HAUSDORFF DIMENSION OF DIVERGENT TRAJECTORIES ON HOMOGENEOUS SPACE

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ABSTRACT. For one parameter subgroup action on a finite volume homogeneous space, we consider the set of points admitting divergent on average trajectories. We show that the Hausdorff dimension of this set is strictly less than the manifold dimension of the homogeneous space. As a corollary we know that the Hausdorff dimension of the set of points admitting divergent trajectories is not full, which proves a conjecture of Y. Cheung [6].

1. Introduction

Let G be a connected Lie group, Γ be a lattice of G[1] and $F = \{f_t : t \in \mathbb{R}\}$ be a one parameter subgroup of G. The action of F on the homogeneous space G/Γ by left translation defines a flow. In this paper we consider the dynamics of the semiflow given by the action of $F^+ \overset{\text{def}}{=} \{f_t : t \geq 0\}$. For $x \in G/\Gamma$ we say the trajectory $F^+x \overset{\text{def}}{=} \{f_t x : t \geq 0\}$ is *divergent* if $f_t x$ leaves any fixed compact subset of G/Γ provided t is sufficiently large. We say F^+x is *divergent on average* if for any characteristic function $\mathbb{1}_K$ of a compact subset K of G/Γ one has

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{1}_K(f_t x)\,\mathrm{d}t = 0.$$

Clearly, if the trajectory F^+x is divergent, then it is divergent on average. The aim of this paper is to understand the set of divergent points

$$\mathfrak{D}'(F^+, G/\Gamma) \overset{\text{def}}{=} \{x \in G/\Gamma : F^+x \text{ is divergent}\},$$

and the set of divergent on average points

$$\mathfrak{D}(F^+, G/\Gamma) \overset{\text{def}}{=} \{x \in G/\Gamma : F^+x \text{ is divergent on average}\},$$

in terms of their Hausdorff dimensions. Here the Hausdorff dimension is defined by attaching G/Γ with a Riemannian metric. It is well-known that different choices of Riemannian metrics will not affect the Hausdorff dimension of subsets of G/Γ. Indeed, specific Riemannian metric will be used later for the sake of convenience.

According to the work of Margulis [22] and Dani [9][10], it is well-known that if F is Ad-unipotent then the space G/Γ admits no divergent on average trajectories of F^+. In other words, the set $\mathfrak{D}(F^+, G/\Gamma)$, hence the set $\mathfrak{D}'(F^+, G/\Gamma)$, is empty. On the other hand, the set $\mathfrak{D}(F^+, G/\Gamma)$ can be complicated when F is

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[1]A discrete subgroup $\Gamma < G$ is called a lattice if there exists a finite G-invariant measure on the homogeneous space G/Γ.

Ad-diagonalizable. For example, it was proved by Cheung in [6] that the Hausdorff dimension of $\mathfrak{D}'(F^+, \mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z}))$ with $F = \{\mathrm{diag}(e^t, e^t, e^{-2t}) : t \in \mathbb{R}\}$ is equal to $7\frac{1}{3}$. Based on his results, Cheung raised the following conjecture in [6].

Conjecture 1.1. *Let Γ be a lattice of a connected Lie group G and let $F = \{f_t : t \in \mathbb{R}\}$ be a one parameter subgroup of G. Then the Hausdorff dimension of $\mathfrak{D}'(F^+, G/\Gamma)$ is strictly less than the manifold dimension of G/Γ.*

The conjecture is known to be true in the following cases where G is a semisimple Lie group without compact factors and F is Ad-diagonalizable:

(1) G is of rank one [11].
(2) $G = \prod_{i=1}^n \mathrm{SO}(n,1)$, $\Gamma = \prod_{i=1}^n \Gamma_i$ with each Γ_i lattice in $\mathrm{SO}(n,1)$ and $F < G$ the is diagonal embedding of any one parameter real split torus A of $\mathrm{SO}(n,1)$ [5][26].
(3) $G/\Gamma = \mathrm{SL}_{m+n}(\mathbb{R})/\mathrm{SL}_{m+n}(\mathbb{Z})$ and $F = F_{n,m} = \{\mathrm{diag}(e^{nt}, \ldots, e^{nt}, e^{-mt}, \ldots, e^{-mt}) : t \in \mathbb{R}\}$ with $m, n \geq 1$ [6][7][18].

Indeed, for all the cases listed above, the Hausdorff dimension of the corresponding $\mathfrak{D}'(F^+, G/\Gamma)$ have been determined.

There are evidences that a stronger version of this conjecture is true. It was proved by Einsiedler-Kadyrov in [13] that the Hausdorff dimension of $\mathfrak{D}(F^+, \mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z}))$ is at most $7\frac{1}{3}$ when $F = F_{1,2}$ as in (3). Using the contraction property of the height function introduced in [16], it was proved by Kadyrov, Kleinbock, Lindenstrauss and Margulis in [18] that for any $m, n \geq 1$, the Hausdorff dimension of $\mathfrak{D}(F^+, \mathrm{SL}_{m+n}(\mathbb{R})/\mathrm{SL}_{m+n}(\mathbb{Z}))$ is at most $\dim G - \frac{mn}{m+n}$ when $F = F_{n,m}$ as in (3). See also [14][17][19][21][12][27] for related results.

Now we state the main result of this paper, from which Cheung's conjecture follows.

Theorem 1.2. *Let Γ be a lattice of a connected Lie group G and let $F = \{f_t : t \in \mathbb{R}\}$ be a one parameter subgroup of G. Then the Hausdorff dimension of $\mathfrak{D}(F^+, G/\Gamma)$ is strictly less than the manifold dimension of G/Γ.*

We will reduce the proof of Theorem 1.2 to the special case where G is a semisimple linear group. Recall that a connected semisimple Lie group G contained in $\mathrm{SL}_k(\mathbb{R})$ has a natural structure of real algebraic group. So terminologies of algebraic groups have natural meanings for G and are independent of the embeddings of G into $\mathrm{SL}_k(\mathbb{R})$. In particular, the one parameter group F has the following real Jordan decomposition which is a special case of [3, Theorem 4.4].

Lemma 1.3. *Let $G \leq \mathrm{SL}_k(\mathbb{R})$ be a connected semisimple Lie group. For any one parameter subgroup $F = \{f_t : t \in \mathbb{R}\}$, there are uniquely determined one parameter subgroups $K_F = \{k_t : t \in \mathbb{R}\}$, $A_F = \{a_t : t \in \mathbb{R}\}$ and $U_F = \{u_t : t \in \mathbb{R}\}$ with the following properties:*

- *$f_t = k_t a_t u_t$.*
- *K_F is bounded, A_F is $\mathbb{R}$-diagonalizable and U_F is unipotent.*
- *All the elements of K_F, A_F and U_F commute with each other.*

The subgroups K_F, A_F and U_F are called compact, diagonal and unipotent parts of F, respectively. In §2 we will reduce the proof of Theorem 1.2 to its following special case which contains the main unknown situations.

Theorem 1.4. *Let $G \leq \mathrm{SL}_k(\mathbb{R})$ be a connected center-free semisimple Lie group without compact factors. Let $F = \{f_t : t \in \mathbb{R}\}$ be a one parameter subgroup of G such that the compact part K_F is trivial but the diagonal part A_F is nontrivial. We assume the followings hold:*

- *$G = \prod_{i=1}^m G_i$ is a direct product of connected normal subgroups G_i.*
- *$\Gamma = \prod_{i=1}^m \Gamma_i$ where each Γ_i is a nonuniform irreducible lattice of G_i.*
- *The group A_F has nontrivial projection to each G_i.*

Then the Hausdorff dimension of $\mathfrak{D}(F^+, G/\Gamma)$ is strictly less than the manifold dimension of G/Γ.

The proof of Theorem 1.4 is from §3 till the end of the paper. Indeed, the upper bound of the Hausdorff dimension in the setting of Theorem 1.4 can be explicitly calculated and we will make this point clear during the proof.

Our main tool will be the Eskin-Margulis height function (abbreviated as EM height function) introduced in [15]. If F is diagonalizable, i.e. $F = A_F$, Theorem 1.2 can be established using the strategy developed in [18] and the contraction property of the proved in [25]. But when F has nontrivial unipotent parts, i.e. U_F is nontrivial, essential new ideas are needed. The following example contains the main difficulties we need to handle in the proof of Theorem 1.4: $G = \mathrm{SL}_4(\mathbb{R}) \times \mathrm{SL}_4(\mathbb{R}), \Gamma = \mathrm{SL}_4(\mathbb{Z}[\sqrt{2}])$ which embeds in G diagonally via Galois conjugates, and

$$f_t = \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

There are two main difficulties. One is caused by the unipotent part of f_t in the first $\mathrm{SL}_4(\mathbb{R})$ factor, and the other is caused by the unipotent part of f_t in the second $\mathrm{SL}_4(\mathbb{R})$ factor. To overcome these difficulties, we will prove a uniform contraction property for a family of one parameter subgroups in §3 and §4 with respect to the EM height function. Then the last two sections are devoted to the proof of Theorem 1.4.

2. Proof of Theorem 1.2

In this section we prove Theorems 1.2 assuming Theorem 1.4. Let G, Γ, F be as in Theorem 1.2. We choose and fix a Euclidean norm $\|\cdot\|$ on the Lie algebra $\mathfrak{g}$ of G, which induces a right invariant Riemannian metric $\mathrm{dist}(\cdot,\cdot)$ on G. Moreover, this metric naturally induce a metric on G/Γ, also denoted by "dist", as follows:

$$\mathrm{dist}(g\Gamma, h\Gamma) = \inf_{\gamma \in \Gamma} \mathrm{dist}(g\gamma, h) \quad \text{where } g, h \in G.$$

Let $\mathfrak{r}$ be the maximal amenable ideal of the Lie algebra $\mathfrak{g}$ of G, i.e. the largest ideal whose analytic subgroup is amenable. The adjoint action of G on $\mathfrak{s} = \mathfrak{g}/\mathfrak{r}$ defines a homomorphism $\pi : G \mapsto \mathrm{Aut}(\mathfrak{s})$. Let S be the connected component of $\mathrm{Aut}(\mathfrak{s})$. It follows from the Levi decomposition of G that $\pi(G) = S$ and S is a center-free semisimple Lie group without compact factors. It is known that $\Gamma \cap \mathrm{Ker}(\pi)$ is a cocompact lattice in $\mathrm{Ker}(\pi)$ and $\pi(\Gamma)$ is a lattice in S, see e.g. [1, Lemma 6.1].

Therefore, the induced map $\overline{\pi} : G/\Gamma \mapsto S/\pi(\Gamma)$ is proper, and consequently

$$\overline{\pi}(\mathfrak{D}(F^+, G/\Gamma)) = \mathfrak{D}(\pi(F^+), S/\pi(\Gamma)). \tag{2.1}$$

Let $\varphi : \mathfrak{s} \to \mathfrak{g}$ be an embedding of Lie algebras such that $\mathrm{d}\pi \circ \varphi$ is the identity map. It follows from (2.1) that for any $x \in G/\Gamma$ and any $v \in \mathfrak{s}$

$$\exp(v)\overline{\pi}(x) \in \mathfrak{D}(\pi(F^+), S/\pi(\Gamma))$$

if and only if

$$\exp(\varphi(v)) \exp(v')x \in \mathfrak{D}(F^+, G/\Gamma) \quad \text{for all } v' \in \mathfrak{r}.$$

By Marstrand's product theorem, the Hausdorff dimension of the product of two sets of Euclidean spaces is bounded from above by the sum of the Hausdorff dimension of one set and the packing dimension of the other, e.g. [4, Theorem 3.2.1]. So to prove Theorem 1.2 it suffices to give a nontrivial upper bound of the Hausdorff dimension of $\mathfrak{D}(\pi(F^+), S/\pi(\Gamma))$.

We summarize what we have obtained as follows.

Lemma 2.1. *Theorem 1.2 is equivalent to its special case where G is a center-free semisimple Lie group without compact factors.*

Proof of Theorem 1.2. According to Lemma 2.1, it suffices to prove the theorem under the additional assumption that G is a center-free semisimple Lie group without compact factors. Under this assumption, the adjoint representation $\mathrm{Ad} : G \to \mathrm{SL}(\mathfrak{g})$ is a closed embedding. According to the real Jordan decomposition in Lemma 1.3, the compact part K_F does not affect the divergence on average property of the trajectories. So we assume without loss of generality that K_F is trivial.

There exist finitely many connected semisimple subgroups G_i such that $G = \prod_i G_i$ and $\Gamma_i \overset{\text{def}}{=} \Gamma \cap G_i$ is an irreducible lattice of G_i for each i. It follows that $\prod_i \Gamma_i$ is a finite index subgroup of Γ and the natural quotient map $G/\prod_i \Gamma_i \to G/\Gamma$ is proper. So we assume moreover that $\Gamma = \prod_i \Gamma_i$.

Denote by π_j the projection of G to $G/G_j = \prod_{i \neq j} G_i$ and denote by $\overline{\pi}_j$ the induced map from G/Γ to $\pi_j(G)/\pi_j(\Gamma) = \prod_{i \neq j} G_i/\Gamma_i$. Here if $G = G_j$ we interpret $\prod_{i \neq j} G_i$ as a trivial group and $\prod_{i \neq j} G_i/\Gamma_i$ as a singe point set. If G_j/Γ_j is compact or the projection of A_F to G_j contains only the neutral element, then

$$\mathfrak{D}(F^+, G/\Gamma) = \overline{\pi}_j^{-1}\Big(\mathfrak{D}\big(\pi_j(F^+), \prod_{i \neq j} G_i/\Gamma_i\big)\Big).$$

So either $\mathfrak{D}(F^+, G/\Gamma)$ is an empty set or we finally can reduce the problem to the setting of Theorem 1.4 where each Γ_i is a nonuniform lattice and the projection of A_F to each G_i is nontrivial. This completes the proof.

□

3. Preliminary on linear representations

From this section, we start the proof of Theorem 1.4. At the beginning of each section we will set up some notation that will be used later. Let G and F be as in Theorem 1.4. Let $A_F = \{a_t : t \in \mathbb{R}\}$ and $U_F = \{u_t : t \in \mathbb{R}\}$ be the diagonal and unipotent parts of F. Let H be the unique connected normal subgroup of G such that $A_F \leq H$ and the projection of A_F to each simple factor of H is nontrivial.

Since A_F is nontrivial and G is center-free, H is (nontrivial) product of some simple factors of G. Hence H is a semisimple Lie group without compact factors. Let S be the product of simple factors of G not contained in H. Then S is also semisimple normal subgroup of G that commutes with H. Moreover, $G = HS$ and $H \cap S = 1_G$, where 1_G is the neutral element of G.

In this section we prove a couple of auxiliary results for a finite dimensional linear representation $\rho : G \to \mathrm{GL}(V)$ on a (nonzero real) normed vector space V. These results will be used in the next section to prove the uniform contracting property of EM height function. We will use $\|\cdot\|$ to denote the norm on V.

For $\lambda \in \mathbb{R}$, we denote the λ-Lyapunov subspace of A_F by

$$V^\lambda = \{v \in V : \rho(a_t)v = e^{\lambda t}v\}.$$

Recall that if $V^\lambda \neq \{0\}$, then λ is a called an Lyapunov exponent of (ρ, V). Since U_F commutes with A_F, every Lyapunov subspace V^λ is U_F-invariant. As A_F is $\mathbb{R}$-diagonalizable, the space V can be decomposed as $V^+ \oplus V^0 \oplus V^-$ where

$$V^+ = \oplus_{\lambda>0} V^\lambda \quad \text{and} \quad V^- = \oplus_{\lambda<0} V^\lambda.$$

Now we consider the adjoint representation of G on the Lie algebra $\mathfrak{g}$ of G. It is easily checked that $\mathfrak{g}^+, \mathfrak{g}^-$ and $\mathfrak{g}^c \overset{\text{def}}{=} \mathfrak{g}^0$ are subalgebras of $\mathfrak{g}$. The connected subgroup G^+ (resp. G^-) with Lie algebras $\mathfrak{g}^+$ (resp. $\mathfrak{g}^-$) is called unstable (resp. stable) horospherical subgroup of a_1. We denote the connected component of the centralizer of a_1 in G by G^c whose Lie algebra is $\mathfrak{g}^c$. Let d, d^c, d^- be the manifold dimensions of G^+, G^c and G^-, respectively. It follows from the nontriviality of A_F that $d > 0$.

For $r \geq 0$ we let $B_r^G = \{h \in G : \mathrm{dist}(h, 1_G) < r\}$, $B_r^\pm = \{h \in G^\pm : \mathrm{dist}(h, 1_G) < r\}$ and $B_r^c = \{h \in G^c : \mathrm{dist}(h, 1_G) < r\}$. By rescaling the Riemannian metric if necessary, we may assume that:

(1) the product map $B_1^- \times B_1^c \times B_1^+ \to G$ is a diffeomorhism onto its image,
(2) and the logarithm map is well-defined on B_1^G and is a diffeomorphism onto its image.

According to (1), it is safe to identity the product $B_1^- \times B_1^c \times B_1^+$ with $B_1^- B_1^c B_1^+$ and we will mainly use the later notation for sake of convenience. The same statement as (2) also holds for $B_1^\pm$ and B_1^c.

We fix a Haar measure μ on G^+ normalized with $\mu(B_1^+) = 1$. Since the metric "dist" is right invariant, any open ball of radius r in G^+ has the form $B_r^+ h$ ($h \in G^+$) and there exits $C_0 \geq 1$ such that

$$C_0^{-1} r^d \leq \mu(B_r^+ h) = \mu(B_r^+) \leq C_0 r^d \quad \text{for all } 0 \leq r \leq 1. \tag{3.1}$$

For $g, h \in G$ we let $g^h = h^{-1}gh$. For $z \in G^c$, let $F_z = \{f_t^z : t \in \mathbb{R}\}$ and $F_z^+ = \{f_t^z : t \geq 0\}$. Note that $f_t^z = a_t u_t^z$ and

$$\{u_1^z : z \in B_1^c\} \quad \text{is relatively compact.} \tag{3.2}$$

Lemma 3.1. *Let $\rho : G \to \mathrm{GL}(V)$ be a representation on a finite dimensional normed vector space V. Let λ be a Lyapunov exponent of (ρ, V). For any $0 < \delta < 1$, there exists $T_\delta > 0$ such that, for all $t \geq T_\delta$, $z \in B_1^c$ and unit vector $v \in V^\lambda$ we have*

$$e^{(1-\delta)\lambda t} \leq \|\rho(f_t^z)v\| \leq e^{(1+\delta)\lambda t}. \tag{3.3}$$

Proof. For all $v \in V^\lambda$ with $\|v\| = 1$ we have $\|\rho(a_t)v\| = e^{\lambda t}$. On the other hand, in view of (3.2), there exists $C > 0$ and $n \in \mathbb{N}^2$ such that

$$\|\rho(u_t^z)\| \le C(|t| + 1)^n$$

for all $z \in B_1^c$ and $t \in \mathbb{R}$. Therefore, for any unit vector $v \in V^\lambda, z \in B_1^c$ and sufficiently large t,

$$\|\rho(f_t^z)v\| \ge \|\rho(u_{-t}^z)\|^{-1}\|\rho(a_t)v\| \ge C^{-1}(|t| + 1)^{-n}e^{\lambda t} \ge e^{(1-\delta)\lambda t},$$
$$\|\rho(f_t^z)v\| \le \|\rho(u_t^z)\|\|\rho(a_t)v\| \le C(|t| + 1)^n e^{\lambda t} \le e^{(1+\delta)\lambda t}.$$

□

From now on till the end of this section, we assume that $\rho : G \to \mathrm{GL}(V)$ is a representation on a finite dimensional normed vector space V which has no nonzero H-invariant vectors. As any two norms on V are equivalent, we also assume that the norm is Euclidean without loss of generality.

Recall that a nonzero H-invariant subspace V' of V is said to be H-irreducible if V' contains no H-invariant subspaces besides $\{0\}$ and itself. The complete reducibility of representations of H implies that there exists a unique decomposition (called H-isotropic decomposition)

$$V = V_1 \oplus \cdots \oplus V_m \tag{3.4}$$

such that irreducible sub-representations of H in the same V_i are isomorphic but irreducible sub-representations in different V_i are non-isomorphic. Since S commutes with H, each V_i is S-invariant, and hence G-invariant. Each V_i is called an H-isotropic subspace of V.

Let λ_i be the top Lyapunov exponent of A_F in (ρ, V_i), i.e.,

$$\lambda_i = \max\{\lambda \in \mathbb{R} : V_i^\lambda \ne \{0\}\}.$$

Since the projection of A_F to each simple factor of H is nontrivial, every λ_i is positive. Let λ be the minimum of top Lyapunov exponents in each V_i, i.e.

$$\lambda = \min\{\lambda_i : 1 \le i \le m\} > 0. \tag{3.5}$$

Let $\pi_i : V_i \to V_i^{\lambda_i}$ be the A_F-equivariant projection.

Lemma 3.2. *For all $v \in V_i \setminus \{0\}$, the map*

$$\varphi_v : G^+ \mapsto \mathbb{R} \quad \textit{where} \quad \varphi_v(h) = \|\pi_i(\rho(h)v)\|^2 \tag{3.6}$$

is not identically zero.

Proof. Suppose φ_v is identically zero. Then $\rho(G^+)v \subset V_i'$ where $V_i' \subset V_i$ is the A_F-invariant complimentary subspace of $V_i^{\lambda_i}$. This implies that $\rho(G^- G^c G^+)v \subset V_i'$. Since $G^- G^c G^+$ contains an open dense subset of G, see e.g. [23, Proposition 2.7], we moreover have that $\rho(G)v \subset V_i'$. This is impossible since the intersection of $V_i^{\lambda_i}$ with each H-invariant subspace of V_i is nonzero. This contradiction completes the proof. □

[2] Here $\mathbb{N} = \{1, 2, 3, \dots\}$.

Lemma 3.3. *For all $v \in V_i \smallsetminus \{0\}$ and $r \geq 0$, let*

$$E(v,r) = \{h \in B_1^+ : \|\pi_i(\rho(h)v)\| \leq r\}.$$

Then there exists $\theta_i > 0$ such that

$$C_i \overset{\text{def}}{=} \sup_{\|v\|=1, v\in V_i} r^{-\theta_i}\mu(E(v,r)) < \infty. \tag{3.7}$$

In particular, $\mu(E(v,0)) = 0$.

Proof. Since G^+ is a unipotent group, it is simply connected and by [8, Theorem 1.2.10 (a)] there is an isomorphism of affine varieties $\mathbb{R}^d \to G^+$ such that the Lebesgue measure of $\mathbb{R}^d$ corresponds to the Haar measure μ. During the proof, we will identify the group G^+ with $\mathbb{R}^d$ for convenience.

By Lemma 3.2, for every nonzero $v \in V_i$ the map φ_v in (3.6) is a nonzero polynomial map. So $\varphi_v|_{B_1^+}$ is nonzero. Note that the degrees of φ_v ($v \in V_i$) are uniformly bounded from above. Therefore, the (C,α)-good property of polynomials in [2, §3] implies that there exist positive constants C and α such that

$$\mu(E(v,r)) \leq C\left(\frac{r^2}{\sup_{h\in B_1^+}\varphi_v(h)}\right)^{\alpha} \tag{3.8}$$

for all nonzero $v \in V_i$. Since the set of unit vectors of V_i is compact,

$$\inf_{\|v\|=1, v\in V_i} \sup_{h\in B_1^+} \varphi_v(h) > 0. \tag{3.9}$$

So (3.7) follows from (3.8) and (3.9) by taking $\theta_i = 2\alpha$. $\square$

Remark 3.4. According to [2, Lemma 3.2] we have $\alpha = \frac{1}{dl}$ where d is the manifold dimension of G^+ and l is a uniform upper bound of the degree of φ_v ($v \in V_i$). So the constant θ_i can be calculated explicitly.

Lemma 3.5. *Let $\theta_0 = \min_{1\leq i\leq m}\theta_i$ where $\theta_i > 0$ so that Lemma 3.3 holds and let λ be as in (3.5). Then for any $0 < \delta < \theta < \theta_0$, there exists $T_{\theta,\delta} > 0$ such that for all $t \geq T_{\theta,\delta}$, $z \in B_1^c$ and $v \in V$ with $\|v\| = 1$, we have*

$$\int_{B_1^+} \|\rho(f_t^z h)v\|^{-\theta} d\mu(h) \leq e^{-(\theta-\delta)\lambda t}. \tag{3.10}$$

Proof. Without loss of generality, we assume further that the Euclidean norm $\|\cdot\|$ on V satisfies the following properties:

- Lyapunov subspaces of A_F are orthogonal to each other.
- H-isotropic subspaces V_i ($1 \leq i \leq m$) are orthogonal to each other.

Let

$$R_i = \sup_{v\in V_i, \|v\|=1, h\in B_1^+} \|\pi_i(\rho(h)v)\| \quad \text{and} \quad R = \max\{R_i : 1 \leq i \leq m\}. \tag{3.11}$$

Let $C = \max\{C_i : 1 \leq i \leq m\}$ where C_i is given in (3.7). Let $\theta' = \max\{\theta_i : 1 \leq i \leq m\}$.

According to Lemma 3.1, there exists $T_{\frac{\delta}{2\theta}} > 0$ such that (3.3) holds for any $t \geq T_{\frac{\delta}{2\theta}}$, any nonzero $v \in V^{\lambda_i}$ $(1 \leq i \leq m)$ and any $z \in B_1^c$, i.e.,

$$\|\rho(f_t^z)v\|^{-\theta} \leq e^{-(1-\frac{\delta}{2\theta})\theta\lambda_i t}\|v\|^{-\theta} \leq e^{-(\theta-\frac{\delta}{2})\lambda t}\|v\|^{-\theta}.$$

This inequality and the assumption of the norm implies that for all nonzero $v \in V_i$ and $t \geq T_{\frac{\delta}{2\theta}}$

$$\|\rho(f_t^z h)v\|^{-\theta} \leq e^{-(\theta-\frac{\delta}{2})\lambda t}\|\pi_i(\rho(h)v)\|^{-\theta}, \tag{3.12}$$

where $\frac{1}{0}$ is interpreted as ∞. Let $T_{\theta,\delta} \geq T_{\frac{\delta}{2\theta}}$ be a large enough real number so that $t \geq T_{\theta,\delta}$ implies

$$\frac{(2m)^{\theta'}CR^{\theta'-\theta}}{1-2^{\theta-\theta_0}}e^{-(\theta-\frac{\delta}{2})\lambda t} \leq e^{-(\theta-\delta)\lambda t}. \tag{3.13}$$

Let v be a unit vector of V. We write $v = v_1 + \cdots + v_m$ where $v_i \in V_i$. Since we assume different V_i are orthogonal to each other, there exists an integer $i \in [1, m]$ such that $m\|v_i\| \geq \|v\| = 1$.

There is a disjoint union decomposition of B_1^+ as

$$E(v_i, 0) \cup \left(\cup_{n\geq 0} E^+(v_i, 2^{-n}R_i)\right),$$

where

$$E^+(v_i, 2^{-n}R_i) = E(v_i, 2^{-n}R_i) \smallsetminus E(v_i, 2^{-n-1}R_i).$$

Since $\mu(E(v_i, 0)) = 0$, for any $z \in B_1^c$ and $t \geq T_{\theta,\delta}$ we have

$$\begin{aligned}
\int_{B_1^+} \|\rho(f_t^z h)v\|^{-\theta} d\mu(h) &\leq \sum_{n=0}^{\infty} \int_{E^+(v_i,2^{-n}R_i)} \|\rho(f_t^z h)v_i\|^{-\theta} d\mu(h) \\
\text{(by (3.12))} \quad &\leq e^{-(\theta-\frac{\delta}{2})\lambda t} \sum_{n=0}^{\infty} \int_{E^+(v_i,2^{-n}R_i)} \|\pi_i(\rho(h)v_i)\|^{-\theta} d\mu(h) \\
\text{(by (3.7))} \quad &\leq e^{-(\theta-\frac{\delta}{2})\lambda t} \sum_{n=0}^{\infty} C_i 2^{\theta}(2^{-n}R_i)^{\theta_i-\theta}\|v_i\|^{-\theta_i} \\
&\leq \frac{m^{\theta'}2^{\theta'}CR^{\theta'-\theta}}{1-2^{\theta-\theta_0}}e^{-(\theta-\frac{\delta}{2})\lambda t} \\
\text{(by (3.13))} \quad &\leq e^{-(\theta-\delta)\lambda t}.
\end{aligned}$$

□

4. Eskin-Margulis height function

Let the notation be as in Theorem 1.4. In this section, we will establish a uniform contraction property of the EM height function on G/Γ with respect to a family of one parameter groups F_z $(z \in B_1^c)$.

Recall that $G/\Gamma = \prod_{i=1}^m G_i/\Gamma_i$ where each G_i/Γ_i is a nonuniform irreducible quotient of a semisimple Lie group without compact factors. Since we assume the projection of A_F to each G_i is nontrivial, we have $H = \prod_{i=1}^m H_i$, where $H_i = G_i \cap H$ is a connected normal subgroup of G_i with positive dimension.

Let us recall the definition of the EM height function from [15]. The EM height function is constructed on each G_i/Γ_i using a finite set Δ_i of Γ_i-rational parabolic subgroups of G_i. Recall that a parabolic subgroup P of G_i is Γ_i-rational if the unipotent radical of P intersects Γ_i in a lattice. If the rank of G_i is bigger than one, then Margulis' arithmeticity theorem implies that there is a $\mathbb{Q}$-structure on G_i such that Γ_i is commensurable with $G_i(\mathbb{Z})$. In this case the set Δ_i consists of standard $\mathbb{Q}$-rational maximal parabolic subgroups of G_i with respect to a fixed $\mathbb{Q}$-split torus and fixed positive roots. So the irreducibility of Γ_i implies that no conjugates of H_i is contained in any $P \in \Delta_i$. The same conclusion holds in the case where G_i has rank one. The reason is that in this case $H_i = G_i$ and $\Delta_i = \{P\}$ where P is a maximal parabolic subgroup defined over $\mathbb{R}$.

For each $P_{i,j} \in \Delta_i$, there exists a representation $\rho_{i,j} : G_i \to \mathrm{GL}(V_{i,j})$ on a normed vector space and a nonzero vector $w_{i,j} \in V_{i,j}$ such that the stabilizer of $\mathbb{R}w_{i,j}$ is $P_{i,j}$. We consider $\rho_{i,j}$ as a representation of G so that $\rho(G_s)$ is the identity linear map if $s \neq i$. Let $V_{i,j}^H$ be the H-invariant subspace of $V_{i,j}$ consisting of H-invariant vectors. Let $\pi_{i,j}$ be the projection of $V_{i,j}$ to the H-invariant subspace $V'_{i,j}$ complementary to $V_{i,j}^H$. Since no conjugates of H_i is contained in $P_{i,j}$ and $G_i = K_i P_{i,j}$ for some maximal compact subgroup K_i of G_i, there exists $C \geq 1$ such that

$$\|v\| \leq C\|\pi_{i,j}(v)\|$$

for all $v \in \rho_{i,j}(G)w_{i,j}$. Note that $V'_{i,j}$ is G-invariant and it has no nonzero H-invariant vectors. Therefore, Lemma 3.5 implies the following lemma which corresponds to **Condition A** in [15].

Lemma 4.1. *For each pair of index i, j there exist positive constants $\theta_0^{i,j}$ and $\lambda^{i,j}$ such that for any $0 < \delta < \theta < \theta_0^{i,j}$, any nonzero $v \in \rho_{i,j}(G)w_{i,j}$ and any $z \in B_1^c$ one has*

$$\int_{B_1^+} \|\rho_{i,j}(f_t^z h)v\|^{-\theta}\, \mathrm{d}h \leq e^{-(\theta-\delta)t\lambda^{i,j}}\|v\|^{-\theta} \tag{4.1}$$

provided $t \geq T_{\theta,\delta}^{i,j}$ where $T_{\theta,\delta}^{i,j} > 0$ is a constant depending on θ and δ.

Proof. We assume without loss of generality that for all $V_{i,j}$ the norm $\|\cdot\|$ is Euclidean and $V_{i,j}^H$ and $V'_{i,j}$ are orthogonal to each other. According to Lemma 3.5, for each representation $\rho_{i,j}|_{V'_{i,j}}$, there exist positive constants $\theta_0^{i,j}$ and $\lambda^{i,j}$ with the following properties: for any $0 < \delta < \theta < \theta_0^{i,j}$ there exists $T_{\theta,\delta} > 0$ such that for any $t \geq T_{\theta,\delta}$, $z \in B_1^c$ and any nonzero $v \in \rho_{i,j}(G)w_{i,j}$, one has

$$\begin{aligned}\int_{B_1^+} \|\rho_{i,j}(f_t^z h)v\|^{-\theta}\, \mathrm{d}h &\leq \int_{B_1^+} \|\rho_{i,j}(f_t^z h)\pi_{i,j}(v)\|^{-\theta}\, \mathrm{d}h \\ &\leq e^{-(\theta-\delta)t\lambda^{i,j}}\|\pi_{i,j}(v)\|^{-\theta} \\ &\leq C^\theta e^{-(\theta-\delta)t\lambda^{i,j}}\|v\|^{-\theta}.\end{aligned}$$

It is not hard to see from above estimate that (4.1) holds for sufficiently large t. □

Besides $\rho_{i,j}$, the EM height function is constructed using positive constants $c_{i,j}$ and $q_{i,j}$ which are combinatorial data determined by the root system, see [15, (3.22),(3.28)]. Let

$$u_{i,j}(g\Gamma) = \max_{\gamma\in\Gamma} \frac{1}{\|\rho_{i,j}(g\gamma)w_{i,j}\|^{1/c_{i,j}q_{i,j}}} \tag{4.2}$$

where $g \in G$.[3] Let

$$\theta_1 = \max\{\theta > 0 : \frac{\theta}{q_{ij}c_{i,j}} \le \theta_0^{i,j} \quad \text{for all } i,j\} \quad \text{and} \quad \alpha_1 = \min_{i,j}\{\frac{\theta_1}{q_{ij}c_{i,j}}\lambda^{i,j}\}, \tag{4.3}$$

where $\theta_0^{i,j}$ and $\lambda^{i,j}$ are constants given by Lemma 4.1. We call α_1 a contraction rate for the dynamical system $(G/\Gamma, F^+)$.

Remark 4.2. We will see in next sections that α_1 plays an important role in bounding the Hausdorff dimension of $\mathfrak{D}(F^+, G/\Gamma)$. We believe that optimal α_1 is possible to give the sharp bound of the dimension. By Remark 3.4, the constant $\theta_{i,j}$ can be explicitly calculated, so are the constants θ_1 and α_1. Consequently, it will be clear in the proof in the next sections that the upper bound of the dimension we obtain can also be explicitly calculated, although not optimal.

Lemma 4.3. *For every $\alpha < \alpha_1$, there exist $0 < \theta < \theta_1$ and $T > 0$ such that for all $t \ge T$ and ϵ sufficiently small depending on t, the EM height function*

$$u : G/\Gamma \to (0,\infty) \quad \text{defined by } u(x) = \sum_{i,j} (\epsilon\, u_{i,j}(x))^\theta \tag{4.4}$$

satisfies the following properties:

(1) *$u(x) \to \infty$ if and only if $x \to \infty$ in G/Γ.*
(2) *For any compact subset K of G, there exists $C \ge 1$ such that $u(hx) \le Cu(x)$ for all $h \in K$ and $x \in G/\Gamma$.*
(3) *There exists $b > 0$ depending on t such that for all $z \in B_1^c$ and $x \in G/\Gamma$ one has*

$$\int_{B_1^+} u(f_t^z hx)\, \mathrm{d}\mu(h) < e^{-\alpha t}u(x) + b. \tag{4.5}$$

(4) *There exists $\ell \ge 1$ such that if $u(x) \ge \ell$, then for all $z \in B_1^c$*

$$\int_{B_1^+} u(f_t^z hx)\, \mathrm{d}\mu(h) < e^{-\alpha t}u(x). \tag{4.6}$$

Proof. It follows from the corresponding results for each G_i/Γ_i proved in [15] that the first two conclusions hold for any choice of θ and ϵ. Note that (4) is a direct corollary of (3).

Now we prove (3). Let n the cardinality of the indices i,j appeared in the definition of u. We fix $\delta > 0$ sufficiently small such that

$$\alpha + \delta + \frac{\delta\lambda^{i,j}}{c_{i,j}q_{i,j}} < \alpha_1 \quad \forall\ i,j.$$

[3] Although only the product $c_{i,j}q_{i,j}$ is used in this paper, the constants $c_{i,j}$ and $q_{i,j}$ are given by different combinatorial data and we use both of them for the consistency with [15].

According to the definitions of θ_1, α_1 and the choice of δ above, there exists $\theta > 0$ such that

$$\theta < \theta_1 \quad \text{and} \quad \frac{(\theta - \delta)\lambda^{i,j}}{c_{i,j} q_{i,j}} \geq \alpha + \delta \quad \forall\, i, j.$$

Let $\delta_{i,j} = \delta / c_{i,j} q_{i,j}$, $\theta_{i,j} = \theta / c_{i,j} q_{i,j}$, then according to Lemma 4.1 there exists $T^{i,j} > 0$ such that for $t \geq T^{i,j}$ one has (4.1) holds with $\delta = \delta_{i,j}$ and $\theta = \theta_{i,j}$. We will show that Lemma 4.3 holds for $T = \frac{\log 2}{\delta} + \max_{i,j} T^{i,j}$.

Now we fix $0 < \epsilon < 1$, $x = g\Gamma \in G/\Gamma, t \geq T$ and i, j. According to the definition of $u_{i,j}(x)$, there exists $\gamma \in \Gamma$ such that $u_{i,j}(x) = \frac{1}{\|\rho(g\gamma) w_{i,j}\|^{1/c_{i,j} q_{i,j}}}$. For any $h \in B_1^+$ and $z \in B_1^c$, if $u_{i,j}(f_t^z h x) = \frac{1}{\|\rho(f_t h g \gamma) w_{i,j}\|^{1/c_{i,j} q_{i,j}}}$, then we can use (4.1). Otherwise, there exist $0 < \kappa < 1$, $b > 0$ and $C' \geq 1$ where b and C' depend on t such that

$$(\epsilon u_{i,j}(f_t^z h x))^\theta \leq C' \epsilon^\kappa (\epsilon u(x))^\theta + \frac{b}{n}.$$

These facts are proved in [15, §3.2]. In summary, we have

$$\begin{aligned}
\int_{B_1^+} (\epsilon u_{i,j}(f_t h x))^\theta \, \mathrm{d}h &\leq \epsilon^\theta \int_{B_1^+} \frac{1}{\|\rho_{i,j}(f_t h g \gamma) w_{i,j}\|^{\theta / c_{i,j} q_{i,j}}} \, \mathrm{d}h + \epsilon^\kappa C' u(x) + \frac{b}{n} \\
&\leq e^{-(\theta - \delta) t \lambda^{i,j} / c_{i,j} q_{i,j}} (\epsilon u_{i,j}(x))^\theta + \epsilon^\kappa C' u(x) + \frac{b}{n} \\
&\leq e^{-(\alpha + \delta) t} (\epsilon u_{i,j}(x))^\theta + \epsilon^\kappa C' u(x) + \frac{b}{n}.
\end{aligned}$$

Therefore, we have

$$\int_{B_1^+} u(f_t h x) \, \mathrm{d}h \leq e^{-(\alpha + \delta) t} u(x) + n \epsilon^\kappa C' u(x) + b.$$

We choose ϵ sufficiently small so that $n \epsilon^\kappa C' \leq e^{-(\alpha + \delta) t}$, then (4.5) holds. $\square$

5. Applications of the uniform contraction property

In this section we will introduce and study some auxiliary sets closely related to $\mathfrak{D}(F^+, G/\Gamma)$ using the uniform contraction property of the EM height function established in Lemma 4.3. To be specific, we will prove some covering results for these auxiliary sets in Proposition 5.1 and these covering results will play an important role in bounding the Hausdorff dimension of $\mathfrak{D}(F^+, G/\Gamma)$.

Let α_1 be a contraction rate of the dynamical system $(G/\Gamma, F^+)$ given by (4.3) and let λ be the top Lyapunov exponent of A_F in the representation $(\mathrm{Ad}, \mathfrak{g})$. We fix $\alpha < \alpha_1, t > 0$ and a EM height function $u : G/\Gamma \to (0, \infty)$ so that Lemma 4.3 holds. Let $\ell \geq 1$ so that (4.6) holds for all $z \in B_1^c$ if $u(x) \geq \ell$. By Lemma 4.3 (3), there exists $C \geq 1$ such that

$$C^{-1} u(x) \leq u(f_s h x) \leq C u(x) \quad \text{for all } 0 \leq s \leq t, h \in B_2^G \text{ and } x \in G/\Gamma. \tag{5.1}$$

We also fix an auxiliary $\delta > 0$ (which will go to zero finally) and assume that t is sufficiently large so that according to Lemma 3.1 for all $r \le 1, z \in B_1^c$

$$B^+_{e^{-(\lambda+\delta)t}r} \subset f^z_{-t} B^+_r f^z_t \subset B^+_{r/4}; \tag{5.2}$$

$$B^c_{e^{-\delta t}r} \subset f^z_{-t} B^c_r f^z_t \subset B^c_{e^{\delta t}r}; \tag{5.3}$$

$$2 < e^{\delta(\alpha+1)t/2}. \tag{5.4}$$

Note that the logarithm map from the metric space (B_1^+, dist) to the Lie algebra $\mathfrak{g}^+$ (with the fixed Euclidean structure) is a bi-Lipschitz homeomorphism to its image. Therefore (B_1^+, dist) is Besicovitch, see [24], namely, for any subset D of B_1^+ and a covering of D by balls centered at D, there is a finite sub-covering such that each element of D is covered by at most E' times. Therefore, there exists $E \ge E'$ such that for all $0 < r \le 1$, the set $B^+_{1/2}$ can be covered by no more than Er^{-d} open balls of radius r, where $d = \dim G^+$.

We use $|I|$ to denote the cardinality of a finite set I. The following is the main result of this section.

Proposition 5.1. *Let $x \in G/\Gamma$. There exists $0 < \sigma < 1$ and $E_0 \ge 1$ such that for $z \in B_1^c$ and $N \in \mathbb{N}$, the set*

$$\mathfrak{D}_x(z, N, \sigma, C^2\ell) \stackrel{\mathrm{def}}{=} \{h \in B^+_{1/2} : |\{1 \le n \le N : u(f^z_{nt} hx) \ge C^2\ell\}| \ge \sigma N\} \tag{5.5}$$

can be covered by no more than $E_0 e^{(d\lambda - \alpha + \delta(d+\alpha))tN}$ open balls of radius $e^{-(\lambda+\delta)tN}$ in B_1^+.

The rest of this section is devoted to show that Proposition 5.1 holds for

$$\sigma = \frac{(1-\delta/2)\alpha t + \log C}{\alpha t + \log C} \tag{5.6}$$

In the rest of this section we fix $z \in B_1^c$ and $N \in \mathbb{N}$. We begin with the following simple observation.

Lemma 5.2. *If $B \subset G^+$ is a ball of radius $e^{-(\lambda+\delta)tN}$ centered at $\mathfrak{D}_x(z, N, \sigma, C^2\ell)$, then $B \subset \mathfrak{D}_x(z, N, \sigma, C\ell)$.*

Proof. Let h_0 be the center of B and $h \in B$. It suffices to show that for all $1 \le n \le N$ if $u(f^z_{nt} h_0 x) \ge C^2\ell$ then $u(f^z_{nt} hx) \ge C\ell$. By (5.2) we have

$$\mathrm{dist}(f^z_{nt} h_0, f^z_{nt} h) = \mathrm{dist}(1_G, f^z_{nt} h h_0^{-1} f^z_{-nt}) < 1.$$

By (5.1)

$$u(f_{nt} hx) = u(f_{nt} h h_0^{-1} f_{-nt} \cdot f_{nt} h_0 x) \ge C^{-1} u(f_{nt} h_0 x) \ge C^{-1} \cdot C^2 \ell = C\ell.$$

$\square$

For a subset $I \subset \{1, \dots, N\}$, we let

$$\mathfrak{D}_x(z, I, C\ell) = \{h \in B^+_{1/2} : u(f^z_{nt} hx) \ge C\ell \text{ for all } n \in I\}. \tag{5.7}$$

Since $\mathfrak{D}_x(z, N, \sigma, C\ell) = \bigcup_{|I| \ge \sigma N} \mathfrak{D}_x(z, I, C\ell)$, one has

$$\mu(\mathfrak{D}_x(z, N, \sigma, C\ell)) \le \sum_{|I| \ge \sigma N} \mu(\mathfrak{D}_x(z, I, C\ell)). \tag{5.8}$$

The following lemma will play an important role in the proof of Proposition 5.1.

Lemma 5.3. *Suppose that $I \subset \{1, \dots, N\}$ and $|I| \geq \sigma N$. Then*

$$\mu(\mathfrak{D}_x(z, I, C\ell)) \leq C^2 u(x) e^{-(1-\delta/2)\alpha t N}. \tag{5.9}$$

We fix I as in the statement of Lemma 5.3. Our strategy is to estimate the measure of $\mathfrak{D}_x(z, I, C\ell)$ by relating it to a subset coming from random walks on G/Γ with alphabet $f_t^z B_1^+$. Let $p = \sup I$ and for $1 \leq k \leq p$ let

$$Z_k = \{(h_1, \dots, h_k) \in (B_1^+)^k : u(f_t^z h_n \dots f_t^z h_1 x) \geq \ell \ \forall\ n \in (I \cap [1, k])\}.$$

Define $\eta : (B_1^+)^p \to G^+$ by

$$\eta(h_1, \dots, h_p) = \tilde{h}_p \cdots \tilde{h}_1, \text{ where } \tilde{h}_n = f_{-(n-1)t}^z h_n f_{(n-1)t}^z. \tag{5.10}$$

We remark here that the image of η is contained in B_2^+ by (5.2). The following two lemmas are needed in the proof of Lemma 5.3.

Lemma 5.4. *For all $h \in \mathfrak{D}_x(z, I, C\ell)$ one has $\eta^{-1}(h) \subset Z_p$.*

Proof. Suppose that $\eta(h_1, \dots, h_p) = h$ where $h_i \in B_1^+$. Then for all $n \leq p$

$$\operatorname{dist}(f_{nt}^z h, f_t^z h_n \cdots f_t^z h_1) = \operatorname{dist}(f_{nt}^z \tilde{h}_p \cdots \tilde{h}_{n+1} f_{-nt}^z, 1_G) < 2,$$

where we use (5.2), (5.10) and the right invariance of $\operatorname{dist}(\cdot, \cdot)$. Therefore by (5.1) we have for $n \in I$

$$u(f_t^z h_n \cdots f_t^z h_1 x) \geq C^{-1} u(f_{nt}^z h x) \geq \ell.$$

So $(h_1, \dots, h_p) \in Z_p$ and the proof is complete. $\square$

Let $\widetilde{\mu}_n$ be the Radon measure on G^+ defined by

$$\int_{G^+} \varphi(h)\, \mathrm{d}\widetilde{\mu}_n(h) = \int_{B_1^+} \varphi(f_{-nt}^z h f_{nt}^z))\, \mathrm{d}h \tag{5.11}$$

for all $\varphi \in C_c(G^+)$. For any positive integer n let $\mu_n = \widetilde{\mu}_{n-1} * \cdots * \widetilde{\mu}_1 * \widetilde{\mu}_0$ be the measure on G^+ defined by the n convolutions. Clearly, μ_n is absolutely continuous with respect to μ and μ_p is the pushforward of $(\mu|_{B_1^+})^{\otimes p}$ by the map η. The following lemma shows that μ_n has density bigger than or equal to one at every $h \in B_{1/2}^+$.

Lemma 5.5. *For all $n \leq N$ and $h \in B_{1/2}^+$ we have $\frac{d\mu_n}{d\mu}(h) \geq 1$.*

Proof. The conclusion is clear if $n = 1$. Now we assume $n > 1$ and let

$$\nu = \widetilde{\mu}_{n-1} * \widetilde{\mu}_{n-2} * \cdots * \widetilde{\mu}_1.$$

It follows from (5.2) and (5.11) that for $k > 0$ the probability measure $\widetilde{\mu}_k$ is supported on $B_{1/4^k}^+$. Since the metric on G^+ is right invariant, the measure ν is supported on $B_{1/2}^+$. Suppose $\nu = \varphi\ \mathrm{d}\mu$, then $\mu_n = \nu * \widetilde{\mu}_0 = \varphi * \mathbb{1}_{B_1^+}\ \mathrm{d}\mu$. So for any $h \in B_{1/2}^+$, we have

$$\varphi * \mathbb{1}_{B_1^+}(h) = \int_{G^+} \varphi(h_1) \mathbb{1}_{B_1^+}(h_1^{-1} h) d\mu(h_1) \geq \int_{B_{1/2}^+} \varphi(h_1) d\mu(h_1) = 1.$$

$\square$

Now we are ready to prove Lemma 5.3.

Proof of Lemma 5.3. By Lemmas 5.4 and 5.5,

$$\mu(\mathfrak{D}_x(z,I,C\ell)) \le \mu_\ell(\mathfrak{D}_x(z,I,C\ell)) \le \mu_p(Z_p). \tag{5.12}$$

Now we are left to estimate $\mu_p(Z_p)$. For $1 \le k \le p$ let

$$s(k) = \int_{Z_k} u(f_t^z h_k \cdots f_t^z h_1 x) d\mu^{\otimes k}(h_1, \cdots, h_k).$$

Let

$$s(p+1) = \int_{Z_p} \left[\int_{B_1^+} u(f_t^z h_{p+1} f_t^z h_p \cdots f_t^z h_1 x) \, \mathrm{d}\mu(h_{p+1}) \right] d\mu^{\otimes p}(h_1, \cdots, h_p). \tag{5.13}$$

Then for every $1 < k \le p+1$,

$$s(k) \le \int_{Z_{k-1}} \left[\int_{B_1^+} u(f_t^z h_k f_t^z h_{k-1} \cdots f_t^z h_1 x) \, \mathrm{d}\mu(h_k) \right] d\mu^{\otimes (k-1)}(h_1, \cdots, h_{k-1}).$$

If $k-1 \in I$, then $s(k) \le e^{-\alpha t} s(k-1)$ by (4.6). If $k-1 \notin I$, then by (5.1) we have $s(k) \le C s(k-1)$. We apply this estimate to $k = p+1, p, \cdots, 2$, then we have

$$s(p+1) \le C^{(N-|I|)} e^{-|I|\alpha t} \int_{B_1^+} u(f_t h x \, \mathrm{d}\mu(h)) \le C^{1+(1-\sigma)N} e^{-\sigma\alpha t N} u(x).$$

The choice of σ in (5.6) implies that

$$s(p+1) \le C e^{-(1-\delta/2)\alpha t N} u(x). \tag{5.14}$$

On the other hand, in view of (5.13), (5.1) and the fact $p = \sup I$ we have

$$s(p+1) \ge C^{-1} s(p) \ge C^{-1} \ell \cdot \mu_p(Z_p). \tag{5.15}$$

Therefore, (5.9) follows from (5.12), (5.14) and (5.15) and the observation $\ell \ge 1$. □

Proof of Proposition 5.1. As before we fix z and N as in the statement. Let σ be as in (5.6). Since (B_1^+, dist) is Besicovitch, there exists a covering $\mathfrak{U}$ of $\mathfrak{D}_x(z,N,\sigma,C^2\ell)$ by open balls of radius $e^{-(\lambda+\delta)tN}$ centered at $\mathfrak{D}_x(z,N,\sigma,C^2\ell)$ such that each element of $\mathfrak{D}_x(z,N,\sigma,C^2\ell)$ is covered by at most E times. By Lemma 5.2, each $B \in \mathfrak{U}$ is contained in $\mathfrak{D}_x(z,N,\sigma,C\ell)$, so in view of (3.1)

$$\mu(\mathfrak{D}_x(z,N,\sigma,C\ell)) \ge \frac{|\mathfrak{U}|}{E} \mu(B^+_{e^{-(\lambda+\delta)tN}}) \ge \frac{|\mathfrak{U}|}{C_0 E} e^{-(\lambda+\delta)dtN}. \tag{5.16}$$

On the other hand, since there are 2^N subsets $I \subset \{1, \dots, N\}$, by (5.8), (5.4) and Lemma 5.3, we have

$$\mu\big(\mathfrak{D}_x(z,N,\sigma,C\ell)\big) \le C^2 2^N e^{-(1-\delta/2)\alpha t N} u(x) \le e^{-(1-\delta)\alpha t N} u(x). \tag{5.17}$$

By (5.16) and (5.17),

$$|\mathfrak{U}| \le u(x) C_0 C^2 E \cdot e^{(d\lambda - \alpha + \delta(d+\alpha))tN}.$$

The conclusion now follows by taking $E_0 = u(x) C_0 C^2 E$. □

6. Upper bound of Hausdorff dimension

In this section, we finish the proof of Theorem 1.4. We will use the same notation as in §5 prior to Proposition 5.1. For $(z,h) \in B_1^c B_1^+$, $\ell' > 0$ and $N \in \mathbb{N}$, let $I_{N,\ell'}(z,h)$ denote the set of $n \in \{1,\ldots,N\}$ satisfying $u(f_{nt}zhx) \geq \ell'$. For $x \in G/\Gamma$, let

$$\mathfrak{D}_x^0(F^+, N, \sigma, \ell') = \{(z,h) \in B_{1/2}^c B_{1/2}^+ : |I_{N,\ell'}(z,h)| \geq \sigma N\}. \tag{6.1}$$

Lemma 6.1. *Let $x \in G/\Gamma$. Then there exist $0 < \sigma < 1$ and $E_2 \geq 1$ such that for any $N \in \mathbb{N}$ the set $\mathfrak{D}_x^0(F^+, N, \sigma, C^4\ell)$ can be covered by no more than $E_2 e^{(d^c\lambda + d\lambda - \alpha + \delta(d^c + d + \alpha))tN}$ open balls of radius $e^{-(\lambda+\delta)tN}$ in G^cG^+.*

Proof. Let $0 < \sigma < 1$ and $E_0 \geq 1$ so that Proposition 5.1 holds. We fix $N \in \mathbb{N}$.

We claim that: for $W = B^c_{e^{-(\lambda+\delta)tN}} \cdot z \subset B_1^c$, we have

$$\Big(\mathfrak{D}_x^0(F^+, N, \sigma, C^4\ell) \cap (WB_1^+)\Big) \subset \Big(W\mathfrak{D}_x(z, N, \sigma, C^2\ell)\Big). \tag{6.2}$$

Let $(z_1, h_1) \in WB_1^+$. Suppose that $1 \leq n \leq N$ and $u(f_{nt}z_1h_1x) \geq C^4\ell$. In view of (5.3) and (5.1) we have

$$\begin{aligned} u(f_{nt}^z h_1 x) &= u(z^{-1} f_{nt} z h_1 x) \geq C^{-1} u(f_{nt} z h_1 x) \\ &= C^{-1} u(f_{nt}(z z_1^{-1}) f_{-nt} \cdot f_{nt} z_1 h_1 x) \geq C^{-2} u(f_{nt} z_1 h_1 x) \geq C^2 \ell. \end{aligned}$$

In other words, we have proved that if $n \in I_{N,C^4\ell}(z_1,h_1)$, then $u(f_{nt}^z h_1 x) \geq C^2\ell$. Therefore, if (z_1,h_1) belongs to the left hand side of (6.2) then it also belongs to the right hand side.

Since (B_1^c, dist) is also Besicovitch, there exists $E_1 \geq 1$ such that for all $0 < r \leq 1$, $B_{1/2}^c$ can be covered by no more than $E_1 r^{-d^c}$ open balls of radius r. We fix a cover $\mathfrak{U}^c$ of $B_{1/2}^c$ that consists of open balls of radius $e^{-(\lambda+\delta)Nt}$ with $|\mathfrak{U}^c| \leq E_1 e^{d^c(\lambda+\delta)Nt}$. We assume each element of $\mathfrak{U}^c$ has nonempty intersection with $B_{1/2}^c$, then it is contained in B_1^c in view of (5.4). Let $W_z \in \mathfrak{U}^c$ be a ball centered at $z \in B_1^c$. Proposition 5.1 implies that there exists a covering $\mathfrak{U}_z$ of $\mathfrak{D}_x(z, N, \sigma, C^2\ell)$ by open balls of radius $e^{-(\lambda+\delta)tN}$ such that

$$|\mathfrak{U}_z| \leq E_0 e^{d\lambda - \alpha + \delta(d+\alpha)}.$$

In view of claim (6.2), the following class of sets

$$\{W_z B : W_z \in \mathfrak{U}^c, B \in \mathfrak{U}_z\} \tag{6.3}$$

forms an open cover of $\mathfrak{D}_x^0(F^+, N, \sigma, C^4\ell)$. It is easily checked that there exists $E_1' \geq 1$ not depending on N such that each element W_zB of (6.3) can be covered by E_1' open balls of radius $e^{-(\lambda+\delta)Nt}$ in G^cG^+. Therefore the lemma holds with $E_2 = E_0E_1E_1'$. □

Theorem 6.2. *For any $x \in G/\Gamma$, the Hausdorff dimension of $\mathfrak{D}_x^0 \stackrel{\text{def}}{=} \{(z,h) \in B_{1/2}^c B_{1/2}^+ : zhx \in \mathfrak{D}(F^+, G/\Gamma)\}$ is at most $d^c + d - \frac{\alpha_1}{\lambda}$.*

Proof. For each $\alpha < \alpha_1$ and $0 < \delta < 1$ we first choose $t > 0$, a height function u and $\ell, C \geq 1$ so that Lemma 4.3, (5.1), (5.2), (5.3) and (5.4) hold. Then there exists $0 < \sigma < 1$ and $E_2 \geq 1$ so that Lemma 6.1 holds.

It follows from Lemma 4.3 (1)(2) and the definition of $\mathfrak{D}(F^+, G/\Gamma)$ that

$$\mathfrak{D}_x^0 \subset \bigcup_{M\geq 1} W_M \quad \text{where} \quad W_M = \bigcap_{N\geq M} \mathfrak{D}_x^0(F^+, N, \sigma, C^4\ell).$$

Recall that for any metric space S,

$$\dim_H S = \inf\left\{ s>0 : \inf_{\{B_i\}} \sum \rho(B_i)^s = 0 \right\},$$

where the latter "inf" is taken over all the countable coverings $\{B_i\}$ of S that consist of open metric balls. Then in view of Lemma 6.1, we have

$$\begin{aligned}\dim_H W_M &\leq \liminf_{N\to\infty} \frac{[d^c\lambda + d\lambda - \alpha + \delta(d+d^c+\alpha)]tN + \log E_2}{\lambda t N} \\ &= d^c + d - \frac{\alpha}{\lambda} + \delta\frac{d+d^c+\alpha}{\lambda}.\end{aligned}$$

Therefore

$$\dim_H \mathfrak{D}_x^0 \leq d^c + d - \frac{\alpha}{\lambda} + \delta\frac{d+d^c+\alpha}{\lambda}.$$

The conclusion follows by first letting $\delta \to 0$ and then letting $\alpha \to \alpha_1$. □

Lemma 6.3. *If $x \in \mathfrak{D}(F^+, G/\Gamma)$ and $h \in G^-$, then $hx \in \mathfrak{D}(F^+, G/\Gamma)$.*

Proof. Note that by Lemma 3.1,

$$\mathrm{dist}(f_t hx, f_t x) \leq \mathrm{dist}(f_t h f_{-t}, 1_G) \to 0$$

as $t \to \infty$. Therefore the lemma holds. □

Proof of Theorem 1.4. We will show that

$$\dim_H \mathfrak{D}(F^+, G/\Gamma) \leq d^- + d^c + d - \frac{\alpha_1}{\lambda}.$$

In view of the local nature of Hausdorff dimension and the definition of the metric on G/Γ, it suffices to prove that for any $x \in G/\Gamma$

$$\dim_H \{g \in B_r^G : gx \in \mathfrak{D}(F^+, G/\Gamma)\} \leq d^- + d^c + d - \frac{\alpha_1}{\lambda}.$$

where $r < 1$ so that $B_r^G \subset B_1^- B_{1/2}^c B_{1/2}^+$. By Lemma 6.3,

$$\{g \in B_r^G : gx \in \mathfrak{D}(F^+, G/\Gamma)\} \subset B_1^- \mathfrak{D}_x^0,$$

whose Hausdorff dimension is bounded from above by $\dim_H \mathfrak{D}_x^0 + d^-$.[4] In view of Theorem 6.2, the Hausdorff dimension of $\mathfrak{D}(F^+, G/\Gamma)$ is less than $d + d^c + d^- - \frac{\alpha_1}{\lambda}$ which is strictly less than the manifold dimension of G/Γ. □

Remark 6.4. It is worth to mention that, if $F = A_F$, then the contraction property of the Benoist-Quint height function proved in [25] will allow us to prove a stronger result. Namely, we can get a nontrivial upper bound of the Hausdorff dimension of the intersection of $\mathfrak{D}(F^+, G/\Gamma)$ and orbits of the so-called (F^+, Γ)-expanding subgroups introduced in [20]. But unfortunately, we are not able to prove a uniform

[4]Here we are using Marstrand's product theorem again.

contracting property for the Benoist-Quint height functions even in the example mentioned at the end of the introduction due to the existence of the unipotent part in the second $\mathrm{SL}_4(\mathbb{R})$ factor.

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