

Localized mirror functor constructed from a Lagrangian torus

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Fixing a weakly unobstructed Lagrangian torus in a symplectic manifold X , we define a holomorphic function W known as the Floer potential. We construct a canonical A_∞ -functor from the Fukaya category of X to the category of matrix factorizations of W . It provides a unified way to construct matrix factorizations from Lagrangian Floer theory. The technique is applied to toric Fano manifolds to transform Lagrangian branes to matrix factorizations. Using the method, we also obtain an explicit expression of the matrix factorization mirror to the real locus of the complex projective space.

1 Introduction

Homological mirror symmetry conjecture by Kontsevich [31] asserts that for a pair of mirror manifolds (X, \check{X}) , the derived Fukaya category of Lagrangian submanifolds in X is equivalent to the derived category of coherent sheaves on \check{X} . The study of homological mirror symmetry leads to many new insights to Fukaya categories and computational techniques for proving the conjecture in various cases.

More generally when X is not required to be Calabi-Yau, the mirror of X is a Landau-Ginzburg model W , which is a holomorphic function rather than a manifold. Intuitively, the singular locus of W (which is not necessarily smooth nor connected) is the space mirror to X . Homological mirror symmetry can still be stated by using the category of matrix factorizations of W [20, 10, 35] (in place of the derived category of coherent sheaves), or the Fukaya-Seidel category of W [37] (in place of the Fukaya category).

In [15], we proposed and constructed a functor to realize homological mirror symmetry using immersed Lagrangian Floer theory. We used the formal deformations and obstructions coming from the self-intersections of a fixed Lagrangian immersion \mathbb{L} in X to construct a Floer potential which serves as a Landau-Ginzburg mirror W . Given any Lagrangian L_1 in the Fukaya category, the Lagrangian intersection theory between the immersion \mathbb{L} and L_1 is used to construct the mirror object R of L_1 . By the result of Orlov [35], for a Landau-Ginzburg model W , the appropriate objects to consider for singularity theory of W are matrix factorizations, which are endomorphisms R of vector bundles satisfying $R^2 = (W - c) \cdot \text{Id}$ for some constant c . In [15] the theory was applied to the orbifold spheres $\mathbb{P}_{(a,b,c)}^1$, and an inductive method was found in [16] to deduce an explicit expression of the Landau-Ginzburg mirror W (which contains infinitely many terms).

In this paper, we fix the reference to be a smooth Lagrangian torus \mathbb{L} rather than an immersed Lagrangian.* For simplicity we make the technical assumption that (X, \mathbb{L}) is positive (see Assumption 1), although this is not necessary if one uses the full machinery of Lagrangian Floer theory. Similar to [15], we use Lagrangian intersection theory to construct a Landau-Ginzburg model $W = W(\mathbb{L})$ and an A_∞ -functor from the Fukaya category to the category of matrix factorizations of W . Flat \mathbb{C}^\times connections plays a key role in this setup, since they serve as the formal (complexified) deformations of \mathbb{L} . The essential issue coming from considering flat \mathbb{C}^\times connections, rather than self-intersections of an immersion, is the choice of gauge. Different choices of gauge for the same connection result in different expressions of the functor, and we need to make a consistent choice to make sure the functor is well-defined, and study the effect of gauge change.

The fundamental idea of constructing a mirror functor for a Lagrangian torus fibration (with mild singularities) goes back to the work of Fukaya [24], [25], who introduced family Floer homology of Lagrangian torus fibers under certain assumptions. More recently, Abouzaid [1, 2] studied the family Floer theory for Lagrangian torus fibration without singular fibers. The mirror functor constructed in this paper is a local piece of the family Floer functor near a Lagrangian torus fiber \mathbb{L} , which has the advantage that it can be explicitly computed. In the absence of singular fibers such as in the case of toric manifolds, these local pieces can be glued together to give the global functor.

We summarize the construction as follows. First fix a weakly unobstructed smooth Lagrangian torus \mathbb{L} in a symplectic manifold X . We define a holomorphic function W on the space $(\mathbb{C}^\times)^n$ of flat \mathbb{C}^\times -connections ∇ by using the m_0 -term of the A_∞ algebra $\text{CF}^*((\mathbb{L}, \nabla), (\mathbb{L}, \nabla))$. Geometrically W is obtained from counting holomorphic discs of Maslov index two bounded by (\mathbb{L}, ∇) (see Definition 2.1). In general W should serve as a part of a global Landau-Ginzburg mirror to X . This method was used by the joint work [18] of the first author with Oh, and Fukaya-Oh-Ohta-Ono [27] to construct the mirrors of toric manifolds.

Now comes the main construction of this paper. To transform a (weakly unobstructed) Lagrangian L_1 to a matrix factorization, we take the Lagrangian Floer ‘complex’ between (\mathbb{L}, ∇) and L_1 . The differential does not square to zero; indeed it follows from the A_∞ relations that the differential squares to $W - \lambda$, where λ is given by $m_0^{L_1} = \lambda \cdot \mathbf{1}_{L_1}$, and thereby the Lagrangian Floer ‘complex’ is indeed a matrix factorization of W . The strategy of constructing matrix factorizations using Lagrangian Floer theory was found by Oh [33] and [26].

Note that the same flat connection ∇ admits different choices of gauge. The resulting matrix factorizations depend on such a choice. Moreover terms like z^a for $a \notin \mathbb{Z}$ could appear if the gauge is chosen arbitrarily. To make sure the resulting matrix factorizations are still defined over the Laurent series ring, we make the following gauge choice for the flat connections over \mathbb{L} . Namely, we always require that the flat connections are trivial away from small neighborhoods of certain fixed codimension-one tori (called hyper-tori). Then holonomy of a flat connection over a path in \mathbb{L} can be expressed in terms of the number of intersections of the path with the hyper-tori, which are integer-valued. It ensures that we still stay inside the Laurent series ring. Moreover, we can show that the resulting matrix factorization does not depend on the choice of hyper-tori, nor a representative in the Hamiltonian isotopy class of L_1 (see Section 5). As a result, we have the following.

Theorem 1.1. Fix a smooth Lagrangian submanifold $\mathbb{L} \subset X$. There exists a Floer potential W defined over the space of flat \mathbb{C}^\times connections over \mathbb{L} , and an A_∞ functor from $\text{Fuk}_\lambda(X)$ to $\text{MF}(W - \lambda)$ for each $\lambda \in \Lambda$, where $\text{Fuk}_\lambda(X)$ is the Fukaya category of weakly unobstructed Lagrangian submanifolds L with $m_0^L = \lambda \mathbf{1}_L$, and Λ denotes the Novikov field. \square

The mirror functor is computable by using *pearl complex* introduced by Biran-Cornea [8] (decorated with flat complex line bundles for the purpose of this paper), which is explained in Section 7.

In Section 8, we apply our construction to toric Fano manifolds, and transform Lagrangian torus fibers (decorated by flat connections) to matrix factorizations of the mirror. \dagger However, the mirror matrix factorizations are hard to be fully computed.

On the other hand, we find that the leading-order terms of such a matrix factorization R always form another matrix factorization R_0 . We deduce an explicit closed formula for R_0 and show that it is of *wedge-contraction type*, and hence it is a split-generator of the category of matrix factorizations by the result of Dyckerhoff [19]. By spectral-sequence argument of Polishchuk-Vaintrob [36], we deduce that terms in $R - R_0$ do not contribute to cokernel, and hence R is also a split-generator of the category.

We summarize the main result as follows. We switch from the Novikov field Λ to the complex field \mathbb{C} by substituting the Novikov formal variable T by e^{-1} , in order to match with Hori-Vafa mirrors of toric Fano manifolds. It is valid since both W and the matrix factorization R have finitely many terms in the Fano case. For non-Fano cases one should stick with the Novikov field Λ .

*The assumption that \mathbb{L} is a torus is actually not essential. We concentrate on torus because it plays a central role in SYZ mirror symmetry.

\dagger We expect that the method in this paper works for general compact toric manifolds. To avoid technical issues in Lagrangian Floer theory, we restrict to the Fano case.

Theorem 1.2. Let X be a toric Fano n -fold whose moment map polytope is given by

$$\{u : \langle v_i, u \rangle - \lambda_i \geq 0 \text{ for all } i = 1, \dots, m\}$$

where m is the number of primitive generators v_i of the corresponding fan and $\lambda_i \in \mathbb{R}$ are some fixed constants. Without loss of generality we assume v_1, \dots, v_n form an integral basis. Let

$$W = \sum_{i=1}^m e^{\lambda_i} t^{v_i} \in \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$$

be the Hori-Vafa mirror. [‡] Write $t_i = z_i \cdot e^{-\langle u, v_i \rangle}$ for $i = 1, \dots, n$. Let L_u be a moment-map fiber decorated by a flat \mathbb{C}^\times connection $\nabla_{\underline{z}}$ (where $\underline{z} = (z_1, \dots, z_n)$), and $R(\underline{z})$ the matrix factorization mirror to $(L_u, \nabla_{\underline{z}})$ under the functor in Theorem 1.1.

1. $R(\underline{z})$ takes the form $(\wedge^* \underline{\mathbb{C}}^n, d)$ where $\underline{\mathbb{C}}^n$ denotes the module $(\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}])^{\oplus n}$, and $d = \sum_{k=0}^n d_{-(2\lfloor (k+1)/2 \rfloor - 1)}$ with $d^2 = W - W(\underline{z})$.
2. $R_0(\underline{z}) := (\wedge^* \underline{\mathbb{C}}^n, d_1 + d_{-1})$ itself is a matrix factorization of $W - W(\underline{z})$.
3. $d_1 + d_{-1}$ takes the form

$$\left(\sum_{i=1}^n (z_i - \underline{z}_i) e_i \wedge \right) + \left(\sum_{i=1}^n c_i \iota_{e_i} \right) + \left(\sum_{i=n+1}^m c_i \sum_{j=1}^n \alpha_j^i(z, \underline{z}) \iota_{e_j} \right)$$

where $c_i = e^{-\langle u, v_i \rangle - \lambda_i}$ and the explicit formula for $\alpha_j^i(z, \underline{z})$ is given in Theorem 8.9.

4. Both $R(\underline{z})$ and $R_0(\underline{z})$ are split-generators of $DMF(W - W(\underline{z}))$ (which is non-trivial only when $(t_i = z_i \cdot e^{-\langle u, v_i \rangle})_{i=1}^n$ is a critical point of W).

□

We construct an explicit isomorphism between R and R_0 for $\dim X \leq 4$.

The above theorem generalizes the results of Chan-Leung [11] for \mathbb{P}^2 and Tu [38] for toric Fano surfaces. Given a Lagrangian torus fibration, the work of Tu constructed a functor (away from singular fibers) based on Fourier-Mukai transform from the Fukaya category of smooth torus fibers to the category of sheaves of modules (over a certain A_∞ -algebra), whose objects can be interpreted as matrix factorizations. Moreover Abouzaid-Fukaya-Oh-Ohta-Ono announced a proof of homological mirror symmetry conjecture for toric manifolds by showing that the toric fibers generate. In general the functor is difficult to write down, and this paper provides a method to compute it by localizing to each torus fiber.

As another application, we transform the real locus $\mathbb{R}\mathbb{P}^n$ in $\mathbb{C}\mathbb{P}^n$ to a matrix factorization of the mirror, by using the result of [6]. We only consider n being odd so that $\mathbb{R}\mathbb{P}^n$ is orientable. The result is the following.

Theorem 1.3 (Theorem 10.1). Let $W = T^k \left(z_1 + \dots + z_n + \frac{1}{z_1 \dots z_n} \right)$ be the Landau-Ginzburg mirror of \mathbb{P}^n where n is odd. (The base point in the moment polytope is chosen suitably so that W takes this form. Moreover the Kähler form is taken such that $k \in \mathbb{Z}_{>0}$. T denotes the Novikov variable.) Denote the matrix factorization of W mirror to $\mathbb{R}\mathbb{P}^n \subset \mathbb{C}\mathbb{P}^n$ by (E, d) .

Then E is given by the trivial bundle with a basis labelled by $[\pm 1 : \dots : \pm 1] \in (\mathbb{Z}_2)^{n+1} / \mathbb{Z}_2$, where the quotient is given by the diagonal action of \mathbb{Z}_2 on $(\mathbb{Z}_2)^{n+1}$. The differential d is determined by $d p = \sum_q m_{qp} q$ where $p, q \in (\mathbb{Z}_2)^{n+1} / \mathbb{Z}_2$, and m_{qp} are given as follows.

1. When

$$p = [a_0 : \dots : a_{i-1} : -1 : a_{i+1} : \dots : a_n] \text{ and } q = [a_0 : \dots : a_{i-1} : 1 : a_{i+1} : \dots : a_n]$$

where the number of $a_j = -1$ (for $j \neq i$) is even,

$$m_{qp} = \frac{T^{k/2}}{\prod_{1 \leq j \leq n} z_j^{\delta(a_j, -1)}} \text{ and } m_{pq} = \frac{T^{k/2}}{\prod_{1 \leq j \leq n} z_j^{\delta(a_j, 1)}} \text{ if } i = 0,$$

$$m_{qp} = T^{k/2} z_i \text{ and } m_{pq} = T^{k/2} \text{ if } i \neq 0.$$

$\delta(a, b) = 1$ when $a = b$ and zero otherwise.

2. $m_{qp} = m_{pq} = 0$ otherwise.

□

[‡]Indeed the Kähler structure needs to be complexified to match the moduli spaces. Simply put, λ_i should be replaced by $\lambda_i + \sqrt{-1}\theta_i$ for some fixed $\theta_i \in \mathbb{R}/2\pi\mathbb{Z}$.

Acknowledgement

The authors are grateful to Jonathan David Evans and Yanki Lekili for informing them about their very useful results on generation of Fukaya categories of Hamiltonian G-manifolds, and in particular toric Fano manifolds. Combining with the results in this paper, it gives a proof of homological mirror symmetry for toric Fano manifolds. C.H. Cho and S.-C. Lau thank The Chinese University of Hong Kong for its hospitality, where part of the work was carried out. C.H. Cho thanks Yong-Geun Oh for helpful discussions. S.-C. Lau expresses his gratitude to Kwokwai Chan and Junwu Tu for useful explanations of their works. The work of S.-C. Lau was supported by Harvard University and Boston University.

2 Localized Lagrangian Floer potential and Lagrangian Floer complex

Let X be a symplectic manifold. To avoid technical issues of transversality, we make the assumption that X and the Lagrangian submanifolds under consideration are *positive*. The assumption is not necessary if one is willing to handle the issue of transversality by using more advanced machinery of Lagrangian Floer theory.

Assumption 1. Let X be a symplectic manifold and \mathbb{L} be a Lagrangian submanifold. The pair (X, \mathbb{L}) is said to be *positive* if there exists an almost complex structure J such that

1. any non-constant J -holomorphic sphere in X has a positive Chern number;
2. any non-constant J -holomorphic disc with boundary on \mathbb{L} has a positive Maslov index;
3. J -holomorphic discs of Maslov index two with boundary on \mathbb{L} are Fredholm regular.

□

The assumption holds for monotone Lagrangian submanifolds[§]. It also holds for Lagrangian torus fibers of a toric Fano manifold.

Now consider a Lagrangian torus \mathbb{L} satisfying Assumption 1, and we shall fix the almost complex structure J satisfying the assumption. We define a Lagrangian Floer potential in this section, using formal deformations brought by flat \mathbb{C}^\times connections on \mathbb{L} .

Fix a basis $\{E_i\}_{i=1}^n$ of $H_1(\mathbb{L}, \mathbb{Z})$, and its dual basis $\{E_i^*\}_{i=1}^n$ of $H^1(\mathbb{L}, \mathbb{Z}) \subset H^1(\mathbb{L}, \mathbb{C})$. Consider

$$b := x_i E_i^* \in H^1(\mathbb{L}, \mathbb{C})/H^1(\mathbb{L}, \mathbb{Z})$$

which is interpreted as a flat \mathbb{C}^\times connection on \mathbb{L} via the associated representation

$$\rho^b : \pi_1(\mathbb{L}) \rightarrow \mathbb{C}^\times; \gamma \mapsto \exp 2\pi\sqrt{-1}(b, \gamma). \quad (2.1)$$

Define the complex coordinates

$$z_i := \rho^b(E_i) \quad (2.2)$$

of the moduli space of flat (possibly non-unitary) complex line bundles. We denote by \mathcal{L}_z the flat line bundle with the flat connection ∇_z parametrized by $z = (z_1, \dots, z_n)$.

Note that there is a subtle difference between the definition of the variables z_i here and that in the conventional Strominger-Yau-Zaslow (SYZ) approach (see for instance [7] or [26]). In the SYZ setting, the mirror variables are defined using Lagrangian torus fibration and they parametrize locations and flat $U(1)$ connections of a Lagrangian torus fiber. In our setting here, \mathbb{L} is fixed and the mirror variables parametrize flat \mathbb{C}^\times (instead of $U(1)$) connections. In Section 8 for toric manifolds we will relate the two by a change of variables.

Since there are infinitely many holomorphic discs in general, the Floer potential is defined over the Novikov field

$$\Lambda = \left\{ \sum_i a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \lim_{i \rightarrow \infty} \lambda_i = \infty \right\}.$$

Here T is a formal parameter, and Λ has a natural energy filtration considering only elements with $\lambda_i \geq \lambda_0$ for some λ_0 . We also set $\Lambda_0 = \{\sum_i a_i T^{\lambda_i} \in \Lambda \mid \lambda_i \geq 0\}$ which is known as the Novikov ring. We will sometimes use the notation $\Lambda_{\mathbb{C}}$ to emphasize that $a_i \in \mathbb{C}$, and we can define $\Lambda_{\mathbb{Z}}$ in a similar way.

Let $\mathcal{M}_1(\mathbb{L}, J, \beta)$ be the moduli space of stable J -holomorphic discs in a homotopy class $\beta \in \pi_2(X, \mathbb{L})$. By using Assumption 1, to define the Floer potential it is enough to consider those β with Maslov index two. For such a β (which is of the minimal Maslov index) the moduli space of holomorphic discs is itself compact and does not require compactification by stable discs. Moreover $\dim \mathcal{M}_1(\mathbb{L}, J, \beta) = n$. Hence the image of the evaluation map $ev_\beta : \mathcal{M}_1(\mathbb{L}, J, \beta) \rightarrow \mathbb{L}$ induced on homology is a constant multiple of the fundamental class of \mathbb{L} , and we denote this multiple by $n_\beta(\mathbb{L})$.

[§]A compact Lagrangian submanifold $\mathbb{L} \subset M$ is called monotone if $I_\omega = \lambda I_\mu$ for some $\lambda \geq 0$, where $I_\omega, I_\mu : \pi_2(M, \mathbb{L}) \rightarrow \mathbb{R}, \mathbb{Z}$ are symplectic area and Maslov index homomorphism respectively.

Definition 2.1. The Floer potential of (\mathbb{L}, ∇_z) is defined as

$$W(\mathbb{L})(z_1, \dots, z_n) := \sum_{\beta, \mu(\beta)=2} n_\beta(\mathbb{L}) T^{\omega(\beta)} \rho^b(\partial\beta)$$

where each $\rho^b(\partial\beta)$ can be expressed as a monomial in $z_i^{\pm 1}$ by Equation (2.2). \square

In general the above expression could be an infinite series. Thus W is a Laurent series in z_i 's whose coefficients are Novikov elements. In case when the sum is finite, we can simply put T to be the constant e^{-1} , and so W is a Laurent polynomial with complex coefficients. For simplicity of notations we will assume that the sum is finite from now on, and write $W \in \Lambda[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$.

Remark 2.2. In general there are infinitely many β contributing to W . The potential W belongs to $\Lambda \ll z_1^{\pm 1}, \dots, z_n^{\pm 1} \gg$, which is the completion of the Laurent polynomial ring with respect to the energy filtration of Λ . \square

For the purpose of the next section, we now recall the Lagrangian Floer complex between two positive Lagrangian submanifolds. Let L_0 and L_1 be two oriented, spin, positive Lagrangian submanifolds L_0 and L_1 in a symplectic manifold (X^{2n}, ω) . Lagrangian Floer homology $HF(L_0, L_1)$ in this setting was first defined by Oh [33] in the monotone cases and was later generalized by Fukaya-Oh-Ohta-Ono [26] (see also Biran-Cornea [8] for the notion of pearl complex when $L_0 = L_1$). The definition below assumes that L_0 and L_1 intersect transversely.

The Floer complex $CF^*(L_0, L_1)$ is a free $\mathbb{Z}/2$ -graded Λ -module generated by the intersection points $p \in L_0 \cap L_1$. The Floer differential δ is defined by counting J -holomorphic strips: for $p \in L_0 \cap L_1$, we have

$$\delta(\langle p \rangle) = \sum_{q \in L_0 \cap L_1} n(p, q) \langle q \rangle, \quad (2.3)$$

where $n(p, q)$ is the signed number of isolated J -holomorphic strips $u : \mathbb{R} \times [0, 1] \rightarrow M$ modulo time translation weighted by the symplectic area $T^{\omega(u)}$ (we refer readers to [26] for the details on signs):

$$u(\mathbb{R} \times \{1\}) \subset L_0, \quad u(\mathbb{R} \times \{0\}) \subset L_1, \quad \bar{\partial}_J u = 0.$$

Here, $J = \{J_t\}$ is a time-dependent generic compatible almost complex structure, where J_0 and J_1 satisfy the positivity assumption of L_0 and L_1 respectively. To define a signed counting, we need to fix an orientation of the orientation spaces associated to intersection points p, q . We refer the readers to [26] for the detail.

As before, we denote by $n_\beta(L_i)$ the number of Maslov index two J_i -holomorphic discs with a boundary on L_i passing through a point $p \in L_i$. We set $\Phi(L_i) = \sum_{\beta, \mu(\beta)=2} n_\beta(L_i) T^{\omega(\beta)}$. Standard Floer theory argument (Gromov-compactness and gluing theorem) produces the identity

$$\delta^2(x) = (\Phi(L_1) - \Phi(L_0))x \quad \text{for any } x \in CF^*(L_0, L_1). \quad (2.4)$$

If $\Phi(L_1) = \Phi(L_0)$, then $\delta^2 = 0$ and hence Floer cohomology can be defined.

As in [31] (or [14]), we consider a slight generalization by introducing flat (possibly non-unitary) complex line bundles $\mathcal{L}_0 \rightarrow L_0$ and $\mathcal{L}_1 \rightarrow L_1$. For $i = 0, 1$, consider a representation $\rho_i : \pi_1(L_i) \rightarrow \mathbb{C} \setminus \{0\}$, and we take a flat connection ∇_i for \mathcal{L}_i whose holonomy representation is given by ρ_i . Note that the complex given below depends on the choice of gauge of ∇_i .

The Floer complex $CF^*((L_0, \mathcal{L}_0), (L_1, \mathcal{L}_1))$ is a free $\mathbb{Z}/2$ -graded Λ -module generated by the intersection points $L_0 \cap L_1$. For each $p \in L_0 \cap L_1$, we consider the vector space $\text{Hom}((\mathcal{L}_0)_p, (\mathcal{L}_1)_p)$, which is identified with \mathbb{C} by fixing the isomorphisms $(\mathcal{L}_0)_p \cong \mathbb{C}, (\mathcal{L}_1)_p \cong \mathbb{C}$. Then we tensor with the Novikov field Λ to obtain the Floer complex.

The differential $\delta^{\mathcal{L}_0, \mathcal{L}_1}$ also takes account of holonomies of the flat connections $\mathcal{L}_0, \mathcal{L}_1$. Given a J -holomorphic strip u from p to q , by taking the boundary we obtain a path $\partial_i u$ from p to q in L_i . Parallel transport along $\partial_i u$ gives $\text{Pal}_{\partial_i u} : (\mathcal{L}_i)_p \rightarrow (\mathcal{L}_i)_q$. Using the identifications $(\mathcal{L}_i)_p \cong \mathbb{C}$ fixed before, $\text{Pal}_{\partial_i u}$ is identified with multiplication by a nonzero complex number. Each strip u from p to q contributes to the differential as

$$(-1)^{a(u)} \text{Pal}_{\partial_0 u}^{-1} \cdot \text{Pal}_{\partial_1 u} \cdot T^{\omega(u)}, \quad (2.5)$$

where $(-1)^{a(u)}$ is the sign of u from [26], and summing all u defines $n(p, q)$ in the definition of $\delta(\langle p \rangle)$ (2.3).

We also define $\Phi(\mathcal{L}_i) = \sum_{\beta, \mu(\beta)=2} n_\beta(L_i) \rho_i(\partial\beta) T^{\omega(\beta)}$, which is a Novikov constant. Then Floer equation (2.4) becomes

$$(\delta^{\mathcal{L}_0, \mathcal{L}_1})^2 x = (\Phi(\mathcal{L}_1) - \Phi(\mathcal{L}_0))x, \quad \text{for any } x \in CF((L_0, \mathcal{L}_0), (L_1, \mathcal{L}_1)). \quad (2.6)$$

Later on we shall put \mathbb{L} (with a flat line bundle \mathcal{L}_z whose holonomy varies in z) in place of L_0 . As we have emphasized, the above definition of differential $\delta^{\mathcal{L}_0, \mathcal{L}_1}$ depends on the choice of gauges of \mathcal{L}_0 and \mathcal{L}_1 . On the other hand z_i 's are parametrizing the isomorphism classes of flat connections over \mathbb{L} . In order to identify the Floer complex as a matrix factorization over $\mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$, we need to make a specific choice of gauge in each isomorphism class. In the next section we will introduce the notion of gauge hypertori in order to fix this choice.

3 Mirror matrix factorization via Floer complex

3.1 Gauge hypertori

Fix a Lagrangian torus \mathbb{L} equipped with a flat line bundle \mathcal{L}_z and denote its Floer potential by $W(\mathbb{L})$. We want to transform a positive Lagrangian submanifold $L_1 \subset X$ together with a flat connection \mathcal{L}_1 to a matrix factorization (P^0, P^1, d) of $W(\mathbb{L})$. This matrix factorization is given by a Floer complex. P^0 (resp. P^1) are defined as free $\Lambda \ll z_1^{\pm 1}, \dots, z_n^{\pm 1} \gg$ -modules generated by even (resp. odd) intersection points of $\mathbb{L} \cap L_1$, and

$$d(\cdot) := (-1)^{\deg(\cdot)} \delta^{\mathcal{L}_z, \mathcal{L}_1}(\cdot). \quad (3.1)$$

Equation (2.6) gives

$$d^2 = W(\mathbb{L}) - \lambda \quad (3.2)$$

where $\lambda \in \Lambda$ is the potential value $\Phi(\mathcal{L}_1)$ of the Lagrangian L_1 .

However the differential d defined using Equation (2.5) is not a Laurent polynomial of z for general choice of gauge of the flat connection on \mathbb{L} . In what follows we fix a uniform choice of gauge for each isomorphism class of flat \mathbb{C}^\times connections on \mathbb{L} , which is parametrized by the mirror variables z_i 's, such that $\delta^{\mathcal{L}_z, \mathcal{L}_1}$ is expressed in Laurent polynomials of z .

For each basic vector $E_i^* \in H^1(\mathbb{L}, \mathbb{Z})$, we fix an oriented hyper-torus $H_i \subset \mathbb{L}$ (which means a codimension 1 submanifold diffeomorphic to $(S^1)^{n-1}$) whose class is Poincaré dual to E_i^* .

Definition 3.1. We fix an identification $\mathbb{L} \cong (\mathbb{R}/\mathbb{Z})^n$, and define H_i to be

$$(\mathbb{R}/\mathbb{Z})^{i-1} \times \{0\} \times (\mathbb{R}/\mathbb{Z})^{n-i}$$

for $i = 1 \dots, n$. Fix $p = (p_1, \dots, p_n) \in (\mathbb{R}/\mathbb{Z})^n$. $p + H_i = p_i + H_i$ for each i is called a *gauge hyper-torus*. We orient H_i so that the intersection $E_i \cap H_i$ is positive. \square

Remark 3.2. The gauge hypertorus plays a similar role of bounding cochain in the work of Fukaya-Oh-Ohta-Ono [26]. A standard bounding cochain b have strictly positive exponents in the Novikov variable T in order to have e^b well-defined. A constant term (corresponding to T^0) can be added formally to b using flat non-unitary line bundles ([27], [14]). The above gauge hypertori will be used to define a flat connection below which corresponds to a constant term of b . \square

We define a flat connection ∇ with a prescribed holonomy ρ which is trivial away from tubular neighborhoods of the gauge hypertori $H_i + p$. When crossing $H_i + p$ in the positive transverse orientation (i.e. along the direction of E_i), the flat connection acts on the fiber of the line bundle by multiplication of $\rho(E_i)$.

Lemma 3.3. Given gauge hypertori $\{p + H_i\}_{i=1}^n$ and $\rho : \pi_1(\mathbb{L}) \rightarrow \mathbb{C} \setminus \{0\}$, there exists a flat connection ∇ for the trivial complex line bundle \mathcal{L}_0 over \mathbb{L} , such that ρ is trivial outside any given small neighborhood of the gauge hypertori, and has the associated holonomy representation ρ . \square

Proof. For simplicity, we consider the case of the trivial bundle $\mathbb{R} \times \mathbb{C}$ over \mathbb{R} and we leave the case of \mathbb{L} as an exercise. Let $p_i \in \mathbb{R}$, and we define its connection ∇ to be trivial ($\nabla = d$) outside $(p_i - \epsilon, p_i + \epsilon)$, and in the interval $(p_i - \epsilon, p_i + \epsilon)$, ∇ is defined to be

$$d - 2\pi i b_i \delta(t - p_i) dt$$

where $\delta(t)$ is a smooth function whose support is in the interval $(-\epsilon, +\epsilon)$ with $\int_{-\epsilon}^{+\epsilon} \delta(t) dt = 1$. ∇ defines the desired flat connection over \mathbb{R} . \blacksquare

The parallel transport of the above chosen connection along a path γ in \mathbb{L} can be expressed in terms of intersection number between $H_i + p$ and γ . Namely, for a path γ whose endpoints are away from a small neighborhood of $H_i + p$ and transversal to $H_i + p$, the parallel transport along γ is given by

$$\prod_{i=1}^n \rho(E_i)^{k_i},$$

when k_i is the signed intersection number between γ and $H_i + p$.

Now, we express the Floer differential $\delta^{\mathcal{L}_z, \mathcal{L}_1}$ in terms of z . Let $\rho^b : \pi_1(\mathbb{L}) \rightarrow \mathbb{C} \setminus \{0\}$ (see (2.1)) be the holonomy presentation with $\rho^b(E_i) = z_i$. Then we fix the flat connection ∇_z as in Lemma 3.3. Denote by $U_H \subset \mathbb{L}$ a small neighborhood of $\cup_i(p_i + H_i)$ so that the parallel transport for ∇_z is trivial outside U_H . Given

another positive Lagrangian submanifold L_1 which intersects \mathbb{L} transversely, we may assume that each point in $\mathbb{L} \cap L_1$ does not lie in U_H by shrinking the open neighborhood and changing the base point p of the gauge hypertori.

Given two intersection points $p, q \in \mathbb{L} \cap L_1$, consider a J -holomorphic strip $u : \mathbb{R} \times [0, 1] \rightarrow M$ contributing to the differential $\delta^{\mathcal{L}_z, \mathcal{L}_1}$. Consider the boundary path $\partial_i u$ from p to q in \mathbb{L} for $i = 0$ and L_1 for $i = 1$. Recall that the contribution of u to the differential $\delta^{\mathcal{L}_z, \mathcal{L}_1}$ was given by $(-1)^{a(u)} \text{Pal}_{\partial_0 u}^{-1} \text{Pal}_{\partial_1 u} T^{\omega(u)}$. Here, $\text{Pal}_{\partial_1 u}$ is a complex number.

We claim that the holonomy factor $\text{Pal}_{\partial_0 u}^{-1}$ along the path $\partial_0 u$ is indeed a monomial in $\Lambda[z_1^\pm, \dots, z_n^\pm]$. Since $\partial_0 u$ is a path starting and ending away from U_H , the holonomy factor $\text{Pal}_{\partial_0 u}^{-1}$ is given by

$$\prod_{i=1}^n z_i^{-k_i},$$

where k_i is the signed intersection number between $\partial_0 u$ and the hyper-torus $p_i + H_i$. Hence, $\delta^{\mathcal{L}_z, \mathcal{L}_1}$ from p to q gives an element of $\Lambda[z_1^\pm, \dots, z_n^\pm]$ (or $\Lambda \ll z_1^{\pm 1}, \dots, z_n^{\pm 1} \gg$ for an infinite sum) taking a sum of all such contributions from isolated J -holomorphic strips from p to q .

Intuitively, we are recording how many times the \mathbb{L} -edge of a J -holomorphic strip crosses the gauge hypertori in \mathbb{L} in terms of (z_1, \dots, z_n) variables (see Figure 1).

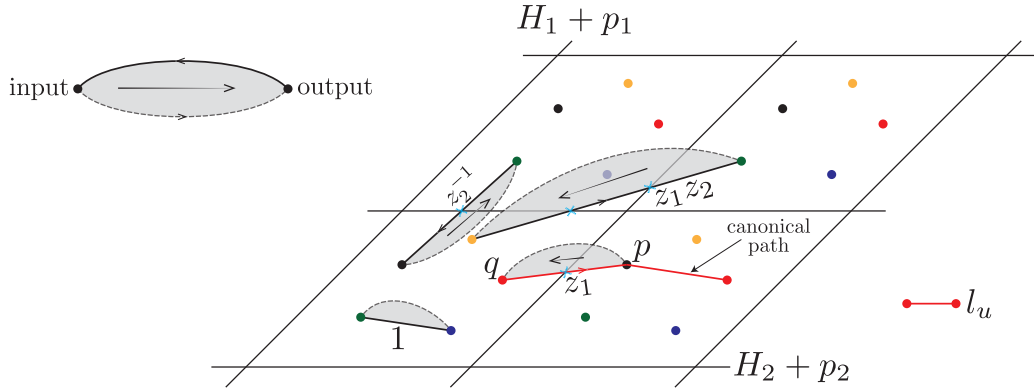


Fig. 1: J -holomorphic strips and corresponding monomials. The diagram shows the universal cover of the torus \mathbb{L} . The dots represent intersection points of \mathbb{L} and L_1 .

Remark 3.4. Here is another interpretation of the above construction. Note that by removing the gauge hypertori from \mathbb{L} , we get a simply connected region U . Hence there is a unique path (up to homotopy) for any pair of intersection points of \mathbb{L} and L_1 contained in this simply connected region. Then the path $\partial_0 u$ from p to q coming from a J -holomorphic strip u can be concatenated with this unique path (up to homotopy) from q to p to obtain a loop l_u (drawn as the red line in Figure 1), starting and ending at p . In this way, our construction can be regarded as a Fourier transform, in which the counting of J -holomorphic strips from p to q whose boundaries correspond to loops (up to homotopy) in $\pi_1(\mathbb{L}) \cong \mathbb{Z}^n$ transforms into a (Laurent) polynomial in $\Lambda[z_1^\pm, \dots, z_n^\pm]$. \square

Let P^0 (resp. P^1) be the free $\Lambda[z_1^\pm, \dots, z_n^\pm]$ -module generated by even (resp. odd) intersection points of $\mathbb{L} \cap L_1$. Here, we are using the canonical $\mathbb{Z}/2$ -grading of the Lagrangian Floer complex. The degree of $p \in CF(\mathbb{L}, L_1)$ is defined using a loop in the Lagrangian Grassmannian of $T_p X$ constructed in the following way. We choose any path from the oriented Lagrangian subspace $T_p L_1$ to the oriented Lagrangian subspace $T_p \mathbb{L}$ in the oriented Lagrangian Grassmannian of $T_p M$. We compose this path with a canonical path (see [5, Section 3]) from $T_p \mathbb{L}$ to $T_p L_1$ (without considering orientation). Hence we obtain a loop in the Lagrangian Grassmannian of $T_p M$, starting and ending at $T_p L_1$. The winding number of this loop gives the canonical $\mathbb{Z}/2$ -grading, since the different choices of oriented path from $T_p L_1$ to $T_p \mathbb{L}$ change the winding number of this loop by $2\mathbb{Z}$. $\delta^{\mathcal{L}_z, \mathcal{L}_1}$ is an odd map with respect to this $\mathbb{Z}/2$ -grading.

$\delta^{\mathcal{L}_z, \mathcal{L}_1}$ can be linearly extended to $\Lambda[z_1^\pm, \dots, z_n^\pm]$ -module homomorphisms $P^0 \rightarrow P^1$ and $P^1 \rightarrow P^0$, which is still denoted as $\delta^{\mathcal{L}_z, \mathcal{L}_1}$. We define $d_0 : P^0 \rightarrow P^1$ by $\delta^{\mathcal{L}_z, \mathcal{L}_1}$, and $d_1 : P^1 \rightarrow P^0$ by $-\delta^{\mathcal{L}_z, \mathcal{L}_1}$. Then Floer's equation (2.6) can be rewritten as a matrix factorization identity:

$$d_0 \circ d_1 = d_1 \circ d_0 = (W(\mathbb{L}) - \Phi(\mathcal{L}_1)) \cdot Id.$$

Definition 3.5. The matrix factorization of $W(\mathbb{L})$ mirror to a Lagrangian brane (L_1, \mathcal{L}_1) is defined as (P^0, P^1, d) given by the above construction. \square

The above construction depends on the choice of the base point p of the gauge hypertori. We will show in Lemma 5.3 that different choices of gauge hypertori and base point give rise to matrix factorizations in the same isomorphism class.

3.2 Generalizations

In this subsection, we briefly discuss how the construction goes in more general situations (without Assumption 1) using the machinery of [26].

First recall the definition of weakly unobstructed Lagrangian submanifold. Let \mathbb{L} be a Lagrangian torus in a general symplectic manifold. Denote by $(C(\mathbb{L}), \{m_k\})$ a unital filtered A_∞ -algebra of \mathbb{L} constructed in [26] or in [27] (which can be made unital by taking a canonical model). Recall that we have

$$m_k = \sum_{\beta \in H_2(X, \mathbb{L})} m_{k, \beta} \otimes T^{\omega(\beta)} \quad (3.3)$$

Denote by $F^+C(\mathbb{L})$ the elements of $C(\mathbb{L})$ whose coefficients have positive T -exponents. An element $b_+ \in F^+C(\mathbb{L})$ is called a *weak bounding cochain* if $m(e^b) = \sum_{k=0}^{\infty} m_k(b, \dots, b)$ is a multiple of a unit.

Choose a flat connection ∇^{b_0} of a complex line bundle \mathcal{L} over \mathbb{L} whose holonomy is ρ^{b_0} for $b_0 = \sum x_i E_i^*$ as in Section 2. We can modify (3.3) to define

$$m_k^{\rho} = \sum_{\beta \in H_2(X, \mathbb{L})} \rho^b(\partial\beta) m_{k, \beta} \otimes T^{\omega(\beta)}. \quad (3.4)$$

As explained in [27], it has the effect of adding a constant term b_0 to b_+ .

We need to make the following assumption on \mathbb{L} in this general setting.

Assumption 2. We require that there exists $b_+ \in F^+C(\mathbb{L})$ such that $b = b_+ + b_0$ is a weak bounding cochain for every $b_0 \in H^1(\mathbb{L}, \mathbb{C})/H^1(\mathbb{L}, \mathbb{Z})$. \square

As we require the existence of a family of weak bounding cochains, this is stronger than the standard weakly unobstructed condition on b_+ .

It was shown in [27, Section 4] that a Lagrangian torus fiber in a compact toric manifold satisfies this assumption (with $b_+ = 0$). Hence a Floer potential for a torus fiber of any toric manifold can be defined. In this case, the Floer equation (2.6) was shown in Lemma 12.7 of [27].

With Assumption 2 on \mathbb{L} , the previous construction of mirror matrix factorization generalizes as follows. Let L_1 be a weakly unobstructed Lagrangian submanifold of X with a weak bounding cochain b' (L_1 does not have to satisfy the stronger assumption 2). In addition, we assume that L_1 intersects transversely with \mathbb{L} (by using Hamiltonian isotopy of X if necessary) and the intersection is away from the neighborhood of the chosen hyper-tori.

Denote by $b_z = b_0 + b_+$ to emphasize the dependence on z . Then the Lagrangian Floer complex between (\mathbb{L}, b_z) and (L_1, b') satisfies the Floer equation (2.6) ([26]), and hence it is easy to see that we can find the mirror matrix factorization of (L_1, b') in the same way.

On the other hand in actual computations, we find that the pearl complex given by [8] behaves better, since generators of the complex are given by the critical points of a Morse function which can be chosen away from the hyper-tori. We will explain them in Section 7.

4 Examples

In this section, we explain the Floer potentials and mirror matrix factorizations through a couple of monotone examples. We will perform the construction systematically for toric fibers of toric manifolds in Section 8.

4.1 \mathbb{P}^1

Consider \mathbb{P}^1 with total symplectic area k . Take \mathbb{L} to be an oriented great circle S^1 in \mathbb{P}^1 . Fix a point $h \in \mathbb{L}$, and a flat connection ∇_z (on a complex line bundle \mathcal{L}_z) which is trivial away from a small neighborhood $U_h \subset \mathbb{L}$ of h and has holonomy $z \in \mathbb{C}^\times$ along \mathbb{L} .

There are two holomorphic discs bounded by \mathbb{L} , namely the upper and lower hemispheres. The Floer potential equals to

$$W(\mathbb{L}) = T^{k/2}(z + z^{-1}).$$

(It is equivalent to the well-known Hori-Vafa mirror $z' + T^k/z'$ by the change of coordinate $z' = T^{k/2}z$.)

Take L_1 to be another great circle intersecting transversely with \mathbb{L} at the two antipodal points $p, q \in \mathbb{L} \cap L_1$. It is assumed that h and U_h are taken such that $p, q \notin U_h$. Equip L_1 with a trivial flat line bundle \mathcal{L}_1 with holonomy $\lambda \in \mathbb{C}^\times$. (It is known from [27] that the object (L_1, \mathcal{L}_1) is non-trivial in the Fukaya category if and only if $\lambda = \pm 1$.)

We also fix the gauge of \mathcal{L}_1 by fixing a point $h_1 \in L_1$ and requiring that the flat connection on \mathcal{L}_1 is trivial away from a small neighborhood $U_{h_1} \subset L_1$ of h_1 . (Again h_1, U_{h_1} are chosen such that $p, q \notin U_{h_1}$.) Orient \mathbb{L}, L_1 as in Figure 2. Then p and q have even and odd degree respectively. The differential is given as

$$\delta^{\mathcal{L}_z, \mathcal{L}_1}(p) = T^{k_1} \left(z - \frac{1}{\lambda} \right) q, \quad \delta^{\mathcal{L}_z, \mathcal{L}_1}(q) = T^{k_2} \left(-1 + \frac{\lambda}{z} \right) p$$

Hence,

$$(\delta^{\mathcal{L}_z, \mathcal{L}_1})^2 = T^{k/2} \left(\lambda + \frac{1}{\lambda} \right) - T^{k/2} \left(z + \frac{1}{z} \right)$$

If we change the location of h or h' , we get a different but isomorphic matrix factorization. This can be checked directly, and indeed it is a general fact (Lemma 5.3).

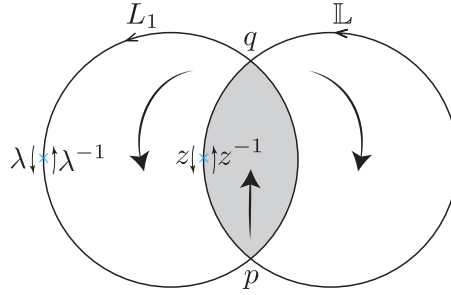


Fig. 2: \mathbb{L} and L_1 in $\mathbb{C}P^1 \setminus \{\infty\}$

4.2 \mathbb{P}^1 with another Lagrangian

L_1 in the last subsection is obtained from the great circle \mathbb{L} by rotation of \mathbb{P}^1 , which is a Hamiltonian isotopy. Let us take another Hamiltonian isotope L_2 of \mathbb{L} as shown in Figure 3. In particular the areas $\alpha, \beta, \gamma, \delta$ in Figure 3 satisfy the relation $\alpha + \beta = \gamma + \delta$. For simplicity equip L_2 with the trivial holonomy.

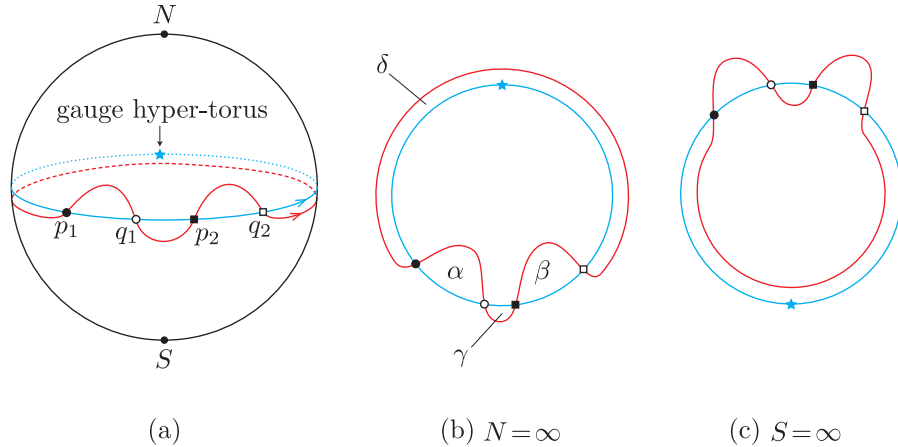


Fig. 3: $L_2 \subset \mathbb{P}^1$

By Definition 3.5 and counting holomorphic strips, it is easy to obtain the matrix factorization mirror to L_2 :

$$\begin{array}{c} p_1 \\ q_1 \end{array} \begin{pmatrix} T^{\frac{k}{2}-\alpha}(z-1) & T^{\frac{k}{2}-\gamma}(1-\frac{1}{z}) \\ T^{\frac{k}{2}-\alpha-\beta+\gamma}(z-1) & T^{\frac{k}{2}-\beta}(z-1) \end{pmatrix}, \quad \begin{array}{c} q_1 \\ p_2 \end{array} \begin{pmatrix} T^\alpha & -T^{\alpha+\beta-\gamma}\frac{1}{z} \\ -T^\gamma & T^\beta \end{pmatrix}. \quad (4.1)$$

Since L_2 is Hamiltonian isotopic to L_1 in the previous section, one naturally expect that (4.1) would give an equivalent object in the mirror of \mathbb{P}^1 . We will prove this fact in more general setting in Section 5.2.

4.3 $\mathbb{P}^1 \times \mathbb{P}^1$

Consider the symplectic manifold $\mathbb{P}^1 \times \mathbb{P}^1$ whose moment map image is $[-\frac{k}{2\pi}, \frac{k}{2\pi}]^2$. We compute matrix factorizations for two specific Lagrangian submanifolds, namely the toric fiber at $0 \in [-\frac{k}{2\pi}, \frac{k}{2\pi}]^2$ and the anti-diagonal A .

Note that toric fibers split-generates the Fukaya category of $\mathbb{P}^1 \times \mathbb{P}^1$ by the result of [3]. In 9.2, we will see that their mirror matrix factorizations are also split-generators. The anti-diagonal A is another interesting object in the Fukaya category, which appears (as a Lagrangian) only when two \mathbb{P}^1 -factors have the same symplectic forms. Similar phenomenon happens on the mirror side explained in [29, 8.2]. $A \oplus A[1]$ is isomorphic to the sum of two toric fibers with holonomies $(\pm 1, \mp 1)$ (which is conjectured in [11] and proven in [17]).

Let S_0 and S_1 be two distinct oriented great circles of \mathbb{P}^1 which intersect transversely with each other at two antipodal points. Denote the two intersection points of S_0 and S_1 by p (which has odd-degree) and q (which has even degree). Set $\mathbb{L} = S_0 \times S_1$, which is Hamiltonian isotopic to the toric fiber at $0 \in [-\frac{k}{2\pi}, \frac{k}{2\pi}]^2$. Let $\mathcal{L}_z \rightarrow \mathbb{L}$ be the flat line bundle with holonomy (z_1, z_2) .

\mathbb{L} is monotone, and its Floer potential is given by

$$W(\mathbb{L})(z_1, z_2) = T^{k/2} \left(z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} \right). \quad (4.2)$$

Let $L_1 = S_1 \times S_0$ and it is equipped with the flat line bundle \mathcal{L}_1 with holonomy (λ_1, λ_2) . Note that L_1 is Hamiltonian isotopic to \mathbb{L} and hence the toric fiber at $0 \in [-\frac{k}{2\pi}, \frac{k}{2\pi}]^2$.

The matrix factorization corresponding to L_1 can be found by using the previous calculation for \mathbb{P}^1 as follows. The intersection $\mathbb{L} \cap L_1$ consists of 4 points, namely two even intersections $(p, p), (q, q)$ and two odd intersections $(p, q), (q, p)$. The differential $\delta^{\mathcal{L}_z, \mathcal{L}_1}$ is represented by the matrices

$$A := \begin{array}{c} (p, p) \\ (p, q) \\ (q, p) \end{array} \begin{pmatrix} T^{k_1}(\lambda_2 z_2 - 1) & T^{k_2}(-\frac{1}{\lambda_1} + \frac{1}{z_1}) \\ T^{k_1}(\lambda_1 z_1 - 1) & -T^{k_2}(-\frac{1}{\lambda_2} + \frac{1}{z_2}) \end{pmatrix}, \quad B := \begin{array}{c} (p, q) \\ (q, q) \end{array} \begin{pmatrix} T^{k_2}(-\frac{1}{\lambda_2} + \frac{1}{z_2}) & T^{k_2}(-\frac{1}{\lambda_1} + \frac{1}{z_1}) \\ T^{k_1}(\lambda_1 z_1 - 1) & -T^{k_1}(\lambda_2 z_2 - 1) \end{pmatrix} \quad (4.3)$$

with $k_1 + k_2 = k/2$. The additional signs at (2,2) positions of the matrices come from Koszul sign convention. One can check that this gives a matrix factorization of (4.2). (Compare it with the one given in [17, Remark 4.3].) Hence we obtain

Proposition 4.1. The matrix factorization mirror to L_1 is given by rank 2 free modules P_0, P_1 with $d_0 = A, d_1 = -B$, which satisfies

$$d_0 d_1 = d_1 d_0 = (W(\mathbb{L})(z_1, z_2) - W(\mathbb{L})(\lambda_1, \lambda_2)) \text{Id}_{2 \times 2}.$$

□

In the following we compute the matrix factorization mirror to the anti-diagonal which was first studied in [17]. The anti-diagonal A is defined by

$$A := \{([z : w], [\bar{z} : \bar{w}]) \mid [z : w] \in \mathbb{P}^1\} \subset \mathbb{P}^1 \times \mathbb{P}^1.$$

We remark that $\Phi(A) = 0$ since the minimal Maslov index for A is 4. We fix \mathcal{L}_A to be the flat line bundle on A with trivial holonomy. The intersection $\mathbb{L} \cap A$ consists of 2 points $(p, p), (q, q)$, which generate $CF((\mathbb{L}, \mathcal{L}_z), (A, \mathcal{L}_A))$.

Recall the following doubling argument from [17, Proposition 4.1].

Lemma 4.2. There is an one-to-one correspondence between

$$\begin{array}{l} \{\text{holomorphic strips in } \mathbb{P}^1 \times \mathbb{P}^1 \text{ bounded by } \mathbb{L} \text{ and } A\} \text{ and} \\ \{\text{holomorphic strips in } \mathbb{P}^1 \text{ bounded by } S_0 \text{ and } S_1\}. \end{array}$$

Moreover, the correspondence preserves symplectic areas. \square

Proof. Let $u = (u_1, u_2) : \mathbb{R} \times [0, 1] \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be a holomorphic strip between $S_0 \times S_1$ and A . u_1 and u_2 can be glued as in (a) of Figure 4 to give a holomorphic strip between S_0 and S_1 . For more details, see [17]. \blacksquare

Therefore the differential from (p, p) to (q, q) is $T^{k_1}(1 + z_1 z_2)$, and the differential from (q, q) to (p, p) is $T^{k_2}(\frac{1}{z_1} + \frac{1}{z_2})$ (see (b) of Figure 4). Hence we obtain

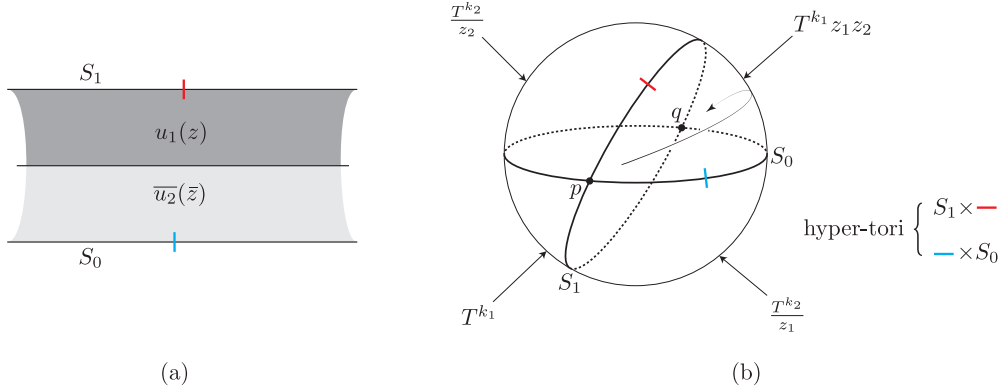


Fig. 4: Gluing argument for $(S_1 \times S_0, A)$

Proposition 4.3. The anti-diagonal A is mirror to the matrix factorization

$$W(\mathbb{L}) = (T^{k_1}(1 + z_1 z_2)) \cdot \left(T^{k_2} \left(\frac{1}{z_1} + \frac{1}{z_2} \right) \right).$$

\square

The result agrees with the one in [17] which was obtained by using SYZ-fibration structure.

5 Invariance of matrix factorizations under various choices

We have made a choice of gauge hypertori and Hamiltonian perturbations on Lagrangian submanifolds. In this section, we show that a different choice of gauge hypertori gives rise to an isomorphic matrix factorization, and Hamiltonian isotopic Lagrangians induce homotopic matrix factorizations.

5.1 Infinitesimal gauge equivalences and a choice of gauge hypertori

We first show that the isomorphism class of mirror matrix factorization does not depend on the choice of gauge hypertori. Indeed, we may interpret changing gauge hypertori as an infinitesimal gauge equivalence. For a trivial line bundle $E := \mathbb{L} \times \mathbb{C}$ on \mathbb{L} and two flat connection ∇_i on E ($i = 0, 1$), we define the infinitesimal gauge equivalence relation between two flat connections ∇_0 and ∇_1 as follows.

Definition 5.1. ∇_0 and ∇_1 are said to be infinitesimally gauge equivalent if we have a trivial bundle $E \times (-\epsilon', 1 + \epsilon') \rightarrow \mathbb{L} \times (-\epsilon', 1 + \epsilon')$ and a flat connection ∇ on $E \times (-\epsilon', 1 + \epsilon')$ such that ∇ when restricted to $E \times \{0\} \rightarrow \mathbb{L} \times \{0\}$ becomes ∇_0 and when restricted to $E \times \{1\} \rightarrow \mathbb{L} \times \{1\}$ becomes ∇_1 . Here ϵ' is a small positive real number. \square

Now consider gauge hypertori $\{H_i + p\}$ for two different choices of $p \in (\mathbb{R}/\mathbb{Z})^n$.

Lemma 5.2. Flat connections constructed in Lemma 3.3 from two different choices of gauge hypertori $\{H_i + p^j\}_{i=1}^n$ (for $j = 0, 1$) are infinitesimally gauge equivalent. \square

Proof. We prove it for a trivial bundle $\mathbb{R} \times \mathbb{C}$ over \mathbb{R} , and the proof for a torus \mathbb{L} is similar. Consider $p^0, p^1 \in \mathbb{R}$, and the corresponding flat connections ∇_j defined by

$$\nabla_j = d - 2\pi i b \delta(t - p^j) dt$$

for $j = 0, 1$. Then the desired flat connection ∇ on $\mathbb{R} \times \mathbb{C} \times (-\epsilon', 1 + \epsilon') \rightarrow \mathbb{R} \times (-\epsilon', 1 + \epsilon')$ can be defined as

$$\nabla = d - 2\pi i b \delta(t - p^0 + (p^0 - p^1)s) dt - 2(p^0 - p^1)\pi i b \delta(t - p^0 + (p^0 - p^1)s) ds,$$

where s is a coordinate on $(-\epsilon', 1 + \epsilon')$. One can easily check that ∇ is flat, and it gives infinitesimal gauge equivalence between ∇_0 and ∇_1 . \blacksquare

Let $(P_{p^j}^0, P_{p^j}^1, d^j)$ be the matrix factorization corresponding to the gauge hypertori $\{H_i + p^j\}_{i=1}^n$. Using the infinitesimal gauge equivalence, we define the chain isomorphism between these two matrix factorizations.

Lemma 5.3. There exists a chain isomorphism between two matrix factorizations $(P_{p^j}^0, P_{p^j}^1, d^j)$ for $j = 1, 2$, that is, we have an isomorphism $\phi : P_{p^0}^\bullet \rightarrow P_{p^1}^\bullet$ such that $\phi \circ d = d \circ \phi$. \square

Proof. Let $(\tilde{\mathcal{L}}, \tilde{\nabla})$ be the trivial bundle over $\mathbb{L} \times (-\epsilon', 1 + \epsilon')$, which defines the infinitesimal gauge equivalence between two flat connections on \mathcal{L} from two different gauge hypertori.

We define ϕ for each generator $p \in \mathbb{L} \cap L_1$ to be an identity map, multiplied by a holonomy of the bundle $\tilde{\mathcal{L}}$ along an interval $p \times [0, 1]$. To see that this gives a chain isomorphism between two matrix factorizations, observe that the parallel transport with respect to the flat connection $\tilde{\nabla}$ does not depend on the choice of a path with fixed end points. This directly leads to the chain map property $\phi \circ d = d \circ \phi$. One can construct its inverse in a similar way. \blacksquare

In particular, this prove that the isomorphism class of the mirror matrix factorization under our construction is independent of a choice of gauge hypertori to define the connection ∇ .

5.2 Hamiltonian isotopy

Consider a Hamiltonian isotopy ψ of X , and two Lagrangian submanifolds, L_1 and $L_2 := \psi(L_1)$. For a positive Lagrangian torus \mathbb{L} , with Floer potential $W(\mathbb{L})$, we obtain two matrix factorizations, $(P^0(L_i), P^1(L_i), d)$ from L_i for $i = 1, 2$.

Proposition 5.4. Two matrix factorizations $(P^0(L_i), P^1(L_i), d)$ from L_i for $i = 1, 2$ of $W(\mathbb{L})$ are homotopic to each other. Namely, there exist even maps $f : P^*(L_1) \rightarrow P^*(L_2)$, $g : P^*(L_2) \rightarrow P^*(L_1)$ such that

$$g \circ f \sim \text{id}, \quad f \circ g \sim \text{id}.$$

\square

Proof. The standard continuation maps in Floer theory are given by

$$f_{\#} : CF(\mathbb{L}, L_1) \rightarrow CF(\mathbb{L}, \psi(L_1)), \quad g_{\#} : CF(\mathbb{L}, \psi(L_1)) \rightarrow CF(\mathbb{L}, L_1),$$

(see for example [22], [33]) which satisfies

$$g_{\#} \circ f_{\#} = \delta \circ H_1 - H_1 \circ \delta, \quad f_{\#} \circ g_{\#} = \delta \circ H_2 - H_2 \circ \delta.$$

Here, homotopy H_i is also constructed by counting parametrized version of J -holomorphic strips.

Now, to prove the proposition, we introduce flat line bundles $\mathcal{L}_z, \mathcal{L}_1$ with connections ∇_z, ∇_1 on \mathbb{L}, L_1 . ($L_2 = \psi(L_1)$ is equipped with a flat line bundle $(\psi^{-1})^*(\mathcal{L}_1)$). Then the same construction with an addition of holonomies yields required maps between matrix factorizations: we denote the corresponding maps as $f := f_{\#}^b, g := g_{\#}^b, h_1 := H_1^b$ and $h_2 := H_2^b$, which satisfies

$$g \circ f = d \circ h_1 - h_1 \circ d, \quad f \circ g = d \circ h_2 - h_2 \circ d.$$

\blacksquare

Consequently, if L_2 is a Hamiltonian isotopy image of a Lagrangian submanifold L_1 , then the resulting matrix factorization of L_1 and L_2 are quasi-isomorphic.

6 A_∞ -functor

In this section, we extend the previous construction of the correspondence between Lagrangians and matrix factorizations in the object level to an A_∞ -functor from the Fukaya category of X to the matrix factorization category of $W(\mathbb{L})$.

6.1 Preliminaries

Let us first recall the definition of a filtered A_∞ -category to set up notations (see [23] for details).

Definition 6.1. A filtered A_∞ -category \mathcal{C} consists of a collection of objects $Ob(\mathcal{C})$ together with morphisms $\mathcal{C}(C_1, C_2)$ for $C_1, C_2 \in Ob(\mathcal{C})$ given by a graded torsion-free filtered Λ_0 -module, and degree-1 operations m_k 's on morphisms

$$m_k : \mathcal{C}[1](C_1, C_2) \otimes \cdots \mathcal{C}[1](C_{k-1}, C_k) \rightarrow \mathcal{C}[1](C_0, C_k)$$

for $k = 0, 1, \dots$, which respects the filtration and satisfy the following A_∞ -equations: for $x_i \in \mathcal{C}[1](C_{i-1}, C_i)$ ($i = 1, \dots, n$), we have

$$\sum_{k_1+k_2=k+1} \sum_{i=1}^{k_1} (-1)^\epsilon m_{k_1}(x_1, \dots, x_{i-1}, m_{k_2}(x_i, \dots, x_{i+k_2-1}), \dots, x_k) = 0$$

where $\epsilon = \sum_{j=1}^{i-1} (|x_j| + 1)$. □

A *filtered differential graded category* \mathcal{C} is a filtered A_∞ -category with $m_0 = m_{\geq 3} \equiv 0$.

An A_∞ -functor \mathcal{F} between two filtered A_∞ -categories $\mathcal{C}_1, \mathcal{C}_2$ can be defined as follows. First, we have a map between objects $\mathcal{F}_0 : Ob(\mathcal{C}_1) \rightarrow Ob(\mathcal{C}_2)$. And given $A, B \in Ob(\mathcal{C}_1)$, we have homomorphism of degree 0:

$$\mathcal{F}_j(A, B) : \bigoplus_{A=C_0, C_1, \dots, C_j=B} \mathcal{C}[1](C_0, C_1) \otimes \cdots \mathcal{C}[1](C_{j-1}, C_j) \rightarrow \mathcal{C}_2[1](\mathcal{F}_0(A), \mathcal{F}_0(B)),$$

for each $j = 1, 2, \dots$ which satisfies A_∞ -functor equation (see [23]).

Our main concerns are the following two A_∞ -categories: one is the Fukaya category of a symplectic manifold X , and the other is a filtered differential graded category $\mathcal{MF}(W)$, of matrix factorizations of $W(\mathbb{L})$.

Let us first define the matrix factorization category which is a differential graded category.

Definition 6.2. Let \mathcal{O} be the algebra $\Lambda[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_n^{\pm 1}]$. The category of matrix factorization $\mathcal{MF}(W)$ of $W \in \mathcal{O}$ is defined as follows. An object of $\mathcal{MF}(W)$ is a finite dimensional free $\mathbb{Z}/2$ -graded \mathcal{O} -modules $P = P^0 \oplus P^1$, together with an odd map $d : P \rightarrow P$ such that

$$d^2 = (W - \lambda) \cdot \text{Id}_P$$

for some $\lambda \in \mathbb{C}$.

A morphism between two matrix factorizations $(P, d_P), (Q, d_Q)$ is given by an \mathcal{O} -module homomorphism $f : P \rightarrow Q$. A differential d on a morphism f is given as

$$d(f) = d_Q \circ f + (-1)^{\deg f} f \circ d_P,$$

and composition of morphisms are defined as usual. □

Recall that differential graded category \mathcal{C} with differential d , and composition \circ gives rise to an A_∞ -category with the following sign convention.

$$m_1(x) = (-1)^{\deg x} d(x), m_2(x_1, x_2) = (-1)^{\deg x_1 (\deg x_2 + 1)} x_2 \circ x_1.$$

From now on, we regard $\mathcal{MF}(W)$ as a filtered A_∞ -category with vanishing $m_0, m_{\geq 3}$.

The Fukaya category $\mathcal{Fuk}(X)$ of a symplectic manifold X is defined as follows. We only sketch the setup briefly, and leave detailed construction to [23] and [26]. An object of Fukaya category $\mathcal{Fuk}(X)$ is a spin (oriented) Lagrangian submanifold with a flat complex line bundle, and in addition, it is assumed to be *positive* (or in general weakly unobstructed in the sense of [26]). We remark that the grading datum is not included, as we use $\mathbb{Z}/2$ -grading. We require an object to be spin (not relatively spin) so that Lagrangian Floer complex with L_0 is defined over a characteristic-0 field Λ . (But we can work in more general setting as illustrated in Section 10.)

Morphisms and m_1 between two objects L_0, L_1 are given by the Lagrangian Floer complex $CF^*(L_0, L_1)$ as explained before. (Here we omit the notation of flat line bundles on L_i for simplicity.)

Higher morphism m_k is defined by counting J -holomorphic polygons: For distinct Lagrangian submanifolds L_0, \dots, L_k , the A_∞ -operation m_k is defined as

$$m_k : CF(L_0, L_1) \times \dots \times CF(L_{k-1}, L_k) \rightarrow CF(L_0, L_k) \quad (6.1)$$

$$m_k(p_1, \dots, p_k) = \sum_q n(p_1, \dots, p_k, y) y$$

where $n(p_1, \dots, p_k, y)$ are the contributions from signed counting of J -holomorphic polygon u 's together an symplectic area $T^{u(u)}$ with holonomy effects from flat complex line bundles along their boundary $u(\partial D^2)$. (see Definition 3.26 [23]). Here, J -holomorphic polygon above is a map u from D^2 with $k+1$ punctures $z_0, \dots, z_k \in \partial D^2$ such that a part of the boundary ∂D^2 between z_i, z_{i+1} is required to map into L_i for $i = 0, \dots, k$ and the map u limits to the intersection point p_i at the puncture z_i for $i = 1, \dots, k$ whereas u limits to \bar{y} at the puncture z_0 . Here, \bar{y} is the intersection point y regarded as an element of $CF(L_k, L_0)$. Since we assumed that Lagrangians are positive (Assumption 1), we can use domain-dependent perturbations (as in [37]) to make the above operation transversal, and satisfy A_∞ -operations between transversal Lagrangians. Lagrangians here can have nontrivial m_0 given by Maslov index disc two contributions.

When Lagrangians are not distinct, the construction of m_k needs more advanced machinery such as Kuranishi structures, and we refer readers to [23] for details. We remark that one may instead work with A_∞ -pre-categories defined by Kontsevich-Soibelman [32]. The construction of A_∞ -functor will resemble the Yoneda embedding, and hence functor is well-defined once the Fukaya category itself is well-defined, which we assume from now on.

6.2 Localized mirror functor

Let us fix a reference positive Lagrangian torus \mathbb{L} in a symplectic manifold X of real dimension $2n$. Let us denote its Floer potential by W ($:= W(\mathbb{L})$). We regard the potential W as an element of $\Lambda \ll z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_n^{\pm 1} \gg$.

We fix gauge hypertori $\{p_i + H_i\}_{i=1}^n$ of \mathbb{L} , and its sufficiently small neighborhood $U \subset \mathbb{L}$. For the Fukaya category $\mathcal{Fuk}(X)$ of X , we suppose that it has only countably many objects (L_i, \mathcal{L}_i) for $i = 1, 2, \dots$. We may assume that each Lagrangian submanifold L_i is transverse to \mathbb{L} . Furthermore, we may assume that the intersection $L_i \cap \mathbb{L}$ is away from the gauge hypertori and in particular disjoint from U . This can be done by taking a suitable Hamiltonian isotopy of L_i 's in the following way if necessary. For any finite, say k points of \mathbb{L} , we can move it to another configuration of k points by a Hamiltonian isotopy preserving \mathbb{L} (see Lemma 2.7 [39]). Hence, for any L_i , we can move points in $L_i \cap \mathbb{L}$ away from U by a Hamiltonian isotopy ϕ_H , and we take $\phi_H(L_i)$ as an object of the Fukaya category $\mathcal{Fuk}(X)$ instead of L_i .

In particular, we do not take the reference Lagrangian \mathbb{L} itself to be an object of this Fukaya category $\mathcal{Fuk}(X)$, but take a suitable Hamiltonian isotopy image of \mathbb{L} (to be one of L_i 's). The above step is essential to define the mirror functor. In fact, we first consider slighted extended version of Fukaya category $\mathcal{Fuk}_{+0}(X)$ whose objects are $\{(\mathbb{L}, \mathcal{L})\} \cup \{(L_i, \mathcal{L}_i) : i = 1, 2, \dots\}$, satisfying the above conditions, and restrict to the objects of $i = 1, 2, \dots$ to obtain Fukaya category $\mathcal{Fuk}(X)$.

Let us denote by $\{m_k\}_{k=1}^\infty$ the A_∞ -operations on $\mathcal{Fuk}_{+0}(X)$ (and then those on $\mathcal{Fuk}(X)$ by restriction). Now, recall that for \mathbb{L} , the holonomy of the bundle \mathcal{L}_z is written in mirror parameters $z_i = \rho^b(E_i)$ (see equation (2.1)), where $b = \sum x_i E_i^* \in H^1(\mathbb{L}, \mathbb{C})/H^1(\mathbb{L}, \mathbb{Z})$. To highlight the commonly used notation b of deformation parameter, we write \mathcal{L}_z^b for \mathcal{L}_z in this section.

In what follows, we will always put \mathbb{L} at the first slot in the A_∞ -operation (6.1). Then, we will modify the definition m_k by incorporating the effect of \mathcal{L}_z^b . Namely, for a relevant J -holomorphic polygon u , we will record the holonomy contribution of \mathcal{L}^b along the arc $u(\partial D^2)$ between 0-th marked point p_0 and 1-st marked point p_1 . Like in the formula (3.4), we modify m_k by multiplying this holonomy effect along $p_0 p_1$ -arc, and denote it as $m_k^{b,0,\dots,0}$. This is not exactly the same as $m_k^{b,0,\dots,0}$ defined in [26], but it should be considered as its line bundle analogue. (Here, we do not define m_k -operation for E_i^* , and this is the reason of potential confusion. For the proper comparison, we can set a geometric representative $\tilde{b} = \sum z_i(p + H_i)$, and then what we define may be considered as $m_k^{\tilde{b},0,\dots,0}$ defined in [26].)

In any case, what is important is that we obtain the correct the A_∞ -equation for $m_k^{b,0,\dots,0}$ from the Gromov-Floer compactification of the moduli space of J -holomorphic polygons, by tracking the arc that we take holonomy along.

From our assumption that the intersection points are away from the chosen gauge hypertori for $\mathcal{Fuk}(X)$, such a holonomy for $m_k^{b,0,\dots,0}$ of $\mathcal{Fuk}(X)$ is always given as a Laurent monomial in z_1, \dots, z_n . For example, the differential $\delta^{\mathcal{L}_z^b, \mathcal{L}_i}$ of the Floer complex $CF((\mathbb{L}, \mathcal{L}_z^b), (L_i, \mathcal{L}_i))$ is nothing but $m_1^{b,0}$ between $(\mathbb{L}, \mathcal{L}_z^b)$ and (L_i, \mathcal{L}_i) .

Definition 6.3. The A_∞ -functor

$$\mathcal{LM} : \mathcal{Fuk}(X) \rightarrow \mathcal{MF}(W).$$

is defined as follows.

1. For objects, \mathcal{LM}_0 sends an object (L_i, \mathcal{L}_i) of the Fukaya category to the matrix factorization obtained by the Lagrangian Floer complex

$$MF(L_1) := (CF(\mathbb{L}, \mathcal{L}_z^b), (L_i, \mathcal{L}_i), m_1^{b,0})$$

2. For $x_1 \in CF(L_i, L_j)$,

$$\mathcal{LM}_1(x_1) : CF((\mathbb{L}, \mathcal{L}_z^b), L_i) \rightarrow CF((\mathbb{L}, \mathcal{L}_z^b), L_j)$$

is defined as follows. For $y \in CF((\mathbb{L}, \mathcal{L}_z^b), L_i)$,

$$\mathcal{LM}_1(x_1)(y) = (-1)^{(\deg x_1 + 1)(\deg p + 1)} m_2^{b,0,0}(y, x_1).$$

3. Similarly, \mathcal{LM}_k is defined as follows. For $x_j \in CF(L_{i_j}, L_{i_{j+1}})$ for $j = 1, \dots, k$,

$$\mathcal{LM}_k(x_1, \dots, x_k) : CF((\mathbb{L}, \mathcal{L}_z^b), L_{i_1}) \rightarrow CF((\mathbb{L}, \mathcal{L}_z^b), L_{i_k})$$

sends $y \in CF((\mathbb{L}, \mathcal{L}_z^b), L_{i_1})$ to

$$\mathcal{LM}_k(x_1, x_2, \dots, x_k)(y) = (-1)^{(k + \sum_i \deg x_i)(\deg p + 1)} m_k^{b,0,\dots,0}(y, x_1, \dots, x_k).$$

□

We refer readers to Section 2 of [15] for the algebraic formalism of mirror functor. The proof of Theorem 2.18 in [15] carries over to the setting here and give

Theorem 6.4. The collection of maps $\{\mathcal{LM}_k\}$ defines an A_∞ -functor. □

We remark that the above construction naturally generalizes to weakly unobstructed Lagrangians. Namely, the condition that L_i 's are positive Lagrangians in a Fano manifold can be relaxed to the condition that L_i 's are weakly unobstructed Lagrangians in a symplectic manifold in the sense of [26]. In this case, we consider (L_i, b_i) for the weak bounding cochain b_i of L_i as an object of $\mathcal{Fuk}(X)$, and then replace $m_k^{b,0,\dots,0}$ by m_k^{b,b_1,\dots,b_k} . The rest of the procedure is the same, and we leave the details as an exercise (see also Theorem 2.19 [15]).

7 Matrix factorizations and pearl complex

Let \mathbb{L} be a positive Lagrangian torus with a Floer potential $W(\mathbb{L})$. In the last section, we have defined an A_∞ -functor transforming Lagrangian branes (L_1, \mathcal{L}_1) to a matrix factorization (see also Definition 3.5). In this section, we consider the case of transforming \mathbb{L} itself, namely $L_1 = \mathbb{L}$ with a fixed line bundle \mathcal{L}_1 on it.

From Proposition 5.4 on Hamiltonian invariance, we may take a Hamiltonian isotopy ϕ such that $\phi(\mathbb{L})$ intersects \mathbb{L} transversely and define its mirror matrix factorization by the Floer complex of $(\mathbb{L}, \phi(\mathbb{L}))$. On the other hand, Bott-Morse Floer theory is very useful for computation. In this section, we define the mirror matrix factorization by using pearl complex defined by Biran-Cornea [8] (the idea of such a complex appeared in [34] earlier). As an application we compute the matrix factorization mirror to the Clifford torus in \mathbb{P}^2 , which agrees with the result of [11] from a different approach. Later in Section 8.1, we use pearl complex to construct mirror matrix factorizations for general toric Fano manifolds.

7.1 Pearl complex with decoration by flat bundles

We first recall from [8] the set-up of a pearl complex for a positive spin Lagrangian submanifold \mathbb{L} . We fix a generic Morse-Smale function h on \mathbb{L} . The pearl chain complex $C^{\text{pearl}}(\mathbb{L}, h)$ is a $\mathbb{Z}/2$ -graded free $\Lambda_{\mathbb{Z}}$ -module generated by critical points of h , where the $\mathbb{Z}/2$ -grading come from Morse indices of h . The differential of the pearl complex is given by counting pearl trajectories.

First, for each pair of critical points p, q of Morse function h , let $\mathcal{M}_2(p, q)$ be the moduli space of pearl trajectories as in Figure 5, where they consist of gradient trajectories of $-h$ together with J -holomorphic discs. More precisely, we have a collection of gradient trajectories

$$\gamma_0 : (-\infty, t_0] \rightarrow \mathbb{L}, \gamma_1 : [t_0, t_1] \rightarrow \mathbb{L}, \dots, \gamma_{k-1} : [t_{k-2}, t_{k-1}] \rightarrow \mathbb{L}, \gamma_k : [t_{k-1}, \infty) \rightarrow \mathbb{L},$$

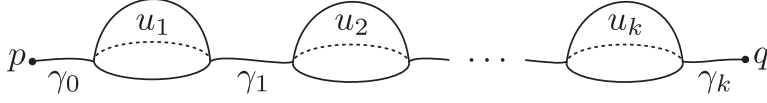


Fig. 5: A pearl trajectory

satisfying

$$\lim_{t \rightarrow -\infty} \gamma_0(t) = p, \lim_{t \rightarrow \infty} \gamma_k(t) = q, \quad \frac{d\gamma_i}{dt} = -\nabla h(\gamma_i(t)) \text{ for all } i = 0, 1, \dots, k,$$

and a collection of somewhere injective J -holomorphic discs $u_1, \dots, u_k : (D^2, \partial D^2) \rightarrow (X, \mathbb{L})$ satisfying

$$\gamma_j(t_j) = u_{j+1}(-1), j = 0, 1, \dots, k-1,$$

$$\gamma_j(t_{j-1}) = u_j(+1), j = 1, 2, \dots, k.$$

If we define the total Maslov index μ by $\sum_{j=1}^n \mu(u_j)$, then the expected dimension of $\mathcal{M}_2(p, q)$ is given by $\text{ind}(p) - \text{ind}(q) + \mu - 1$. A pearl trajectory is a collection $\{\gamma_j\}_{j=0}^k \cup \{u_j\}_{j=1}^k$ satisfying the above conditions, and we denote it by u .

Now the differential for the pearl complex is given by

$$\delta_{\text{pearl}}(\langle p \rangle) = \delta_{\text{Morse}}(\langle p \rangle) + \sum_{q, \text{ind}(q) - \text{ind}(p) = \mu - 1 > 1} (-1)^{\text{ind}(q)} n(p, q) \langle q \rangle, \quad (7.1)$$

where δ_{Morse} is the usual Morse differential, and $n(p, q)$ is the signed count of isolated pearl trajectories denoted by u between p and q weighted by the symplectic area $T^{\sum_{j=1}^k \omega(u_j)}$. It is proved in [8, Lemma 5.1.3] that $\delta_{\text{pearl}}^2 = 0$.

It is easy to see that δ_{pearl} in fact can be decomposed in terms of the total Maslov index $\mu = \sum_{j=1}^n \mu(u_j)$, since we know that $\mu \geq 0$ (the equality holds for Morse trajectories). That is, we can write

$$\delta_{\text{pearl}} = (\delta_{\text{pearl}})_1 + (\delta_{\text{pearl}})_{-1} + (\delta_{\text{pearl}})_{-3} + \dots \quad (7.2)$$

where $(\delta_{\text{pearl}})_i$ are contributed by the trajectories from p to q with $\text{ind}(q) = \text{ind}(p) - i$. Thus, $(\delta_{\text{pearl}})_i$ increases the degree by i , where $\text{deg} = \text{dim} - \text{ind}$. In particular, $(\delta_{\text{pearl}})_1$ equals to δ_{Morse} in (7.1). The above sum is finite since the index of a critical point is at most the dimension of the manifold.

To generalize it to the case where \mathbb{L} is equipped with flat bundles, let us review the proof of $(\delta_{\text{pearl}})^2 = 0$. The main scheme of the proof is to consider the compactification of one dimensional moduli space of pearl trajectories, and show that the only non-trivial contribution is $(\delta_{\text{pearl}})^2$, and hence obtaining $(\delta_{\text{pearl}})^2 = 0$. Namely, the limit where one of the gradient trajectories γ_j ($0 < j < k$) contracts to a point has a canceling partner obtained by a disc-bubbling of a corresponding family of pearl trajectories.

Denote by $\mathcal{M}_2(p, p; \beta)$ the moduli space of pearl trajectories of the type $(\gamma_0, u_1, \gamma_1)$ from a critical point p to itself, where the pearl u_1 is a J -holomorphic disc of Maslov index two of homotopy class β . It is easy to see that the dimension of $\mathcal{M}(p, p, \beta)$ is one, and the possible degenerations are given as follows. The gradient trajectory γ_0 may degenerate to a broken trajectory γ_{01} and γ_{02} so that γ_{01} is a gradient trajectory connecting p to another critical point q ($\text{ind}(q) = \text{ind}(p) - 1$) and $(\gamma_{02}, u_1, \gamma_1)$ is an isolated pearl trajectory from q to p . Similarly γ_1 can degenerate to a broken trajectory $\{\gamma_{11}, \gamma_{12}\}$. These two types of degenerations give $(\delta_{\text{pearl}})^2$.

There are two additional types of degeneration (see Figure 6), which are in fact not found in the discussion of Lemma 5.1.3 [8] (in the case of [8] their contributions cancel each other and so do not affect the result). They play an important role in our story.

Such a degeneration is given by a pearl trajectory $(\gamma_0, u_{11}, \gamma_1)$ with a disc bubble u_{12} attached either at the upper semi-circle or at the lower semi-circle of u_{11} , and γ_0, γ_1 are constant gradient trajectories attached to the component u_{11} which is a constant disc. These two contributions cancel each other, and as a result we have $(\delta_{\text{pearl}})^2 = (\Phi(L_0) - \Phi(L_0)) = 0$.

Now, we consider the same complex decorated with two flat complex line bundles $\mathcal{L}_0, \mathcal{L}_1$ over \mathbb{L} . Namely, we consider a pearl complex for the Floer homology $HF((\mathbb{L}, \mathcal{L}_0), (\mathbb{L}, \mathcal{L}_1))$. The pearl chain complex $C^{\text{pearl}}(\mathbb{L}, \mathcal{L}_0, \mathcal{L}_1, h)$ is a $\mathbb{Z}/2$ -graded free $\Lambda_{\mathbb{C}}$ -module generated by critical points of h , where $\mathbb{Z}/2$ -grading come from Morse indices of h as before. The differential of the pearl complex is given by counting pearl trajectories weighted by holonomy and areas.

Given an isolated pearl trajectory u from p to q , we have a path $\partial_0 u$ (resp. $\partial_1 u$) from p to q obtained by traveling along gradient trajectories and images of upper (resp. lower) semi-circle of ∂D^2 of the holomorphic

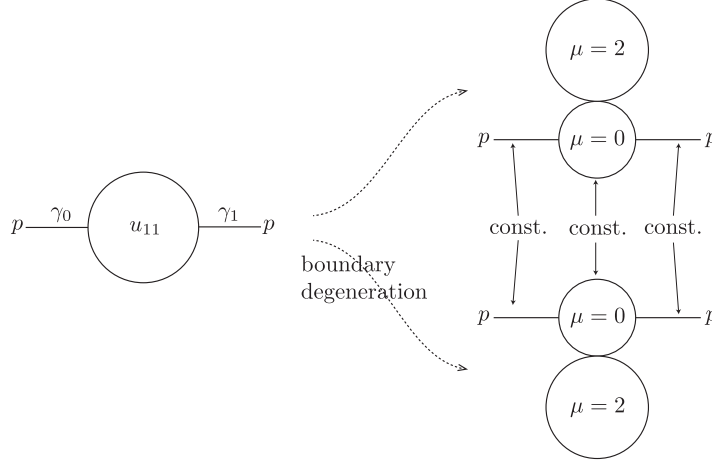
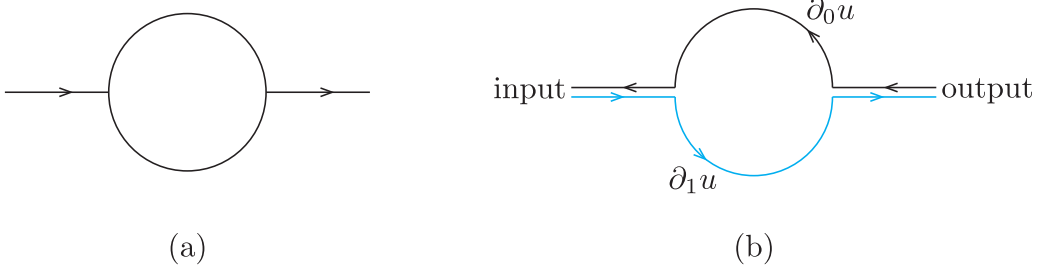


Fig. 6: Additional types of degenerations

discs u_1, \dots, u_k . We denote by $\text{Pal}_{\partial_i u}(\mathcal{L}_i)$ the holonomy of \mathcal{L}_i along $\partial_i u$ from p to q . Then each pearl trajectory from p to q contributes to the differential $\delta_{\text{pearl}}^{\mathcal{L}_0, \mathcal{L}_1}$ as

$$(-1)^{a(u)} (\text{Pal}_{\partial_0 u}(\mathcal{L}_0))^{-1} \text{Pal}_{\partial_1 u}(\mathcal{L}_1),$$

where $(-1)^{a(u)}$ is the sign for the pearl trajectory and sum of such contributions together with the area $T \sum_{j=1}^k \omega(u_j)$ define $n(p, q)$ with holonomy effects for (7.1). Note that gradient trajectories in a pearl trajectory contribute to both $\text{Pal}_{\partial_i u}(\mathcal{L}_i)$ for $i = 0, 1$ with opposite directions, but their holonomies may not cancel out since \mathcal{L}_0 is not equal to \mathcal{L}_1 . Here, the holonomy contribution from a gradient flow should be analyzed carefully since it is not clear which part of the flow lies on $(\mathbb{L}, \mathcal{L}_0)$ or $(\mathbb{L}, \mathcal{L}_1)$ from the picture of the pearl trajectory itself ((a) of Figure 7). For this, we use the schematic picture of the trajectory drawn as in (b) of Figure 7.

Fig. 7: Paths $\partial_0 u, \partial_1 u$ from a pearl trajectory

As in (2.6), we obtain

$$(\delta_{\text{pearl}}^{\mathcal{L}_0, \mathcal{L}_1})^2 = (\Phi(\mathcal{L}_1) - \Phi(\mathcal{L}_0)), \quad (7.3)$$

where the right hand side comes from the two contributions drawn in Figure 6. The precise sign of the above formula will be proved in Appendix A, Lemma A.1.

Now, we vary flat connections on a line bundle to obtain a matrix factorization. i.e. instead of considering \mathcal{L}_0 with a fixed flat connection, we use a family of flat line bundles \mathcal{L}_z whose holonomies are parametrized by z . As in (3.1), we make the sign change

$$d := (-1)^{\text{deg}} \delta_{\text{pearl}}$$

to obtain a matrix factorization of W :

$$d^2 = W - \lambda$$

Note that (7.3) implies that with the original δ_{pearl} of [8], we instead have $(\delta_{\text{pearl}})^2 = \lambda - W$.

[8, Proposition 5.6.2] shows that the homology of the pearl complex $(C^{\text{pearl}}(\mathbb{L}, h), \delta)$ is isomorphic to the Lagrangian Floer homology $HF(\mathbb{L}, \phi(\mathbb{L}))$ for a Hamiltonian isotopy ϕ . Such an isomorphism is constructed by a Lagrangian version of Piuniknin-Salamon-Schwarz morphism (see for example [4], [30],[8],[26]). Namely, a chain

map which induces an isomorphism is given by counting another version of pearl trajectory: It is given by a pearl trajectory $\gamma_0, u_1, \gamma_1, u_2, \dots, u_k, \gamma_k$ where the last $\gamma_k : [t_{k-1}, \infty) \rightarrow \mathbb{L}$ is replaced by $\gamma_k : [t_{k-1}, t_k] \rightarrow \mathbb{L}$ with an additional strip component $u_{k+1} : (\mathbb{R} \times [0, 1], \mathbb{R} \times \{0, 1\}) \rightarrow (M, \mathbb{L})$ satisfying

$$\partial_s u_{k+1} + J(u_{k+1})\partial_t u_{k+1} + \beta(s)\nabla H_t(u_{k+1}) = 0$$

where $u(-\infty)$ equals $\gamma_k(t_k)$ and $u(+\infty)$ maps to the Lagrangian intersection point $q \in \mathbb{L} \cap \phi(\mathbb{L})$. (See Figure 5.) Here, H_t is the Hamiltonian function for ϕ , and $\beta(s)$ is a cut-off function which vanishes for $s \leq 0$, and has value 1 for $s \geq 0$. Given such a trajectory, say u , we can similarly define a path $\partial_0 u$ (resp. $\partial_1 u$) from p to q as before, traveling along gradient trajectories and upper (resp. lower) semi-circles and $\mathbb{R} \times \{1\}$ (resp. $\mathbb{R} \times \{0\}$) in the last component.

Hence, by incorporating holonomy $(\text{Pal}_{\partial_0 u}(\mathcal{L}_z))^{-1} \text{Pal}_{\partial_1 u}(\mathcal{L}_1)$ as before, and proceeding as in Proposition 5.4, we can prove that the matrix factorization obtained from the pearl complex $(C^{\text{pearl}}(\mathbb{L}, \mathcal{L}_z, \mathcal{L}_1, h), \delta_{\text{pearl}}^{\mathcal{L}_z, \mathcal{L}_1})$ is equivalent to the one obtained from the Lagrangian Floer complex $(CF((\mathbb{L}, \mathcal{L}_z), (\phi(\mathbb{L}), (\phi^{-1})^* \mathcal{L}_1)), \delta_{\text{pearl}}^{\mathcal{L}_z, \mathcal{L}_1})$.

7.2 The projective plane

As an example, we employ a pearl complex to compute the matrix factorization mirror to the Clifford torus of the projective plane. Let $\mathbb{L} = \{[u_0, u_1, u_2] \in \mathbb{P}^2 \mid |u_0| = |u_1| = |u_2|\}$ be the Clifford torus in \mathbb{P}^2 . From [13], there are only three holomorphic discs with boundary on \mathbb{L} (up to $\text{Aut}(D^2)$ and T^2 -action) whose Maslov index is 2, and these are given by

$$D_0 := [z, 1, 1], \quad D_1 := [1, z, 1], \quad D_2 := [1, 1, z],$$

for $z \in D^2$. Denoting their common symplectic areas as k , the Floer potential $W(\mathbb{L})$ is

$$W(\mathbb{L}) = T^k \left(z_1 + z_2 + \frac{1}{z_1 z_2} \right).$$

We choose a Morse function $f : \mathbb{L} \rightarrow \mathbb{R}$ such that critical points and gradient flow lines of f are as shown in Figure 8. Such a Morse function can be chosen as follows. Since \mathbb{L} is a torus, we identify \mathbb{L} with $(\mathbb{R}/\mathbb{Z})^n$, and we choose $f(x_1, \dots, x_n) = \sum_{i=1}^n (\sin(\pi x_i))^2$, and compose f with a diffeomorphism h of \mathbb{L} so that the gradient flow lines of $f \circ h$ are as in Figure 8. We use a diffeomorphism h so that a Maslov index-2 disc do not meet two critical points of index difference 2.

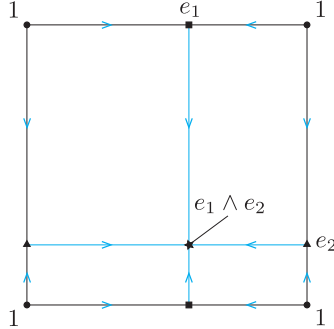


Fig. 8: Morse function on \mathbb{L}

It is convenient to identify the vector space generated by critical points with the exterior algebra with two generators e_1, e_2 so that four generators $1, e_1, e_2, e_{12}$ correspond to the critical points where 1 is the maximum and e_{12} is the minimum of f as in Figure 8. Here, $e_{12} = e_1 \wedge e_2$.

Now we choose gauge hypertori as in Figure 9. Namely, for \mathcal{L}_z , we choose hyper-tori $\{H_i^0\}$ as

$$H_1^0 = H_1 + (0, p_2), \quad H_2^0 = H_2 + (p_1, 0),$$

for sufficiently small p_1, p_2 where H_1, H_2 is defined in Definition 3.1.

We also choose a gauge hypertori for \mathcal{L}_1 (over \mathbb{L}) as

$$H_1^1 = H_1 + (0, 1 - p_2), \quad H_2^1 = H_2 + (1 - p_1, 0).$$

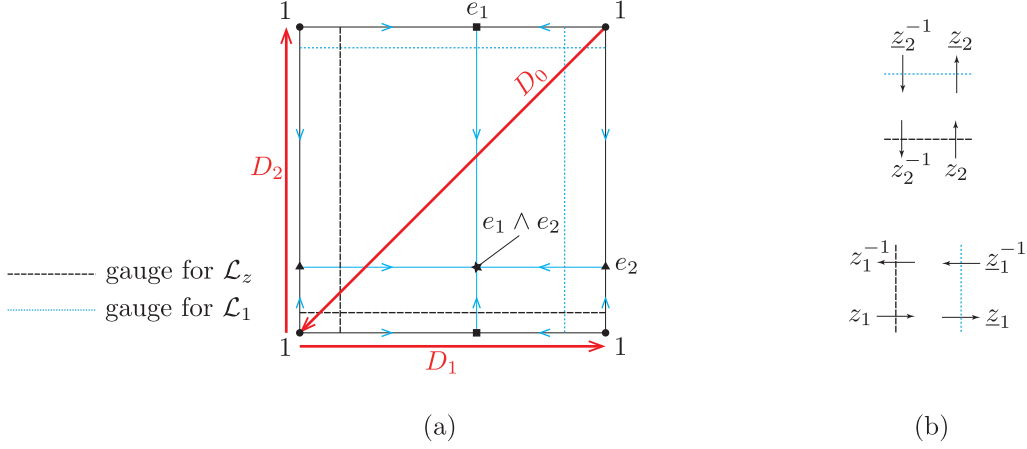


Fig. 9: The choice of gauge hypertori

Thus, \mathcal{L}_1 is a flat complex line bundle with fixed holonomy whose connection is trivial away from gauge hypertori $\{H_i^1\}$. Note that critical points of f are away from gauge hypertori.

Given an isolated pearl trajectory u between two critical points of h , we will compute the signed intersection number, say m_1, m_2 of a path $\partial_0 u$ with gauge hypertori H_1^0, H_2^0 respectively, and the corresponding holonomy factor will be given by $z_1^{m_1} z_2^{m_2}$ for the mirror variables z_1, z_2 . Also from u , we compute the signed intersection number, say m'_1, m'_2 of a path $\partial_1 u$ with gauge hypertori H_1^1, H_2^1 respectively, and the corresponding holonomy factor is given by $\underline{z}_1^{m'_1} \underline{z}_2^{m'_2}$, where $\underline{z}_1, \underline{z}_2 \in \mathbb{C} \setminus \{0\}$ are fixed complex numbers. Hence the total contribution of u to the differential $\delta^{\mathcal{L}_z, \mathcal{L}_1}$ is given as (up to sign)

$$z_1^{-m_1} z_2^{-m_2} \underline{z}_1^{m'_1} \underline{z}_2^{m'_2}.$$

Let us first consider Morse differentials contained in $\delta^{\mathcal{L}_z, \mathcal{L}_1}_{\text{pearl}}$. For each pair of critical points of index difference one, we have two Morse trajectories of opposite directions. From our choice of gauge hypertori, one can check that

$$\begin{aligned} \delta^{\mathcal{L}_z, \mathcal{L}_1}_{\text{pearl}}(1) &= (z_1 - \underline{z}_1)e_1 + (z_2 - \underline{z}_2)e_2, & \delta^{\mathcal{L}_z, \mathcal{L}_1}_{\text{pearl}}(e_1 \wedge e_2) &= 0, \\ \delta^{\mathcal{L}_z, \mathcal{L}_1}_{\text{pearl}}(e_1) &= (z_2 - \underline{z}_2)e_1 \wedge e_2, & \delta^{\mathcal{L}_z, \mathcal{L}_1}_{\text{pearl}}(e_2) &= -(z_1 - \underline{z}_1)e_1 \wedge e_2. \end{aligned}$$

Hence $(-1)^{\deg} \delta^{\mathcal{L}_z, \mathcal{L}_1}_{\text{pearl}}(\cdot)$ may be written as

$$(z_1 - \underline{z}_1)e_1 \wedge (\cdot) + (z_2 - \underline{z}_2)e_2 \wedge (\cdot).$$

The precise sign will be discussed in Section A.3.

In what follows, we compute the contribution from a pearl trajectory with a single holomorphic disc of Maslov index two, which will be called a *single pearl (trajectory)* for short.

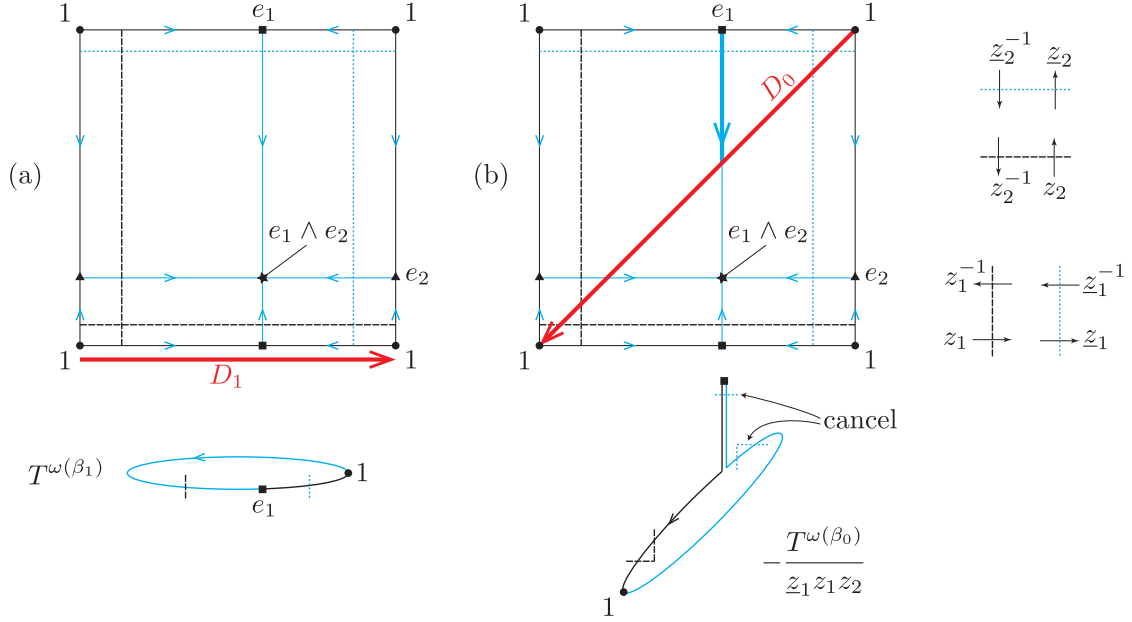
7.2.1 D_1 disc

There are two single D_1 -pearl trajectories, one from e_{12} to e_2 and one from e_1 to 1. Denote by $u = (\gamma_0, D_1, \gamma_1)$ the pearl trajectory from e_1 to 1. In this case, γ_0, γ_1 are constant trajectories (see Remark 7.1). Then, paths $\partial_0 u, \partial_1 u$ in \mathbb{L} from e_1 to 1 are given by part of the boundary of D_1 disc.

As illustrated in Figure 10 (a), $\partial_0 u$ does not intersect gauge hypertori H^0 (but only intersect H^1) and $\partial_1 u$ does not intersect gauge hypertori H^1 (but only intersect H^0). Hence the holonomy contribution for u is trivial.

The same argument works for a pearl trajectory from e_{12} to e_2 . Hence we may write the D_1 disc contributions as $T^k_{\iota_{e_1}}$, where ι_{e_1} is the contraction which sends e_1 to 1 and e_{12} to e_2 .

Remark 7.1. In fact, in the construction in [8] using generic f and J , “constant(flow)-disc-constant(flow)” configuration does not appear as discs do not meet two critical point at once generically. But we show in Lemma 8.11 that this configuration is also transversal in toric cases, and justifies the use of the standard J_0 , and our choice of the Morse function. Alternatively, one may choose a different Morse function corresponding to another \mathbb{Z} -basis of \mathbb{R}^2 , which do not contain any normal vector to the facets of moment polytope, then such configuration will not appear also. \square

Fig. 10: Thin strips from e_1 to 1

7.2.2 D_2 disc

There are two single D_2 -pearl trajectories, one from e_{12} to e_1 and one from e_2 to 1, whose contributions are given by $T^k \iota_{e_2}$ as in D_1 -case.

7.2.3 D_0 disc

There are four single D_0 -pearl trajectories, two from e_{12} to e_1 and e_2 , two from e_1, e_2 to 1. Denote by $u = (\gamma_0, D_0, \gamma_1)$ the pearl trajectory from e_1 to 1, illustrated in Figure 10 (b). In this case, γ_0 is a non-trivial gradient trajectory from e_1 to D_0 disc, and γ_1 is a constant trajectory. Now, $\partial_0 u$ passes through both H_1^0, H_2^0 (contributing $z_1 z_2$) and $\partial_1 u$ passes through H_1^1 (contributing z_1) once and H_2^1 twice (but with opposite orientations contributing 1), and hence the total contribution from e_1 to 1 is

$$-\frac{T^k}{z_1 z_1 z_2}.$$

How to obtain the precise sign will be discussed in Section A.4. One can check that D_0 pearl trajectory from e_{12} to e_2 has the same holonomy contribution as above up to sign. Hence, we may write them as $-\frac{T^k}{z_1 z_1 z_2} \iota_{e_1}$.

Similarly the pearl trajectory from e_2 to 1 is illustrated in Figure 11 (b), whose contribution is

$$-\frac{T^k}{z_1 z_1 z_2},$$

and the same goes for the trajectory from e_{12} to e_1 . Hence we may write them as

$$-\frac{T^k}{z_1 z_1 z_2} \iota_{e_2}.$$

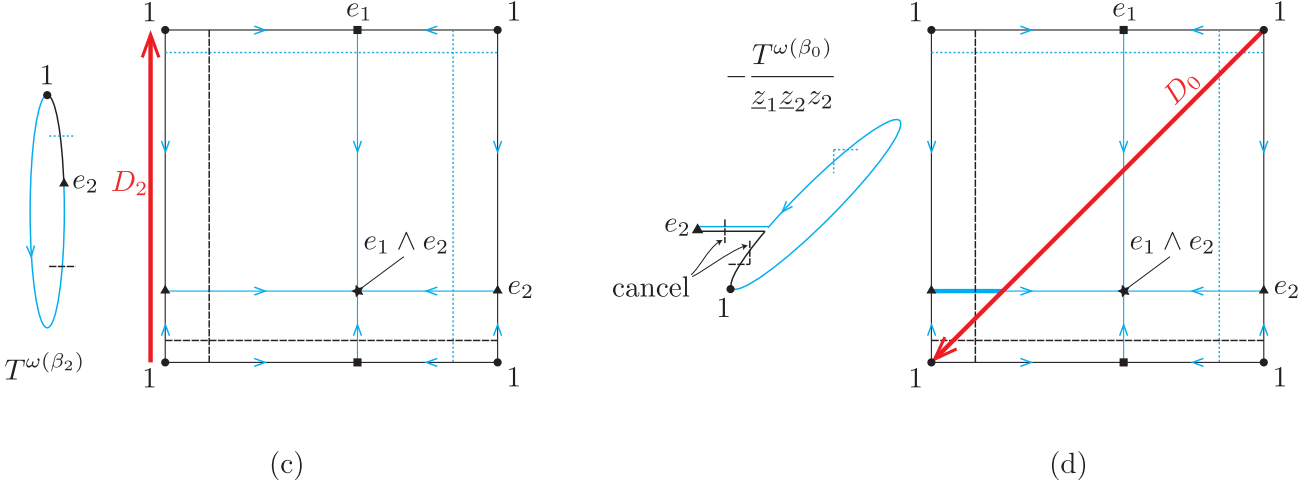
Therefore, the pearl differential $(-1)^{\deg} \delta_{\text{pearl}}^{\mathcal{L}_z, \mathcal{L}_1}$ gives the mirror matrix factorization which can be written as

$$(z_1 - z_1) e_1 \wedge + (z_2 - z_2) e_2 \wedge + \left(T^k - \frac{T^k}{z_1 z_1 z_2} \right) \iota_{e_1} + \left(T^k - \frac{T^k}{z_1 z_2 z_2} \right) \iota_{e_2}. \quad (7.4)$$

The square of the above becomes

$$T^k \left(z_1 + z_2 + \frac{1}{z_1 z_2} \right) - T^k \left(z_1 + z_2 + \frac{1}{z_1 z_2} \right) = W(z) - W(z).$$

By writing the above in a matrix form, we obtain the following

Fig. 11: Thin strips from e_2 to 1

Proposition 7.2. The matrix factorization mirror to the Clifford torus T^2 with holonomy (z_1, z_2) is given by

$$\begin{matrix} & 1 & e_{12} & e_1 & e_2 \\ \begin{matrix} 1 \\ e_{12} \\ e_1 \\ e_2 \end{matrix} & \begin{pmatrix} 0 & 0 & T^k - \frac{T^k}{z_1 z_1 z_2} & T^k - \frac{T^k}{z_1 z_2 z_2} \\ 0 & 0 & -(z_2 - z_2) & (z_1 - z_1) \\ z_1 - z_1 & -T^k + \frac{T^k}{z_1 z_2 z_2} & 0 & 0 \\ z_2 - z_2 & T^k - \frac{T^k}{z_1 z_1 z_2} & 0 & 0 \end{pmatrix} \end{matrix}. \quad (7.5)$$

□

In [11], Chan-Leung computed the matrix factorization mirror to the Clifford torus in \mathbb{P}^2 from a sketch of arguments based on SYZ. Let us show that by setting $(z_1, z_2) = (1, 1)$, the above matrix factorization agrees with that in [11] up to a change of coordinates.

First, the Givental-Hori-Vafa superpotential of $\mathbb{C}P^2$ is

$$W(z') = z'_1 + z'_2 + \frac{q}{z'_1 z'_2}$$

To identify $W(z')$ and $W(z)$, we make the following change of variables

$$q = T^{3k}, \quad z'_1 = q^{1/3} z_1, \quad z'_2 = q^{1/3} z_2.$$

As we consider the case $(z_1, z_2) = (1, 1)$, we set $z'_1 = z'_2 = q^{1/3}$.

We take the basis of $C^{pearl}(\mathbb{L}, f)$ in the following way:

$$\text{odd generators : } p_1 := e_2, \quad p_2 := -z'_1 e_1,$$

$$\text{even generators : } q_1 := q^{1/3}, \quad q_2 := q^{-1/3} z'_1 e_{12}.$$

Then, the above matrix factorization (7.5) can be written as

$$\begin{matrix} & p_1 & q^{1/3} p_2 & q^{1/3} q_1 & q_2 \\ \begin{matrix} p_1 \\ q^{1/3} p_2 \\ q^{1/3} q_1 \\ q_2 \end{matrix} & \begin{pmatrix} 0 & 0 & z'_2 - q^{1/3} & z'_1 - \frac{q^{2/3}}{z'_2} \\ 0 & 0 & -\left(1 - \frac{q^{1/3}}{z'_1}\right) & 1 - \frac{q^{1/3}}{z'_2} \\ 1 - \frac{q^{1/3}}{z'_2} & -\left(z'_1 - \frac{q^{2/3}}{z'_2}\right) & 0 & 0 \\ 1 - \frac{q^{1/3}}{z'_1} & z'_2 - q^{1/3} & 0 & 0 \end{pmatrix} \end{matrix}. \quad (7.6)$$

After switching z'_1 and z'_2 , (7.6) is precisely the matrix factorization appearing in [11],

8 Toric Fano manifolds

In this section, we compute the mirror matrix factorizations of Lagrangian torus fibers of toric Fano manifolds. We shall use pearl complexes discussed in the last section. We expect that the same method would work for semi-Fano toric manifolds when we incorporate virtual perturbation techniques to deal with sphere bubbles of indices zero.

We take \mathbb{L} to be a Lagrangian torus fiber (with non-trivial Floer cohomology) in a toric Fano manifold (X, ω) and define the mirror potential $W(\mathbb{L})$ (also denoted as W_z) using family of flat line bundles \mathcal{L}_z . A weakly unobstructed Lagrangian L corresponds to a matrix factorization of $W(\mathbb{L}) - c$ via the mirror functor (where $m_0^L = c \mathbf{1}_L$). We are particularly interested in the mirror matrix factorization of a Lagrangian torus fiber \underline{L} equipped with a flat line bundle $\mathcal{L}_{\underline{z}}$ (where $\underline{z} \in (\mathbb{C}^\times)^n$ is fixed).

For $\underline{L} \neq \mathbb{L}$, $\mathbb{L} \cap \underline{L} = \emptyset$ and hence the corresponding matrix factorization is trivial. Thus we only consider the case when \underline{L} and \mathbb{L} are the same Lagrangian torus fiber (equipped with possibly different flat line bundles). We will use a pearl complex as in the example given in Section 7.2. The actual computations require much more effort though.

Recall that d in our case is defined by $(-1)^{\deg} \delta_{\text{pearl}}$ (we omit upper indices indicating line bundles in $\delta_{\text{pearl}}^{\mathcal{L}_z, \mathcal{L}_{\underline{z}}}$ for simplicity). The main result is the following. The matrix factorization mirror to $(\underline{L}, \mathcal{L}_{\underline{z}})$ takes the form $(\bigwedge^* \underline{\Lambda}^n, d)$ (Λ is the Novikov ring),

$$d = d_1 + d_{-1} + \cdots + d_{-(2\lfloor (n+1)/2 \rfloor - 1)} \quad (8.1)$$

where an element in $\bigwedge^l \underline{\Lambda}^n$ has degree l (or Morse index $n - l$), and d_{-k} sends $\bigwedge^l \underline{\Lambda}^n \rightarrow \bigwedge^{l-k} \underline{\Lambda}^n$. (The decomposition of d directly comes from that of δ_{pearl} in Equation (7.2).) By definition $d^2 = W_z - W_{\underline{z}}$. While it is difficult to write down all the pearl trajectories contributing to d , we deduce an explicit formula for the following ‘approximation’ of d (Theorem 8.9):

$$\tilde{d} := d_1 + d_{-1}.$$

We prove that \tilde{d} itself is a matrix factorization of $W_z - W_{\underline{z}}$ (Theorem 8.8) which is of wedge-contraction type whose definition is given below (8.2).

Even though the explicit expression for d is unknown, we can prove that $(\bigwedge^* \underline{\Lambda}^n, d)$ generates the category of matrix factorizations of $W_z - W_{\underline{z}}$, see Theorem 9.1. It uses the method of spectral sequence by Polishchuk-Vaintrob [36]. When $n \leq 3$, d simply equals to \tilde{d} . In general, we expect that d and \tilde{d} are equivalent by some quantum change of coordinates of $\bigwedge^* \underline{\Lambda}^n$. We deduce such a change of coordinate for $n = 4$ in the end of Section 9.

Let us recall the definition of wedge-contraction type matrix factorizations. Let R be the formal power series ring on n variables x_1, \dots, x_n and W an element of R . Suppose that the origin is a unique critical point of W in $W^{-1}(0)$, and W can be written as $W = \sum_{i=1}^n x_i w_i$ for some series w_i in x_1, \dots, x_n . Consider the exterior algebra generated by e_1, \dots, e_n over R , which has an obvious $\mathbb{Z}/2$ -grading. Then a wedge-contraction of matrix factorization d_{wc} is defined by

$$d_{wc} = \sum_i x_i e_i \wedge (-) + \sum_i w_i \iota_{e_i}. \quad (8.2)$$

Dyckerhoff has shown in [19, Theorem 4.1] that if $W^{-1}(0)$ has a unique singularity at the origin, then (M, d_{wc}) is a split-generator of $D^\pi \mathcal{MF}(W)$.

We will first recall the Floer potentials of toric Fano manifolds. Then we deduce regularity of pearl trajectories and compute the mirror matrix factorizations.

8.1 Localized Floer potential in the toric Fano cases

Lagrangian Floer theory has been actively developed in the last decade, and Floer cohomology of Lagrangian torus fibers has been computed from the classification of all holomorphic discs with boundary on Lagrangian torus fibers ([14],[18]), and in much more generality by Fukaya-Oh-Ohta-Ono [27], [28] by introducing (bulk) deformation theories and T^n -equivariant perturbations on the moduli space of holomorphic discs.

Let us first recall the Floer potential $W(X)$ for toric Fano manifold X introduced by Cho-Oh [18] (which were generalized significantly in [27] and also in [7], [12] based on Strominger-Yau-Zaslow methods to understand mirror symmetry). In the Fano case, $W(X)$ can be identified with the Givental-Hori-Vafa mirror Landau-Ginzburg potential. And we compare it with the Floer potential $W(\mathbb{L})$ for a Lagrangian torus fiber \mathbb{L} given in Definition 2.1. Here, $W(X)$ depends on X only, but $W(\mathbb{L})$ depends on the particular Lagrangian torus fiber \mathbb{L} in X as well as X itself.

Let (X, ω) be a n -dimensional toric Fano manifold with a moment polytope P , defined in $M_{\mathbb{R}}$ by the set of inequalities

$$\langle u, v_i \rangle \geq \lambda_i \text{ for } i = 1, \dots, m$$

for $u \in M_{\mathbb{R}}$ and inner normal vectors $v_i \in N$ to facets of P . For each u in the interior of P , the inverse image of the moment map $\mu^{-1}(u)$ gives a Lagrangian torus $\mathbb{L} = L(u)$, which satisfies Assumption 1 ([18]).

For each normal vector v_i of the moment polytope P , there exists a unique holomorphic disc passing through a generic point p (up to $\text{Aut}(D^2)$), whose homotopy class is denoted as $\beta_i \in \pi_2(X, L(u))$. Hence the number n_{β_i} of such discs is one, and its symplectic area is given by $2\pi(\langle u, v_i \rangle - \lambda_i)$. Let us also denote $\partial\beta_i = \sum_{j=1}^n v_{ij}e_j$ for a basis $\{e_i\}_{i=1}^n$ of $H_1(\mathbb{L}; \mathbb{Z})$. Consider holonomy parameters $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{R}^n$ which is used to consider flat unitary line bundle \mathcal{L} over $L(u)$ with holonomy $\exp(2\pi\sqrt{-1}\nu_i)$ along e_i (see Section 12 [18] for more details).

$$\begin{aligned} W(X) &= \sum_{\beta, \mu(\beta)=2} n_{\beta} \exp\left(-\frac{1}{2\pi} \int_{\beta} \omega\right) \text{Hol}_{\mathcal{L}}(\partial\beta) \\ &= \sum_{i=1}^m e^{-(\langle u, v_i \rangle - \lambda_i)} \exp(2\pi\sqrt{-1}\langle \nu, v_i \rangle) \end{aligned}$$

Hence, it is natural to introduce mirror variables depending on the positions $u_i = \langle u, v_i \rangle$ and the holonomies $\nu_i = \langle \nu, v_i \rangle$

$$t_i = e^{-u_i + 2\pi\sqrt{-1}\nu_i} \text{ for } i = 1, \dots, n.$$

If we denote

$$t^{v_i} = t_1^{v_{i1}} t_2^{v_{i2}} \cdots t_n^{v_{in}},$$

then $W(X)$ can be written in terms of (t_1, \dots, t_n) as

$$W(X) = \sum_{i=1}^m e^{\lambda_i} t^{v_i} \quad (8.3)$$

Now, let us compare $W(X)$ and the Floer potential $W(\mathbb{L})$. For this, we choose $\mathbb{L} = L(u_0)$ for a fixed u_0 in the interior of the polytope P . Recall that in our setting, mirror variable is given by the holonomy (see (2.1))

$$z_i = \exp(2\pi\sqrt{-1}\nu_i).$$

By setting $z^{v_i} = z_1^{v_{i1}} z_2^{v_{i2}} \cdots z_n^{v_{in}}$, and from the Definition 2.1, Floer potential is

$$W(\mathbb{L}) = \sum_{i=1}^m T^{\omega(\beta)} \rho^b(\partial\beta) = \sum_{i=1}^m c_i z^{v_i} \quad (8.4)$$

where $c_i := T^{2\pi(\langle u, v_i \rangle - \lambda_i)}$.

Remark 8.1. The Floer potential in semi-Fano case is given by

$$W(\mathbb{L}) = \sum_{i=1}^m \left(\sum_{\alpha} n_{\beta_i + \alpha} T^{\int_{\beta_i + \alpha} \omega} \right) \text{Hol}_{\mathcal{L}}(\partial\beta_i) = \sum_{i=1}^m c_i z^{v_i}$$

where $c_i = \sum_{\alpha} n_{\beta_i + \alpha} T^{\int_{\beta_i + \alpha} \omega}$ and $z^{v_i} = \prod_{j=1}^n z_j^{v_{i,j}} = \text{Hol}_{\mathcal{L}}(\partial\beta_i)$. The sum is over all $\alpha \in H_2^{\text{eff}}(X)$ with $c_1(\alpha) = 0$, and $n_{\beta_i} = 1$. \square

Comparing (8.3) and (8.4), we obtain the following lemma which is well-known from [27].

Lemma 8.2. For toric Fano manifolds, the substitution $t_i = z_i \cdot e^{-\langle u, v_i \rangle}$ and $T^{2\pi} = e^{-1}$ gives

$$W(X) = W(\mathbb{L}).$$

\square

8.2 The matrix factorizations

Let X be a compact toric Fano manifold of dimension n defined by a fan supported in $N_{\mathbb{R}}$ and fix a reference torus fiber \mathbb{L} . In this section we compute the mirror matrix factorization of $\underline{\mathbb{L}}$ induced by the mirror functor given in Section 3 using pearl complexes (Section 7), where $\underline{\mathbb{L}}$ is a torus fiber together with a fixed flat line bundle $\mathcal{L}_{\underline{z}}$ with a \mathbb{C}^{\times} connection $\nabla_{\underline{z}}$. Only when $\underline{\mathbb{L}}$ and \mathbb{L} are fiber over the same point, the resulting matrix factorization is non-trivial (or otherwise $\underline{\mathbb{L}} \cap \mathbb{L} = \emptyset$), and so from now on we assume this is the case. Thus $\nabla_{\underline{z}}$ belongs to the mirror space $M_{\mathbb{C}^{\times}} \cong (\mathbb{C}^{\times})^n$, and corresponds to a certain value $\underline{z} \in (\mathbb{C}^{\times})^n$. We have $m_0^{(\underline{\mathbb{L}}, \nabla_{\underline{z}})} = W_{\underline{z}} \cdot \mathbf{1}_{\underline{\mathbb{L}}}$.

Let m be the number of rays in the fan. Without loss of generality we assume $E_i := v_i = \partial\beta_i$ for $i = 1, \dots, n$ form a basis of N , and this gives a coordinate system $z_i := \text{Hol}_{\mathcal{L}}(\partial\beta_i)$, $i = 1, \dots, n$ on $M_{\mathbb{C}^{\times}} \cong (\mathbb{C}^{\times})^n$. As before, we write $v_i = \partial\beta_i = \sum_{j=1}^n v_{i,j} E_j$ for any $i = 1, \dots, m$ in terms of this basis. Then $v_{i,j} = \delta_{ij}$ when $i = 1, \dots, n$. We set $s_{i,j}$ to be the sign of $v_{i,j}$.

We use the basis E_1, \dots, E_n to identify \mathbb{L} with the standard torus $\mathbb{R}^n/\mathbb{Z}^n$. Choose $(a_1, \dots, a_n) \in (0, 1)^n$ satisfying the following condition:

- a_j/a_k is irrational for all $j \neq k$.
 - $|v_{i,k}|a_j - |v_{i,j}|a_k > 0$ for every $i = 1, \dots, m$ and $j < k$ with $s_{i,j} = s_{i,k}$
 - a_j is sufficiently close to 0 for each j
- (8.5)

Then take a Morse function $f_{(a_1, \dots, a_n)}$ whose critical points are in one-to-one correspondence with subsets $I \subset \{1, \dots, n\}$. The critical points are denoted by e_I for $I \subset \{1, \dots, n\}$, and $f_{(a_1, \dots, a_n)}$ is taken such that e_I has coordinates (c_1, \dots, c_n) where $c_i = 0$ when $i \notin I$ and $c_i = a_i$ when $i \in I$. We will write $e_i := e_{\{i\}}$, $e_0 := e_{\emptyset}$, $e_{top} := e_{\{1, \dots, n\}}$ for notational convenience. Also for $I = \{i_1, \dots, i_k\}$, we identify e_I with $e_{i_1} \wedge \dots \wedge e_{i_k}$ with $i_1 < \dots < i_k$. Using this terminology, we can define various Λ -linear endomorphisms on $\bigwedge^* \underline{\Lambda}^n$ such as

$$e_j \wedge e_I = (-1)^i e_{I'}, \quad \iota_{e_i} e_I = (-1)^{l-1} e_{I''}$$

where $I = \{i_1 < \dots < i_k\}$, $I' = \{j\} \cup I = \{i_1 < \dots < i_l < j < i_{l+1} < \dots < i_k\}$ and $I'' = \{i_1 < \dots < i_{l-1} < i_{l+1} < \dots < i_k\}$.

The flat connections on \mathcal{L}_z and $\mathcal{L}_{\underline{z}}$ are specified by the values z and \underline{z} and the gauge hypertori $H_i + p$ and $H_i + \underline{p}$ in \mathbb{L} , respectively. The points $p = ([p_1], \dots, [p_n])$, $\underline{p} = ([\underline{p}_1], \dots, [\underline{p}_n]) \in \mathbb{R}^n/\mathbb{Z}^n$ are taken such that $0 < p_1, \dots, p_n < 1$, $p_1, \dots, p_n \ll 1$, $0 < \underline{p}_1, \dots, \underline{p}_n < 1$ and $1 - p_1, \dots, 1 - p_n \ll 1$. These choices of gauge of the flat connections are used to fix the mirror matrix factorization. Certainly we can take other choices, and we will get another matrix factorization which is equivalent to the original one by Lemma 5.3.

The matrix factorization $(\bigwedge^* \underline{\Lambda}^n, d)$ transformed from $(\mathbb{L}, \nabla_{\underline{z}})$ is defined by $d := (-1)^{\deg} \delta_{\text{pearl}}$ where δ_{pearl} counts pearl trajectories connecting every pair of critical points $e_I, e_J \in \mathbb{L}$ (weighted by area and holonomy). One has $d^2 = (W_z - W_{\underline{z}}) \cdot \text{Id}$. Recall that

$$\delta_{\text{pearl}} = (\delta_{\text{pearl}})_1 + (\delta_{\text{pearl}})_{-1} + \dots + (\delta_{\text{pearl}})_{-(2\lfloor (n+1)/2 \rfloor - 1)}$$

where $(\delta_{\text{pearl}})_{-(2k-1)}$ takes the form

$$(\delta_{\text{pearl}})_{-(2k-1)} : e_I \mapsto \sum_{\substack{|J|=|I|-(2k-1) \\ \Gamma(e_I \rightarrow e_J)}} \left(\text{sign}(\Gamma(e_I \rightarrow e_J)) T^{-\frac{1}{2\pi} \sum_i \int_{D_i} \omega} \text{Hol}(\Gamma(e_I \rightarrow e_J)) \right) e_J.$$

Here, $\Gamma(e_I \rightarrow e_J)$ is a pearl trajectory from e_I to e_J with holomorphic disc components D_i with total Maslov index $\sum_i \mu(D_i) = 2k$, and Hol denotes the holonomy.

Remark 8.3. Note that the above expression for $(\delta_{\text{pearl}})_{-(2k-1)}$ is a finite sum, since for a toric Fano manifold a non-constant holomorphic disc class has at least Maslov index two. Thus we can substitute the Novikov formal variable T by e^{-1} and work over complex numbers. \square

We shall deduce an explicit formula for $\tilde{d} := (-1)^{\deg} (\delta_{\text{pearl}})_1 + (-1)^{\deg} (\delta_{\text{pearl}})_{-1}$ and prove that \tilde{d} itself is a matrix factorization: $\tilde{d}^2 = (W_z - W_{\underline{z}}) \cdot \text{Id}$. First we introduce the following terminology for later convenience.

Definition 8.4. Consider a pearl trajectory from a critical point p to another critical point q which only consists of two flow-line components and a disc component D . The point in ∂D that the flow line from p is glued to is called to be an *entry point*, and the point in ∂D that is glued to the flow line to q is called to be an *exit point*. \square

Lemma 8.5.

$$d_1 = (-1)^{\deg}(\delta_{\text{pearl}})_1 = \sum_{i=1}^m (z_i - \underline{z}_i) e_i \wedge (\cdot). \quad (8.6)$$

□

Proof. Given a critical point e_I , $(\delta_{\text{pearl}})_1 \cdot e_I$ is a linear combination of $e_{I \cup \{j\}}$'s for $j \notin I$ whose coefficients count flow lines from e_I to $e_{\{j\} \cup I}$'s, which is standard in Morse homology theory. There are two such flow lines, one passing through the gauge torus $H_j + p$ and one passing through $H_j + \underline{p}$. Since both of them are positive intersections (as holonomies of pearl trajectories), one contributes z_i and one contributes \underline{z}_i . What it remains to check that the signs of z_i and \underline{z}_i are given precisely as in (8.6), which we postpone to Section A.3. ■

Before computing $d_{-1} = (-1)^{\deg}(\delta_{\text{pearl}})_{-1}$, we show that there does not exist a pearl trajectory from e_I to e_J with $|J| = |I| - 1$, but $J \not\subset I$. The question is equivalent to ask when there exists a Maslov index two disc which connects $W^u(e_I)$ and $W^s(e_J)$, and the following lemma proves that this is impossible unless $J \subset I$.

Lemma 8.6. If J is not a subset of I , then there does not exist a trajectories with a single pearl from e_I to e_J . □

Proof. Without loss of generality, one may assume that $I = \{1, 2, \dots, k\}$ and $J = \{l+1, l+2, \dots, l+(k-1)\}$ for $l \geq 2$. Recall that we have taken our Morse function such that

$$e_{\text{top}} = (a_1, \dots, a_n)$$

with an irrational slope a_i/a_j for $1 \leq i \neq j \leq n$. In terms of coordinates of the Lagrangian torus, $W^u(e_I)$ and $W^s(e_J)$ are given modulo \mathbb{Z}^n as

$$W^u(e_I) = \{(a_1, \dots, a_k, t_1, \dots, t_{n-k} \mid 0 \leq t_i \leq a_i \text{ for } k+1 \leq i \leq n)\}$$

$$W^s(e_J) = \{(0, \dots, 0, s_{l+1}, \dots, s_{l+(k-1)}, 0, \dots, 0 \mid 0 \leq s_i \leq a_i \text{ for } l+1 \leq i \leq l+(k-1)\}.$$

Now suppose there is a Maslov index two disc whose boundary image connects $W^u(e_I)$ and $W^s(e_J)$. Note that for toric manifolds, such a boundary image has an integral direction (which is normal to a facet of the moment polytope). However, if there is a vector from $W^u(e_I)$ to $W^s(e_J)$, then it should be of the form

$$(a_1, a_2, \dots, \dots) \pmod{\mathbb{Z}^n}$$

which can not be made integral by a multiplication of any scalar because a_1/a_2 is irrational. This gives a contradiction. ■

Lemma 8.7.

$$d_{-1} = (-1)^{\deg}(\delta_{\text{pearl}})_{-1} = \sum_{j=1}^n \left(\sum_{i=1}^m c_i \alpha_j^i \right) \iota_{e_j}$$

where $c_i = \int_{\beta_i} \omega$ and α_j^i takes the form

$$\alpha_j^i = s_{i,j} \sum_{\gamma_j \cup D_i} Z_{\gamma_j} \cdot Z_{\partial D_i}$$

for $s_{i,j}$ = the sign of $v_{i,j}$. The sum is over the finitely many flow-disc trajectories from e_j to 1, where the flow part γ_j is a segment of a flow line from e_j to e_{top} and the disc part D_i is the basic disc representing β_i passing through the point 1 once. $Z_{\gamma_j} \in \mathbb{C}^\times$ and $Z_{\partial D_i} \in \mathbb{C}^\times$ denote the holonomy contributions from γ_j and ∂D_i respectively. Moreover $\alpha_j^i = \delta_j^i$ when $i = 1, \dots, n$. □

Proof. Let e_I and e_J be two critical points with $|J| = |I| - 1 \geq 0$. By Lemma 8.6, it suffices to consider the case when $J = I - \{j\}$ for some $1 \leq j \leq n$. In such a case, we prove that there is an one-to-one correspondence between pearl trajectories from e_I to $e_{I-\{j\}}$ and those from e_j to 1 both of which involves a single Maslov-2 disc of class β_i .

A pearl trajectory Γ_1 from e_I to $e_{I-\{j\}}$ consists of two flow line components and one disc component. Given such a Γ_1 , we can attach it with any chosen flow line from e_j to e_I and obtain a (degenerate) pearl trajectory $\tilde{\Gamma}_1$ from e_j to $e_{I-\{j\}}$. A pearl trajectory Γ_2 from e_j to 1 consists of two components: a flow component from e_j ,

which is glued to a holomorphic disc component representing β_i at the entry point such that the disc boundary passes through $1 \in \mathbb{L}$ at the exit point. Given such a Γ_2 , we can attach it with any chosen flow line from 1 to $e_{I-\{j\}}$ and obtain a (degenerate) pearl trajectory $\tilde{\Gamma}_2$ from e_j to $e_{I-\{j\}}$.

Given Γ_2 , we construct an one-parameter family of pearl trajectories $\tilde{\Gamma}^t$ from e_j to $e_{I-\{j\}}$ for $t \in [1, 2]$ such that $\tilde{\Gamma}^2$ is Γ_2 attached with a flow line $F(1 \rightarrow e_{I-\{j\}})$ from 1 to $e_{I-\{j\}}$, and $\tilde{\Gamma}^1$ is a pearl trajectory Γ_1 from e_I to $e_{I-\{j\}}$ attached with a flow line $F(e_j \rightarrow e_I)$ from e_j to e_I . (If this is the situation, we will associate Γ_1 with Γ_2 .)

The construction is as follows. The torus $T' \subset \mathbb{L}$ passing through the entry point $p \in \mathbb{L}$ generated by the directions $\{E_k : k \in I - \{j\}\}$ intersects the unstable torus of e_I at a unique point p' in the unstable torus of e_j . Let $\alpha : [1, 2] \rightarrow T'$ be a straight line segment with $\alpha(1) = p'$, $\alpha(2) = p$. For each t , we have a flow line from e_j to $\alpha(t)$ (chosen to be continuously depending on t) and a unique holomorphic disc representing β_i whose boundary passes through $\alpha(t)$. Moreover as we vary t from 2 to 1 the exit point of the disc varies continuously from $1 \in \mathbb{L}$ to other points in the stable torus of $e_{I-\{j\}}$, and there exists a flow line (continuously depending on t) from the exit point to $e_{I-\{j\}}$. Thus for each t we have a pearl trajectory $\tilde{\Gamma}^t$ from e_j to $e_{I-\{j\}}$ with $\alpha(t)$ to be the entry point. At $t = 1$, since $\alpha(1) = p'$ lies in the unstable torus of e_I , the flow component from e_j to $\alpha(t)$ actually degenerates to union of a flow line $F(e_j \rightarrow e_I)$ from e_j to e_I and a flow segment from e_I to $\alpha(t)$. Thus the pearl trajectory at $t = 1$ is a pearl trajectory Γ_1 from e_I to $e_{I-\{j\}}$ attached with $F(e_j \rightarrow e_I)$.

Conversely given Γ_1 , we can construct a one-parameter family of pearl trajectories $\tilde{\Gamma}^t$ with the same property as above in a similar way, and obtain a corresponding pearl trajectory Γ_2 . The constructions are inverses to each other, and hence give the desired one-to-one correspondence.

Since $\tilde{\Gamma}^t$ is a continuous family of pearl trajectories with fixed input e_j and output $e_{I-\{j\}}$, their boundaries give the same holonomy. Moreover, the flow lines $F(1 \rightarrow e_{I-\{j\}})$ and $F(e_j \rightarrow e_I)$ give exactly the same holonomy. This implies Γ_1 and Γ_2 give exactly the same holonomy.

In order to show that d_{-1} is of the form $\sum_{j=1}^n w_j \iota_{e_j}$ for some Laurent series w_j 's, we have to additionally check that the sign difference between $\Gamma_1(e_I \rightarrow e_{I-\{j\}})$ and $\Gamma_2(e_j \rightarrow 1)$ equals that of e_I and $e_j \wedge e_{I-\{j\}}$ (i.e. s^* such that $\iota_{e_j} e_I = s^* e_{I-\{j\}}$). Here, we simply assume that they are equal, and postpone the proof of this to Section A.4.

Consequently, we only need to consider pearl trajectories from e_j to 1 whose disc component represents β_i in order to compute the coefficients α_j^i . Such pearl trajectories take the form $\gamma_j \cup D_i$ as stated. If $v_{i,j} = 0$, any disc representing β_i passing through $1 \in \mathbb{L}$ cannot intersect the unstable torus of e_j , and hence there is no such pearl trajectory $\gamma_j \cup D_i$ i.e., $\alpha_j^i = 0$.

Suppose $v_{i,j} \neq 0$. There are just finitely many pearl trajectories (parametrized by the finitely many entry points), and the disc component has area $\int_{\beta_i} \omega$ which contributes the factor c_i , and the holonomy contribution is $Z_{\gamma_j} \cdot Z_{\partial D_i}$. In Section A.4, the sign of this contribution will turn out to be $s_{i,j}$, which is 1 (or -1) when ∂D_i passes through the unstable torus of e_j positively (or negatively). Note that $\alpha_j^i = \delta_j^i$ for $i = 1, \dots, n$ due to our special choice of the basis $E_i = \partial \beta_i$ for $i = 1, \dots, n$.

Finally, the regularity of pearl trajectories contributing to $(\delta_{\text{pearl}})_{-1}$ will be shown in Section 8.4. We will also prove in Lemma 8.12 that there is an almost complex structure which makes all trajectories for δ_{pearl} regular, but does not changes $(\delta_{\text{pearl}})_{-1}$, which justifies our computations in this section. \blacksquare

Combining the above two lemmas,

$$\tilde{d} = d_1 + d_{-1} = \sum_{i=1}^m (z_i - \underline{z}_i) e_i \wedge (\cdot) + \sum_{i=1}^m c_i \sum_{j=1}^n \alpha_j^i \iota_{e_j}.$$

We next prove that $(\wedge^* \underline{\Lambda}^n, \tilde{d})$ indeed forms a matrix factorization:

Theorem 8.8. $\tilde{d}^2 = W_z - W_{\underline{z}}$. \square

Proof. We have

$$\tilde{d}^2 = \sum_{i=1}^n c_i (z_i - \underline{z}_i) + \sum_{i=n+1}^m c_i \sum_{j=1}^n \alpha_j^i (z_j - \underline{z}_j)$$

and, comparing with (8.4), we need to prove that $\sum_{j=1}^n \alpha_j^i (z_j - \underline{z}_j) = z^{v_i} - \underline{z}^{v_i}$ where

$$\begin{aligned} \sum_{j=1}^n \alpha_j^i (z_j - \underline{z}_j) &= \sum_j \sum_{\gamma_j \cup D_i} Z_{\gamma_j} \cdot Z_{\partial D_i} (s_{i,j} (z_j - \underline{z}_j)) \\ &= \sum_{\text{entry point } \zeta} Z_{\gamma_j(\zeta)}^\zeta \cdot Z_{\partial D_i}^\zeta (s_{i,j(\zeta)} (z_{j(\zeta)} - \underline{z}_{j(\zeta)})). \end{aligned} \tag{8.7}$$

Here, the summation is over all the entry points $\zeta \in \partial D_i$, and we order the entry points counterclockwise. The entry point counterclockwise closest to $1 \in \partial D \subset T$ is said to be the first entry point. Each entry point ζ is an intersection of a flow line from $e_{j(\zeta)}$ and ∂D_i (we will write $j = j(\zeta)$ for simplicity). Each summand $Z_{\gamma_j} \cdot Z_{\partial D_i}(s_{i,j}(z_j - \underline{z}_j))$ is a difference of two terms. We want to prove that for every two adjacent summands, the second term of the first one cancels with the first term of the second one. Hence only the first term of the first summand and the last term of the last summand remain, and we claim that those equal to z^{v_i} and \underline{z}^{v_i} respectively. This will finish the proof that $d^2 = W_z - W_{\underline{z}}$.

Consider two consecutive entry points $\zeta_1, \zeta_2 \in \partial D$, where ζ_1 is flowing from e_j and ζ_2 is flowing from e_k (j may equal to k). The unstable submanifolds of e_j and e_k are two hypertori intersecting each other along a subtorus $T^\perp = T/\langle \{E_1, \dots, E_n\} - \{E_j, E_k\} \rangle$ passing through e_{top} . (Here we use E_1, \dots, E_n to denote the standard basis of \mathbb{R}^n and e_1, \dots, e_n are critical points in T with index $n - 1$.) Write the holonomy terms $Z_{\gamma_j}^{\zeta_1} \cdot Z_{\partial D_i}^{\zeta_1}$ and $Z_{\gamma_k}^{\zeta_2} \cdot Z_{\partial D_i}^{\zeta_2}$ (which are monomials in $z_l^{\pm 1}$'s and $\underline{z}_l^{\pm 1}$'s) as products of two factors $Z_{\zeta_1}^{jk} Z_{\zeta_1}^\perp$ and $Z_{\zeta_2}^{jk} Z_{\zeta_2}^\perp$ respectively, where $Z_{\zeta_1}^{jk}$ only has variables $z_j, z_k, \underline{z}_j, \underline{z}_k$ and $Z_{\zeta_1}^\perp$ has variables z_a, \underline{z}_a 's for all $a \neq j, k$. Then $Z^\perp := Z_{\zeta_1}^\perp = Z_{\zeta_2}^\perp$, and we only need to compare $Z_{\zeta_1}^{jk}$ and $Z_{\zeta_2}^{jk}$ in order to compute

$$Z_{\gamma_j}^{\zeta_1} Z_{\partial D_i}^{\zeta_1}(s_{i,j}(z_j - \underline{z}_j)) + Z_{\gamma_k}^{\zeta_2} Z_{\partial D_i}^{\zeta_2}(s_{i,k}(z_k - \underline{z}_k)) = Z^\perp (Z_{\zeta_1}^{jk}(s_{i,j}(z_j - \underline{z}_j)) + Z_{\zeta_2}^{jk}(s_{i,k}(z_k - \underline{z}_k))).$$

To analyze $Z_{\zeta_1}^{jk}$ and $Z_{\zeta_2}^{jk}$, we can project $\gamma_j, \gamma_k, \partial D_j, \partial D_k \subset T$ to T/T^\perp , which is one-dimensional when $j = k$ and two-dimensional when $j \neq k$. When $j = k$, we choose another direction E_l other than E_j and project to $T/(T^\perp/T\langle E_l \rangle)$ instead. Thus in any case the computation goes back to dimension two.

First consider the case $j = k$. $Z_{\zeta_1}^{jk}$ and $Z_{\zeta_2}^{jk}$ are monomials in z_j and \underline{z}_j . See Figure 12a, where the flow lines are shown by dotted lines and ∂D_i is shown by a solid line. In this case ∂D passes through the gauge hypertori $p + H_j$ and $\underline{p} + H_j$ once in between the two entry points ζ_1 and ζ_2 . For $s_{i,j}$ = the sign of $v_{i,j} = 1$, $Z_{\zeta_1}^{jk}$ has one more factor of z_j and one less factor of \underline{z}_j than $Z_{\zeta_2}^{jk}$. Thus $\underline{z}_j Z_{\zeta_1}^{jk} = z_j Z_{\zeta_2}^{jk}$. This shows that the two terms in the middle cancel each other and only the first and last ones are left:

$$Z_{\gamma_j}^{\zeta_1} Z_{\partial D_i}^{\zeta_1}(s_{i,j}(z_j - \underline{z}_j)) + Z_{\gamma_k}^{\zeta_2} Z_{\partial D_i}^{\zeta_2}(s_{i,k}(z_k - \underline{z}_k)) = Z_{\gamma_j}^{\zeta_1} Z_{\partial D_i}^{\zeta_1} s_{i,j} z_j - Z_{\gamma_k}^{\zeta_2} Z_{\partial D_i}^{\zeta_2} s_{i,k} \underline{z}_k. \quad (8.8)$$

For $s_{i,j} = -1$, $Z_{\zeta_1}^{jk}$ has one more factor of \underline{z}_j^{-1} and one less factor of z_j^{-1} than $Z_{\zeta_2}^{jk}$. Thus $z_j Z_{\zeta_1}^{jk} = \underline{z}_j Z_{\zeta_2}^{jk}$. This implies the two terms in the middle cancel each other and the same equation holds.

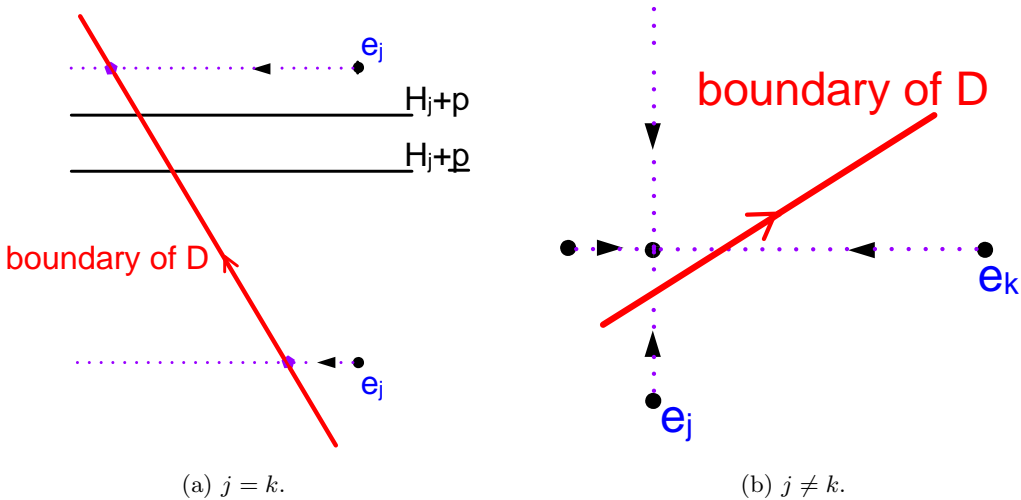


Fig. 12: Two consecutive entry points in ∂D .

Now consider the case $j \neq k$. See Figure 12b. Consider the case $s_{i,j} = s_{i,k} = 1$, and the other three cases $(s_{i,j}, s_{i,k}) = (1, -1), (-1, 1), (-1, -1)$ are similar. $Z_{\zeta_1}^{jk}$ has one more z_k factor than $Z_{\zeta_2}^{jk}$, because the flow segment γ_j from e_j hit the gauge hypertorus $H_k + p$ once while the flow segment γ_k from e_k does not. Similarly, $Z_{\zeta_2}^{jk}$ has one more \underline{z}_j factor than $Z_{\zeta_1}^{jk}$ because the flow segment γ_k from e_k hit the gauge hypertorus $H_j + \underline{p}$ once while the flow segment γ_j from e_j does not. Hence $\underline{z}_j Z_{\zeta_1}^{jk} = z_k Z_{\zeta_2}^{jk}$. This implies Equation (8.8) also holds in this case.

In conclusion, the right hand side of Equation (8.7) equals to first term - last term because all intermediate terms cancel. The first term is $Z_{\gamma_j(\zeta^F)}^{\zeta^F} Z_{\partial D_i}^{\zeta^F} z_j(\zeta^F)$ when $s_{i,j(\zeta^F)} = 1$ and is $Z_{\gamma_j(\zeta^F)}^{\zeta^F} Z_{\partial D_i}^{\zeta^F} \underline{z}_j(\zeta^F)$ when $s_{i,j(\zeta^F)} = -1$

where ζ^F is the first entry point. The last term is $Z_{\gamma_j(\zeta^L)}^{\zeta^L} Z_{\partial D_i}^{\zeta^L} z_j(\zeta^L)$ when $s_{i,j(\zeta^L)} = 1$ and is $Z_{\gamma_j(\zeta^L)}^{\zeta^L} Z_{\partial D_i}^{\zeta^L} z_j(\zeta^L)$ when $s_{i,j(\zeta^L)} = -1$ where ζ^L is the last entry point.

Now consider the first term which corresponds to the entry point ζ anti-clockwisely closest to $1 \in T$ along ∂D , and let $j = j(\zeta^F)$. The flow segment from e_j never hits the gauge hypertorus $H_j + p$ nor $H_j + \underline{p}$. For $k \neq j$, it hits the gauge hypertorus $H_k + p$ if and only if $s_{i,k} = 1$, and hits $H_k + \underline{p}$ if and only if $s_{i,k} = -1$. Hence $Z_{\gamma_j(\zeta^F)}^{\zeta^F} = \prod_{k \neq j} z_k^{\delta(s_{i,k},1)} z_k^{\delta(s_{i,k},-1)}$, where $\delta(a,b) := 1$ if $a = b$ and zero otherwise. For the disc component, for every k the arc from 1 to ζ (counterclockwisely) hits $H_k + p$ if and only if $s_{i,k} = 1$, and hits $H_k + \underline{p}$ if and only if $s_{i,k} = -1$. Also ∂D hits $H_k + p$ and $H_k + \underline{p}$ $|v_{i,k}|$ times respectively. Recall that on the arc from 1 to ζ (which is mapped to $\tilde{\mathbb{L}}$), only intersection with $H_k + \underline{p}$ (but not $H_k + p$) contributes; on the opposite arc from ζ to 1 (which is mapped to \mathbb{L}), only intersection with $H_k + p$ contributes. Thus

$$Z_{\partial D_i}^{\zeta^F} = \prod_{k=1}^n \left(z_k^{-\delta(s_{i,k},-1)} \frac{z^{v_{i,k}}}{z_k^{\delta(s_{i,k},1)}} \right)$$

and so $Z_{\gamma_j(\zeta^F)}^{\zeta^F} Z_{\partial D_i}^{\zeta^F} = z_j^{-\delta(s_{i,j},-1)} z_j^{-\delta(s_{i,j},1)} \prod_{k=1}^n z^{v_{i,k}}$. Thus the first term is $z^{v_i} = \prod_{k=1}^n z^{v_{i,k}}$.

The derivation of the last term to be z^{v_i} is very similar and left to the reader. This proves $\sum_{j=1}^n \alpha_j^i(z_j - z_j) = z^{v_i} - z^{v_i}$. \blacksquare

8.3 Computation of the main terms

We now derive an explicit expression of \tilde{d} by computing the coefficients

$$\alpha_j^i = s_{i,j} \sum_{\gamma_j \cup D_i} Z_{\gamma_j} \cdot Z_{\partial D_i}.$$

By Lemma 8.7, it suffices to compute the holonomy contribution from a single pearl trajectory from e_j to 1 involving the D_i -disc. ∂D_i is a circle spanned by the direction v_i in \mathbb{L} , which hits the unstable submanifold of e_j $|v_{i,j}|$ number of times. Hence there are in total $|v_{i,j}|$ number of pearl trajectories, which are parametrized by the entry points along ∂D_i .

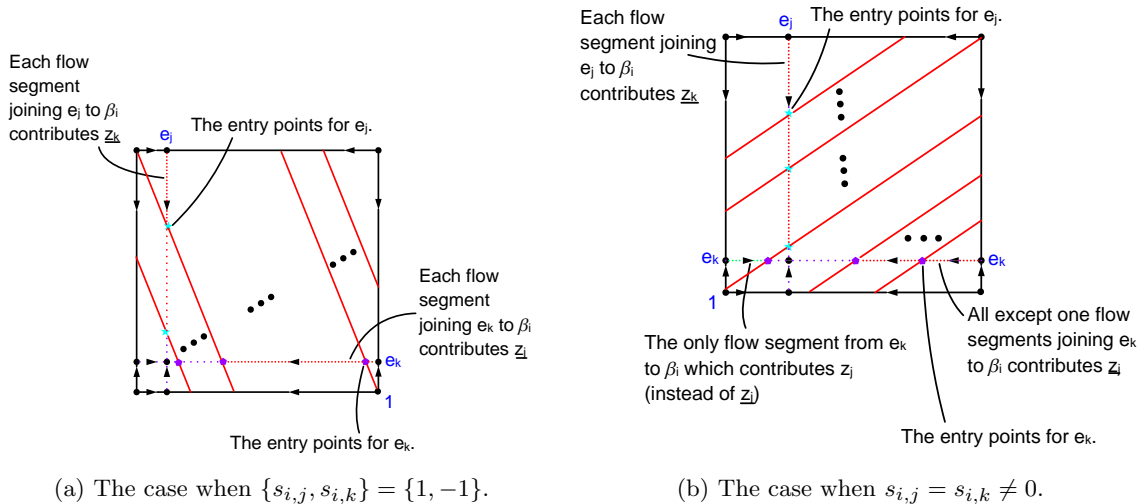


Fig. 13: The jk -plane, $j < k$.

First let's analyze the holonomy contribution Z_{γ_j} from the flow segment γ_j . It is easier to do so by projecting to the jk -plane as in Figure 13, where the assumption (8.5) on the slope of ∂D_i and a_j/a_k is used. Since the flow is contained in a hypertorus normal to e_j , it never hits the gauge hypertori $p + H_j$ and $\underline{p} + H_j$. Now consider gauge hypertori $p + H_k$ and $\underline{p} + H_k$ for $k \neq j$. When $s_{i,k} = 0$, the whole pearl trajectory is contained in the hypertorus containing $1 \in T$ parallel to $p + H_k$, and hence it never hits the gauge hypertori $p + H_k$ and $\underline{p} + H_k$. When $s_{i,k} \neq 0$ and $s_{i,k} \neq s_{i,j}$, the flow segment hits the gauge hypertorus $\underline{p} + H_k$ but not $p + H_k$ in the negative transverse orientation (see Figure 13a). It contributes z_k to the holonomy. When $s_{i,k} = s_{i,j} \neq 0$, we further divide into two cases: $j < k$ and $j > k$ (see Figure 13b). When $j < k$, the flow segment hits the gauge

hypertorus $\underline{p} + H_k$ but not $p + H_k$ in the negative transverse orientation as in the previous case; it contributes \underline{z}_k to the holonomy. When $j > k$, all but one flow-disc configurations have their flow segments hitting $\underline{p} + H_k$ but not $p + H_k$ in the negative transverse orientation, which contributes \underline{z}_k to the holonomy; the exceptional flow-disc configuration has its flow segment hitting $p + H_k$ but not $\underline{p} + H_k$ in the positive transverse orientation, contributing z_k to the holonomy.

Second, consider the holonomy contribution $Z_{\partial D_i}$ from the disc component D_i , which is the basic holomorphic disc whose boundary passes through the critical point 1 whose class is β_i . The holonomy is computed by considering the intersection points of ∂D with the gauge hypertori $H_l + p$ and $H_l + \underline{p}$ for $l = 1, \dots, n$. When $s_{i,l} = 0$, there is no intersection with $H_l + p$ nor $H_l + \underline{p}$. When $s_{i,l} > 0$, ∂D_i passes through $H_l + p$ (or $H_l + \underline{p}$) in the positive transverse direction, and so each of the intersections is marked by z_l (or \underline{z}_l resp.); otherwise when $s_{i,l} < 0$, each of the intersections is marked by z_l^{-1} (or \underline{z}_l^{-1} resp.). The number of points on ∂D_i marked as z_l^\pm is the same as that marked as \underline{z}_l^\pm , which is $|v_{i,l}|$. Figure 14 shows the disc component D_i . Walking along the circle in positive orientation starting from 1, if $s_{i,l} = 1$ then we first encounter $H_l + p$, marked as z_l , and later $H_l + \underline{p}$, marked as \underline{z}_l ; if $s_{i,l} = -1$, then we first encounter \underline{z}_l^{-1} and later z_l^{-1} .

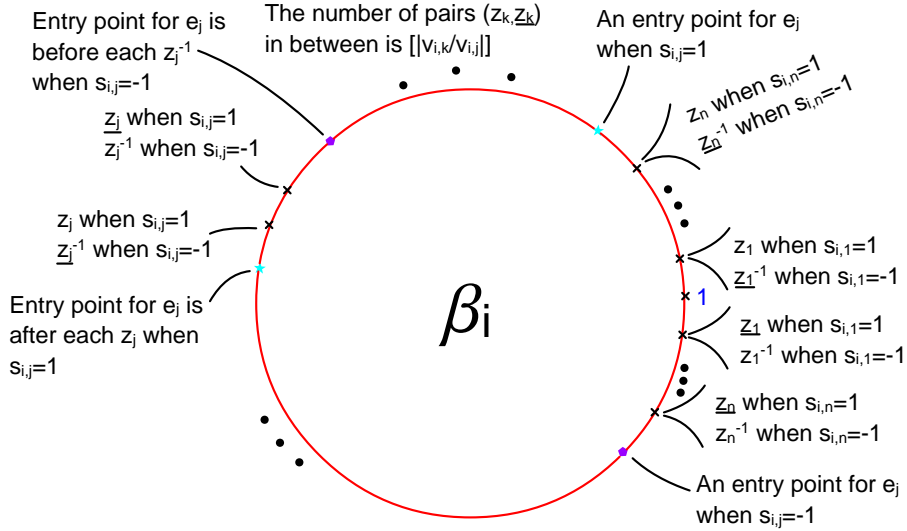


Fig. 14: The basic holomorphic disc passing through the critical point $1 \in T$ representing β_i . When the boundary passes through a gauge hypertorus, there is a holonomy term as marked in the diagram.

For each possible entry position, we need to count the number of markings $z_k^{\pm 1}$ in the arc of boundary circle counterclockwise from 1 to the entry position, and the number of $\underline{z}_k^{\pm 1}$ in the arc from the entry position to 1 (which is the part mapped to \mathbb{L}). There are $\lfloor p|v_{i,k}/v_{i,j}| \rfloor$ pairs of $(z_k^{s_{i,k}}, \underline{z}_k^{s_{i,k}})$ between the first and $(p+1)$ -th entry positions.

From now on we assume $s_{i,j} = 1$, and the case $s_{i,j} = -1$ can be analyzed in a similar way. For $k = j$, the arc from 1 to the first entry position (counterclockwise) contains one z_j and no \underline{z}_j . Then the opposite arc from the first entry position to 1 contains $(|v_{i,j}| - 1)$ number of z_j 's and $|v_{i,j}|$ number of \underline{z}_j 's. Moreover, the flow segment for such a configuration contributes no z_j nor \underline{z}_j . Thus the holonomy contribution is $z_j^{|v_{i,j}|-1} = z_j^{v_{i,j}-1}$.

For $k < j$ with $s_{i,k} = 1$, the arc from 1 to the first entry position contains one z_k and no \underline{z}_k (see Figure 13b with roles of j and k switched). Then the opposite arc from the first entry position to 1 contains $(|v_{i,k}| - 1)$ number of z_k 's and $|v_{i,k}|$ number of \underline{z}_k 's. Moreover, the flow segment for such a configuration contributes z_k . Thus the holonomy contribution is $z_k^{|v_{i,k}|-1} \cdot z_k = z_k^{v_{i,k}}$.

For $k > j$ with $s_{i,k} = 1$, the arc from 1 to the first entry position contains one z_k and no \underline{z}_k (see Figure 13b). Then the opposite arc from the first entry position to 1 contains $(|v_{i,k}| - 1)$ number of z_k 's and $|v_{i,k}|$ number of \underline{z}_k 's. Moreover, the flow segment for such a configuration contributes \underline{z}_k . Thus the holonomy contribution is $z_k^{|v_{i,k}|-1} \cdot \underline{z}_k = z_k^{v_{i,k}-1} \underline{z}_k$.

For $k \neq j$ with $s_{i,k} = -1$, the arc from 1 to the first entry position contains one \underline{z}_k^{-1} and no z_k^{-1} (see Figure 13a). Then the opposite arc from the first entry position to 1 contains $|v_{i,k}|$ number of z_k^{-1} 's and $(|v_{i,k}| - 1)$ number of \underline{z}_k^{-1} 's. Moreover, the flow segment for such a configuration contributes \underline{z}_k . Thus the holonomy contribution is $z_k^{v_{i,k}} \cdot \underline{z}_k^{-1} \cdot \underline{z}_k = z_k^{v_{i,k}}$.

Multiplying all holonomies for $k = 1, \dots, n$ together, the total holonomy of the configuration corresponding to the first entry position is

$$z_1^{v_{i,1}} \dots z_n^{v_{i,n}} z_j^{-1} \left(\prod_{k \neq j} z_k^{-\delta(s_{i,k},1)} \right) \left(\prod_{k \neq j} \underline{z}_k^{-\delta(s_{i,k},-1)} \right) \left(\prod_{k > j} \underline{z}_k^{|s_{i,k}|} \right) \left(\prod_{k < j} \underline{z}_k^{\delta(s_{i,k},-1)} z_k^{\delta(s_{i,k},1)} \right).$$

The factor $z_1^{v_{i,1}} \dots z_n^{v_{i,n}} z_j^{-1} \left(\prod_{k \neq j} z_k^{-\delta(s_{i,k},1)} \right)$ comes from the arc from first entry position to 1; the factor $\left(\prod_{k \neq j} \underline{z}_k^{-\delta(s_{i,k},-1)} \right)$ comes from the arc from 1 to the first entry position; the remaining factor $\left(\prod_{k > j} \underline{z}_k^{|s_{i,k}|} \right) \left(\prod_{k < j} \underline{z}_k^{\delta(s_{i,k},-1)} z_k^{\delta(s_{i,k},1)} \right)$ comes from the flow segment.

Now we compute holonomies of configurations corresponding to the $(p+1)$ -th entry position, $p = 1, \dots, v_{i,j} - 1$. For $k = j$ or $s_{i,k} = 0$, the flow segment contributes nothing; otherwise the flow segment always contributes \underline{z}_k . Thus the holonomy contribution of the flow segment is

$$\prod_{k \neq j} \underline{z}_k^{|s_{i,k}|}.$$

For the arc from 1 to the $(p+1)$ -th entry position, the number of $\underline{z}_k^{s_{i,k}}$'s is $\lfloor p|v_{i,k}/v_{i,j}| \rfloor$ plus that for the arc from 1 to the first entry position. Thus the holonomy contribution of the arc from 1 to the $(p+1)$ -th entry position is

$$\left(\prod_{k=1}^n \underline{z}_k^{s_{i,k} \lfloor p|v_{i,k}/v_{i,j}| \rfloor} \right) \left(\prod_{k \neq j} \underline{z}_k^{-\delta(s_{i,k},-1)} \right).$$

Similarly the number of $z_k^{s_{i,k}}$'s for the arc from the $(p+1)$ -th entry position to 1 is $\lfloor p|v_{i,k}/v_{i,j}| \rfloor$ less than that for the arc from the first entry position to 1. Thus the holonomy contribution of the arc from 1 to the $(p+1)$ -th entry position is

$$\left(\prod_{k=1}^n z_k^{-s_{i,k} \lfloor p|v_{i,k}/v_{i,j}| \rfloor} \right) z_1^{v_{i,1}} \dots z_n^{v_{i,n}} z_j^{-1} \left(\prod_{k \neq j} z_k^{-\delta(s_{i,k},1)} \right).$$

Hence the total holonomy from the flow segment and the disc boundary is

$$z_1^{v_{i,1}} \dots z_n^{v_{i,n}} z_j^{-1} \left(\prod_{l \neq j} \underline{z}_l^{-\delta(s_{i,l},-1)} \right) \left(\prod_{l \neq j} z_l^{-\delta(s_{i,l},1)} \right) \left(\prod_{l \neq j} \underline{z}_l^{|s_{i,l}|} \right) \prod_{l=1}^n \left(\frac{\underline{z}_l}{z_l} \right)^{s_{i,l} \lfloor p \left| \frac{v_{i,l}}{v_{i,j}} \right| \rfloor}.$$

The other case $s_{i,j} = -1$ can be analyzed similarly. We obtain

Theorem 8.9. The matrix factorization $(\wedge^* \underline{\Lambda}^n, \tilde{d})$ is

$$\tilde{d} = \left(\sum_{i=1}^n (z_i - \underline{z}_i) e_i \wedge \right) + \left(\sum_{i=1}^n c_i t e_i \right) + \left(\sum_{i=n+1}^m c_i \sum_{j=1}^n \alpha_j^i t e_j \right) \quad (8.9)$$

where $\alpha_j^i = 0$ when $v_{i,j} = 0$,

$$\begin{aligned} \alpha_j^i = & z_1^{v_{i,1}} \dots z_n^{v_{i,n}} z_j^{-1} \left(\prod_{l \neq j} \underline{z}_l^{-\delta(s_{i,l},-1)} \right) \left(\prod_{l \neq j} z_l^{-\delta(s_{i,l},1)} \right) \left(\left(\prod_{l > j} \underline{z}_l^{|s_{i,l}|} \right) \left(\prod_{l < j} \underline{z}_l^{\delta(s_{i,l},-1)} z_l^{\delta(s_{i,l},1)} \right) \right) \\ & + \left(\prod_{l \neq j} \underline{z}_l^{|s_{i,l}|} \right) \sum_{p=1}^{v_{i,j}-1} \left(\frac{\underline{z}_j}{z_j} \right)^p \prod_{l \neq j} \left(\frac{\underline{z}_l}{z_l} \right)^{s_{i,l} \lfloor p \left| \frac{v_{i,l}}{v_{i,j}} \right| \rfloor} \end{aligned}$$

when $s_{i,j} = 1$, and

$$\begin{aligned} \alpha_j^i = & \underline{z}_1^{v_{i,1}} \dots \underline{z}_n^{v_{i,n}} z_j^{-1} \left(\prod_{l \neq j} z_l^{-\delta(s_{i,l},-1)} \right) \left(\prod_{l \neq j} \underline{z}_l^{-\delta(s_{i,l},1)} \right) \left(\left(\prod_{l > j} \underline{z}_l^{|s_{i,l}|} \right) \left(\prod_{l < j} \underline{z}_l^{\delta(s_{i,l},1)} z_l^{\delta(s_{i,l},-1)} \right) \right) \\ & + \left(\prod_{l \neq j} \underline{z}_l^{|s_{i,l}|} \right) \sum_{p=1}^{|v_{i,j}|-1} \left(\frac{\underline{z}_j}{z_j} \right)^p \prod_{l \neq j} \left(\frac{\underline{z}_l}{z_l} \right)^{s_{i,l} \lfloor p \left| \frac{v_{i,l}}{v_{i,j}} \right| \rfloor} \end{aligned}$$

when $s_{i,j} = -1$. □

As a simple application, the wedge-contraction type matrix factorization $(\wedge^* \underline{\Lambda}^n, \tilde{d})$ for $X = \mathbb{C}P^n$ is given as follows:

Corollary 8.10. The matrix factorization \tilde{d} corresponding to the Clifford torus with the holonomy (z_1, \dots, z_n) in $\mathbb{C}P^n$ is

$$\tilde{d} = \sum_{i=1}^n (z_i - \bar{z}_i) e_i \wedge + \sum_{i=1}^n \left(T^k - \frac{T^k}{z_1 \cdots z_i z_i \cdots z_n} \right) \iota_{e_i}$$

where k is the (common) area of $(n+1)$ Maslov-2 discs bounding the central fiber. \square

8.4 Transversality

In this section, we discuss the regularity of relevant moduli space of pearl trajectories which appeared throughout the section. We first show that the moduli spaces $\mathcal{M}_2(e_I, e_J)$ of pearl trajectories in X from e_I to e_J for $|J| = |I| - 1$ are regular. Because of degree reason, this moduli space consists of pearl trajectories with a unique disc of Maslov index two.

The moduli of Maslov-2 holomorphic discs are known to be regular in toric Fano case by Cho-Oh [18]. Also the Morse functions that we have chosen are Morse-Smale and hence, satisfy the transversality condition. As the moduli space $\mathcal{M}_2(e_I, e_J)$ is given by $ev_{\bar{\beta}}^{-1}(W^u(e_I) \times W^s(e_J))$ for the unstable manifold $W^u(e_I)$ of e_I and the stable manifold $W^s(e_J)$ of e_J , it only remains to prove that the map $ev_{\beta} = (ev_1, ev_0) : \mathcal{M}_2(\mathbb{L}, J, \beta) \rightarrow \mathbb{L} \times \mathbb{L}$ is transversal to $W^u(e_I) \times W^s(e_J)$.

Lemma 8.11. With the setting as above, $ev_{\beta} : \mathcal{M}_2(\mathbb{L}, J, \beta) \rightarrow \mathbb{L} \times \mathbb{L}$ is transversal to $W^u(e_I) \times W^s(e_J)$. \square

Proof. From the condition on I and J , we may assume that $I = \{1, 2, \dots, k\}$ and $J = \{1, 2, \dots, k-1\}$. At an intersection point p of ev_{β} and $W^u(e_I) \times W^s(e_J)$, directions of flow lines in $W^u(e_I) \times W^s(e_J)$ generates

$$\mathbb{R}\langle (E_{k+1}, 0), (E_{k+2}, 0) \cdots, (E_n, 0) \rangle \oplus \mathbb{R}\langle (0, E_1), (0, E_2), \dots, (0, E_{k-1}) \rangle$$

in $T_p(\mathbb{L} \times \mathbb{L})$.

Moreover, the translations in $\mathcal{M}_2(\mathbb{L}, J, \beta)$ due to the torus action give rise to $\bigoplus_{i=1}^n \mathbb{R}\langle (E_i, E_i) \rangle$. Combining these two, ev_{β} and $W^u(e_I) \times W^s(e_J)$ already generate

$$\mathbb{R}\{(E_i, E_j) : i \neq k, j \neq k\} \oplus \mathbb{R}\langle (E_k, E_k) \rangle \leq T_p(\mathbb{L} \times \mathbb{L}), \quad (8.10)$$

and hence, it suffices to prove that the movement of markings in $\mathcal{M}_2(\mathbb{L}, J, \beta)$ induces the $(E_k, 0)$ -direction. To see this, recall that

$$\begin{aligned} W^u(e_I) &\subset \{(a_1, \dots, a_{k-1}, a_k, t_1, t_2, \dots, t_{n-k}) : t_i \in \mathbb{R}\} \\ W^s(e_J) &\subset \{(s_1 a_1, \dots, s_{k-1} a_{k-1}, 0, 0, \dots, 0) : s_j \in \mathbb{R}\} \end{aligned}$$

(modulo \mathbb{Z}^n). Since a_k is not an integer, the boundary image of the disc associated with the intersection point p should have a nontrivial e_k -component. Therefore, if we vary the location of the first marking, we obtain a vector $(v, 0)$ in $T_p(\mathbb{L} \times \mathbb{L})$ where v has a nontrivial E_k -component. This together with (8.10) proves the lemma. \blacksquare

In [8], Biran and Cornea proved that there exists a second category J_{reg} of compatible almost complex structures on X whose moduli space of pearl trajectories is a smooth manifold of expected dimension. It is not clear that the standard complex structure J_0 , which is regular for a trajectory with a single pearl, belongs to J_{reg} . But we can show that the single pearl computation with a generic J sufficiently close to J_0 is the same as that with J_0 , basically due to the fact that single pearl trajectories for J_0 are already Fredholm regular.

Lemma 8.12. There exists an almost complex structure $J \in J_{reg}$ sufficiently close to J_0 such that the single pearl contribution of the matrix factorization (of J) is identical to that of J_0 . \square

Proof. Let p, q be two critical points of the chosen Morse function, and consider the one-dimensional moduli space of single pearl trajectories $\mathcal{M} := \cup_{t \in [0,1]} \{t\} \times \mathcal{M}_2(p, q, \beta, J_t)$ for one parameter family of compatible almost complex structures $\{J_t\}_{t \in [0,1]}$. By standard argument as in [8], we can find such an one parameter family of almost complex structures $\{J_t\}_{t \in [0,1]}$ starting from the standard complex structure J_0 such that \mathcal{M} is a smooth one-dimensional manifold with boundary. We can choose $t_{p,q}$ such that in the part $0 < t < t_{p,q}$, there exists no creation or cancellation phenomenon of the cobordism \mathcal{M} . We take a minimum of $t_{p,q}$ for all such p, q and denote it by t_e . Recall that the matrix factorization is given from the relative homotopy class of a path (with fixed end points p, q) for a pearl trajectory by considering intersection data with hyper-tori. It is easy to see that for sufficiently small $0 < t < t_e$ with $J_t \in J_{reg}$, single pearl trajectories for J_t and J_0 give rise to the same homotopy class of paths, and hence provides the identical matrix factorization. \blacksquare

9 Applications

From the previous section, the matrix factorization mirror to a Lagrangian torus fiber $(\underline{L}, \mathcal{L}_{\underline{z}})$ takes the form $d = d_1 + d_{-1} + \cdots + d_{-(2\lfloor(n+1)/2\rfloor-1)}$ on $\wedge^* \underline{\mathbb{C}}^n$ where $d^2 = W_z - W_{\underline{z}}$, see Equation (8.1). We derived an explicit expression for the approximated matrix factorization $\tilde{d} = d_1 + d_{-1}$ in Theorem 8.9.

In this section, we prove that d generates $DMF(W(z) - W(\underline{z}))$ by using the spectral-sequence method by Polishchuk-Vaintrob [36] and the result of Dyckerhoff [19]. The key is that the higher order terms d_l for $l < -1$ have no contribution to the spectral sequence which computes the cokernel of d , and hence cokernel of d is just the same as that of its approximation \tilde{d} which generates the category. We also deduce a change of coordinates which brings d to \tilde{d} in low dimensions $n \leq 4$.

9.1 Generating the category of matrix factorizations

Let N be a lattice of rank n , and let $\{E_1, \dots, E_n\}$ be a basis of N . For $v_i \in N$ for $i = 1, \dots, m$, let $W = \sum_{i=1}^m c_i z^{v_i}$ be a Laurent polynomial, where $c_i \in \mathbb{C}$ are constants. Here z^{v_i} denotes the monomial $\prod_{l=1}^n z_l^{v_{i,l}}$, where $v_i = \sum_{l=1}^n v_{i,l} E_l$ is written in terms of the basis $\{E_l\}$. Let $\text{Jac}(W) = \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}] / (z_1 \partial_{z_1} W, \dots, z_n \partial_{z_n} W)$ be the Jacobian ring of W , which can be regarded as the deformation space of W . (We change from Λ to \mathbb{C} by setting $T = e^{-1}$ in this section.)

The number of critical points of W (counted with multiplicities) equals to $\dim \text{Jac}(W)$. The category of matrix factorizations of W can be written as a direct sum:

$$\bigoplus_l DMF(W(z) - W(\underline{z}^{(l)}))$$

where l labels the critical points. For each l , $W(z) - W(\underline{z}^{(l)})$ is identified as an element in $\mathbb{C}[[z_1 - \underline{z}_1^{(l)}, \dots, z_n - \underline{z}_n^{(l)}]]$, which has an isolated critical point at $z = \underline{z}^{(l)}$.

Fix a critical point $\underline{z} = \underline{z}^{(l)}$. We have the matrix factorization $(\wedge^* \underline{\mathbb{C}}^n, R_0(\underline{z}))$ given in Theorem 8.9:

$$R_0(\underline{z}) = \left(\sum_{i=1}^n (z_i - \underline{z}_i) E_i \wedge \right) + \left(\sum_{i=1}^m c_i \sum_{j=1}^n \alpha_j^i E_j \right)$$

where α_j^i 's are as stated there. All terms involved in the definition are combinatorial and only depend on W . This matrix factorization is of wedge-contraction type, and hence serves as a split-generator of $DMF(W(z) - W(\underline{z}^{(l)}))$ by [19]. At this point we do not need to assume W to be the Landau-Ginzburg mirror of a toric manifold. For instance W could be mirror to a Grassmannian. These split-generators should be helpful for proving homological mirror symmetry.

Now suppose W is mirror to a toric Fano manifold. A critical point \underline{z} corresponds to a non-displaceable Lagrangian torus fiber [27]. The matrix factorization $R^{(l)}$ given above is the approximated matrix factorization $\tilde{d} = d_1 + d_{-1}$ mirror to a critical Lagrangian torus fiber. By spectral sequence technique below, we see that the higher order terms d_k for $k \leq -3$ are useless to generation, and so the mirror matrix factorizations $(\wedge^* \underline{\mathbb{C}}^n, d)$ generates as well.

Theorem 9.1. The matrix factorization $R = (\wedge^* \underline{\mathbb{C}}^n, d)$ split generates $DMF(W(z) - W(\underline{z}))$. □

Proof. We closely follow the proof given in [36, Proposition 2.3.1]. $W(z) - W(\underline{z})$ (where \underline{z} is constant) has an isolated singularity at $z = \underline{z}$. Let $R = \Lambda[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ and $I = (z_1 - \underline{z}_1, \dots, z_n - \underline{z}_n)$ a maximal ideal. d consists of $d^{1 \rightarrow 0} : \wedge^{\text{odd}} \underline{\mathbb{C}}^n \rightarrow \wedge^{\text{even}} \underline{\mathbb{C}}^n$ and $d^{0 \rightarrow 1} : \wedge^{\text{even}} \underline{\mathbb{C}}^n \rightarrow \wedge^{\text{odd}} \underline{\mathbb{C}}^n$.

Suppose n is even. We shall show that the cokernel of $d^{1 \rightarrow 0}$ has the $R/(W - W(\underline{z}))$ -module R/I (which is geometrically the point \underline{z} of the hypersurface $\{W(z) - W(\underline{z}) = 0\}$) as a direct summand. Since R/I split generates the category, $d^{1 \rightarrow 0}$ also split generates. For n is odd, we shall replace $d^{0 \rightarrow 1}$ in the above statement.

The cokernel is the cohomology of the second term of the sequence

$$0 \rightarrow \wedge^{\text{odd}} \underline{\mathbb{C}}^n \rightarrow \wedge^{\text{even}} \underline{\mathbb{C}}^n \rightarrow 0$$

where the middle arrow is $d^{1 \rightarrow 0}$. Since both \wedge^{odd} and \wedge^{even} are graded, it gives the following spectral sequence whose total cohomology compute the cokernel of $d^{1 \rightarrow 0}$.

$$\begin{array}{ccc}
\begin{array}{c} \wedge^n \underline{\mathbb{C}}^n \\ \uparrow d_1 \\ \wedge^{n-1} \underline{\mathbb{C}}^n \xrightarrow{d_{-1}} \wedge^{n-2} \underline{\mathbb{C}}^n \\ \uparrow d_{-3} \\ \wedge^{n-3} \underline{\mathbb{C}}^n \xrightarrow{d_{-3}} \wedge^{n-4} \underline{\mathbb{C}}^n \\ \vdots \\ \vdots \end{array} & R/I & \begin{array}{c} \ker(d_1|_{\wedge^{n-1}}) \xrightarrow{d_{-1}} \text{coker}(d_1|_{\wedge^{n-3}}) \\ \uparrow d_{-3} \\ \ker(d_1|_{\wedge^{n-3}}) \xrightarrow{d_{-3}} \text{coker}(d_1|_{\wedge^{n-5}}) \\ \vdots \\ \vdots \end{array} \\
E^0 \text{ page} & & E^1 \text{ page}
\end{array}$$

By Lemma 8.5, $d_1 = \sum_{i=1}^m (z_i - \underline{z}_i) e_i \wedge (\cdot)$. Thus $\text{coker}(d_1 : \wedge^{n-1} \underline{\mathbb{C}}^n \rightarrow \wedge^n \underline{\mathbb{C}}^n) = R/I$ which is the top left corner of the E^1 page. Moreover from the exactness of the Koszul complex $(\wedge^* \underline{\mathbb{C}}^n, d_1)$, we have $\ker(d_1|_{\wedge^k}) = \text{im}(d_1|_{\wedge^{k-1}})$ for $k = 0, \dots, n-1$. Since $d_{-1} \circ d_1 = (W_z - W_{\underline{z}}) \cdot \text{Id}$ by Theorem 8.8, the horizontal maps d_{-1} in the E^1 page are identified with the multiplication by $(W_z - W_{\underline{z}})$, which is injective. As a result the E^2 page has only diagonal terms being non-zero and hence stabilizes. We see that R/I is a direct summand of the cokernel of $d^{1 \rightarrow 0}$.

When n is odd, we consider the cokernel of $d^{0 \rightarrow 1}$, which is the cohomology of the second term of the sequence

$$0 \rightarrow \wedge^{\text{even}} \underline{\mathbb{C}}^n \rightarrow \wedge^{\text{odd}} \underline{\mathbb{C}}^n \rightarrow 0.$$

The spectral sequence is the same as above and we have the same conclusion. \blacksquare

9.2 Mirror matrix factorizations in low dimensions

Suppose $\dim_{\mathbb{C}} X \leq 4$. We prove that the matrix factorization R in Theorem 1.2 is exactly of wedge-contraction type in this section.

Theorem 9.2. The mirror matrix factorization of $W(\mathbb{L})$ corresponding to the critical point $(\mathbb{L}, \underline{z}) \in \text{Fuk}(X)$ is of *wedge-contraction type*. Namely, we have the $\mathbb{Z}/2$ -graded free Λ -module $\wedge^* \underline{\Lambda}^n$ generated by e_I^{new} for $I \subset \{1, \dots, n\}$ and

$$d = \sum x_i e_i^{new} \wedge_{new} + \sum_{i \in I} w_i l_i^{new},$$

such that for $\lambda = W(\mathbb{L})(\underline{z})$, we have $d^2 = W(\mathbb{L})(z) - \lambda$. \square

Here, we used the notation e_I^{new} instead of e_I as we need to define a new (quantum) exterior algebra structure to make it of wedge-contraction type (in dimension 4).

Proof. Denote by (P, d) the matrix factorization of $W(\mathbb{L})$ corresponding to $(\mathbb{L}, \underline{z})$ obtained from the pearl complex. By the degree reason (with $\dim(X) \leq 4$), the decomposition of d (8.1) has only three components:

$$d = d_1 + d_{-1} + d_{-3}.$$

First, let us assume that $\dim X \leq 2$. In this case, d_{-3} vanishes by degree reason, and we already show that $d_1 + d_{-1}$ is a wedge-contraction type matrix factorization of $W - \lambda$.

Now, let $\dim X = 3$. We prove that $d_{-3} = 0$. By degree reason, it is enough to show that $d_{-3}(e_{top}) = 0$ for $e_{top} = e_1 e_2 e_3$. We expand d^2 using the above decomposition of d , and the degree $-(-2)$ component of $d^2 = (W - \lambda) \cdot \text{Id}$ gives $d_1 \circ d_{-3} + d_{-3} \circ d_1 = 0$. Applying it for $e_2 \wedge e_3$, we get

$$0 = (d_1 \circ d_{-3} + d_{-3} \circ d_1)(e_2 \wedge e_3) = d_{-3}((z_1 - \underline{z}_1) e_{top}) = (z_1 - \underline{z}_1) d_{-3}(e_{top}).$$

Since P is torsion-free, this implies that $d_{-3}(e_{top}) = 0$ as desired.

Finally, let $\dim X = 4$. For notational convenience, we set $x_i := z_i - \underline{z}_i$, and write $d_1 = \sum_{i=1}^4 x_i e_i \wedge$ and $d_{-1} := \sum_{i=1}^4 w_i \iota_{e_i}$ (then $W = \sum x_i w_i$). Also, for degree three generators, we set the notation as follows:

$$e_{top \setminus 1} := e_2 \wedge e_3 \wedge e_4 \quad e_{top \setminus 2} := e_1 \wedge e_3 \wedge e_4 \quad e_{top \setminus 3} := e_1 \wedge e_2 \wedge e_4 \quad e_{top \setminus 4} := e_1 \wedge e_2 \wedge e_3$$

We do not know if d_{-3} vanishes in this case, but we can show that there is a “new exterior algebra structure” on the pearl complex so that d becomes a wedge-contraction type. For this purpose, we need some calculations.

From $d^2 = (W - \lambda) \cdot Id$, we have the following two identities:

$$d_1 \circ d_{-3} + d_{-3} \circ d_1 \equiv 0 \tag{9.1}$$

$$d_{-1} \circ d_{-3} + d_{-3} \circ d_{-1} \equiv 0 \tag{9.2}$$

Write $d_{-3}(e_{top \setminus i}) := f_i$ which is a function on x_i 's. From (9.1) with $e_{top \setminus i}$, it follows that

$$f_i \cdot \left(\sum x_j e_j \right) + d_{-3}((-1)^{i-1} x_i e_{top}) = 0.$$

(Here, we used $d_{+1}(e_{top \setminus i}) = (-1)^{i-1} x_i e_{top}$.) Since the second term is divisible by x_i , $f_i x_j e_j$ for $i \neq j$ in the first summand should be a multiple of x_i , also. This implies that $f_i = x_i g_i$ for some g_i . Then, one gets

$$d_{-3}(e_{top}) = (-1)^i g_i \left(\sum_j x_j e_j \right),$$

and hence $(-1)^i g_i$ does not depend on i , and we may set $g := (-1)^i g_i$. In summary, we have

$$d_{-3}(e_{top}) = g \left(\sum_j x_j e_j \right) \quad d_{-3}(e_{top \setminus i}) = (-1)^i x_i g \quad \text{for } i = 1, 2, 3, 4. \tag{9.3}$$

We will consider a new “exterior algebra structure” on \wedge^4 . We define a new basis e_I^{new} whose interior and exterior multiplication defined in a standard way: For a disjoint I, J ,

$$e_I^{new} \wedge_{new} e_J^{new} = \pm e_{I \cup J}^{new}.$$

where the sign is determined by identifying e_I^{new} with $e_{i_1}^{new} \wedge_{new} \cdots \wedge_{new} e_{i_k}^{new}$ for $I = \{i_1 < \cdots < i_k\}$. (The wedge product is zero if I, J are not disjoint.) The new interior multiplication $\iota_i^{new} := \iota_{e_i^{new}}^{new}$ is similarly defined.

We would like to define such a new exterior algebra structure so that the differential d becomes a wedge-contraction type as in the statement of the theorem. In the case of $n = 4$, we define a new *exterior algebra structure* by setting

$$e_{top}^{new} := e_{top} - g, \quad e_I^{new} := e_I \quad \text{for } I \neq \{1, 2, 3, 4\}.$$

Then, from the previous setting,

$$d(e_{top \setminus i}^{new}) = d(e_{top \setminus i}) = (d_1 + d_{-1} + d_{-3})(e_{top \setminus i}) = d_{-1}(e_{top \setminus i}) + (d_1 + d_{-3})(e_{top \setminus i}),$$

and one can easily check that

$$(d_1 + d_{-3})(e_{top \setminus i}) = (-1)^{i-1} x_i e_{top}^{new}.$$

Also,

$$d(e_{top}^{new}) = (d_1 + d_{-1} + d_{-3})(e_{top} - g) = d_{-1}(e_{top}) - d_1 g + d_{-3} e_{top} = \sum (-1)^{i-1} w_i e_{top \setminus i} + 0$$

since we have

$$-d_1 g + d_{-3} e_{top} = - \sum_i x_i g e_i + \sum_i x_i g e_i = 0.$$

This provides the desired wedge-contraction type structure obtained from the “quantum correction of exterior algebra structure”. ■

10 The real Lagrangian in the projective space

In this section, we compute the mirror matrix factorization of the real Lagrangian $\mathbb{R}\mathbb{P}^n$ in $\mathbb{C}\mathbb{P}^n$ for $n \geq 2$ when we take the reference to be the Clifford torus $\mathbb{L} = L_u$. Here, u is the center of the moment polytope of $\mathbb{C}\mathbb{P}^n$, and L_u is the fiber of the moment map at u . We will give a detailed description for $\mathbb{R}\mathbb{P}^3$ at the end of the section.

Alston and Amorim [6] presented a comprehensive examination of Lagrangian Floer theory between the torus fiber and $\mathbb{R}\mathbb{P}^n$. They first twisted Floer cohomology by a locally constant sheaf over a characteristic-2 ring instead of a line bundle, and took the Novikov ring

$$\Lambda^{\mathbb{F}_2} := \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{F}_2, \lambda_i \in \mathbb{R}, \lambda_i \rightarrow \infty \right\}$$

over \mathbb{F}_2 as the coefficient ring, where \mathbb{F}_2 is the algebraic closure of \mathbb{Z}_2 . Following their approach, we take our mirror variables to live in \mathbb{F}_2^\times .

From [33], real Lagrangian $\mathbb{R}\mathbb{P}^n$ has a minimal Maslov number $n + 1$, and hence there exists no Maslov index two disc with boundary on $\mathbb{R}\mathbb{P}^n$ for $n \geq 2$. Thus, the real Lagrangian $\mathbb{R}\mathbb{P}^n$ should correspond to the matrix factorization of $W - 0$ for the Floer potential W of \mathbb{L} . However, one can check easily that 0 is not a critical value of W which implies that the real Lagrangian $\mathbb{R}\mathbb{P}^n$ corresponds to a trivial object with the usual Novikov coefficients. The same phenomenon happens in Floer theory also. Namely, Clifford torus \mathbb{L} and $\mathbb{R}\mathbb{P}^n$ have different potential values, and hence, its Floer cohomology cannot be defined. But, if we use a characteristic-2 coefficient ring, 0 is a critical value for n odd, hence $\mathbb{R}\mathbb{P}^n$ can provide a non-trivial matrix factorization in the mirror. We assume that n is odd from now on.

Let us briefly review the construction of the Floer cohomology in [6] which involves locally constant sheaves (analogous to flat connections over the field \mathbb{F}_2). For each homomorphism $\rho : \pi_1(\mathbb{L}) \rightarrow \mathbb{F}_2^\times$, one can equip \mathbb{L} with a locally constant sheaf defined by

$$\mathcal{L}_\rho = \tilde{\mathbb{L}} \times \mathbb{F}_2 / (x \cdot \gamma, v) \sim (x, \rho(\gamma)v) \quad \gamma \in \pi_1(\mathbb{L})$$

where $\tilde{\mathbb{L}}$ is the universal cover of \mathbb{L} . This process is analogous to the construction of \mathbb{C}^\times -flat line bundles from elements of $\text{Hom}(\pi_1(\mathbb{L}), \mathbb{C}^\times)$. As fibers are discrete, one can define a parallel transport in \mathcal{L}_ρ along a path in \mathbb{L} . Then,

$$CF((\mathbb{L}, \mathcal{L}_\rho), \mathbb{R}\mathbb{P}^n) := \bigoplus_{\mathbb{L} \cap \mathbb{R}\mathbb{P}^n} \text{Hom}(\mathcal{L}_\rho|_p, \mathbb{F}_2) \otimes \Lambda^{\mathbb{F}_2}$$

and the Floer differential is defined as in the case of flat complex line bundles (see Section 3).

Let us now choose a hyper-tori for the reference Lagrangian \mathbb{L} . Recall that we fix the gauge of the (flat) connection for \mathcal{L}_z in such a way that we put holonomy effect from ρ whenever the (upper) boundary $\partial_0 u$ of a strip u passes through chosen gauge hypertori. i.e. if $\partial_0 u$ traverses a hyper-torus H_i positively, we have a variable z_i which takes value in \mathbb{F}_2^\times . Here, the gauge hyper-tori are chosen in terms of homogeneous coordinates of $\mathbb{C}\mathbb{P}^n$ as follows: if we write $\mathbb{L} = \{[1 : e^{i\theta_1} : \dots : e^{i\theta_n}] \mid 0 \leq \theta_j < 2\pi\}$ (a part of \mathbb{L} that lies inside the affine chart $\{x_0 \neq 0\}$),

$$H_i := \{[1 : e^{i\theta_1} : \dots : e^{i\theta_{i-1}} : e^{i\epsilon_i} : e^{i\theta_{i+1}} : \dots : e^{i\theta_n}] \mid 0 \leq \theta_j < 2\pi, j \neq i\}$$

for a positive ϵ_i close to 0.

Consequently, the potential

$$W = T^k \left(z_1 + z_2 + \dots + z_n + \frac{1}{z_1 z_2 \dots z_n} \right)$$

is regarded as a function on $(\mathbb{F}_2^\times)^n$ (the potential itself does not depend on the choice of gauge hyper-tori.) Note that $z_i = -z_i$ since now the coefficient ring is of characteristic 2.

To compute the matrix factorization associated with $\mathbb{R}\mathbb{P}^n$, we first need to find intersection points between the torus fiber and $\mathbb{R}\mathbb{P}^n$ and then classify holomorphic strips of Maslov index one among these intersection points. \mathbb{L} and $\mathbb{R}\mathbb{P}^n$ intersect at 2^n points, which can be written in homogeneous coordinates as

$$[\pm 1 : \pm 1 : \dots, \pm 1]$$

whose degrees depend on the number of -1 's in the entries (or equivalently the number of 1 's since n is odd).

Although $1 = -1$ in \mathbb{F}_2 , we put signs so that (10.1) also defines a matrix factorization over a characteristic zero field.

A Orientations

A.1 Orientation conventions

Biran and Cornea has shown that when a Lagrangian submanifold L is spin, the pearl complex can be defined over \mathbb{Z} -coefficient in [9, Appendix A]. We follow their orientation convention for the computations of matrix factorizations in this paper.

First, we recall some of elementary orientation conventions. An intersection $A \cap B$ is oriented as follows (see [9], [26]). For each $x \in A \cap B$, we choose the orientation of the normal bundle $N_A L$ so that we have $o(T_x L) = o(N_{x,A} L) \wedge o(T_x A)$, where we denote the orientation of a vector space V as $o(V)$. If A, B intersect transversely, we define the orientation of $A \cap B$ by

$$o(T_x L) = o(N_{x,A} L) \wedge o(N_{x,B} L) \wedge o(T_x(A \cap B)).$$

For $f : A \rightarrow L, g : B \rightarrow L$, the orientation convention of the fiber product $A \times_L B$ in [9], [26] are the same. If f is submersion, and g is an embedding, then choose orientation of $\text{Ker } f$ so that $o(A) = o(\text{Ker } f) \wedge o(f^{-1}([L]))$. Then we set the orientation of the fiber product to be

$$o(A \times_L B) = o(\text{Ker } f) \wedge o(B),$$

as in Section 7 [13]. When both f and g are embedding, the fiber product becomes the intersection $B \cap A$ as oriented spaces. The key point of this convention is the following identity as oriented spaces, which can be checked easily:

$$\partial(A \times_L B) = \partial A \times_L B \sqcup (-1)^{n+\dim A} A \times_L \partial B$$

A.2 Signs for pearl trajectories

We first fix the orientation of stable manifold $W^s(p)$ for each critical point p , and orient the unstable manifold so that we have $T_p L = T_p W^u(p) \oplus T_p W^s(p)$ as oriented spaces. In particular, the intersection $W^s(p) \cap W^u(p)$ gives a positive intersection number. Hence, we define the orientation of the Morse trajectory from p to q as $W^s(q) \cap W^u(p)$. As usual, the signed count of such trajectory is obtained by comparing this orientation with that of the flow orientation.

Now, consider the moduli space $\mathcal{M}_2(\beta)$ of J -holomorphic discs of class β (of Maslov index two) with two marked points, with an evaluation map

$$ev = (ev_0, ev_1) : \mathcal{M}_2(\beta) \rightarrow L \times L.$$

For the inclusion $i : W^u(p) \times W^s(q) \rightarrow L \times L$, Biran-Cornea [8] defined the moduli space of the pearl complexes with a single pearl from p to q as the fiber product

$$W^u(p) \times_{L, ev_0} (\mathcal{M}_2(\beta)_{ev_1} \times_L W^s(q)).$$

With this sign rule in mind, we find the precise sign in Equation(7.3).

Lemma A.1. We have $(\delta_{\text{pearl}}^{\mathcal{L}_0, \mathcal{L}_1})^2 = (\Phi(\mathcal{L}_1) - \Phi(\mathcal{L}_0))$. □

Proof. We want to show that the sign of the above identity is correct, and the rest of the proof is standard and given in the main body. We write $\delta_{\text{pearl}}^{\mathcal{L}_0, \mathcal{L}_1}$ by δ_{pearl} for simplicity.

Let us consider the moduli space of single pearls from p to p itself, which is given as the fiber product

$$W^u(p) \times \mathcal{M}_2(\beta) \times W^s(p) \tag{A.1}$$

where $\mu(\beta) = 2$ and the product is taken over L . Taking the boundary of (A.1) gives

$$\begin{aligned} & (\partial W^u(p)) \times \mathcal{M}_2(\beta) \times W^s(p) \\ \cup & (-1)^{n+\text{ind}(p)} W^u(p) \times (\partial \mathcal{M}_2(\beta)) \times W^s(p) \\ \cup & (-1)^{(n+\text{ind}(p))+(n+(n-1))} W^u(p) \times \mathcal{M}_2(\beta) \times (\partial W^s(p)). \end{aligned} \tag{A.2}$$

By the orientation convention in [9], parts of boundaries of unstable and stable manifolds are oriented as follows:

$$\partial W^u(p) = m(p, q) \times W^u(q), \quad \partial W^s(p) = (-1)^{n+\text{ind}(p)} m(r, p) \times W^s(r)$$

where $m(p, q)$ denotes the moduli space of gradient flow lines from p to q . Thus, the first and the second components of (A.2) can be rewritten as

$$m(p, q) \times (W^u(q) \times \mathcal{M}_2(\beta) \times W^s(p)) \cup (-1) (W^u(p) \times \mathcal{M}_2(\beta) \times W^s(r)) \times m(r, p)$$

which corresponds to $(-1)^{\text{ind}(p)} (\delta_{\text{pearl}})^2$. On the other hand, the boundary of the middle factor $\mathcal{M}_2(\beta)$ in (A.1) is nontrivial due to disc bubbles, and we have two more terms $\Phi(\mathcal{L}_0)$ and $\Phi(\mathcal{L}_1)$ in addition to $(-1)^{\text{ind}(p)} (\delta_{\text{pearl}})^2$. We only check the sign for $\Phi(\mathcal{L}_0)$, and the sign of $\Phi(\mathcal{L}_1)$ can be similarly proven to be opposite to that of $\Phi(\mathcal{L}_0)$. $\Phi(\mathcal{L}_0)$ comes from a disc bubble attached along the upper boundary of the disc component of the original single pearl, which corresponds $\mathcal{M}_3(\beta_0)_{ev_1} \times_{ev_0} \mathcal{M}_1(\beta)$ where β_0 represents the constant class.

Recall that there is a subtle difference between the orientation conventions in [26] and [9].

(i) Both of them fix two markings on the boundary of discs and consider the action of 1-dimensional automorphism group which preserves these two markings. See for example, [26, (8.3.2)]. However, they used the opposite orientations for this group so that the moduli space of discs has opposite orientations for [26] and [9].

(ii) Moreover, the role of two markings z_0, z_1 used to attach two discs are opposite in [26] and [9], which is equivalent to the switch of positions of two factors in the fiber product $\mathcal{M}_{k_1}(\beta_0) \times_L \mathcal{M}_{k_2}(\beta_1)$.

Let us consider the inclusion of the boundary stratum:

$$\mathcal{M}_{k_1}(\beta_0) \times_L \mathcal{M}_{k_2}(\beta) \hookrightarrow \mathcal{M}_2(\beta).$$

As explained in [9, Remark A.1.1], if $k_1 = k_2 = 2$, then the sign of this inclusion is $(-1)^{n+1}$ for both [26] (see [26, Proposition 8.3.3]) and [9]. In this case, sign differences from (i) and (ii) are both (-1) and cancel each other.

We claim that the sign of this inclusion is $(-1)^n$ when $k_1 = 3, k_2 = 1$. First, the inclusion from [26] has sign $(-1)^{n+1}$, with an additional sign (-1) from (i). Now, in this case, it is not hard to compute the effect of switching of two factors (from (ii)) and find that there is no additional sign contribution from this. This will be needed for the sign of the second term in (A.2).

Here, we orient L so that we have $\mathcal{M}_3(\beta_0) \cong L$. Consequently, the sign of $\Phi(\mathcal{L}_0)$ in the equation (induced by) (A.2) is $(-1)^{(n+\text{ind}(p))} \cdot (-1)^n = (-1)^{\text{ind}(p)}$, according to the sign rule of [9]. Therefore, we have

$$(-1)^{\text{ind}(p)} (\delta_{\text{pearl}})^2 + (-1)^{\text{ind}(p)} (\Phi(\mathcal{L}_0)) - \Phi(\mathcal{L}_1) = 0,$$

or equivalently, $(\delta_{\text{pearl}})^2 = \Phi(\mathcal{L}_1) - \Phi(\mathcal{L}_0)$. ■

A.3 Sign computations for Lemma 8.5

We first make the following the sign convention for the Morse-differential $(\delta_{\text{pearl}})_1 = \delta_{\text{Morse}}$. We first choose the orientation of stable manifolds of e_I as E_I where $E_I = E_{i_1} \wedge \dots \wedge E_{i_{|I|}}$ for $I = \{i_1, \dots, i_{|I|}\}$ with $i_1 < \dots < i_{|I|}$. Hence, unstable manifolds are oriented as

$$o(T_{e_I} \mathbb{L}) = o(T_{e_I} W^u(e_I)) \wedge E_I$$

For $j \notin I$, suppose $E_j \wedge E_I = s E_{\{j\} \cup I}$ for $s \in \{\pm 1\}$. Then, we have

$$o(T_{e_{\{j\} \cup I}} W^u(e_I)) = s \cdot o(T_{e_{\{j\} \cup I}} W^u(e_{\{j\} \cup I})) \wedge E_j.$$

Now consider the moduli space $\mathcal{M}(q' \rightarrow q)$ of negative gradient flow lines from q' to q , which is oriented as an intersection $W^s(q) \cap W^u(q')$ (see subsection A.2 for sign convention). For $\gamma \in \mathcal{M}(q' \rightarrow q)$, let T_γ be the direction of the flow of γ . Then, we have

$$o(T_q W^u(q)) \wedge o(T_{q'} W^s(q')) \wedge T_\gamma \wedge o(\mathcal{M}(q' \rightarrow q)) = o(T_{q'} \mathbb{L}),$$

where we assume $\mathcal{M}(q' \rightarrow q)$ to be oriented 0-dimensional vector spaces. In particular, $\gamma \in \mathcal{M}(q' \rightarrow q)$ has a positive sign if the splitting $(-1)^{\deg q'} o(T_q W^u(q)) \wedge T_\gamma$ equals that of $o(T_{q'} W^u(q'))$ and has a negative sign otherwise.

We now analyze the sign of a flow line γ from e_I to $e_{I \cup \{j\}}$. The one which passes through $H_j + p$ has tangent vector $T_\gamma = E_j$, and the one which passes through $H_j + \underline{p}$ has tangent vector $T_\gamma = -E_j$. First consider the case $j < \min I$ (when $I = \emptyset$, $\min I := +\infty$). By our choice of orientations for unstable submanifolds, the splitting $TW^u(e_I)|_\gamma \cong T_{e_{\{j\} \cup I}} W^u(e_{\{j\} \cup I}) \oplus T_\gamma$ preserves orientation if $T_\gamma = E_j$ and reverses orientation if $T_\gamma = -E_j$. Hence excluding the factor $(-1)^{|I|}$, the flow line has a positive sign if it is the one passing through $H_j + p$, and has a negative sign if it is the one passing through $H_j + \underline{p}$. As a result when $j < \min I$, the coefficient of $e_{\{j\} \cup I} = e_j \wedge e_I$ in $(\delta_{\text{pearl}})_1 \cdot e_I$ is $(-1)^{|I|}(z_i - \underline{z}_i) = (-1)^{\deg(e_I)}(z_i - \underline{z}_i)$.

For the general case $j = 1, \dots, n$, we claim that the flow line which passes through $H_j + p$ has the sign s with $E_j \wedge E_I = sE_{\{j\} \cup I}$, and the one which passes through $H_j + \underline{p}$ has the sign $-s$. This finishes the proof of this lemma. To see this, we compare the orientation of $TW^u(e_I)|_\gamma$ and that of $T_{e_{\{j\} \cup I}} W^u(e_{\{j\} \cup I}) \oplus \langle E_j \rangle$. By definition the orientation forms have the relation

$$o(T_{e_I} \mathbb{L}|_\gamma) = o(TW^u(e_I)|_\gamma) \wedge E_I = o(T_{e_{\{j\} \cup I}} W^u(e_{\{j\} \cup I})) \wedge E_{\{j\} \cup I} = o(T_{e_{\{j\} \cup I}} W^u(e_{\{j\} \cup I})) \wedge (s \cdot E_j \wedge E_I)$$

and hence $o(TW^u(e_I)|_\gamma) = s \cdot o(T_{e_{\{j\} \cup I}} W^u(e_{\{j\} \cup I})) \wedge E_j$. By definition, the flow line γ has the sign $(-1)^{|I|}s = (-1)^{\deg(e_I)}s$.

A.4 Sign computations for Lemma 8.7

We next derive the sign of a single β_i pearl trajectory from e_I to $e_{I - \{j\}}$ appearing in the proof of Lemma 8.7. The moduli space of such pearl trajectories is oriented as $W^u(e_I) \times_{\mathbb{L}, ev_1} (\mathcal{M}_2(\beta_i) \times_{ev_0, \mathbb{L}} W^s(e_{I - \{j\}}))$ (see Section A.2). In the toric cases, we can equip the Lagrangian torus \mathbb{L} with the standard spin structure and if β_i is a basic disc class, then we have that $ev_0 : \mathcal{M}_1(\beta_i) \rightarrow \mathbb{L}$ is an orientation preserving homeomorphism. (This follows from Proposition 8.1 [13] since we take the opposite orientation of both $\mathcal{M}_1(\beta_i)$ and \mathbb{L} compared to that of [13].) Hence, $o(\mathcal{M}_2(\beta_i)_{ev_0}) = (-1)^{n+1}o(\partial D_0^2 \times \mathcal{M}_1(\beta_i))$ and we have

$$o(\mathcal{M}_2(\beta_i)_{ev_0} \times_{\mathbb{L}} W^s(e_{I - \{j\}})) = (-1)^{n+1}o(\partial \beta_i \times W^s(e_{I - \{j\}})),$$

and hence

$$W^u(e_I) \times_{\mathbb{L}} (\mathcal{M}_2(\beta_i) \times_{\mathbb{L}} W^s(e_{I - \{j\}})) = ((-1)^{n+1} \partial \beta_i \times W^s(e_{I - \{j\}})) \cap W^u(e_I).$$

Since $W^s(e_I) \cap W^u(e_I) = +1$, it suffices to compare the orientations on $(-1)^{n+1} \partial \beta_i \times W^s(e_{I - \{j\}})$ and $W^s(e_I)$. Note that the former (after being projected onto E_I -plane) is equivalent to $(-1)^{n+1} s_{i,j} \times E_{I - \{j\}}$ and the latter is simply E_I itself. Thus, the total sign difference between $(-1)^{n+1} \partial \beta_i \times W^s(e_{I - \{j\}})$ and $W^s(e_I)$ is $(-1)^{n+1} s_{i,j} s^*$ for $s^* =$ the sign difference of E_I and $(E_j \wedge E_{I - \{j\}})$. Consequently,

$$\langle (\delta_{\text{pearl}})_{-1}(e_I), e_{I - \{j\}} \rangle = (-1)^{\text{ind}(e_{I - \{j\}})} (-1)^{n+1} s_{i,j} s^* Z_{\gamma_j} \cdot Z_{\partial D_i} = (-1)^{\deg(e_I)} s^* (s_{i,j} Z_{\gamma_j} \cdot Z_{\partial D_i})$$

where $(-1)^{\text{ind}(e_{I - \{j\}})} = (-1)^{n - |I| + 1}$ comes from the sign factor in the second term of (7.1).

Note that $\iota_{e_j} e_I = s^* e_{I - \{j\}}$, and hence we have shown that the sign factor of $(-1)^{\deg} \delta_1$ equals $s_{i,j}$ as claimed.

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