausdorff dimension of $\mathcal{F}(\mathcal{T}, \beta)$ can also be estimated via local data under an additional assumption. We state this observation as the following corollary.

Corollary 2.4. *Let the notation be as in Theorem 2.1. We moreover assume that there exist $k, n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ (ii)-(iv) in Corollary 2.3 hold, $L_{kn}/L_{kn-1} \leq W_n/W_{n-1}$ and $L_{kn_0-1} < W_{n_0-1}$. If*

$$\lim_{n \rightarrow \infty} \frac{\log(L_n C_n / L_{n-1})}{-\log(W_n / W_{n-1})}$$

exists and is equal to $r > 0$, then $\dim_H \mathcal{F}(\mathcal{T}, \beta) \geq 1 + r$.

Proof. Since the sequence $\{L_n\}$ is monotonically decreasing, for $n \geq n_0$ we have

$$\begin{aligned}
L_{kn} &= L_{kn_0-1} \prod_{i=kn_0}^{kn} \frac{L_i}{L_{i-1}} \leq L_{kn_0-1} \prod_{i=n_0}^n \frac{L_{ki}}{L_{ki-1}} \\
&\leq L_{kn_0-1} \prod_{i=n_0}^n \frac{W_i}{W_{i-1}} = W_n \frac{L_{kn_0-1}}{W_{n_0-1}} < W_n.
\end{aligned}$$

So the assumption (i) of Corollary 2.3 holds and we conclude that the Hausdorff dimension of $\mathcal{F}(\mathcal{T}, \beta)$ is bounded from below by $1 + r$. \square

In the rest of this section we keep the notation and assumptions of Theorem 2.1 and give a proof of it. We first develop some tools for the proof of Theorem 2.1. According to the assumptions there is a one-to-one correspondence between $\mathcal{F}(\mathcal{T}, \beta)$ and $\partial\mathcal{T}$. For each $x \in \mathcal{F}(\mathcal{T}, \beta)$ we let $\{\tau_n(x)\}_{n \in \mathbb{N}} \in \partial\mathcal{T}$ such that $\bigcap_{n \in \mathbb{N}} \beta(\tau_n(x)) = \{x\}$. We take $\tau_0(x)$ to be the root of \mathcal{T} for all x . Let μ be the measure on $\mathcal{F}(\mathcal{T}, \beta)$ with the property that for all $y \in \mathcal{F}(\mathcal{T}, \beta)$ and $n \in \mathbb{N}$

$$\frac{\mu(\{x \in \mathcal{F}(\mathcal{T}, \beta) : \tau_n(x) = \tau_n(y)\})}{\mu(\{x \in \mathcal{F}(\mathcal{T}, \beta) : \tau_{n-1}(x) = \tau_{n-1}(y)\})} = \frac{1}{\#\mathcal{T}(\tau_{n-1}(y))} \leq \frac{1}{C_n}.$$

For any $\tau \in \mathcal{T}$, we define *elementary squares* of $\beta(\tau)$ to be the closed squares contained in $\beta(\tau)$ with side-length $W(\tau)$. In the following two lemmas we estimate the measure of an elementary square.

Lemma 2.5. *Suppose $n \in \mathbb{N} \cup \{0\}$ and $D_n > n$. Let $\kappa \in \mathcal{T}_n$ and $\tau \in \mathcal{T}_{i-1}$ where $n+1 \leq i \leq D_n$. Then for any elementary square S of $\beta(\kappa)$ one has*

$$\#\{\tau' \in \mathcal{T}(\tau) : \beta(\tau') \cap S \neq \emptyset\} \leq 72\rho_i^{-1}.$$

Proof. Let $R_0 = \beta(\tau) \cap S$ and

$$\mathcal{S} = \{\beta(\tau') \cap S : \tau' \in \mathcal{T}(\tau), \beta(\tau') \cap S \neq \emptyset\}.$$

Without loss of generality we assume R_0 is nonempty. Then R_0 is a rectangle with size $l_1 \times l_2$ where $l_1 = \min\{W_n, W_{i-1}\} = W_{i-1}$ and $l_2 \leq \min\{W_n, L_{i-1}\} = W_n$. Each $R \in \mathcal{S}$ has size $W_i \times l(R)$ where $l(R) \leq W_n$ and the distance of two different elements of \mathcal{S} is at least $W_{i-1}\rho_i$. For every $R \in \mathcal{S}$ let R' be the rectangle with the same center and size $(W_i + \frac{\rho_i}{4}W_{i-1}) \times W_n$. Similarly, let R'_0 be the rectangle with the same center as R_0 and size $3W_{i-1} \times 3W_n$. Each point of R'_0 is covered by at most two rectangles of $\{R' : R \in \mathcal{S}\}$ (here we use $L_i \geq W_n$) and every R' is contained in R'_0 . Therefore

$$(W_i + \frac{\rho_i}{4}W_{i-1})W_n \cdot \#\mathcal{S} \leq 18 W_n W_{i-1},$$

which implies

$$\#\mathcal{S} \leq 72 \frac{W_n}{W_n} \cdot \frac{W_{i-1}}{W_{i-1}\rho_i} = 72\rho_i^{-1}.$$

□

Lemma 2.6. *Let $n \in \mathbb{N} \cup \{0\}$ and $\kappa \in \mathcal{T}_n$. Then for any elementary square S of $\beta(\kappa)$ one has*

$$(2.3) \quad \mu(S) \leq 72^{D_n-n} P_n^{-1} \prod_{i=n+1}^{D_n} \rho_i^{-1} C_i^{-1}.$$

Proof. It is easy to see that (2.3) holds if $D_n = n$. In the rest of the proof we assume $D_n > n$. Applying Lemma 2.5 for $i = n+1, n+2, \dots, D_n$, we get

$$(2.4) \quad \#\{\tau \in \mathcal{T}_{D_n} : \beta(\tau) \cap S \neq \emptyset\} \leq 72^{D_n-n} \prod_{i=n+1}^{D_n} \rho_i^{-1}.$$

We can cover $S \cap \mathcal{F}(\mathcal{T}, \beta)$ with the rectangles

$$\{\beta(\tau) : \tau \in \mathcal{T}_{D_n}, \beta(\tau) \cap S \neq \emptyset\}.$$

Therefore

$$\begin{aligned} \mu(S) &\leq \sum_{\substack{\tau \in \mathcal{T}_{D_n} \\ \beta(\tau) \cap S \neq \emptyset}} \mu(\beta(\tau)) \\ &\leq \mu(\beta(\kappa)) \prod_{i=n+1}^{D_n} \frac{1}{C_i} \cdot \#\{\tau \in \mathcal{T}_{D_n} : \beta(\tau) \cap S \neq \emptyset\}. \end{aligned}$$

The above inequality, (2.4) and the fact $\mu(\beta(\kappa)) \leq P_n^{-1}$ imply (2.3). \square

Let U be an open subset of \mathbb{R}^2 with $U \cap \mathcal{F}(\mathcal{T}, \beta) \neq \emptyset$. If $U \cap \mathcal{F}(\mathcal{T}, \beta)$ contains at least two points we let $n(U)$ be the largest index $n \geq 0$ such that $U \cap \mathcal{F}(\mathcal{T}, \beta) \subset \beta(\tau)$ for some $\tau \in \mathcal{T}_n$. If $U \cap \mathcal{F}(\mathcal{T}, \beta)$ contains a single point we let $n(U)$ be the largest index $n \geq 0$ such that $\text{diam}(U) \geq \rho_{n+1}W_n$. Then in $\mathcal{T}_{n(U)}$, there is a unique element denoted by $\kappa(U)$, such that $U \cap \mathcal{F}(\mathcal{T}, \beta) \subset \beta(\kappa(U))$.

Lemma 2.7. *Let U be an open subset of \mathbb{R}^2 with $U \cap \mathcal{F}(\mathcal{T}, \beta) \neq \emptyset$. Let $n = n(U)$ and $\kappa = \kappa(U)$. There is a family \mathcal{S} of elementary squares of $\beta(\kappa)$ such that*

$$(2.5) \quad \bigcup_{S \in \mathcal{S}} S \supset U \cap \mathcal{F}(\mathcal{T}, \beta) \quad \text{and} \quad W_n^t \cdot \#\mathcal{S} \leq 2\rho_{n+1}^{-t} \text{diam}(U)^t$$

for all $t \geq 1$.

Proof. We claim that $\text{diam}(U) \geq \rho_{n+1}W_n$. In fact, if $U \cap \mathcal{F}(\mathcal{T}, \beta)$ contains a single point then the claim follows directly from the definition of $n(U)$. Otherwise, there are at least two elements $\tau \neq \tau' \in \mathcal{T}(\kappa)$ for which $\beta(\tau), \beta(\tau')$ intersect U , and hence $\text{diam}(U) \geq \rho_{n+1}W_n$ by the assumption (3) of Theorem 2.1.

If $\text{diam}(U) \leq W_n$, then there is an elementary square S of $\beta(\kappa)$ such that $U \cap \mathcal{F}(\mathcal{T}, \beta) \subset S$ and we may take $\mathcal{S} = \{S\}$. Then

$$W_n^t \cdot \#\mathcal{S} \leq \rho_{n+1}^{-t} \text{diam}(U)^t.$$

If $\text{diam}(U) > W_n$, then there is a cover of $U \cap \mathcal{F}(\mathcal{T}, \beta)$ by $\left\lceil \frac{\text{diam}(U)}{W_n} \right\rceil$ elementary squares. In this case,

$$\begin{aligned} W_n^t \cdot \#\mathcal{S} &= \left\lceil \frac{\text{diam}(U)}{W_n} \right\rceil W_n^t \leq 2 \left(\frac{\text{diam}(U)}{W_n} \right) W_n^t \\ &\leq 2 \left(\frac{\text{diam}(U)}{W_n} \right)^t W_n^t \leq 2\rho_{n+1}^{-t} \text{diam}(U)^t, \end{aligned}$$

where in the last two inequalities we use the assumptions $t \geq 1$ and $\rho_{n+1} \leq 1$ in (3) of Theorem 2.1. \square

Proof of Theorem 2.1. Let t be any real number such that $1 \leq t < s$. By the definition of s there exists $n_0 = n_0(t)$ such that for all $n \geq n_0$ one has

$$(2.6) \quad P_n W_n^t \rho_{n+1}^t \cdot \prod_{i=n+1}^{D_n} \rho_i C_i \geq 72^{\max\{D_n - n, 1\}} \geq 72^{D_n - n}.$$

Suppose \mathcal{U} is an open cover of $\mathcal{F}(\mathcal{T}, \beta)$. We assume that the diameters of all elements in \mathcal{U} are small enough so that $n(U) > n_0$ for all $U \in \mathcal{U}$. Since $\mathcal{F}(\mathcal{T}, \beta)$ is compact, there is a finite subcover \mathcal{U}_0 such that each element of \mathcal{U}_0 has a nonempty intersection with $\mathcal{F}(\mathcal{T}, \beta)$.

Using Lemma 2.7, for every $U \in \mathcal{U}_0$ there is a set \mathcal{S}_U of elementary squares of $\beta(\kappa(U))$ such that (2.5) holds for $\mathcal{S} = \mathcal{S}_U$. Let $\mathcal{Q} = \bigcup_{U \in \mathcal{U}_0} \mathcal{S}_U$ and $n(S) = n(U)$ for $S \in \mathcal{S}_U$. We note here that although it is possible that the same S belongs to different \mathcal{S}_U , the number $n(S)$ is well-defined. Then \mathcal{Q} is a covering of $\mathcal{F}(\mathcal{T}, \beta)$ and

$$\begin{aligned} \sum_{U \in \mathcal{U}} \text{diam}(U)^t &\geq \frac{1}{2} \sum_{S \in \mathcal{Q}} \rho_{n(S)+1}^t W_{n(S)}^t && \text{by (2.5)} \\ &\geq \frac{1}{2} \sum_{S \in \mathcal{Q}} 72^{D_{n(S)} - n(S)} P_{n(S)}^{-1} \prod_{i=n(S)+1}^{D_{n(S)}} \rho_i^{-1} C_i^{-1} && \text{by (2.6)} \\ &\geq \frac{1}{2} \sum_{S \in \mathcal{Q}} \mu(S) && \text{by (2.3)} \\ &\geq \frac{1}{2}. \end{aligned}$$

Therefore, $\dim_H(\mathcal{F}(\mathcal{T}, \beta)) \geq t$. By considering an arbitrary t with $1 \leq t < s$ we have $\dim_H(\mathcal{F}(\mathcal{T}, \beta)) \geq s$. \square

2.3. Fractal relation and upper bound. Let Q be a countable set. We call a subset σ of $Q^2 = Q \times Q$ a relation on Q . For each $\tau \in Q$ we let $\sigma(\tau) = \{\kappa \in Q : (\tau, \kappa) \in \sigma\}$. We write $\kappa < \tau$ if either $\kappa = \tau$ or there exist $\tau_1, \dots, \tau_n \in Q$ such that $\tau_1 = \tau$, $\tau_n = \kappa$ and $(\tau_i, \tau_{i+1}) \in \sigma$ for all $1 \leq i < n$. The boundary of σ is defined as

$$\partial\sigma = \{\{\tau_i\}_{i \in \mathbb{N}} : (\tau_i, \tau_{i+1}) \in \sigma\}.$$

A triple (Q, σ, β) is said to be a fractal relation on a Polish space Y if β is a map from Q to nonempty compact subsets of Y and σ is a relation on Q . Moreover, we say (Q, σ, β) is admissible if $\text{diam} \beta(\kappa) < \text{diam} \beta(\tau)$ for any $(\tau, \kappa) \in \sigma$ and $\text{diam} \beta(\tau_i) \rightarrow 0$ as $i \rightarrow \infty$ for any sequence $\{\tau_i\}_{i \in \mathbb{N}} \in \partial\sigma$. A fractal relation (Q, σ, β) gives a fractal set

$$\mathcal{F}(Q, \sigma, \beta) := \{y \in Y : \{y\} = \bigcap_{i \in \mathbb{N}} \beta(\tau_i) \text{ for some } \{\tau_i\}_{i \in \mathbb{N}} \in \partial\sigma\}.$$

The following lemma is a self-affine version of [6, Theorem 3.1].

Lemma 2.8. *Let (Q, σ, β) be an admissible fractal relation on \mathbb{R}^2 such that for every $\tau \in Q$ the compact set $\beta(\tau)$ is a rectangle with size $W(\tau) \times L(\tau)$ where $W(\tau) \leq L(\tau)$. Suppose s is a positive real number with*

$$(2.7) \quad \sum_{\kappa \in \sigma(\tau)} L(\kappa) \cdot W(\kappa)^{s-1} \leq L(\tau) \cdot W(\tau)^{s-1}$$

for all $\tau \in Q$, then $\dim_H \mathcal{F}(Q, \sigma, \beta) \leq s$.

Proof. For $\tau_0 \in Q$ let $\mathcal{F}(\tau_0) = \{\bigcap_{i \in \mathbb{N}} \beta(\tau_i) : \{\tau_i\}_{i \in \mathbb{N}} \in \partial\sigma, \tau_1 = \tau_0\}$. Since $\mathcal{F}(Q, \sigma, \beta)$ is a countable union of $\mathcal{F}(\tau_0)$ ($\tau_0 \in Q$), it suffices to show that $\dim_H \mathcal{F}(\tau_0) \leq s$ for all $\tau_0 \in Q$.

We fix τ_0 and assume that $\mathcal{F}(\tau_0) \neq \emptyset$. For $0 < \varepsilon < \text{diam} \beta(\tau_0)$ we will find an ε -covering \mathcal{U} of $\mathcal{F}(\tau_0)$ such that $\sum_{U \in \mathcal{U}} \text{diam}(U)^s$ is bounded from above by a finite number independent of ε . This will imply $\dim_H \mathcal{F}(\tau_0) \leq s$.

Let

$$\mathcal{S} = \{\tau \in Q : \text{diam} \beta(\tau) \leq \varepsilon \text{ and } \text{diam} \beta(\kappa) > \varepsilon \text{ for some } \kappa < \tau_0 \text{ with } (\kappa, \tau) \in \sigma\}.$$

Each rectangle $\beta(\tau)$ is covered by $r(\tau) := \lceil L(\tau)/W(\tau) \rceil$ elementary squares with sides $W(\tau)$. Denote these pieces by $\{S_{\tau,i} : 1 \leq i \leq r(\tau)\}$. Since (Q, σ, β) is admissible and $\varepsilon < \text{diam } \beta(\tau_0)$, the set $\mathcal{F}(\tau_0)$ is covered by

$$\bigcup_{\tau \in \mathcal{S}} \beta(\tau) = \bigcup_{\tau \in \mathcal{S}} \bigcup_{1 \leq i \leq r(\tau)} S_{\tau,i},$$

which is an ε -covering. So it suffices to show

$$(2.8) \quad \sum_{\tau \in \mathcal{S}} \sum_{1 \leq i \leq r(\tau)} \text{diam}(S_{\tau,i})^s \leq 2^{s+1} L(\tau_0) W(\tau_0)^{s-1}.$$

We first note that for each $\tau \in \mathcal{T}$

$$\sum_{1 \leq i \leq r(\tau)} \text{diam}(S_{\tau,i})^s = r(\tau) \cdot (\sqrt{2} \cdot W(\tau))^s \leq 2^{s+1} \cdot \frac{L(\tau)}{W(\tau)} \cdot W(\tau)^s.$$

Hence,

$$(2.9) \quad \sum_{\tau \in \mathcal{S}} \sum_{1 \leq i \leq r(\tau)} \text{diam}(S_{\tau,i})^s \leq 2^{s+1} \sum_{\tau \in \mathcal{S}} L(\tau) \cdot W(\tau)^{s-1}.$$

We claim that

$$\sum_{\tau \in \mathcal{S}} L(\tau) \cdot W(\tau)^{s-1} \leq L(\tau_0) \cdot W(\tau_0)^{s-1}.$$

Suppose the contrary, then there exists a finite set $\mathcal{S}' = \{\tau_i : 1 \leq i \leq k\} \subset \mathcal{S}$ such that

$$(2.10) \quad \sum_{\tau \in \mathcal{S}'} L(\tau) \cdot W(\tau)^{s-1} > L(\tau_0) \cdot W(\tau_0)^{s-1}.$$

According to the definition of \mathcal{S} , for each $\tau_i \in \mathcal{S}'$, there exists a finite sequence $\tau_{i,j}$ ($0 \leq j \leq n_i$) such that $(\tau_{i,j-1}, \tau_{i,j}) \in \sigma$ ($1 \leq j \leq n_i$), $\tau_{i,0} = \tau_0$, $\tau_{i,n_i} = \tau_i$ and $\text{diam}(\beta(\tau_{i,n_i-1})) > \varepsilon$.

For each $0 \leq j \leq n := \max_{1 \leq i \leq k} n_i$ let

$$\mathcal{S}_j = \{\tau_{i,j} : 1 \leq i \leq k, n_i \geq j\} \dot{\cup} \{\tau_i : 1 \leq i \leq k, n_i < j\},$$

where $\dot{\cup}$ denotes the disjoint union. Note that for $1 \leq j \leq n$

$$\mathcal{S}_{j-1} = \{\tau_{i,j-1} : 1 \leq i \leq k, n_i \geq j\} \dot{\cup} \{\tau_i : 1 \leq i \leq k, n_i < j\}.$$

The union of $\sigma(\tau)$ for τ runs over $\{\tau_{i,j-1} : 1 \leq i \leq k, n_i \geq j\}$ contains $\{\tau_{i,j} : 1 \leq i \leq k, n_i \geq j\}$. Therefore, (2.7) implies

$$(2.11) \quad \sum_{\tau \in \mathcal{S}_j} L(\tau) \cdot W(\tau)^{s-1} \leq \sum_{\tau \in \mathcal{S}_{j-1}} L(\tau) \cdot W(\tau)^{s-1} \quad (1 \leq j \leq n).$$

Observing $\mathcal{S}_0 = \{\tau_0\}$ and $\mathcal{S}_n = \mathcal{S}'$, we deduce from (2.11) a contradiction to (2.10). The claim then follows. The claim together with (2.9) imply (2.8), which completes the proof. \square

3. COUNTING LATTICE POINTS IN CONVEX SETS

The aim of this section is to develop some tools for counting lattice points in a convex subset of the Euclidean space $\mathcal{E}_d = \mathbb{R}^d$. Although we only need these results in the case where $d \leq 3$, we give some results in general Euclidean space in §3.1. The reason for this is that the proofs are the same and they might be useful in other contexts. Results in §3.2 will only be used in our estimation of the lower bound. Since this section contains technical results that we will use later, the reader can skip this section in the first reading and come back when needed.

3.1. Lattice points counting in \mathbb{R}^d . Let K be a bounded centrally symmetric convex subset of \mathbb{R}^d with nonempty interior and let Λ be a lattice of \mathbb{R}^d . We use $\lambda_i(K, \Lambda)$ ($i = 1, 2, \dots, d+1$) to denote the i -th minimum of Λ with respect to K , i.e. the infimum of those numbers λ such that $\lambda K \cap \Lambda$ contains i linearly independent vectors. We remark here that $\lambda_{d+1}(K, \Lambda) = \infty$. Let $\text{vol}(\cdot)$ be the Lebesgue measure on \mathbb{R}^d . The covolume of Λ , denoted by $\text{cov}(\Lambda)$, is the Lebesgue measure of a fundamental domain of Λ . Write

$$\theta(K, \Lambda) := \frac{\text{vol}(K)}{\text{cov}(\Lambda)}.$$

By Minkowski's (second) theorem (see [5]) one has

$$(3.1) \quad \frac{2^d}{d!} \leq \lambda_1(K, \Lambda) \cdots \lambda_d(K, \Lambda) \cdot \theta(K, \Lambda) \leq 2^d.$$

For an affine subspace H of \mathbb{R}^d we let $\text{vol}_H(\cdot)$ be the Lebesgue measure on H with respect to the subspace Riemannian structure. To simplify the notation we let $\text{vol}_H(S) = \text{vol}_H(S \cap H)$ for a Borel measurable subset S of \mathbb{R}^d . A subspace H of \mathbb{R}^d is said to be Λ -rational if $H \cap \Lambda$ is a lattice of H . The covolume of the lattice $H \cap \Lambda$ in H is denoted by $\text{cov}_H(\Lambda)$. The same notations are used for the dual vector space \mathcal{E}_d^* (the vector space of linear functionals on \mathbb{R}^d) with respect to the standard Euclidean structure. For every $\varphi \in \mathcal{E}_d^*$, denote $H_\varphi = \ker \varphi$.

We use $\|\cdot\|$ for the Euclidean norms on \mathbb{R}^d and \mathcal{E}_d^* . For a normed vector space V we use $B_r(V)$ (or B_r if $V = \mathbb{R}^d$) to denote the ball of radius r centered at $0 \in V$. We will also use K -norms on \mathbb{R}^d and \mathcal{E}_d^* defined by

$$(3.2) \quad \begin{cases} \|\mathbf{v}\|_K = \inf\{r > 0 : \mathbf{v} \in rK\} & \mathbf{v} \in \mathbb{R}^d \\ \|\varphi\|_K = \sup_{\mathbf{v} \in K} |\varphi(\mathbf{v})| & \varphi \in \mathcal{E}_d^* \end{cases}.$$

It can be checked that K -norms satisfy the triangle inequality and other axioms of norm on a real vector space.

Let \mathcal{L}_d be the space of unimodular lattices in \mathbb{R}^d . The group $SL_d(\mathbb{R})$ acts transitively on \mathcal{L}_d via $g\Lambda = \{g\mathbf{v} : \mathbf{v} \in \Lambda\}$. The stabilizer of \mathbb{Z}^d is $SL_d(\mathbb{Z})$, so we can identify \mathcal{L}_d with $SL_d(\mathbb{R})/SL_d(\mathbb{Z})$ as topological spaces. For $g \in SL_d(\mathbb{R})$ we let g^* be the adjoint action on \mathcal{E}_d^* defined by $\varphi \rightarrow \varphi \circ g$. Note that with respect to the standard basis $\mathbf{e}_1, \dots, \mathbf{e}_d$ on \mathbb{R}^d and its dual basis $\mathbf{e}_1^*, \dots, \mathbf{e}_d^*$ on \mathcal{E}_d^* , the matrix g^* is the transpose of g . We define

$$\mathcal{K}_\varepsilon(d) = \{\Lambda \in \mathcal{L}_d : \|\mathbf{v}\| \geq \varepsilon, \forall \mathbf{v} \in \Lambda \setminus \{0\}\} = \{\Lambda \in \mathcal{L}_d : \lambda_1(B_1, \Lambda) \geq \varepsilon\}.$$

The dual lattice of Λ is the lattice in \mathcal{E}_d^* defined by

$$\Lambda^* = \{\varphi \in \mathcal{E}_d^* : \varphi(\mathbf{v}) \in \mathbb{Z}, \forall \mathbf{v} \in \Lambda\}.$$

We also define

$$(3.3) \quad \mathcal{K}_\varepsilon^*(d) = \{\Lambda \in \mathcal{L}_d : \|\varphi\| \geq \varepsilon, \forall \varphi \in \Lambda^* \setminus \{0\}\}.$$

Recall that \mathcal{E}_d^* can be naturally identified with $\wedge_{\mathbb{R}}^{d-1} \mathbb{R}^d$ with the standard Euclidean structure. Under this identification one has $\Lambda^* = \wedge_{\mathbb{Z}}^{d-1} \Lambda$. Therefore, $\mathcal{K}_\varepsilon^*$ is the set of $\Lambda \in \mathcal{L}_d$ with the property that each Λ -rational hyperplane intersects Λ in a lattice of covolume greater than or equal to ε . Using the natural identification $\mathcal{E}_d^{**} = \mathbb{R}^d$ and (3.1) one has

$$\Lambda \in \mathcal{K}_\varepsilon^*(d) \implies \lambda_1(B_1, \Lambda) \gg_d \varepsilon^{d-1} \quad \text{and} \quad \lambda_d(B_1, \Lambda) \ll_d \varepsilon^{-1}.$$

There is a basis $\mathbf{v}_1, \dots, \mathbf{v}_d$ of Λ with the properties

$$\lambda_i(B_1, \Lambda) \leq \|\mathbf{v}_i\| \leq 2^i \lambda_i(B_1, \Lambda) \quad (1 \leq i \leq d),$$

see [13, Lemma X.6.2]. This basis is called a Minkowski reduced basis of Λ .

A nonzero vector $\mathbf{v} \in \Lambda$ is said to be primitive if $\frac{1}{n}\mathbf{v} \notin \Lambda$ for all $n \in \mathbb{N}$. The set of primitive vectors in Λ is denoted by $\hat{\Lambda}$.

Lemma 3.1. *Let $d \geq 2$. For every lattice Λ of \mathbb{R}^d and every bounded centrally symmetric convex subset K of \mathbb{R}^d with $\lambda_d(K, \Lambda) \leq 1$ we have*

$$\#K \cap \hat{\Lambda} = (\zeta(d)^{-1} + \eta(K, \Lambda)) \cdot \theta(K, \Lambda)$$

where ζ is the Riemann ζ -function and

$$|\eta(K, \Lambda)| \ll_d \lambda_d(K, \Lambda) - \lambda_d(K, \Lambda) \log \lambda_1(K, \Lambda).$$

Note that $\lambda_d(K, \Lambda) \leq 1$ implies that the interior of K (denoted by K°) is nonempty. To prove Lemma 3.1 we need a few preparations (Lemmas 3.2-3.5).

Lemma 3.2. *Let $d \geq 1$. For every lattice Λ of \mathbb{R}^d and every bounded centrally symmetric convex subset K of \mathbb{R}^d with $\lambda_d(K, \Lambda) \leq 1$ one has*

$$\#K \cap (\Lambda \setminus \{0\}) = (1 + \alpha(K, \Lambda)) \cdot \theta(K, \Lambda),$$

where $|\alpha(K, \Lambda)| \ll_d \lambda_d(K, \Lambda)$.

Proof. It follows from the definition of successive minima that there is a set of linearly independent vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_d\} \subset \Lambda$ such that $\|\mathbf{v}_i\|_K = \lambda_i(K, \Lambda)$. So there is a fundamental domain Ω of Λ contained in

$$\{s_1 \mathbf{v}_1 + \dots + s_d \mathbf{v}_d : |s_i| \leq 1/2\} \subset d\lambda_d(K, \Lambda)K.$$

It follows that

$$(3.4) \quad \lambda_d(K, \Lambda)^d \theta(K, \Lambda) \gg_d 1.$$

Since $(\#K \cap \Lambda) \cdot \text{cov}(\Lambda) = \text{vol}(K \cap \Lambda + \Omega)$ and $K \cap \Lambda + \Omega \subset (1 + d\lambda_d(K, \Lambda))K$, one has

$$(3.5) \quad (\#K \cap \Lambda) \cdot \text{cov}(\Lambda) \leq \text{vol}((1 + d\lambda_d(K, \Lambda))K).$$

If $\lambda_d(K, \Lambda) \geq 1/d$, then the conclusion of the lemma follows from (3.5). In the case where $\lambda_d(K, \Lambda) < 1/d$ one has $(1 - d\lambda_d(K, \Lambda))K \subset K \cap \Lambda + \Omega$ which implies

$$(3.6) \quad \text{vol}((1 - d\lambda_d(K, \Lambda))K) \leq (\#K \cap \Lambda) \cdot \text{cov}(\Lambda).$$

In view of (3.4) (which takes care of $0 \in \Lambda$), (3.5) and (3.6) the conclusion of the lemma also holds in this case. \square

Lemma 3.3. *Let K and Λ be as in Lemma 3.2. Then*

$$(3.7) \quad \#K \cap \Lambda \asymp_d \theta(K, \Lambda).$$

Proof. It is clear from (3.5) that $\#K \cap \Lambda \ll_d \theta(K, \Lambda)$. On the other hand, it follows from [5, Chapter 3 Theorem II] that

$$\#K \cap \Lambda \geq 2^{-d} \theta(K, \Lambda).$$

□

Lemma 3.4. *Let Λ be a lattice of \mathbb{R}^d ($d \geq 1$) and K be a bounded centrally symmetric convex subset of \mathbb{R}^d with nonempty interior. Then*

$$(3.8) \quad \#K^\circ \cap \Lambda \asymp_d \#K \cap \Lambda \asymp_d \#\overline{K} \cap \Lambda.$$

Proof. Let H be the linear span of $\overline{K} \cap \Lambda$. Suppose H is an i -dimensional real vector space and we assume without loss of generality that $i > 0$. Since an open neighborhood of 0 is contained in K and K is convex, the closure of $K^\circ \cap H$ is equal to $\overline{K} \cap H$. It follows from (3.7) that

$$\#\overline{K} \cap \Lambda \asymp_i \frac{\text{vol}_H(\overline{K})}{\text{cov}_H(\Lambda)} \quad \text{and} \quad \#K^\circ \cap \Lambda \asymp_i \frac{\text{vol}_H(K^\circ)}{\text{cov}_H(\Lambda)}.$$

Since every convex subset of the Euclidean space has negligible boundary, one has $\text{vol}_H(\overline{K}) = \text{vol}_H(K^\circ)$. Since there is no difference between a constant depending on d and on $\{1, \dots, d\}$, (3.8) holds. □

Lemma 3.5. *Let K and Λ be as in Lemma 3.2. Suppose $\lambda_i(K, \Lambda) \leq s \leq s' \leq \lambda_{j+1}(K, \Lambda)$ where $1 \leq i \leq j \leq d$, then*

$$(3.9) \quad \left(\frac{s'}{s}\right)^i \ll_d \frac{\#s'K \cap \Lambda}{\#sK \cap \Lambda} \ll_d \left(\frac{s'}{s}\right)^j.$$

Proof. If $\lambda_i(K, \Lambda) \leq s \leq s' \leq \lambda_{i+1}(K, \Lambda)$, then (3.9) follows from Lemmas 3.3 and 3.4. The general case follows from this special case. □

Proof of Lemma 3.1. The set

$$\widehat{\Lambda} = \Lambda \setminus \cup_{p \text{ prime}} p\Lambda = (\Lambda \setminus \{0\}) \setminus (\cup_{p \text{ prime}} p\Lambda \setminus \{0\}).$$

Let μ be the Möbius function and $\lambda_i = \lambda_i(K, \Lambda)$. It follows from the inclusion-exclusion principle that

$$(3.10) \quad \begin{aligned} & \#K \cap (\widehat{\Lambda} \setminus \{0\}) \\ &= \#K \cap (\Lambda \setminus \{0\}) + \sum_{\substack{p_1 < \dots < p_k \\ \text{primes}}} (-1)^k \#K \cap (p_1 \cdots p_k \Lambda \setminus \{0\}) \\ &= \sum_{n=1}^{\infty} \mu(n) \#K \cap (n\Lambda \setminus \{0\}) \\ &= \sum_{n=1}^{\lfloor \lambda_d^{-1} \rfloor} \mu(n) \frac{\theta(K, \Lambda)}{n^d} (1 + \alpha(K, n\Lambda)) + \sum_{n=\lfloor \lambda_d^{-1} \rfloor + 1}^{\lfloor \lambda_1^{-1} \rfloor} \#K \cap (n\Lambda \setminus \{0\}), \end{aligned}$$

where in the last equality we use the notation of Lemma 3.2 and the observation that for any integer $n > \lambda_1^{-1}$ one has $K \cap n\Lambda = \{0\}$.

It follows from Lemmas 3.3 and 3.5 (with $\lambda_1 \leq s = n^{-1} \leq s' = \lambda_d \leq \lambda_{d+1}$) that

$$(3.11) \quad \sharp K \cap n\Lambda = \sharp n^{-1}K \cap \Lambda \ll_d \lambda_d^{d-1} n^{-1} \theta(K, \Lambda)$$

for $n \in [\lambda_d^{-1}, \lambda_1^{-1}]$. Using (3.10), (3.11) and the estimate of α in Lemma 3.2, we can write

$$\sharp \widehat{\Lambda} \cap K = (\zeta(d)^{-1} + \eta(K, \Lambda)) \theta(K, \Lambda),$$

where

$$\begin{aligned} |\eta(K, \Lambda)| &\ll_d \sum_{n=\lfloor \lambda_d^{-1} + 1 \rfloor}^{\infty} \frac{1}{n^d} + \sum_{n=1}^{\lfloor \lambda_d^{-1} \rfloor} \frac{1}{n^d} |\alpha(K, n\Lambda)| + \sum_{n=\lfloor \lambda_d^{-1} + 1 \rfloor}^{\lfloor \lambda_1^{-1} \rfloor} \lambda_d^{d-1} n^{-1} \\ &\ll_d \lambda_d^{d-1} + \sum_{n=1}^{\lfloor \lambda_d^{-1} \rfloor} \frac{\lambda_d}{n^{d-1}} + \lambda_d^{d-1} \log(\lambda_d \lambda_1^{-1}) \\ &\ll \lambda_d - \lambda_d \log \lambda_1. \end{aligned}$$

□

Lemma 3.6. *Let K be a bounded centrally symmetric convex subset of \mathbb{R}^d ($d \geq 2$) with nonempty interior and let $\varphi \in \mathcal{E}_d^* \setminus \{0\}$. Then $\text{vol}_{H_\varphi}(K) \asymp_d \frac{\|\varphi\| \text{vol}(K)}{\|\varphi\|_K}$.*

Proof. Since $\|\varphi\|/\|\varphi\|_K = \|t\varphi\|/\|t\varphi\|_K$ for every nonzero real number t , we assume without loss of generality that $\|\varphi\| = 1$. Using Fubini's theorem one has

$$(3.12) \quad \text{vol}(K) = \int_{\mathbb{R}} \text{vol}_{\varphi^{-1}(t)}(K) dt = 2 \int_0^{\|\varphi\|_K} \text{vol}_{\varphi^{-1}(t)}(K) dt.$$

Since K is centrally symmetric, for each $t \in \mathbb{R}$ and $\mathbf{v} \in K$ with $\varphi(\mathbf{v}) = t$ one has $-\mathbf{v} \in K$ and the line segment joining $-\mathbf{v}$ and $\varphi^{-1}(t)$ is in K . Therefore

$$\text{vol}_{\varphi^{-1}(t)}(K) \leq 2 \text{vol}_{H_\varphi}(K).$$

This estimate and (3.12) imply

$$(3.13) \quad \text{vol}(K) \leq 4 \|\varphi\|_K \text{vol}_{H_\varphi}(K).$$

For any $0 < \varepsilon < \|\varphi\|_K$, there is a vector $\mathbf{v} \in K$ such that $\varphi(\mathbf{v}) = \|\varphi\|_K - \varepsilon$. Since K is convex, for every t with $0 < t < \|\varphi\|_K - \varepsilon$ one has

$$K \cap \varphi^{-1}(t) \supset \frac{\|\varphi\|_K - \varepsilon - t}{\|\varphi\|_K - \varepsilon} (K \cap H_\varphi) + \frac{t}{\|\varphi\|_K - \varepsilon} \mathbf{v},$$

which implies

$$\text{vol}_{\varphi^{-1}(t)}(K) \geq \left(\frac{\|\varphi\|_K - \varepsilon - t}{\|\varphi\|_K - \varepsilon} \right)^{d-1} \text{vol}_{H_\varphi}(K).$$

This estimate and (3.12) imply

$$\begin{aligned} \text{vol}(K) &\geq 2 \int_0^{\|\varphi\|_K - \varepsilon} \text{vol}_{\varphi^{-1}(t)}(K) dt \\ &\geq 2 \text{vol}_{H_\varphi}(K) \int_0^{\|\varphi\|_K - \varepsilon} \left(\frac{\|\varphi\|_K - \varepsilon - t}{\|\varphi\|_K - \varepsilon} \right)^{d-1} dt \\ &= 2 \text{vol}_{H_\varphi}(K) \frac{\|\varphi\|_K - \varepsilon}{d}. \end{aligned}$$

Taking $\varepsilon = \|\varphi\|_K/2$, we have

$$(3.14) \quad \text{vol}(K) \geq \frac{\text{vol}_{H\varphi}(K)\|\varphi\|_K}{d}.$$

The lemma follows from (3.13) and (3.14). \square

3.2. Lattice points counting in \mathbb{R}^3 . In this subsection we estimate the number of the lattice points in

$$(3.15) \quad M_{\mathbf{r}} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_i| \leq r_i\}$$

for a triple of positive real numbers $\mathbf{r} = (r_1, r_2, r_3)$. Lemmas 3.7, 3.9 and 3.10 will be used in §4.2 for lower bound estimation. The two latter results are deduced from Lemma 3.8 where we prove an upper bound of the number of the lattice points in $M_{\mathbf{r}}$ that lie in certain badly shaped hyperplanes. Let

$$M_{\mathbf{r}}^* = \{\varphi \in \mathcal{E}_3^* : |x_i^\varphi| \leq r_i\}.$$

where $x_1^\varphi, x_2^\varphi, x_3^\varphi \in \mathbb{R}$ are the coordinates of $\varphi \in \mathcal{E}_3^*$, that is, $\varphi = x_1^\varphi \mathbf{e}_1^* + x_2^\varphi \mathbf{e}_2^* + x_3^\varphi \mathbf{e}_3^*$.

Lemma 3.7. *There exists a positive real number $\tilde{c} < 1$ such that for every lattice Λ of \mathbb{R}^3 and every triple of positive real numbers \mathbf{r} with*

$$\lambda_3(M_{\mathbf{r}}, \Lambda) \leq \tilde{c} \quad \text{and} \quad -\lambda_3(M_{\mathbf{r}}, \Lambda) \log \lambda_1(M_{\mathbf{r}}, \Lambda) \leq \tilde{c}$$

one has

$$\frac{4}{5\zeta(3)}\theta(M_{\mathbf{r}}, \Lambda) \leq \#M_{\mathbf{r}} \cap \widehat{\Lambda} \leq \frac{6}{5\zeta(3)}\theta(M_{\mathbf{r}}, \Lambda).$$

Proof. This lemma follows directly from Lemma 3.1. \square

Let us fix $0 < s < 1/2$ and $\mathbf{r} \in \mathbb{R}^3$ with $1 \leq r_1 \leq r_2$ and $r_3 = 1$. Let

$$\|\varphi\|_{\mathbf{r}} = \max\{r_1|x_1^\varphi|, r_2|x_2^\varphi|, |x_3^\varphi|\}.$$

It can be checked directly that

$$(3.16) \quad \|\varphi\|_{\mathbf{r}} \leq \|\varphi\|_{M_{\mathbf{r}}} \leq 3\|\varphi\|_{\mathbf{r}}.$$

For $q > 0$ let

$$(3.17) \quad N_q(\mathbf{r}, s) = \{\varphi \in \mathcal{E}_3^* : |x_1^\varphi| \leq s, |x_2^\varphi| \leq s, \|\varphi\|_{\mathbf{r}} \leq q\}.$$

A direct calculation shows that

$$N_q(\mathbf{r}, s) = \begin{cases} M_{\frac{q}{r_1}, \frac{q}{r_2}, q}^* & q \leq sr_1 \\ M_{s, \frac{q}{r_2}, q}^* & sr_1 \leq q \leq sr_2 \\ M_{s, s, q}^* & sr_2 \leq q \end{cases}.$$

For a lattice Λ of \mathbb{R}^3 let $q_i(\Lambda, \mathbf{r}, s)$ ($i = 1, 2, 3$) be the infimum of those positive numbers q such that $N_q(\mathbf{r}, s) \cap \Lambda^*$ contains i linearly independent vectors. We use $\widehat{\Lambda}^*$ to denote the set of primitive vectors of the dual lattice Λ^* . In the next lemma we give an upper bound of the cardinality of

$$(3.18) \quad \mathcal{S}(\Lambda, \mathbf{r}, s) := \{\mathbf{v} \in M_{\mathbf{r}} \cap \widehat{\Lambda} : \varphi(\mathbf{v}) = 0 \text{ for some } \varphi \in N_{3sr_2}(\mathbf{r}, s) \cap \widehat{\Lambda}^*\}.$$

Lemma 3.8. *Let Λ be a unimodular lattice of \mathbb{R}^3 with $q_1(\Lambda, \mathbf{r}, s) \geq s^{-2}$. The following statements hold:*

(i) *if $r_1 = r_2$ and $q_3(\Lambda, \mathbf{r}, s) \leq 2s^{-1/2}r_2$ then $\#\mathcal{S}(\Lambda, \mathbf{r}, s) \ll s^{1/2} \cdot \text{vol}(M_{\mathbf{r}})$;*

(ii) if $r_1 < r_2$ and $q_3(\Lambda, \mathbf{r}, s) \log q_3(\Lambda, \mathbf{r}, s) \leq sr_2$ then $\#\mathcal{S}(\Lambda, \mathbf{r}, s) \ll s^2 \cdot \text{vol}(M_{\mathbf{r}})$.

Proof. We write $N_q = N_q(\mathbf{r}, s)$, $q_i = q_i(\Lambda, \mathbf{r}, s)$ and $\mathcal{S} = \mathcal{S}(\Lambda, \mathbf{r}, s)$ for simplicity. If $N_{3sr_2} \cap \widehat{\Lambda}^*$ is empty, then there is nothing to prove. So we assume in the remaining of the proof that $N_{3sr_2} \cap \widehat{\Lambda}^* \neq \emptyset$, i.e. $q_1 \leq 3sr_2$. It is clear from the definition that

$$(3.19) \quad \#\mathcal{S} \leq \sum_{\varphi \in N_{3sr_2} \cap \widehat{\Lambda}^*} \#H_\varphi \cap M_{\mathbf{r}} \cap \widehat{\Lambda}.$$

We claim that for all $\varphi \in N_{3sr_2} \cap \widehat{\Lambda}^*$

$$(3.20) \quad \#H_\varphi \cap M_{\mathbf{r}} \cap \widehat{\Lambda} \ll \frac{\text{vol}(M_{\mathbf{r}})}{\|\varphi\|_{M_{\mathbf{r}}}} \leq \frac{\text{vol}(M_{\mathbf{r}})}{\|\varphi\|_{\mathbf{r}}},$$

where the second inequality follows from (3.16). If $\#H_\varphi \cap M_{\mathbf{r}} \cap \widehat{\Lambda} > 2$, then (3.7) and Lemma 3.6 imply

$$\#H_\varphi \cap M_{\mathbf{r}} \cap \widehat{\Lambda} \ll \frac{\text{vol}_{H_\varphi}(M_{\mathbf{r}})}{\text{cov}_{H_\varphi}(\Lambda)} \ll \frac{\|\varphi\| \text{vol}(M_{\mathbf{r}})}{\text{cov}_{H_\varphi}(\Lambda) \|\varphi\|_{M_{\mathbf{r}}}} = \frac{\text{vol}(M_{\mathbf{r}})}{\|\varphi\|_{M_{\mathbf{r}}}}.$$

On the other hand for every $\varphi \in N_{3sr_2} \cap \widehat{\Lambda}^*$ we always have

$$\frac{\text{vol}(M_{\mathbf{r}})}{\|\varphi\|_{M_{\mathbf{r}}}} \geq \frac{8r_1r_2}{9sr_2} \gg 2.$$

This completes the proof of the claim.

In view of (3.19) and (3.20) it suffices to estimate

$$(3.21) \quad \eta := \sum_{\varphi \in N_{3sr_2} \cap \widehat{\Lambda}^*} \|\varphi\|_{\mathbf{r}}^{-1} = \frac{1}{3sr_2} \#N_{3sr_2} \cap \widehat{\Lambda}^* + \sum_{\varphi \in N_{3sr_2} \cap \widehat{\Lambda}^*} \int_{\|\varphi\|_{\mathbf{r}}}^{3sr_2} \frac{1}{q^2} dq.$$

The second term of the right hand side of (3.21) is

$$\eta_2 := \sum_{\varphi \in N_{3sr_2} \cap \widehat{\Lambda}^*} \int_{\|\varphi\|_{\mathbf{r}}}^{3sr_2} \frac{1}{q^2} dq = \sum_{\varphi \in N_{3sr_2} \cap \widehat{\Lambda}^*} \int_{q_1}^{3sr_2} \frac{\mathbb{1}_q(\|\varphi\|_{\mathbf{r}})}{q^2} dq,$$

where $\mathbb{1}_q$ is the indicator function on \mathbb{R} defined by $\mathbb{1}_q(x) = 1$ if $x \leq q$ and 0 otherwise. Using Fubini's theorem one has

$$\begin{aligned} \eta_2 &= \int_{q_1}^{3sr_2} \sum_{\varphi \in N_{3sr_2} \cap \widehat{\Lambda}^*} \frac{\mathbb{1}_q(\|\varphi\|_{\mathbf{r}})}{q^2} dq \\ &\leq \int_{q_1}^{3sr_2} \frac{\#N_q \cap \widehat{\Lambda}^*}{q^2} dq. \end{aligned}$$

If $q_1 \leq q < q_2$ then $\#N_q \cap \widehat{\Lambda}^* = 2$. So

$$(3.22) \quad \int_{q_1}^{q_2} \frac{\#N_q \cap \widehat{\Lambda}^*}{q^2} dq \leq \int_{q_1}^{q_2} \frac{2}{q^2} dq \leq \frac{2}{q_1} \leq 2s^2$$

where in the last inequality we use the assumption $q_1 \geq s^{-2}$.

From here we consider different cases according to the two situations in the statement of the lemma.

Case i: we show $\eta \ll \sqrt{s}$ under the assumption of (i). We first compute

$$(3.23) \quad \int_{sr_2}^{3sr_2} \frac{\#N_q \cap \widehat{\Lambda}^*}{q^2} dq \leq \int_{sr_2}^{3sr_2} \frac{\#N_{3sr_2} \cap \Lambda^*}{q^2} dq \leq \frac{\#N_{3sr_2} \cap \Lambda^*}{sr_2}.$$

Note that $q_3 \leq 2s^{-1/2}r_2$ by assumption and $sr_2 < s^{-1/2}r_2$ since $s < 1$. So by Lemma 3.3 (which is used in the second inequality below)

$$(3.24) \quad \frac{\#N_{3sr_2} \cap \Lambda^*}{sr_2} \leq \frac{\#N_{3s^{-1/2}r_2} \cap \Lambda^*}{sr_2} \ll \frac{\text{vol}(N_{3s^{-1/2}r_2})}{sr_2} \ll s^{1/2}.$$

Case i.1: suppose $q_2 > 3sr_2$. It follows from (3.21), (3.22), the assumption $q_1 \geq s^{-2}$ and the observation

$$\frac{1}{3sr_2} \#N_{3sr_2} \cap \widehat{\Lambda}^* \leq \frac{2}{3sr_2} \leq \frac{2}{q_1}$$

that $\eta \ll s^2 \leq \sqrt{s}$.

Case i.2: suppose $sr_2 \leq q_2 \leq 3sr_2$. It follows from (3.21), (3.22), (3.23) and (3.24) that

$$\eta \leq 2s^2 + \frac{2}{sr_2} \#N_{3sr_2} \cap \Lambda^* \ll s^{1/2}.$$

Case i.3: suppose $q_2 < sr_2$. For all $q_2 < q < sr_2 = sr_1$ one has

$$\#N_q \cap \widehat{\Lambda}^* = \# \left(\frac{q}{sr_2} N_{sr_2} \right) \cap \widehat{\Lambda}^* \leq \# \left(\frac{q}{sr_2} N_{sr_2} \right) \cap \Lambda^* \ll \left(\frac{q}{sr_2} \right)^2 \#N_{sr_2} \cap \Lambda^*,$$

where in the last inequality we use Lemma 3.5. Using this estimate, $sr_2 < s^{-1/2}r_2$ and Lemma 3.3 we get

$$\#N_q \cap \widehat{\Lambda}^* \ll \left(\frac{q}{sr_2} \right)^2 \#N_{s^{-1/2}r_2} \cap \Lambda^* \ll \frac{q^2 \text{vol}(N_{s^{-1/2}r_2})}{s^2 r_2^2} \ll \frac{q^2 s^{1/2}}{sr_2}.$$

So

$$(3.25) \quad \int_{q_2}^{sr_2} \frac{\#N_q \cap \widehat{\Lambda}^*}{q^2} dq \ll \int_{q_2}^{sr_2} \frac{s^{1/2} dq}{sr_2} \leq s^{1/2}.$$

It follows from (3.21), (3.22), (3.23), (3.24) and (3.25) that $\eta \ll \sqrt{s}$.

Case ii: we show $\eta \ll s^2$ under the assumption of (ii). Since $s < 1/2$ and $q_1 \geq s^{-2}$, we have $q_3 \geq q_1 \geq 4$. Therefore $q_3 < sr_2$, since $q_3 \log q_3 \leq sr_2$ by the assumption. So by Lemma 3.3

$$(3.26) \quad \frac{\#N_{3sr_2} \cap \widehat{\Lambda}^*}{3sr_2} \ll \frac{\text{vol}(N_{3sr_2})}{3sr_2} = 8s^2.$$

Similarly, for $sr_2 < q < 3sr_2$ one has $\#N_q \cap \widehat{\Lambda}^* \leq \#N_q \cap \Lambda^* \ll \text{vol}(N_q) \leq 8s^2 q$. So

$$(3.27) \quad \int_{sr_2}^{3sr_2} \frac{\#N_q \cap \widehat{\Lambda}^*}{q^2} dq \ll \int_{sr_2}^{3sr_2} \frac{8s^2 dq}{q} = 8s^2 \log 3.$$

Using Lemma 3.3 for $q_3 < q < sr_2$, one has $\#N_q \cap \widehat{\Lambda}^* \ll \text{vol}(N_q) \ll q^2 s / r_2$. It follows that

$$(3.28) \quad \int_{q_3}^{sr_2} \frac{\#N_q \cap \widehat{\Lambda}^*}{q^2} dq \ll \frac{s(sr_2 - q_3)}{r_2} \leq s^2.$$

Now we estimate $\sharp N_q \cap \widehat{\Lambda}^*$ for $q_2 \leq q < q_3 < sr_2$. Let H be the \mathbb{R} -linear span of $N_{q_2} \cap \Lambda^*$. We claim that

$$\text{vol}_H(N_q) \leq \frac{q}{q_3} \text{vol}_H(N_{q_3}).$$

If $H = \text{span}_{\mathbb{R}}\{\mathbf{e}_2^*, \mathbf{e}_3^*\}$, then the claim follows easily from the definition of N_q for $q < sr_2$. Otherwise, the intersection of H with the affine hyperplane $H_t = t\mathbf{e}_1^* + \text{span}_{\mathbb{R}}\{\mathbf{e}_2^*, \mathbf{e}_3^*\}$ is a line. The length of $H \cap H_t \cap N_q$ is at most $\frac{q}{q_3}$ times the length of $H \cap H_t \cap N_{q_3}$. Since the volume of $H \cap N_q$ is proportional to the integration of the length of $H \cap H_t \cap N_q$ with respect to t , the claim follows. It follows from the claim that for $q_2 \leq q < q_3$

$$(3.29) \quad \sharp N_q \cap \widehat{\Lambda}^* \ll \frac{\text{vol}_H(N_q)}{\text{cov}_H(\Lambda^*)} \leq \frac{q}{q_3} \frac{\text{vol}_H(N_{q_3})}{\text{cov}_H(\Lambda^*)}.$$

Note that

$$(3.30) \quad \begin{aligned} \frac{\text{vol}_H(N_{q_3})}{\text{cov}_H(\Lambda^*)} &\ll \sharp N_{q_3} \cap H \cap \Lambda^* && \text{by (3.7)} \\ &\ll \sharp N_{q_3}^\circ \cap H \cap \Lambda^* && \text{by (3.8)} \\ &= \sharp N_{q_3}^\circ \cap \Lambda^* \\ &\leq \sharp N_{q_3} \cap \Lambda^*. \end{aligned}$$

In view of (3.29) and (3.30), for $q_2 \leq q < q_3$ we have

$$\sharp N_q \cap \widehat{\Lambda}^* \ll \frac{q}{q_3} \sharp N_{q_3} \cap \Lambda^* \ll \frac{q}{q_3} \text{vol}(N_{q_3}) \ll \frac{q_3 q s}{r_2}.$$

Therefore

$$(3.31) \quad \int_{q_2}^{q_3} \frac{\sharp N_q \cap \widehat{\Lambda}^*}{q^2} dq \ll s^2 \frac{q_3 \log q_3}{sr_2} \leq s^2,$$

where in the last inequality we use the assumption $q_3 \log q_3 \leq sr_2$. It follows from (3.21), (3.22), (3.26), (3.27), (3.28) and (3.31) that $\eta \ll s^2$. \square

We will apply Lemma 3.8 in the two concrete cases below. We first introduce some notation. Let $w = (w_1, w_2)$ be as in Theorem 1.1. We moreover assume that $w_1 > w_2$. We fix $C \geq 1$ such that C is an implied constant for the conclusions of Lemma 3.8 (i) and (ii). For a lattice Λ' of \mathbb{R}^3 and fixed \mathbf{r}, s we let $q_i(\Lambda') = q_i(\Lambda', \mathbf{r}, s)$ ($i = 1, 2, 3$) for simplicity. Similarly, we write $N_q = N_q(\mathbf{r}, s)$. Recall that

$$\mathcal{L}'_3 = \{\Lambda \in \mathcal{L}_3 : \Lambda \cap \mathbb{R}\mathbf{e}_3 = r\mathbb{Z}\mathbf{e}_3 \text{ for some } r \text{ with } 1/2 < r \leq 1\}.$$

Lemma 3.9. *Let $s = \varepsilon^2$, $\mathbf{r} = (r_1, r_2, r_3) = (\varepsilon e^t, \varepsilon e^t, 1)$, $\Lambda \in \mathcal{K}_{\varepsilon^2}^* \cap \mathcal{L}'_3$ and $a_t = \text{diag}(e^{w_1 t}, e^{w_2 t}, e^{-t})$. There exists a positive real number $\tilde{\varepsilon} \leq 1$ such that for all $\varepsilon, t > 0$ with $e^{-w_2 t/20} < \varepsilon < \tilde{\varepsilon}$ one has*

$$(3.32) \quad \sharp \mathcal{S}(a_t \Lambda, \mathbf{r}, s) \leq \varepsilon^{1/2} \cdot \text{vol}(M_{\mathbf{r}}).$$

Proof. We will show that the lemma holds for $\tilde{\varepsilon} = \frac{1}{100C^2}$. In view of Lemma 3.8 (i) and the choice of $\tilde{\varepsilon}$, it suffices to prove

$$q_1(a_t \Lambda) \geq s^{-2} \quad \text{and} \quad q_3(a_t \Lambda) \leq 2s^{-1/2} r_2.$$

We need to look at the lattice points in

$$(3.33) \quad N_q \cap (a_t \Lambda)^* = N_q \cap a_{-t}^* \Lambda^* = a_{-t}^* (a_t^* N_q \cap \Lambda^*),$$

where a_t^* is the transpose of a_t with respect to the standard basis of \mathbb{R}^3 and its dual basis.

Since $e^{-w_2 t/20} < \varepsilon$ by assumption, we have

$$(3.34) \quad \varepsilon^{20} e^{w_2 t} > 1.$$

It follows that $s^{-2} \leq r_1 s$. So

$$N_{s^{-2}} = \{\varphi \in \mathcal{E}_3^* : |x_1^\varphi| \leq \varepsilon^{-5} e^{-t}, |x_2^\varphi| \leq \varepsilon^{-5} e^{-t}, |x_3^\varphi| \leq \varepsilon^{-4}\}.$$

Hence

$$a_t^* N_{s^{-2}} = \{\varphi \in \mathcal{E}_3^* : |x_1^\varphi| \leq \varepsilon^{-5} e^{-w_2 t}, |x_2^\varphi| \leq \varepsilon^{-5} e^{-w_1 t}, |x_3^\varphi| \leq \varepsilon^{-4} e^{-t}\}$$

which is contained in the interior of $B_{\varepsilon^2}(\mathcal{E}_3^*)$ by (3.34). Since $\Lambda \in \mathcal{K}_{\varepsilon^2}^*$, the dual lattice Λ^* has no nonzero vectors in $a_t^* N_{s^{-2}}$. Therefore $q_1(a_t \Lambda) > s^{-2}$.

Next we turn to the proof of $q_3(a_t \Lambda) \leq 2s^{-1/2} r_2 = 2e^t$. Since $2e^t > r_2 > r_2 s$ one has

$$(3.35) \quad a_t^* N_{2e^t} = \{\varphi \in \mathcal{E}_3^* : |x_1^\varphi| \leq s e^{w_1 t}, |x_2^\varphi| \leq s e^{w_2 t}, |x_3^\varphi| \leq 2\}.$$

It follows from the assumption $\Lambda \in \mathcal{L}'_3$ that $r \mathbf{e}_3 \in \hat{\Lambda}$ for some r with $1/2 < r \leq 1$. Let $\mathbf{pr} : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the orthogonal projection to the subspace $\mathbb{R} \mathbf{e}_1 + \mathbb{R} \mathbf{e}_2$. It follows that $\mathbf{pr}(\Lambda)$ is a lattice with covolume $1/r$ in \mathbb{R}^2 . Suppose $\mathbf{v} \in \Lambda$ and $\|\mathbf{pr}(\mathbf{v})\| = \lambda_1(B_1(\mathbb{R}^2), \mathbf{pr}(\Lambda))$, then

$$(3.36) \quad \lambda_1(B_1(\mathbb{R}^2), \mathbf{pr}(\Lambda)) \geq r \cdot \lambda_1(B_1(\mathbb{R}^2), \mathbf{pr}(\Lambda)) = \|\mathbf{v} \wedge r \mathbf{e}_3\| \geq \varepsilon^2,$$

where in the last inequality we use $\Lambda \in \mathcal{K}_{\varepsilon^2}^*$. It follows from of (3.1) and (3.36) that

$$\lambda_2(B_1(\mathbb{R}^2), \mathbf{pr}(\Lambda)) \leq 8\varepsilon^{-2}.$$

Recall that there exists a Minkowski reduced basis $v^{(1)}, v^{(2)}$ of $\mathbf{pr}(\Lambda)$ with the property $\|v^{(1)}\| \leq 4\lambda_1(B_1(\mathbb{R}^2), \mathbf{pr}(\Lambda))$ and $\|v^{(2)}\| \leq 4\lambda_2(B_1(\mathbb{R}^2), \mathbf{pr}(\Lambda))$. Let $\mathbf{v}_i \in \Lambda$ ($i = 1, 2$) with $\mathbf{pr}(\mathbf{v}_i) = v^{(i)}$ and $\mathbf{e}_3^*(\mathbf{v}_i) < 1$. It follows that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 := r \mathbf{e}_3$ is a basis of Λ . Recall that Λ^* can be identified with $\wedge^2 \Lambda$ as Euclidean spaces. In view of (3.33) and (3.35) it suffices to show that the coordinates of

$$\mathbf{v}_i \wedge \mathbf{v}_j = x_1 \mathbf{e}_2 \wedge \mathbf{e}_3 + x_2 \mathbf{e}_1 \wedge \mathbf{e}_3 + x_3 \mathbf{e}_1 \wedge \mathbf{e}_2$$

satisfy

$$(3.37) \quad |x_1| \leq s e^{w_1 t}, |x_2| \leq s e^{w_2 t} \quad \text{and} \quad |x_3| \leq 2.$$

It follows from the definition of \mathbf{v}_i that $\|\mathbf{v}_i\| \leq 1 + 32\varepsilon^{-2} \leq 33\varepsilon^{-2}$. So

$$\|\mathbf{v}_i \wedge \mathbf{v}_j\| \leq \|\mathbf{v}_i\| \cdot \|\mathbf{v}_j\| \leq 33^2 \varepsilon^{-4} \leq s e^{w_2 t}$$

where in the last inequality we use (3.34) and the assumption $\varepsilon < \tilde{\varepsilon}$. Therefore the upper bounds of $|x_1|$ and $|x_2|$ in (3.37) hold. Finally note that $x_3 = 0$ unless $\{i, j\} = \{1, 2\}$ where $|x_3| = 1/r \leq 2$. □

Lemma 3.10. *Let $\mathbf{r} = (r_1, r_2, r_3) = (\varepsilon e^{w_1 t}, \varepsilon e^{(w_1+2w_2)t}, 1)$, $\Lambda \in \mathcal{K}_{\varepsilon^2}^*$ and $b_t = \text{diag}(e^{(w_1-w_2)t}, e^{2w_2 t}, e^{-t})$. Then there exists a positive real number $\tilde{s} \leq 1$ such that for all $s > 0$ and $t > 0$ with $e^{-\delta t} < \varepsilon < s < \tilde{s}$ where $\delta = \frac{1}{20} \min\{w_2, w_1 - w_2\}$ one has*

$$(3.38) \quad \sharp \mathcal{S}(b_t \Lambda, \mathbf{r}, s) \leq s \cdot \text{vol}(M_{\mathbf{r}}).$$

Proof. There exists a positive real number $c < 1$ such that if $e^{-\delta t} < c$ then

$$(3.39) \quad e^{w_2 t/20} \geq \left(1 + \frac{w_2}{2}\right)t.$$

We will show that (3.38) holds for $\tilde{s} = \min\{\frac{1}{100c}, c\}$. In view of Lemma 3.8 (ii) it suffices to prove

$$q_1(b_t \Lambda) \geq s^{-2} \quad \text{and} \quad q_3(b_t \Lambda) \log q_3(b_t \Lambda) \leq s \varepsilon e^{(w_1+2w_2)t}.$$

Using the assumption $e^{-\delta t} < \varepsilon < s$ one has

$$(3.40) \quad s^5 \varepsilon^5 \geq e^{-10\delta t},$$

which implies $s^{-2} \leq sr_1$. It follows that

$$b_t^* N_{s^{-2}} = \{\varphi \in \mathcal{E}_3^* : |x_1^\varphi| \leq e^{-w_2 t} \varepsilon^{-1} s^{-2}, |x_2^\varphi| \leq e^{-w_1 t} \varepsilon^{-1} s^{-2}, |x_3^\varphi| \leq e^{-t} s^{-2}\}$$

which in view of (3.40) is contained in the interior of $B_{\varepsilon^2}(\mathcal{E}_3^*)$. Since $\Lambda \in \mathcal{K}_{\varepsilon^2}^*$, one has

$$N_{s^{-2}} \cap (b_t \Lambda)^* = b_{-t}^* (b_t^* N_{s^{-2}} \cap \Lambda^*) = \{0\}.$$

Therefore $q_1(b_t \Lambda) \geq s^{-2}$.

We claim that $q_3(b_t \Lambda) \leq e^{(1+w_2/2)t}$. Note that $r_1 s < e^{(1+w_2/2)t} < r_2 s$ by (3.40). Therefore

$$b_t^* N_{e^{(1+w_2/2)t}} = \{\varphi \in \mathcal{E}_3^* : |x_1^\varphi| \leq s e^{(w_1-w_2)t}, |x_2^\varphi| \leq e^{3w_2 t/2} \varepsilon^{-1}, |x_3^\varphi| \leq e^{w_2 t/2}\}.$$

Since $\Lambda \in \mathcal{K}_{\varepsilon^2}^*$, Minkowski's second theorem (3.1) implies $\lambda_3(B_1(\mathcal{E}_3^*), \Lambda^*) \leq 2\varepsilon^{-4}$. Therefore there exists Minkowski reduced basis $\varphi_1, \varphi_2, \varphi_3$ of Λ^* such that $\|\varphi_i\| \leq 16\varepsilon^{-4} \leq \varepsilon^{-5}$. Using (3.40) it is not hard to see that $\varphi_1, \varphi_2, \varphi_3 \in b_t^* N_{e^{(1+w_2/2)t}}$. Therefore $q_3(b_t \Lambda) \leq e^{(1+w_2/2)t}$. This completes the proof of the claim. Finally we have

$$\begin{aligned} q_3(b_t \Lambda) \log q_3(b_t \Lambda) &\leq e^{(1+w_2/2)t} (1 + w_2/2)t && \text{by the claim} \\ &\leq e^{(1+w_2/2)t} e^{w_2 t/20} && \text{by (3.39)} \\ &\leq s \varepsilon e^{(w_1+2w_2)t} && \text{by (3.40)}. \end{aligned}$$

□

4. LOWER BOUND

Recall that \mathcal{L}_3 is the space of unimodular lattices in \mathbb{R}^3 and $\mathbb{N} = \{1, 2, 3, 4, \dots\}$. Let a_t and $h(x)$ be as in (1.2) and (1.3) respectively. A vector $x \in \mathbb{R}^2$ is w -singular if and only if the trajectory $\{a_t h(x) \mathbb{Z}^3 : t \geq 0\}$ is divergent, i.e. for any compact subset \mathcal{K} of \mathcal{L}_3 , there exists $T_0 > 0$ such that $a_t h(x) \mathbb{Z}^3 \notin \mathcal{K}$ for all $t \geq T_0$.

In this section we will construct a fractal subset of $\text{Sing}(w)$ whose Hausdorff dimension is equal to that of $\text{Sing}(w)$ using the above dynamical interpretation and the idea of shadowing. Roughly speaking shadowing means the following: given $t_0 \in \mathbb{R}$, if $x, y \in \mathbb{R}^2$ are close to each other (depending on t_0), then $a_{t_0+t} h(x) \mathbb{Z}^3$ and

$a_{t_0+t}h(y)\mathbb{Z}^3$ are close to each other for $t \in \mathbb{R}$ with $|t| \leq C$ where C is a constant depending on x and y .

The construction of the fractal structure starts with the lattice \mathbb{Z}^3 . But all the results and proofs remain valid if \mathbb{Z}^3 is replaced by a lattice in \mathcal{L}'_3 , the subset of \mathcal{L}_3 defined in (1.5). This observation allows us to give a proof of Theorem 1.5 at the end of this section.

4.1. Construction of the fractal set. We define a fractal structure (\mathcal{T}', β) on \mathbb{R}^2 inductively for any choice of sequences of positive real numbers $\{\varepsilon_n\}_{n \in \mathbb{N}}$ and $\{t_n\}_{n \in \mathbb{N}}$ with the following properties:

$$(4.1) \quad \varepsilon_n \leq \varepsilon_{n-1} \text{ for all } n \in \mathbb{N} \text{ and } \varepsilon_n \rightarrow 0 \text{ as } n \rightarrow \infty,$$

$$(4.2) \quad t_n \geq t_{n-1} + 1 \text{ for all } n \in \mathbb{N} \text{ and } t_{n+1} - t_n \rightarrow \infty \text{ as } n \rightarrow \infty,$$

where we set $t_0 = 0$ and $\varepsilon_0 = 1$ for convenience.

For $x = (x_1, x_2) \in \mathbb{R}^2$ and $r_1, r_2 > 0$ we let

$$I(x; r_1, r_2) = [x_1 - r_1, x_1 + r_1] \times [x_2 - r_2, x_2 + r_2] \subset \mathbb{R}^2.$$

We remark that $\mathbb{Z}^3 \in \mathcal{L}'_3$ and elements of \mathcal{L}'_3 will play the role of \mathbb{Z}^3 in our inductive construction of the fractal structure.

The tree \mathcal{T}' will have vertices in the set of rational vectors \mathbb{Q}^2 . We take the root of \mathcal{T}' to be $\tau_0 = (0, 0)$ and define

$$\beta(\tau_0) = I(\tau_0; \varepsilon_0 e^{-w_1 t_1}, \varepsilon_0 e^{-w_2 t_1}).$$

Suppose we have defined the tree structure and the map β till height $(n-1)$ of \mathcal{T}' . For each vertex $\tau_{n-1} \in \mathcal{T}'_{n-1}$ we want to define the set $\mathcal{T}'(\tau_{n-1})$ and the map β on it. This will complete the construction of the fractal structure. We define

$$(4.3) \quad \mathcal{T}'(\tau_{n-1}) = \{\tau \in \beta(\tau_{n-1}) : a_{t_n} h(\tau) \mathbb{Z}^3 \in \mathcal{L}'_3\}.$$

It is clear from the definition of \mathcal{L}'_3 and the assumption $t_n \geq t_{n-1} + 1$ that $\mathcal{T}'(\tau_{n-1})$ has empty intersection with $\bigcup_{0 \leq i \leq n-1} \mathcal{T}'_i$. For $\tau \in \mathcal{T}'(\tau_{n-1})$ we define

$$(4.4) \quad \beta(\tau) = I(\tau; \varepsilon_n e^{-w_1 t_{n+1} - t_n}, \varepsilon_n e^{-w_2 t_{n+1} - t_n}).^2$$

It follows from (4.3) that for every $\tau \in \mathcal{T}'_n$ ($n \geq 0$) there is a unique vector

$$(4.5) \quad \mathbf{v}(\tau) \in \{r\mathbf{e}_3 : 1/2 < r \leq 1\} \cap a_{t_n} h(\tau) \mathbb{Z}^3.$$

This property will be used several times below. We end up this subsection by proving the following lemma.

Lemma 4.1. $\mathcal{F}(\mathcal{T}', \beta) \subset \text{Sing}(w)$.

Proof. Let $n \in \mathbb{N} \cup \{0\}$, $\tau \in \mathcal{T}'_n$ and $x \in \beta(\tau)$. Let $\mathbf{v}(\tau) \in a_{t_n} h(\tau) \mathbb{Z}^3$ be as in (4.5). Then for $t \in \mathbb{R}$ the lattice

$$a_t h(x) \mathbb{Z}^3 = a_t h(x - \tau) a_{t_n}^{-1} \cdot a_{t_n} h(\tau) \mathbb{Z}^3$$

contains the primitive vector

$$a_t h(x - \tau) a_{t_n}^{-1} \mathbf{v}(\tau)$$

²Recall that β is a map from \mathcal{T}' which is identified with the vertices of the tree to compact subsets of \mathbb{R}^2 . Each vertex $\tau \in \mathcal{T}'$ has a height n and our definition of $\beta(\tau)$ also depends on n . Similar concerns apply in the definition of $\tilde{\beta}$ below.

whose norm is less than or equal to

$$3 \max\{\varepsilon_n e^{-w_1(t_{n+1}-t)}, \varepsilon_n e^{-w_2(t_{n+1}-t)}, e^{-(t-t_n)}\}.$$

Recall that we assume $w_1 \geq w_2$. So for $n \in \mathbb{N}$ we solve the equation

$$\varepsilon_{n-1} e^{w_1(t-t_n)} = e^{-(t-t_n)}$$

to get a unique solution $t = l_n$ with

$$l_n - t_n = \frac{\log \varepsilon_{n-1}^{-1}}{1 + w_1} \geq 0.$$

Since $\varepsilon_n \rightarrow 0$ one has $l_n - t_n \rightarrow \infty$ as $n \rightarrow \infty$.

Suppose $x \in \bigcap_{n \in \mathbb{N} \cup \{0\}} \beta(\tau_n)$ where $\{\tau_n\} \in \partial \mathcal{T}'$. For $n \in \mathbb{N}$ and $t \in [t_n, l_n]$, the lattice $a_t h(x) \mathbb{Z}^3$ contains the primitive vector

$$a_t h(x - \tau_{n-1}) a_{t_{n-1}}^{-1} \mathbf{v}(\tau_{n-1})$$

whose norm is less than or equal to

$$(4.6) \quad 3 \max\{e^{-(l_n - t_n)}, e^{-(t_n - t_{n-1})}\}.$$

For $t \in [l_n, t_{n+1}]$, the lattice $a_t h(x) \mathbb{Z}^3$ contains the primitive vector

$$a_t h(x - \tau_n) a_{t_n}^{-1} \mathbf{v}(\tau_n)$$

whose norm is less than or equal to

$$(4.7) \quad 3 \max\{\varepsilon_n, e^{-(l_n - t_n)}\}.$$

As the numbers in (4.6) and (4.7) tend to zero as $n \rightarrow \infty$, Mahler's compactness criterion (see [5, Chapter V]) implies $x \in \text{Sing}(w)$. □

4.2. Refinement of the fractal structure. We make explicit choices of the sequences $\{\varepsilon_n\}, \{t_n\}$ and refine the tree \mathcal{T}' associated to them to get a subtree \mathcal{T} so that (\mathcal{T}, β) is a regular self-affine structure satisfying the assumptions of Corollary 2.4. In this subsection we assume in addition that $w_1 > w_2$, although our method also works in unweighted case where we use first two conditions of (4.9) below to define the subtree structure. In the lower bound estimate we will not go into details of unweighted case where the Hausdorff dimension of $\text{Sing}(w)$ is known.

Let $\tilde{c}, \tilde{\varepsilon}, \tilde{s} \leq 1$ be positive real numbers as in Lemmas 3.7, 3.9 and 3.10 respectively. We fix $\varepsilon, t, r > 0$ with the following properties

- (i) $0 < \varepsilon < r < \frac{1}{10^4} \min\{\tilde{\varepsilon}, \tilde{s}, \tilde{c}, w_2, w_1 - w_2\}$;
- (ii) $t = \frac{100}{\varepsilon^2}$.

The sequence $\{\varepsilon_n\}$ and $\{t_n\}$ are defined by

- (iii) $\varepsilon_n = \frac{\varepsilon}{n}$ for $n \in \mathbb{N}$;
- (iv) $t_n - t_{n-1} = nt$ for $n \in \mathbb{N}$.³

It is not hard to see that for any integer $n \geq 0$ one has

$$(4.8) \quad \varepsilon_n^{-100} \leq \min\{e^{w_2 nt}, e^{(w_1 - w_2) nt}\}.$$

³Recall that $t_0 = 0$ and $\varepsilon_0 = 1$.

It can be checked directly that (4.1) and (4.2) hold for the sequences $\{\varepsilon_n\}_{n \in \mathbb{N}}$ and $\{t_n\}_{n \in \mathbb{N}}$. Hence they define a fractal structure (\mathcal{T}', β) with $\mathcal{F}(\mathcal{T}', \beta) \subset \text{Sing}(w)$ by Lemma 4.1. For $n \in \mathbb{N} \cup \{0\}$ we let

$$b_n = \text{diag}(e^{-w_2 n t}, e^{w_2 n t}, 1) \in SL_3(\mathbb{R})$$

$$\tilde{\beta}(\tau) = I(\tau; \varepsilon_{n+1} e^{-w_1 t_{n+1} - t_n}, \varepsilon_{n+1} e^{-w_2 t_{n+1} - t_n}) \quad \text{where } \tau \in \mathcal{T}_n.$$

From the definition, it is evident that $\tilde{\beta}(\tau) \subset \beta(\tau)$ for $\tau \in \mathcal{T}_n$.

Let \mathcal{T} be the rooted subtree of \mathcal{T}' defined in the following way: $\tau_0 = (0, 0)$ is the root of \mathcal{T} and $\mathcal{T}(\kappa)$ ($\kappa \in \mathcal{T}_{n-1}$) consists of all the $\tau \in \tilde{\beta}(\kappa)$ with the following properties:

$$(4.9) \quad \begin{aligned} a_{t_n} h(\tau) \mathbb{Z}^3 &\in \mathcal{L}'_3, \\ a_{t_n} h(\tau) \mathbb{Z}^3 &\in \mathcal{K}_{\varepsilon_n^2}^*, \\ b_n a_{t_n} h(\tau) \mathbb{Z}^3 &\in \mathcal{K}_r^*, \end{aligned}$$

where \mathcal{K}_r^* is defined in (3.3). It can be checked directly that $\beta(\tau) \subset \beta(\kappa)$ for all $\tau \in \mathcal{T}(\kappa)$ (this is the main reason for using $\tilde{\beta}$). It will follow from Lemma 4.2 below that each vertex of \mathcal{T} has nonempty set of sons. Therefore (\mathcal{T}, β) is a regular self-affine structure.

Lemma 4.2. *For every $n \in \mathbb{N}$ and $y \in \mathcal{T}_{n-1}$ one has*

$$\frac{1}{100} \varepsilon_n^2 e^{2nt} \leq \#\mathcal{T}(y) \leq 10 \varepsilon_n^2 e^{2nt}.$$

Let us fix $n \in \mathbb{N}$, $y \in \mathcal{T}_{n-1}$. We first reduce the calculation of $\#\mathcal{T}(y)$ to lattice points counting in \mathbb{R}^3 so that we can use the results of §3.2. We set

$$(4.10) \quad \begin{aligned} \Lambda &= a_{t_{n-1}} h(y) \mathbb{Z}^3 \in \mathcal{L}'_3 \cap \mathcal{K}_{\varepsilon_{n-1}^2}^* \subset \mathcal{L}'_3 \cap \mathcal{K}_{\varepsilon_n^2}^*, \\ \Lambda_1 &= a_{t_n} h(y) \mathbb{Z}^3 = a_{nt} \Lambda, \\ \Lambda_2 &= b_n a_{t_n} h(y) \mathbb{Z}^3 = b_n a_{nt} \Lambda. \end{aligned}$$

Given $x \in \tilde{\beta}(y)$, to have $x \in \mathcal{T}(y)$ the lattices

$$(4.11) \quad \begin{aligned} \Lambda_1(x) &= a_{t_n} h(x) \mathbb{Z}^3 = a_{t_n} h(x - y) a_{t_n}^{-1} \Lambda_1 \quad \text{and} \\ \Lambda_2(x) &= b_n a_{t_n} h(x) \mathbb{Z}^3 = b_n a_{t_n} h(x - y) a_{t_n}^{-1} b_n^{-1} \Lambda_2, \end{aligned}$$

must satisfy $\Lambda_1(x) \in \mathcal{K}_{\varepsilon_n^2}^*$, $\Lambda_2(x) \in \mathcal{K}_r^*$ and $\Lambda_1(x) \in \mathcal{L}'_3$ (which implies $\Lambda_2(x) \in \mathcal{L}'_3$). Therefore Lemma 4.2 follows from the following lemma.

Lemma 4.3. *Let $n \in \mathbb{N}$ and $y \in \mathcal{T}_{n-1}$. Then*

$$(4.12) \quad \frac{1}{10} \varepsilon_n^2 e^{2nt} \leq \#\{x \in \tilde{\beta}(y) : \Lambda_1(x) \in \mathcal{L}'_3\} \leq 10 \varepsilon_n^2 e^{2nt};$$

$$(4.13) \quad \#\{x \in \tilde{\beta}(y) : \Lambda_1(x) \in \mathcal{L}'_3 \setminus \mathcal{K}_{\varepsilon_n^2}^*\} \leq \frac{8}{100} \varepsilon_n^2 e^{2nt};$$

$$(4.14) \quad \#\{x \in \tilde{\beta}(y) : \Lambda_2(x) \in \mathcal{L}'_3 \setminus \mathcal{K}_r^*\} \leq \frac{1}{100} \varepsilon_n^2 e^{2nt}.$$

Proof. We first prove (4.12). Suppose $\Lambda_1(x) \in \mathcal{L}'_3$ where $x \in \tilde{\beta}(y)$. Then there exists s_x with $1/2 < s_x \leq 1$ such that $\Lambda_1(x)$ contains a primitive vector $\mathbf{v}(x) =$

$s_x \mathbf{e}_3$. It follows from the definition of $\tilde{\beta}(y)$ and a direct calculation that the vector $a_{t_n} h(y-x) a_{t_n}^{-1} s_x \mathbf{e}_3$ belongs to

$$M = \{(z_1, z_2, z_3) : \max\{|z_1|, |z_2|\} \leq \varepsilon_n e^{nt} |z_3| \text{ and } 1/2 < |z_3| \leq 1\}.$$

It is not hard to see that the map $x \rightarrow a_{t_n} h(y-x) a_{t_n}^{-1} \mathbf{v}(x)$ is a bijection between the sets $\{x \in \tilde{\beta}(y) : \Lambda_1(x) \in \mathcal{L}'_3\}$ and $M \cap \hat{\Lambda}_1$. Let

$$\begin{aligned} M^{(1)} &= \{(z_1, z_2, z_3) \in \mathbb{R}^3 : \max\{|z_1|, |z_2|\} \leq \frac{1}{2} \varepsilon_n e^{nt}, |z_3| \leq 1\} \\ M^{(2)} &= \{(z_1, z_2, z_3) \in \mathbb{R}^3 : \max\{|z_1|, |z_2|\} \leq \frac{1}{2} \varepsilon_n e^{nt}, |z_3| \leq \frac{1}{2}\}. \end{aligned}$$

Then $(M^{(1)} \setminus M^{(2)}) \subset M \subset 2M^{(2)}$. It follows that

$$(4.15) \quad \#M^{(1)} \cap \hat{\Lambda}_1 - \#M^{(2)} \cap \hat{\Lambda}_1 \leq \#(M \cap \hat{\Lambda}_1) \leq \#(2M^{(2)}) \cap \hat{\Lambda}_1.$$

Using (3.1) and $\Lambda \in \mathcal{K}_{\varepsilon_n^*}^*$ one has

$$\lambda_1(B_1, \Lambda) \geq 100^{-1} \varepsilon_n^4 \quad \text{and} \quad \lambda_3(B_1, \Lambda) \leq 100 \varepsilon_n^{-2}$$

where $B_r = B_r(\mathbb{R}^3)$ in this section. Recall that $a_{nt}^{-1} \Lambda_1 = \Lambda$ by (4.10). For $i = 1, 2$ we have

$$\begin{aligned} \lambda_1(M^{(i)}, \Lambda_1) &= \lambda_1(a_{nt}^{-1} M^{(i)}, \Lambda) \\ &\geq \lambda_1(a_{nt}^{-1} M^{(1)}, \Lambda) \geq \lambda_1(B_{3e^{nt}}, \Lambda) \geq (300)^{-1} e^{-nt} \varepsilon_n^4 \end{aligned}$$

and

$$(4.16) \quad \begin{aligned} \lambda_3(M^{(i)}, \Lambda_1) &= \lambda_3(a_{nt}^{-1} M^{(i)}, \Lambda) \\ &\leq \lambda_3(a_{nt}^{-1} M^{(2)}, \Lambda) \leq \lambda_3(B_{\frac{1}{2}\varepsilon_n e^{w_2 nt}}, \Lambda) \leq 200 e^{-w_2 nt} \varepsilon_n^{-3}. \end{aligned}$$

By these estimates and (4.8) it is straightforward to check that the assumptions of Lemma 3.7 for $M^{(1)}$, $M^{(2)}$, $2M^{(2)}$ and Λ are satisfied. Therefore (4.15) and Lemma 3.7 imply

$$(5\zeta(3))^{-1} \varepsilon_n^2 e^{2nt} \leq \#(M \cap \hat{\Lambda}_1) \leq 48(5\zeta(3))^{-1} \varepsilon_n^2 e^{2nt}.$$

To complete the proof of (4.12), it suffices to note that $1 < \zeta(3) < 2$.

Next we prove (4.13) and (4.14) together. Let $s_1 = \varepsilon_n^2$, $s_2 = r$, $a^{(1)} = a_{nt}$, $a^{(2)} = b_n a_{nt}$ and for $i = 1, 2$

$$\mathcal{S}_i = \{x \in \tilde{\beta}(y) : \Lambda_i(x) \in \mathcal{L}'_3 \setminus \mathcal{K}_{s_i}^*\}.$$

We will show that

$$(4.17) \quad \#\mathcal{S}_1 \leq 8\sqrt{\varepsilon_n} \varepsilon_n^2 e^{2nt} \quad \text{and} \quad \#\mathcal{S}_2 \leq 8r \varepsilon_n^2 e^{2nt}.$$

In view of the definitions of ε_n and r , this will complete the proof.

Let Λ_i and $\Lambda_i(x)$ be as in (4.10) and (4.11) respectively. Let $\mathbf{v}(x) \in \Lambda_1(x) \cap \Lambda_2(x)$ be as in (4.5). Let $x^{(i)} \in \mathbb{R}^2$ be such that $h(x^{(i)}) = a^{(i)} a_{t_{n-1}} h(y-x) (a^{(i)} a_{t_{n-1}})^{-1}$. Then

$$\mathbf{w}_i(x) := h(x^{(i)}) \mathbf{v}(x) \in \Lambda_i.$$

It can be calculated that for all $x \in \tilde{\beta}(y)$

$$(4.18) \quad \begin{aligned} \max\{|x_1^{(1)}|, |x_2^{(1)}|\} &\leq \varepsilon_n e^{nt}, \\ |x_1^{(2)}| &\leq \varepsilon_n e^{w_1 nt}, \quad |x_2^{(2)}| \leq \varepsilon_n e^{(w_1 + 2w_2)nt}. \end{aligned}$$

Let $M_i = M_{\mathbf{r}_i}$ (see (3.15) for the definition) where

$$\begin{aligned}\mathbf{r}_1 &= (\varepsilon_n e^{nt}, \varepsilon_n e^{nt}, 1) = (r_{11}, r_{12}, r_{13}) \\ \mathbf{r}_2 &= (\varepsilon_n e^{w_1 nt}, \varepsilon_n e^{(w_1+2w_2)nt}, 1) = (r_{21}, r_{22}, r_{23}).\end{aligned}$$

The map $\mathcal{S}_i \rightarrow M_i \cap \widehat{\Lambda}_i$ with $x \rightarrow \mathbf{w}_i(x)$ is injective. If for all $x \in \mathcal{S}_i$ there exists $\varphi_i \in N_{3s_i r_{i2}}(\mathbf{r}_i, s_i) \cap \widehat{\Lambda}_i^*$ (see (3.17) for the definition of $N_q(\mathbf{r}, s)$) such that $\varphi_i(\mathbf{w}_i(x)) = 0$, then

$$\#\mathcal{S}_i \leq \#\mathcal{S}(\Lambda_i, \mathbf{r}_i, s_i) = \#\mathcal{S}(a^{(i)}\Lambda, \mathbf{r}_i, s_i)$$

where $\mathcal{S}(\Lambda, \mathbf{r}, s)$ is defined in (3.18). Therefore the two estimates of (4.17) will follow from Lemmas 3.9 and 3.10 respectively. Here the assumptions of these two lemmas can be checked easily using (4.10) and the assumptions (i)-(iv) at the beginning of this section.

Suppose $x \in \mathcal{S}_i$. We prove that $\varphi_i(\mathbf{w}_i(x)) = 0$ for some $\varphi_i \in N_{3s_i r_{i2}}(\mathbf{r}_i, s_i) \cap \widehat{\Lambda}_i^*$. It follows from the definition of \mathcal{S}_i that $a^{(i)}a_{t_{n-1}}h(x)\mathbb{Z}^3 \notin \mathcal{K}_{s_i}^*$. So there exists $\varphi_i \in \widehat{\Lambda}_i^*$ such that $\|h(x^{(i)})^*\varphi_i\| < s_i$ where $h(x^{(i)})^*$ is the adjoint action defined by $g^*\varphi(\mathbf{v}) = \varphi(g\mathbf{v})$ for all $g \in SL_3(\mathbb{R})$ and $\mathbf{v} \in \mathbb{R}^3$. We claim that $\varphi_i(\mathbf{w}_i(x)) = 0$. Note that $\mathbf{w}_i(x) \in \Lambda_i$ and $\varphi_i \in \widehat{\Lambda}_i$ implies $\varphi_i(\mathbf{w}_i(x)) \in \mathbb{Z}$. Then the claim follows from

$$\begin{aligned}|\varphi_i(\mathbf{w}_i(x))| &= |h(x^{(i)})^*\varphi_i(h(-x^{(i)})\mathbf{w}_i(x))| = |h(x^{(i)})^*\varphi_i(\mathbf{v}(x))| \\ &\leq \|h(x^{(i)})^*\varphi_i(\mathbf{e}_3)\| \leq \|h(x^{(i)})^*\varphi_i\| < s_i < 1.\end{aligned}$$

From direct calculations we have

$$h(x^{(i)})^*\varphi_i = (\varphi_i(\mathbf{e}_1), \varphi_i(\mathbf{e}_2), x_1^{(i)}\varphi_i(\mathbf{e}_1) + x_2^{(i)}\varphi_i(\mathbf{e}_2) + \varphi_i(\mathbf{e}_3)).$$

It follows from (4.18) and the fact $\|h(x^{(i)})^*\varphi_i\| < s_i$ that

$$\max\{|\varphi_i(\mathbf{e}_1)|, |\varphi_i(\mathbf{e}_2)|\} < s_i \quad \text{and} \quad |\varphi_i(\mathbf{e}_3)| < 3s_i r_{i2}.$$

Therefore $\varphi_i \in N_{3s_i r_{i2}}(\mathbf{r}_i, s_i)$ by (3.17) and this completes the proof. \square

4.3. The lower bound calculation. In this subsection we complete the proof of the lower bound.

Theorem 4.4. *Let $w = (w_1, w_2)$ where $w_1 > w_2 > 0$ and $w_1 + w_2 = 1$. Then $\dim_H \text{Sing}(w) \geq 2 - \frac{1}{1+w_1}$.*

Proposition 4.5. *Suppose $w_1 > w_2 > 0$ and let (\mathcal{T}, β) be the self-affine structure on \mathbb{R}^2 defined in the previous section. Then*

$$\dim_H \mathcal{F}(\mathcal{T}, \beta) \geq 2 - \frac{1}{1+w_1}.$$

Our tool is Corollary 2.4. Let t and ε be constants fixed at the beginning of §4.2. It is clear from its constructions that (\mathcal{T}, β) is a regular self-affine structure satisfying assumptions (1) and (2) of Theorem 2.1 with

$$C_n = \frac{\varepsilon^2}{100n^2} e^{2nt}, \quad W_n = \frac{2\varepsilon}{n} e^{-w_1 t_{n+1} - t_n} \quad \text{and} \quad L_n = \frac{2\varepsilon}{n} e^{-w_2 t_{n+1} - t_n}$$

where $n \geq 1$ and $t_n = \sum_{i=0}^n it = n(1+n)t/2$.⁴ We will see from the following lemma about well-separated property of the fractal structure that assumption (3)

⁴For $n = 0$ we take $W_0 = e^{-w_1 t}$ and $L_n = e^{-w_2 t}$ and $C_0 = 1$.

of Theorem 2.1 holds for

$$\rho_n = e^{-w_1 nt}$$

provided n is sufficiently large.

Lemma 4.6. *Let $\tau \in \mathcal{T}_{n-1}$ ($n \in \mathbb{N}$). Then for all different $x, y \in \mathcal{T}(\tau)$ one has*

$$\text{dist}(\beta(x), \beta(y)) \geq W_{n-1} \cdot \frac{r}{8\varepsilon_{n-1}} \min\{e^{-w_1 nt}, e^{(w_1-w_2)t_n-(1+w_2)nt}\}.$$

Remark 4.7. It will be clear from the proof that for n sufficiently large either $\beta(x)$ and $\beta(y)$ have horizontal distance at least $e^{-w_1 nt}W_{n-1}$ (which is $\gg W_n$) or they have vertical distance at least $L_{n-1}e^{-nt-w_2 nt}$ (which is $\approx L_n$). If we do not assume the third condition of (4.9), the same argument below will give the corresponding horizontal (resp. vertical) separation $e^{-nt}\varepsilon_n^2 W_n$ (resp. $e^{-nt}\varepsilon_n^2 L_n$). But then the possible horizontal separation is too small and the assumption (iv) of Corollary 2.3 no longer holds. The validity of Corollary 2.3 (iv) means that, roughly speaking, nearby $\beta(x)$ and $\beta(y)$ have either horizontal distance $e^{-w_1 nt}W_{n-1}$ or vertical distance $L_{n-1}e^{-nt-w_2 nt}$.

Proof. Since $x, y \in \mathcal{T}(\tau)$, there are $1/2 < s, l \leq 1$ such that

$$s\mathbf{e}_3 \in b_n a_{t_n} h(x)\mathbb{Z}^3 \quad \text{and} \quad l\mathbf{e}_3 \in b_n a_{t_n} h(y)\mathbb{Z}^3.$$

Let

$$\mathbf{v} = b_n a_{t_n} h(y-x)(b_n a_{t_n})^{-1} s\mathbf{e}_3 \in b_n a_{t_n} h(y)\mathbb{Z}^3.$$

Since $b_n a_{t_n} h(y)\mathbb{Z}^3 \in \mathcal{K}_r^*$, one has

$$\|\mathbf{v} \wedge l\mathbf{e}_3\| = ls\|((y_1 - x_1)e^{t_n+w_1 t_n-w_2 nt}, (y_2 - x_2)e^{t_n+w_2 t_n+w_2 nt})\| \geq r.$$

Then either

- (i) $|y_1 - x_1|e^{t_n+w_1 t_n-w_2 nt} \geq r/2$ or
- (ii) $|y_2 - x_2|e^{t_n+w_2 t_n+w_2 nt} \geq r/2$.

Let $x' \in \beta(x)$ and $y' \in \beta(y)$. If (i) holds then

$$\begin{aligned} \|y' - x'\| &\geq |y'_1 - x'_1| \\ &\geq |y_1 - x_1| - |x_1 - x'_1| - |y_1 - y'_1| \\ &\geq e^{-t_n-w_1 t_n+w_2 nt}(r/2 - 2\varepsilon_n e^{-nt}) \\ &\geq e^{-t_n-w_1 t_n+w_2 nt}r/4 \\ &= W_{n-1} \cdot \frac{r}{8\varepsilon_{n-1}} e^{-w_1 nt}. \end{aligned}$$

If (ii) holds, then

$$\begin{aligned} \|y' - x'\| &\geq |y'_2 - x'_2| \\ &\geq |y_2 - x_2| - |x_2 - x'_2| - |y_2 - y'_2| \\ &\geq e^{-t_n-w_2 t_n-w_2 nt}(r/2 - 2\varepsilon_n e^{-w_2 t}) \\ &\geq e^{-t_n-w_2 t_n-w_2 nt}r/4 \\ &= W_{n-1} \cdot \frac{r}{8\varepsilon_{n-1}} e^{(w_1-w_2)t_n-(1+w_2)nt}. \end{aligned}$$

This completes the proof. \square

Proof of Proposition 4.5. We will apply Corollary 2.4 which uses the local data. We have

$$\begin{aligned} W_n/W_{n-1} &= \frac{n-1}{n} e^{-(n+1)tw_1-nt} \\ L_n/L_{n-1} &= \frac{n-1}{n} e^{-(n+1)tw_2-nt} \end{aligned}$$

for $n \geq 2$. It can be checked directly that for any integer k with $k \geq \frac{w_1}{w_2} + 10$ the assumptions of Corollary 2.4 hold. Moreover, we have

$$\lim_{n \rightarrow \infty} \frac{\log(L_n C_n / L_{n-1})}{-\log(W_n / W_{n-1})} = \frac{w_1}{1+w_1}.$$

Therefore Corollary 2.4 implies $\dim_H \mathcal{F}(\mathcal{T}, \beta) \geq 1 + \frac{w_1}{1+w_1} = 2 - \frac{1}{1+w_1}$. \square

Proof of Theorem 4.4. If $w > w_2$, then the conclusion follows from Proposition 4.5 and Lemma 4.1. If $w_1 = w_2$ then the conclusion follows from [6, Theorem 1.1]. \square

Proof of Theorem 1.5 (sketch). Note that the set

$$Q_\Lambda := \{y \in \mathbb{R}^2 : h(y)\Lambda \cap \mathbb{R}e_3 \neq \{0\}\}$$

is dense in \mathbb{R}^2 . We fix $y \in U \cap Q_\Lambda$ and $s \in \mathbb{R}$ such that $a_s h(y)\Lambda \in \mathcal{L}'_3$. In our construction of the fractal structure (\mathcal{T}, β) in §4.1 and §4.2 we only use the property $\mathbb{Z}^3 \in \mathcal{L}'_3$. So the same construction will give a fractal structure (\mathcal{T}'', β'') such that the Hausdorff dimension of $\mathcal{F}(\mathcal{T}'', \beta'') \subset I(0; e^{-w_1 t}, e^{-w_2 t})$ is at least $2 - \frac{1}{1+w_1}$ and for any $x \in \mathcal{F}(\mathcal{T}'', \beta'')$ the trajectory $\mathcal{A}^+ h(x) a_s h(y)\Lambda$ is divergent. By taking t sufficiently large, we can make sure that

$$a_s^{-1} h(\mathcal{F}(\mathcal{T}'', \beta'')) a_s h(y) \subset h(U).$$

This implies that $\{x \in U : \mathcal{A}^+ h(x)\Lambda \text{ is divergent}\}$ contains $y + g\mathcal{F}(\mathcal{T}'', \beta'')$ for some nonsingular linear transformation g of \mathbb{R}^2 . Therefore the conclusion holds. \square

5. BEST APPROXIMATION AND UPPER BOUND

We first review the definition of w -weighted best approximation and use it to construct a self-affine covering of

$$\text{Sing}(w)^* := \{x \in \text{Sing}(w) : 1, x_1, x_2 \text{ are linearly independent over } \mathbb{Q}\}$$

in §5.1. By Khintchine's transference principle ([12, Chapter IV, §5]), it is not hard to see that all $x \in \mathbb{R}^2$ with $1, x_1, x_2$ linearly dependent over \mathbb{Q} are w -singular. Note that the set of these x is a countable union of lines in \mathbb{R}^2 . Thus the Hausdorff dimension of $\text{Sing}(w) \setminus \text{Sing}(w)^*$ is one. We prove in §5.2 that the upper bound of the Hausdorff dimension of the fractal associated to the self-affine structure constructed in §5.1 can be arbitrarily close to $2 - \frac{1}{1+w_1} > 1$. Therefore the Hausdorff dimension of $\text{Sing}(w)$ is bounded from above by $2 - \frac{1}{1+w_1}$.

5.1. Best approximation and self-affine covering. We define w -weighted quasi-norm on \mathbb{R}^2 by

$$\|(x_1, x_2)\|_w = \max\{|x_1|^{1/w_1}, |x_2|^{1/w_2}\}.$$

Although it does not satisfy the triangle inequality, using convexity of the function $s \rightarrow s^{1/w_i}$ we have

$$(5.1) \quad \|x + y\|_w \leq 2^{w_1/w_2} (\|x\|_w + \|y\|_w) \quad \text{for all } x, y \in \mathbb{R}^2.$$

We say $(p, q) \in \mathbb{Z}^2 \times \mathbb{N}$ is a best approximation vector of $x \in \mathbb{R}^2$ with respect to the w -weighted norm if

- (i) $\|qx - p\|_w < \|q'x - p'\|_w$ for any $(p', q') \in \mathbb{Z}^2 \times \mathbb{N}$ with $q' < q$;
- (ii) $\|qx - p\|_w \leq \|qx - p'\|_w$ for any $p' \in \mathbb{Z}^2$.

For simplicity we call (p, q) a w -best approximate of x .

There is a naturally defined bijection between \mathbb{Q}^2 and

$$Q = \{(p, q) = (p_1, p_2, q) \in \mathbb{Z}^2 \times \mathbb{N} : \gcd(p_1, p_2, q) = 1\} \subset \widehat{\mathbb{Z}}^3,$$

namely, every $\mathbf{u} = (p, q) \in Q$ corresponds to $\hat{\mathbf{u}} = \frac{p}{q}$. For $x \in \mathbb{R}^2$ and $\mathbf{u} = (p, q) \in \mathbb{Z}^2 \times \mathbb{Z}$ we let

$$|\mathbf{u}| = q \quad \text{and} \quad A(x, \mathbf{u}) = \|qx - p\|_w.$$

Let

$$(5.2) \quad r(\mathbf{u}) = \min_{\mathbf{v} \in Q, \mathbf{v} \neq \mathbf{u}} A(\hat{\mathbf{u}}, \mathbf{v})$$

which is the best approximation of $\hat{\mathbf{u}}$ by $\mathbf{v} \in Q \setminus \{\mathbf{u}\}$. We will see in (5.15) that $r(\mathbf{u})$ is the smallest w -weighted norm of nonzero vectors of a lattice in \mathbb{R}^2 . For every $x \in \mathbb{R}^2 \setminus \mathbb{Q}^2$ we associate a sequence $\Sigma_x = \{\mathbf{u}_i\}_{i \in \mathbb{N}} \subset Q$ of w -best approximates of x with the following properties:

- $|\mathbf{u}_1| > 1$;
- $|\mathbf{u}_i| < |\mathbf{u}_{i+1}|$ for all $i \in \mathbb{N}$;
- there is no w -best approximate (p, q) of x with $|\mathbf{u}_i| < q < |\mathbf{u}_{i+1}|$.

Lemma 5.1. *Let $\mathbf{u} \in Q$ with $|\mathbf{u}| > 1$. Then for any $\mathbf{v} \in Q$ with $|\mathbf{v}| < |\mathbf{u}|$ one has $A(\hat{\mathbf{u}}, \mathbf{v}) = A(\hat{\mathbf{u}}, \mathbf{u} - \mathbf{v})$.*

Proof. Suppose $\mathbf{u} = (p, q) \in \mathbb{Z}^2 \times \mathbb{N}$ and $\mathbf{v} = (s, l) \in \mathbb{Z}^2 \times \mathbb{N}$. Then $\mathbf{u} - \mathbf{v} = (p - s, q - l)$ and

$$A(\hat{\mathbf{u}}, \mathbf{u} - \mathbf{v}) = \|(q - l)pq^{-1} - (p - s)\|_w = \|(lpq^{-1} - s)\|_w = A(\hat{\mathbf{u}}, \mathbf{v}).$$

□

Corollary 5.2. *Let $\mathbf{u} \in Q$ with $|\mathbf{u}| > 1$. Then there exists $\mathbf{v} \in Q$ with $|\mathbf{v}| \leq \frac{|\mathbf{u}|}{2}$ such that $r(\mathbf{u}) = A(\hat{\mathbf{u}}, \mathbf{v})$.*

Proof. Since $|\mathbf{u}| > 1$ one has $r(\mathbf{u}) = \min_{\mathbf{v} \in Q, |\mathbf{v}| < |\mathbf{u}|} A(\hat{\mathbf{u}}, \mathbf{v})$. So the corollary follows from Lemma 5.1. □

Lemma 5.3. *Let $\mathbf{u} \in Q$ with $|\mathbf{u}| > 1$ be a w -best approximate of $x \in \mathbb{R}^2$. Then $A(x, \mathbf{v}) < 2^{1/w_2} A(\hat{\mathbf{u}}, \mathbf{v})$ for all $\mathbf{v} \in Q$ with $|\mathbf{v}| \leq |\mathbf{u}|/2$.*

Proof. It follows from the definition of best approximate that

$$(5.3) \quad A(x, \mathbf{u}) < A(x, \mathbf{v}).$$

Let $\mathbf{u} = (p_1, p_2, q)$, $\mathbf{v} = (s_1, s_2, l)$ and choose $i \in \{1, 2\}$ such that $A(x, \mathbf{v}) = |lx_i - s_i|^{1/w_i}$. The choice of i implies

$$\begin{aligned}
A(x, \mathbf{v})^{w_i} &= l|x_i - s_i/l| \\
&\leq l(|x_i - p_i/q| + |p_i/q - s_i/l|) \\
&= \frac{l}{q}|qx_i - p_i| + |lp_i/q - s_i| \\
&\leq \frac{1}{2}A(x, \mathbf{u})^{w_i} + A(\hat{\mathbf{u}}, \mathbf{v})^{w_i} \\
&< \frac{1}{2}A(x, \mathbf{v})^{w_i} + A(\hat{\mathbf{u}}, \mathbf{v})^{w_i}. \tag{by (5.3)}
\end{aligned}$$

Therefore

$$A(x, \mathbf{v}) < 2^{1/w_2} A(\hat{\mathbf{u}}, \mathbf{v}).$$

□

Let $\mathbf{u} \in Q$ and $B_w(\mathbf{u}, r) = \{x \in \mathbb{R}^2 : A(x, \mathbf{u}) < r\}$. It is not hard to see that $B_w(\mathbf{u}, r)$ is an open rectangle with center $\hat{\mathbf{u}}$. The set of $x \in \mathbb{R}^2$ which has $\mathbf{u} \in Q$ as a w -best approximate is

$$(5.4) \quad \Delta(\mathbf{u}) = \left(\bigcap_{|\mathbf{v}| < |\mathbf{u}|} \Delta_{\mathbf{v}}(\mathbf{u}) \right) \cap \left(\bigcap_{|\mathbf{v}| = |\mathbf{u}|, \mathbf{u} \neq \mathbf{v}} \overline{\Delta_{\mathbf{v}}(\mathbf{u})} \right).$$

where

$$\Delta_{\mathbf{v}}(\mathbf{u}) = \{x \in \mathbb{R}^2 : A(x, \mathbf{v}) > A(x, \mathbf{u})\}.$$

The following lemma says that $\Delta(\mathbf{u})$ is roughly the rectangle $B_w(\mathbf{u}, r(\mathbf{u}))$.

Lemma 5.4. *For any $\mathbf{u} \in Q$ with $|\mathbf{u}| > 1$ one has*

$$B_w(\mathbf{u}, 2^{-1/w_2} r(\mathbf{u})) \subset \Delta(\mathbf{u}) \subset B_w(\mathbf{u}, 2^{1/w_2} r(\mathbf{u})).$$

Proof. We write $r = r(\mathbf{u})$ to simplify the notation. Let $\mathbf{u} = (p_1, p_2, q)$ and suppose $x = (x_1, x_2) \in B_w(\mathbf{u}, 2^{-1/w_2} r)$. Let $\mathbf{v} = (s_1, s_2, l) \in Q$ with $|\mathbf{v}| \leq |\mathbf{u}|$ and $\mathbf{v} \neq \mathbf{u}$. It follows from the definitions of $B_w(\mathbf{u}, 2^{-1/w_2} r)$ and $r(\mathbf{u})$ that

$$(5.5) \quad A(x, \mathbf{u}) < 2^{-1/w_2} r \leq 2^{-1/w_2} A(\hat{\mathbf{u}}, \mathbf{v}).$$

We choose $i \in \{1, 2\}$ such that

$$(5.6) \quad A(\hat{\mathbf{u}}, \mathbf{v}) = |lp_i/q - s_i|^{1/w_i}.$$

Then

$$\begin{aligned}
A(x, \mathbf{v})^{w_i} &\geq |lx_i - s_i| \\
&= l|x_i - s_i/l| \\
&\geq l(|p_i/q - s_i/l| - |x_i - p_i/q|) \\
(5.7) \quad &= |lp_i/q - s_i| - \frac{l}{q}|qx_i - p_i| \\
&\geq A(\hat{\mathbf{u}}, \mathbf{v})^{w_i} - A(x, \mathbf{u})^{w_i},
\end{aligned}$$

where in the last inequality we use (5.6). In view of (5.7) and (5.5) one has $A(x, \mathbf{v}) > A(x, \mathbf{u})$, from which one has $x \in \Delta_{\mathbf{v}}(\mathbf{u})$. Since \mathbf{v} is an arbitrary element of $Q \setminus \{\mathbf{u}\}$ with $|\mathbf{v}| \leq |\mathbf{u}|$, the definition of $\Delta(\mathbf{u})$ in (5.4) implies $x \in \Delta(\mathbf{u})$.

Now suppose $x \in \Delta(\mathbf{u})$. Corollary 5.2 implies that there exists \mathbf{v} with $|\mathbf{v}| \leq |\mathbf{u}|/2$ such that $r = A(\hat{\mathbf{u}}, \mathbf{v})$. It follows from Lemma 5.3 that

$$A(x, \mathbf{u}) < A(x, \mathbf{v}) < 2^{1/w_2} r.$$

□

Lemma 5.5. *Let $x \in \mathbb{R}^2 \setminus \mathbb{Q}^2$ and $\Sigma_x = \{\mathbf{u}_i\}_{i \in \mathbb{N}}$. Then for all $i, j \in \mathbb{N}$ one has*

$$(5.8) \quad 2^{-1/w_2} A(\hat{\mathbf{u}}_{i+j}, \mathbf{u}_i) < A(x, \mathbf{u}_i) < 2^{1/w_2} r(\mathbf{u}_{i+1}).$$

Proof. By Corollary 5.2 there exists $\mathbf{v} \in Q$ with $|\mathbf{v}| \leq |\mathbf{u}_{i+1}|/2$ such that $r(\mathbf{u}_{i+1}) = A(\hat{\mathbf{u}}_{i+1}, \mathbf{v})$. It follows from Lemma 5.3 that

$$(5.9) \quad A(x, \mathbf{v}) < 2^{1/w_2} r(\mathbf{u}_{i+1}).$$

On the other hand the definition of w -best approximate implies

$$(5.10) \quad A(x, \mathbf{v}) \geq A(x, \mathbf{u}_i).$$

The second inequality of (5.8) follows from (5.9) and (5.10).

Let $\mathbf{u}_{i+j} = (p_1, p_2, q)$, $\mathbf{u}_i = (s_1, s_2, l)$ and choose $k \in \{1, 2\}$ such that $A(\hat{\mathbf{u}}_{i+j}, \mathbf{u}_i) = |lp_k/q - s_k|^{1/w_k}$. We have

$$\begin{aligned} A(x, \mathbf{u}_i)^{w_k} &\geq |lx_k - s_k| \\ &= l|x_k - s_k/l| \\ &\geq l(|p_k/q - s_k/l| - |x_k - p_k/q|) \\ &= |lp_k/q - s_k| - (l/q)|x_k q - p_k| \\ &\geq A(\hat{\mathbf{u}}_{i+j}, \mathbf{u}_i)^{w_k} - A(x, \mathbf{u}_{i+j})^{w_k} \\ &> A(\hat{\mathbf{u}}_{i+j}, \mathbf{u}_i)^{w_k} - A(x, \mathbf{u}_i)^{w_k}. \end{aligned}$$

Therefore $A(x, \mathbf{u}_i) > 2^{-1/w_2} A(\hat{\mathbf{u}}_{i+j}, \mathbf{u}_i)$. □

Note that $r(\mathbf{u}_{i+1}) \leq A(\hat{\mathbf{u}}_{i+1}, \mathbf{u}_i)$. So Lemma 5.5 implies the following corollary.

Corollary 5.6. *Let $x \in \mathbb{R}^2 \setminus \mathbb{Q}^2$ and $\Sigma_x = \{\mathbf{u}_i\}_{i \in \mathbb{N}}$. Then for all $i \in \mathbb{N}$*

$$A(x, \mathbf{u}_i) \asymp r(\mathbf{u}_{i+1}) \asymp A(\hat{\mathbf{u}}_{i+1}, \mathbf{u}_i)$$

where the implied constants do not depend on i .

The following lemma gives a description of a w -singular vector using its associated best approximation sequence and it follows directly from the definition.

Lemma 5.7. *Let $x \in \mathbb{R}^2 \setminus \mathbb{Q}^2$ and $\Sigma_x = \{\mathbf{u}_i\}_{i \in \mathbb{N}}$. Then $x \in \text{Sing}(w)$ if and only if*

$$A(x, \mathbf{u}_i)|\mathbf{u}_{i+1}| \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

In view of Corollary 5.6 and Lemma 5.7 one has the following corollary.

Corollary 5.8. *Let $x \in \mathbb{R}^2 \setminus \mathbb{Q}^2$ and $\Sigma_x = \{\mathbf{u}_i\}_{i \in \mathbb{N}}$. Then $x \in \text{Sing}(w)$ if and only if*

$$r(\mathbf{u}_i)|\mathbf{u}_i| \rightarrow 0.$$

In view of Corollary 5.8, for $\varepsilon > 0$ the set

$$Q_\varepsilon = \{\mathbf{u} \in Q : r(\mathbf{u})|\mathbf{u}| < \varepsilon, |\mathbf{u}| > 1\}$$

consists of best approximates of almost (or ε -close) w -singular vectors. We are going to define a relation σ_ε on Q_ε so that together with the map β on Q defined by

$$(5.11) \quad \beta(\mathbf{u}) = \overline{B_w(\mathbf{u}, |\mathbf{u}|^{-1})} = I(\hat{\mathbf{u}}; |\mathbf{u}|^{-(w_1+1)}, |\mathbf{u}|^{-(w_2+1)})$$

we get an admissible fractal relation $(Q_\varepsilon, \sigma_\varepsilon, \beta)$ such that the corresponding fractal contains $\text{Sing}(w)^*$.

Now we fix $\mathbf{u} \in Q_\varepsilon$ and define the set $\sigma_\varepsilon(\mathbf{u})$. We choose $\mathbf{u}' \in Q$ with the property $|\mathbf{u}'| \leq |\mathbf{u}|/2$ and $r(\mathbf{u}) = A(\hat{\mathbf{u}}, \mathbf{u}')$. Let $H_{\mathbf{u}}$ be the hyperplane in \mathbb{R}^3 generated by \mathbf{u} and \mathbf{u}' . Let

$$D(\mathbf{u}, \varepsilon) = \{\mathbf{v} \in H_{\mathbf{u}} \cap Q_\varepsilon : |\mathbf{v}| \geq |\mathbf{u}|, A(\hat{\mathbf{v}}, \mathbf{u}) < 2^{2/w_2} r(\mathbf{u})\}.$$

We note here that all the $\mathbf{v} \in H_{\mathbf{u}} \cap Q_\varepsilon$ (including \mathbf{u}) with $\hat{\mathbf{v}}$ close to $\hat{\mathbf{u}}$ belong to $D(\mathbf{u}, \varepsilon)$. For every $\mathbf{v} \in D(\mathbf{u}, \varepsilon)$ let

$$E(\mathbf{u}, \mathbf{v}, \varepsilon) = \{\mathbf{w} \in Q_\varepsilon : |\mathbf{w}| > |\mathbf{v}|, \mathbf{w} \notin H_{\mathbf{u}} \text{ and } A(\hat{\mathbf{w}}, \mathbf{v}) < \varepsilon |\mathbf{w}|^{-1}\}.$$

We define

$$\sigma_\varepsilon(\mathbf{u}) = \bigcup_{\mathbf{v} \in D(\mathbf{u}, \varepsilon)} E(\mathbf{u}, \mathbf{v}, \varepsilon).$$

Lemma 5.9. *For every $0 < \varepsilon < 1$ one has $\text{Sing}(w)^* \subset \mathcal{F}(Q_\varepsilon, \sigma_\varepsilon, \beta)$.*

Proof. Let $x \in \text{Sing}(w)^*$ and $\Sigma_x = \{\mathbf{u}_i\}_{i \in \mathbb{N}}$. We are going to construct a subsequence $\{\mathbf{u}_{i_n}\}_{n \in \mathbb{N}}$ such that $(\mathbf{u}_{i_n}, \mathbf{u}_{i_{n+1}}) \in \sigma_\varepsilon$ and $x \in \beta(\mathbf{u}_{i_n})$ for all $n \in \mathbb{N}$. This will complete the proof.

According to Corollary 5.8 there exists $i_0 \in \mathbb{N}$ such that for $i \geq i_0$ one has

$$(5.12) \quad r(\mathbf{u}_i) < \frac{\varepsilon 2^{-2/w_2}}{|\mathbf{u}_i|}.$$

By Lemma 5.5

$$A(x, \mathbf{u}_i) \leq 2^{1/w_2} r(\mathbf{u}_{i+1}) < \frac{\varepsilon 2^{-1/w_2}}{|\mathbf{u}_{i+1}|} < \frac{1}{|\mathbf{u}_i|},$$

which implies that $x \in B_w(\mathbf{u}_i, |\mathbf{u}_i|^{-1})$. Let $i_1 = i_0$ and we inductively define i_{n+1} to be smallest integer of

$$\{m \in \mathbb{N} : m > i_n, \mathbf{u}_m \notin H_{\mathbf{u}_{i_n}}\}$$

which is nonempty since $1, x_1, x_2$ are linearly independent over \mathbb{Q} .

To simplify the notation we take $\mathbf{u} = \mathbf{u}_{i_n}$, $\mathbf{v} = \mathbf{u}_{i_{n+1}-1}$ and $\mathbf{w} = \mathbf{u}_{i_{n+1}}$. It suffices to show $\mathbf{v} \in D(\mathbf{u}, \varepsilon)$ and $\mathbf{w} \in E(\mathbf{u}, \mathbf{v}, \varepsilon)$. Using (5.8) we have

$$\begin{aligned} A(\hat{\mathbf{v}}, \mathbf{u}) &< 2^{1/w_2} A(x, \mathbf{u}) \\ &\leq 2^{1/w_2} A(x, \mathbf{u}_{i_n-1}) \\ &< 2^{2/w_2} r(\mathbf{u}), \end{aligned}$$

which implies $\mathbf{v} \in D(\mathbf{u}, \varepsilon)$. Using (5.8) again and (5.12) we have

$$A(\hat{\mathbf{w}}, \mathbf{v}) < 2^{2/w_2} r(\mathbf{w}) < \varepsilon / |\mathbf{w}|.$$

Therefore $\mathbf{w} \in E(\mathbf{u}, \mathbf{v}, \varepsilon)$. □

5.2. Upper bound. For every $\varepsilon > 0$ sufficiently small we estimate the Hausdorff dimension of $\mathcal{F}(Q_\varepsilon, \sigma_\varepsilon, \beta)$. In view of Lemma 5.9 and the discussion at the beginning of §5 this will give an upper bound of the Hausdorff dimension of $\text{Sing}(w)^*$ and $\text{Sing}(w)$.

For $\mathbf{u} = (p, q) \in Q$ we let $\pi_{\mathbf{u}} : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the projection along $\mathbb{R}\mathbf{u}$ defined by

$$(5.13) \quad \pi_{\mathbf{u}}((x, x_3)) = x - \frac{x_3}{q}p.$$

The kernel of $\pi_{\mathbf{u}}$ is $\mathbb{R}\mathbf{u}$ and $\mathbb{R}\mathbf{u} \cap \mathbb{Z}^3 = \mathbb{Z}\mathbf{u}$. The set

$$(5.14) \quad \Lambda_{\mathbf{u}} := \pi_{\mathbf{u}}(\mathbb{Z}^3)$$

is a lattice in \mathbb{R}^2 with covolume $1/|\mathbf{u}|$. It is easy to see that $A(\hat{\mathbf{u}}, \mathbf{v}) = \|\pi_{\mathbf{u}}(\mathbf{v})\|_w$ for every $\mathbf{v} \in Q$. Therefore

$$(5.15) \quad r(\mathbf{u}) = \inf_{y \in \Lambda_{\mathbf{u}} \setminus \{0\}} \|y\|_w.$$

Lemma 5.10. *Let $0 < \varepsilon < 1$ and $\mathbf{u} \in Q_\varepsilon$. For every real number t with $2 < t \leq 3$ we have*

$$(5.16) \quad \sum_{\mathbf{v} \in D(\mathbf{u}, \varepsilon)} \left(\frac{|\mathbf{u}|}{|\mathbf{v}|} \right)^t \ll \frac{1}{t-2}.$$

Proof. For each $k \in \mathbb{N}$, let

$$D_k = \{\mathbf{v} \in D(\mathbf{u}, \varepsilon) : k|\mathbf{u}| \leq |\mathbf{v}| < (k+1)|\mathbf{u}|\}.$$

Since

$$\sum_{\mathbf{v} \in D(\mathbf{u}, \varepsilon)} \left(\frac{|\mathbf{u}|}{|\mathbf{v}|} \right)^t = \sum_{k=1}^{\infty} \sum_{\mathbf{v} \in D_k} \left(\frac{|\mathbf{u}|}{|\mathbf{v}|} \right)^t \leq \sum_{k=1}^{\infty} \frac{\#D_k}{k^t},$$

it suffices to show $\#D_k \ll k$.

Let $\pi_{\mathbf{u}}$ and $\Lambda_{\mathbf{u}}$ be as in (5.13) and (5.14) respectively. Since \mathbf{u} is a primitive vector of \mathbb{Z}^3 , the projection $\pi_{\mathbf{u}}$ induces a bijection between

$$\{\mathbf{v} \in \mathbb{Z}^3 : k|\mathbf{u}| \leq |\mathbf{v}| < (k+1)|\mathbf{u}|\}$$

and $\Lambda_{\mathbf{u}}$ (note that $\pi_{\mathbf{u}}(\mathbf{v}) = \pi_{\mathbf{u}}(\mathbf{w})$ implies $|\mathbf{v}| \equiv |\mathbf{w}| \pmod{|\mathbf{u}|}$). It follows that $\pi_{\mathbf{u}}$ induces a bijection between

$$\{\mathbf{v} \in H_{\mathbf{u}} \cap \mathbb{Z}^3 : k|\mathbf{u}| \leq |\mathbf{v}| < (k+1)|\mathbf{u}|\}$$

and $\Lambda'_{\mathbf{u}} := \Lambda_{\mathbf{u}} \cap \pi_{\mathbf{u}}(H_{\mathbf{u}})$. According to the definition of $D(\mathbf{u}, \varepsilon)$ and (5.15) there exists $x \in \Lambda_{\mathbf{u}}$ such that $\|x\|_w = r(\mathbf{u})$ and $\Lambda'_{\mathbf{u}} = \mathbb{Z}x$. Let $\mathbf{v} \in D_k$, then $\pi_{\mathbf{u}}(\mathbf{v}) = sx$ for some $s \in \mathbb{Z}$. Let $i \in \{1, 2\}$ such that $\|x\|_w = |x_i|^{1/w_i}$. Then

$$\begin{aligned} A(\hat{\mathbf{v}}, \mathbf{u}) &= \|\mathbf{u}|\hat{\mathbf{v}} - \mathbf{u}|\hat{\mathbf{u}}\|_w = \|\mathbf{u}\|\mathbf{v}^{-1} \cdot (|\mathbf{v}|\hat{\mathbf{v}} - |\mathbf{v}|\hat{\mathbf{u}})\|_w \\ &= \|\mathbf{u}\|\mathbf{v}^{-1} \|sx\|_w \geq \left(\frac{|s|}{k+1} \right)^{1/w_i} r(\mathbf{u}). \end{aligned}$$

On the other hand, we have $A(\hat{\mathbf{v}}, \mathbf{u}) < 2^{2/w_2} r(\mathbf{u})$ since $\mathbf{v} \in D(\mathbf{u}, \varepsilon)$. It follows that $|s| \ll k$ and hence $\#D_k \ll k$. □

Lemma 5.11. *Let $0 < \varepsilon \leq 2^{-2/w_2}$. For all $\mathbf{u} \in Q_\varepsilon$ and $\mathbf{v} \in D(\mathbf{u}, \varepsilon)$ one has $\pi_{\mathbf{u}}(H_{\mathbf{u}}) = \pi_{\mathbf{v}}(H_{\mathbf{v}})$ and $H_{\mathbf{u}} = H_{\mathbf{v}}$.*

Proof. Since $\mathbf{v} \in H_{\mathbf{u}}$, $\pi_{\mathbf{u}}(H_{\mathbf{u}}) = \pi_{\mathbf{v}}(H_{\mathbf{v}})$ implies $H_{\mathbf{u}} = H_{\mathbf{v}}$. So it suffices to prove the former. According to (5.15) and the assumption $\mathbf{u} \in Q_\varepsilon$, there exists $y \in \Lambda_{\mathbf{u}} \cap \pi_{\mathbf{u}}(H_{\mathbf{u}})$ such that

$$\|y\|_w = r(\mathbf{u}) \leq \frac{\varepsilon}{|\mathbf{u}|}.$$

It follows that

$$\lambda_1(K, \Lambda_{\mathbf{u}}) \leq \varepsilon^{w_2} \quad \text{where} \quad K = \{x \in \mathbb{R}^2 : \|x\|_w \leq |\mathbf{u}|^{-1}\}.$$

Since $\text{cov}(\Lambda_{\mathbf{u}}) = \frac{1}{4} \text{vol}(K) = 1/|\mathbf{u}|$, Minkowski's second theorem (see (3.1)) implies that

$$\lambda_2(K, \Lambda_{\mathbf{u}}) \geq \frac{2^2}{2!} \cdot \lambda_1^{-1} \cdot \frac{\text{cov}(\Lambda_{\mathbf{u}})}{\text{vol}(K)} \geq 2^{-1} \varepsilon^{-w_2}.$$

Similarly, there exists $z \in \Lambda_{\mathbf{v}} \cap \pi_{\mathbf{v}}(H_{\mathbf{v}})$ such that

$$(5.17) \quad \|z\|_w = r(\mathbf{v}) \leq \frac{\varepsilon}{|\mathbf{v}|} \leq \frac{\varepsilon}{|\mathbf{u}|}.$$

Since $\mathbf{v} \in H_{\mathbf{u}}$ one has $\pi_{\mathbf{u}}(\mathbf{x}) - \pi_{\mathbf{v}}(\mathbf{x}) \in \mathbb{R}y$ for all $\mathbf{x} \in \mathbb{R}^3$. It follows that $\Lambda_{\mathbf{v}} \subset \Lambda_{\mathbf{u}} + \mathbb{R}y$, and hence $z \in \Lambda_{\mathbf{u}} + \mathbb{R}y$. We will show that $z \in \mathbb{R}y$, which will imply $\pi_{\mathbf{u}}(H_{\mathbf{u}}) = \pi_{\mathbf{v}}(H_{\mathbf{v}})$ and complete the proof.

Suppose $z \notin \mathbb{R}y$, then there exists $0 \leq s < 1$ such that $z + sy \in \Lambda_{\mathbf{u}} \setminus \mathbb{R}y$. So

$$\|z\|_K \geq \|z + sy\|_K - \|y\|_K \geq 2^{-1} \varepsilon^{-w_2} - \varepsilon^{w_2} \geq 1$$

where $\|\cdot\|_K$ be the norm on \mathbb{R}^2 defined in (3.2). This contradicts (5.17). Therefore $z \in \mathbb{R}y$. \square

Lemma 5.12. *Let $0 < \varepsilon \leq 2^{-2/w_2}$. For all $\mathbf{u} \in Q_\varepsilon$, $\mathbf{v} \in D(\mathbf{u}, \varepsilon)$ and $2 < t \leq 3$ one has*

$$(5.18) \quad \sum_{\mathbf{w} \in E(\mathbf{u}, \mathbf{v}, \varepsilon)} \left(\frac{|\mathbf{v}|}{|\mathbf{w}|} \right)^t \ll \frac{\varepsilon}{t-2}.$$

Proof. Let $\Lambda_{\mathbf{v}} = \pi_{\mathbf{v}}(\mathbb{Z}^3)$ and $\Lambda_{\mathbf{v}}^\circ = \Lambda_{\mathbf{v}} \setminus \pi_{\mathbf{v}}(H_{\mathbf{u}})$. Since $H_{\mathbf{u}} = H_{\mathbf{v}}$ according to Lemma 5.11, we have $\Lambda_{\mathbf{v}}^\circ = \Lambda_{\mathbf{v}} \setminus \pi_{\mathbf{v}}(H_{\mathbf{v}})$. Let $y \in \Lambda_{\mathbf{v}} \cap \pi_{\mathbf{v}}(H_{\mathbf{v}})$ with $\|y\|_w = r(\mathbf{v}) \leq \varepsilon/|\mathbf{v}|$. For each $k \in \mathbb{N}$, let

$$E_k = \{\mathbf{w} \in E(\mathbf{u}, \mathbf{v}, \varepsilon) : k|\mathbf{v}| \leq |\mathbf{w}| < (k+1)|\mathbf{v}|\}.$$

Since

$$\sum_{\mathbf{w} \in E(\mathbf{u}, \mathbf{v}, \varepsilon)} \left(\frac{|\mathbf{v}|}{|\mathbf{w}|} \right)^t = \sum_{k=1}^{\infty} \sum_{\mathbf{w} \in E_k} \left(\frac{|\mathbf{v}|}{|\mathbf{w}|} \right)^t \leq \sum_{k=1}^{\infty} \frac{\#E_k}{k^t},$$

it suffices to show that $\#E_k \ll \varepsilon k$. Let $\mathbf{w} \in E_k$ and write $z_{\mathbf{w}} = \pi_{\mathbf{v}}(\mathbf{w})$. Then $\mathbf{w} = (z_{\mathbf{w}} + |\mathbf{w}|\hat{\mathbf{v}}, |\mathbf{w}|)$ and hence by the definition of $E(\mathbf{u}, \mathbf{v}, \varepsilon)$

$$(5.19) \quad A(\hat{\mathbf{w}}, \mathbf{v}) = \|\mathbf{v}\|\|\mathbf{w}\|^{-1}z_{\mathbf{w}}\|_w \leq \varepsilon|\mathbf{w}|^{-1}.$$

Consider the convex set

$$M_k = \left\{ x \in \mathbb{R}^2 : \left\| \frac{1}{k+1}x \right\|_w \leq \frac{\varepsilon}{k|\mathbf{v}|} \right\}.$$

In view of (5.19) and the definition of E_k we have the inclusion $\pi_{\mathbf{v}}(E_k) \subseteq \Lambda_{\mathbf{v}} \cap M_k$. So $\#E_k \leq \#M_k \cap \Lambda_{\mathbf{v}}$ since $\pi_{\mathbf{v}}|_{E_k}$ is injective.

Note that $y \in \Lambda_{\mathbf{v}}$ is always in M_k . So if E_k is nonempty we have $\lambda_2(M_k, \Lambda_{\mathbf{v}}) \leq 1$. Hence by Lemma 3.3 we get

$$\#M_k \cap \Lambda_{\mathbf{v}} \ll \frac{\text{vol } M_k}{\text{cov } \Lambda_{\mathbf{v}}}.$$

Note that $\text{vol } M_k = \frac{4\varepsilon}{k|\mathbf{v}|}(k+1)^2$ and $\text{cov } \Lambda_{\mathbf{v}} = \frac{1}{|\mathbf{v}|}$. Therefore $\#E_k \leq \#M_k \cap \Lambda_{\mathbf{v}} \ll \varepsilon k$ as desired. \square

Now we estimate the upper bound of the Hausdorff dimension of $\text{Sing}(w)^*$.

Theorem 5.13. *There exists $C > 0$ such that for all $0 < \varepsilon \leq 2^{-2/w_2}$ the Hausdorff dimension of $\mathcal{F}(Q_\varepsilon, \sigma_\varepsilon, \beta)$ is less than or equal to*

$$(5.20) \quad 2 - \frac{1}{1+w_1} + C\sqrt{\varepsilon}.$$

Therefore the Hausdorff dimension of $\text{Sing}(w)^*$ and $\text{Sing}(w)$ is less than or equal to $2 - \frac{1}{1+w_1}$.

Proof. Let C' be the product of implied constants of (5.16) and (5.18). By Lemma 5.9 and the discussion at the beginning of §5 it suffices to show that the Hausdorff dimension of $\mathcal{F}(Q_\varepsilon, \sigma_\varepsilon, \beta)$ is less than or equal to (5.20) for $C = \frac{1}{1+w_1}\sqrt{C'}$. In view of Lemma 2.8 it suffices to show that for all

$$(5.21) \quad s > 2 - \frac{1}{1+w_1} + C\sqrt{\varepsilon}$$

one has

$$(5.22) \quad \sum_{\mathbf{w} \in \sigma_\varepsilon(\mathbf{u})} L(\mathbf{w}) \cdot W(\mathbf{w})^{s-1} \leq L(\mathbf{u}) \cdot W(\mathbf{u})^{s-1}$$

where $L(\mathbf{w}) = 2|\mathbf{w}|^{-w_2-1}$ and $W(\mathbf{w}) = 2|\mathbf{w}|^{-w_1-1}$.

Note that (5.22) is equivalent to

$$(5.23) \quad \sum_{\mathbf{w} \in \sigma_\varepsilon(\mathbf{u})} \left(\frac{|\mathbf{u}|}{|\mathbf{w}|} \right)^{(s-1)(w_1+1)+w_2+1} \leq 1.$$

By Lemmas 5.10 and 5.12, for all $t > 2$ one has

$$\sum_{\mathbf{w} \in \sigma_\varepsilon(\mathbf{u})} \left(\frac{|\mathbf{u}|}{|\mathbf{w}|} \right)^t \leq \sum_{\mathbf{v} \in D(\mathbf{u}, \varepsilon)} \left(\frac{|\mathbf{u}|}{|\mathbf{v}|} \right)^t \sum_{\mathbf{w} \in E(\mathbf{u}, \mathbf{v}, \varepsilon)} \left(\frac{|\mathbf{v}|}{|\mathbf{w}|} \right)^t \leq \frac{C' \cdot \varepsilon}{(t-2)^2}.$$

Plugging in $t = (s-1)(w_1+1) + w_2 + 1$ which is $> 2 + \sqrt{C'\varepsilon}$ by (5.21) in the above inequality, we get (5.23). \square

Proof of Theorem 1.1. The authentic weighted cases ($w_1 > w_2$) follow from Theorems 4.4 and 5.13 and the unweighted case ($w_1 = w_2$) is proved in [6]. \square

Proof of Theorem 1.3. We will use notations of §5.1. Let $\text{DI}(w, \varepsilon)^*$ be the set of $x \in \text{DI}(w, \varepsilon)$ such that $1, x_1, x_2$ are linearly independent over \mathbb{Q} . Let $x \in \text{DI}(w, \varepsilon)^*$ and let $\Sigma_x = \{\mathbf{u}_i\}_{i \in \mathbb{N}}$ be the fixed sequence of w -best approximates of x . It follows from definition that there exists $i_1 \in \mathbb{N}$ such that for $i \geq i_1$ one has

$$(5.24) \quad A(x, \mathbf{u}_i) < \frac{\varepsilon}{|\mathbf{u}_{i+1}|}.$$

On the other hand the first inequality of (5.8) implies that for all $i \in \mathbb{N}$

$$(5.25) \quad 2^{-1/w_2} r(\mathbf{u}_{i+1}) < A(x, \mathbf{u}_i).$$

It follows from (5.24) and (5.25) that for all $i \geq i_1$

$$r(\mathbf{u}_{i+1}) < \frac{2^{1/w_2} \varepsilon}{|\mathbf{u}_{i+1}|}.$$

Note that in the proof of Lemma 5.9, we only use (5.12) and the fact that $1, x_1, x_2$ are linearly independent over \mathbb{Q} . Therefore the same argument implies

$$x \in \mathcal{F}(Q_{\varepsilon 2^{3/w_2}}, \sigma_{\varepsilon 2^{3/w_2}}, \beta).$$

So we have

$$\text{DI}(w, \varepsilon)^* \subset \mathcal{F}(Q_{\varepsilon 2^{3/w_2}}, \sigma_{\varepsilon 2^{3/w_2}}, \beta).$$

By Theorem 5.13

$$(5.26) \quad \dim_H \text{DI}(w, \varepsilon)^* \leq 2 - \frac{1}{1 + w_1} + C\sqrt{\varepsilon}$$

where the constant C is independent of ε . The conclusion of Theorem 1.3 follows from (5.26) and the observation that $\text{DI}(w, \varepsilon) \setminus \text{DI}(w, \varepsilon)^*$ is contained in a countable union of lines in \mathbb{R}^2 . □

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LAMA UMR 8050 CNRS, UNIVERSITÉ PARIS-EST CRÉTEIL, 61 AVENUE DU GÉNÉRAL DE GAULLE, 94010 CRÉTEIL CEDEX, FRANCE

E-mail address: `lingmin.liao@u-pec.fr`

SHANGHAI CENTER FOR MATHEMATICAL SCIENCES, FUDAN UNIVERSITY, SHANGHAI 200433, PR CHINA

E-mail address: `ronggang@fudan.edu.cn`

SCHOOL OF MATHEMATICAL SCIENCES, TEL AVIV UNIVERSITY, TEL AVIV 69978, ISRAEL

E-mail address: `omrisola@mail.tau.ac.il`

SCHOOL OF MATHEMATICAL SCIENCES, TEL AVIV UNIVERSITY, TEL AVIV 69978, ISRAEL

E-mail address: `natalita@post.tau.ac.il`