

OPEN GROMOV-WITTEN INVARIANTS AND SYZ UNDER LOCAL CONIFOLD TRANSITIONS

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ABSTRACT. For a local non-toric Calabi-Yau manifold which arises as smoothing of a toric Gorenstein singularity, this paper derives the open Gromov-Witten invariants of a generic fiber of the special Lagrangian fibration constructed by Gross and thereby constructs its SYZ mirror. Moreover it proves that the SYZ mirrors and disc potentials vary smoothly under conifold transitions, giving a global picture of SYZ mirror symmetry.

1. INTRODUCTION

Let N be a lattice and P a lattice polytope in $N_{\mathbb{R}}$. It is an interesting classical problem in combinatorial geometry to construct Minkowski decompositions of P .

On the other hand, consider the family of polynomials $\sum_{v \in P \cap N} c_v z^v$ for $c_v \in \mathbb{C}$ with $c_v \neq 0$ when v is a vertex of P . All these polynomials have P as their Newton polytopes. This establishes a relation between geometry and algebra, and the geometric problem of finding Minkowski decompositions is transformed to the algebraic problem of finding polynomial factorizations.

One purpose of this paper is to show that the beautiful algebro-geometric correspondence between Minkowski decompositions and polynomial factorizations can be realized via *SYZ mirror symmetry*.

A key step leading to the miracle is brought by a result of Altmann [3]. First of all, a lattice polytope corresponds to a toric Gorenstein singularity X . Altmann showed that Minkowski decompositions of the lattice polytope correspond to (partial) smoothings of X . From string-theoretic point of view different smoothings of X belong to different sectors of the same stringy Kähler moduli. Under mirror symmetry, this should correspond to a complex family of local Calabi-Yau manifolds, and the various sectors (coming from Minkowski decompositions) correspond to various (conifold) limits of the complex family.

We show that such a complex family can be realized via SYZ construction by using special Lagrangian fibrations on smoothings of toric Gorenstein singularities constructed by Gross [19]. This paper follows the framework of SYZ given in [10] which uses open Gromov-Witten invariants rather than their tropical analogs. *All the relevant open Gromov-Witten invariants are computed explicitly* (Theorem 4.7), which gives a more direct understanding to symplectic geometry.

From the computations we obtain an explicit expression of the SYZ mirror:

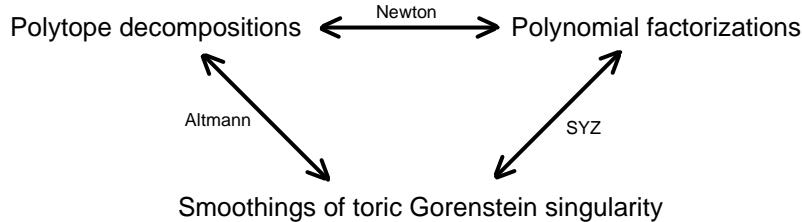
Theorem 1.1 (see Theorem 4.8 for the detailed statement). *For a smoothing of a toric Gorenstein singularity coming from a Minkowski decomposition of the corresponding polytope,*

its SYZ mirror is

$$uv = \prod_{i=0}^p \left(1 + \sum_{l=1, \dots, k_i} z^{u_l^i} \right)$$

where u_l^i are vertices of simplices appearing in the Minkowski decomposition.

See Section 3 and 4 for more details on Minkowski decomposition and SYZ construction. Thus the SYZ mirror gives a factorization of a polynomial corresponding to a Minkowski decomposition of P . The following diagram summarizes the dualities that we have and their relations:



Another motivation of this paper comes from *global SYZ mirror symmetry*. Currently most studies in SYZ mirror symmetry focus on the large complex structure limit, and other limit points of the moduli are less understood. Open Gromov-Witten invariants and SYZ for toric Calabi-Yau manifolds were studied in [10, 12, 8], which are around the large complex structure limit. This paper studies SYZ for conifold transitions of toric Calabi-Yau manifolds, which are other limit points of the moduli. This gives a more global understanding of SYZ mirror symmetry.

More concretely, we prove the following:

Theorem 1.2 (see Theorem 5.2 for the complete statement). *Let Y be a toric Calabi-Yau manifold associated to a triangulation of a lattice polytope P , and \mathcal{X}_t be the conifold transition of Y induced by a Minkowski decomposition of P .*

Then their SYZ mirrors $\check{\mathcal{X}}$ and \check{Y}_q are connected by an analytic continuation: there exists an invertible change of coordinates $q(\check{q})$ and a specialization of parameters $\check{q} = \underline{\check{q}}$ such that

$$\check{\mathcal{X}} = \check{Y}_{q(\check{q})}|_{\check{q}=\underline{\check{q}}}.$$

The above theorem is proved by combining Theorem 1.1 and the toric open mirror theorem given by [8], which gives explicit expressions of the disc potential and SYZ mirror of a toric Calabi-Yau manifold in terms of the mirror map. This is similar to the strategy of proving Ruan's crepant resolution conjecture via mirror symmetry [21, 17].

Notice that *the smoothings \mathcal{X}_t are no longer toric*. Open Gromov-Witten invariants for non-toric cases are usually difficult to compute and only known for certain isolated cases (such as the real Lagrangian in the quintic [25]). This paper gives a class of non-toric manifolds that the open Gromov-Witten invariants can be explicitly computed.

In this paper we do not need Kuranishi's obstruction theory to define the invariants: there is no non-constant holomorphic sphere (Lemma 4.2), and hence no sphere bubbling can occur. (And no disc-bubbling occurs since we consider discs with the minimal Maslov index.) This

makes the theory of open Gromov-Witten invariants easier to deal with. On the other hand, the local Calabi-Yau here is non-toric, and it involves new geometric arguments to compute the invariants.

One can also construct the mirror via Gross-Siebert program [20] using tropical geometry of the base of the fibration, which will give the same answer. In this case Gross-Siebert program is rather easy to run since all the walls are parallel and so they do not interact with each other. Nevertheless, since the correspondence between tropical and symplectic geometry in the open sector is still conjectural, this paper takes the more geometric approach based on symplectic geometry.

There are several interesting related works to the author's knowledge. The work by Castano-Bernard and Matessi [6] gave a detailed treatment of Lagrangian fibrations and affine geometry of the base under conifold transitions, and studied mirror symmetry in terms of tropical geometry along the line of Gross-Siebert. For A_n -type surface singularities and the three-dimensional local conifold singularity $\{xy = zw\}$, SYZ construction by tropical wall-crossing was studied by Chan-Pomerleano-Ueda [7, 14, 13]. Auroux [4, 5] studied wall-crossing of open Gromov-Witten invariants and the SYZ mirror of the complement of an anti-canonical divisor and gave several beautiful and illustrative examples. Abouzaid-Auroux-Katzarkov [1] gave a beautiful treatment of SYZ for blowups of toric varieties which is useful for studying mirror symmetry for hypersurfaces in toric varieties. Relation between open Gromov-Witten invariants of the Hirzebruch surface \mathbb{F}_2 and its conifold transition was studied by Fukaya-Oh-Ohta-Ono [18] with an emphasis on non-displaceable Lagrangian tori (which are not fibers).

A related story in the Fano setting was studied by Akhtar-Coates-Galkin-Kasprzyk [2], where they considered Minkowski polynomials and mutations in relation with mirror symmetry for Fano manifolds. This paper works with local Calabi-Yau manifolds instead, and stresses more on open Gromov-Witten invariants and SYZ constructions.

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2. SMOOTHING OF TORIC GORENSTEIN SINGULARITY BY MINKOWSKI DECOMPOSITION

Let N be a lattice, M be the dual lattice, and $\underline{\nu} \in M$ be a primitive vector. Let P be a lattice polytope in the affine hyperplane $\{v \in N_{\mathbb{R}} : \underline{\nu}(v) = 1\}$. Then $\sigma = \text{Cone}(P) \subset N_{\mathbb{R}}$ is a Gorenstein cone. We denote by m the number of corners of the polytope P , and by \tilde{m} the number of lattice points contained in the (closed) polytope P . The corresponding toric variety $X = X_{\sigma}$ using σ as the fan is a toric Gorenstein singularity. We assume that the singularity is an isolated point.

We choose a lattice point v_0 in the (closed) lattice polytope P to translate it to a polytope in the hyperplane $\underline{\nu}_{\mathbb{R}}^{\perp} \subset N_{\mathbb{R}}$, and by abuse of notation we still denote this by P .

By Altmann [3], from a Minkowski decomposition

$$P = R_0 + R_1 + \dots + R_p$$

where R_i 's are convex subsets in $\underline{\nu}_{\mathbb{R}}^{\perp}$ and $p \in \mathbb{N}$, one obtains a (partial) smoothing of X as follows. Let $\hat{N} := (\underline{\nu}^{\perp}) \oplus \mathbb{Z}^{p+1}$. Define $\hat{\sigma}$ to be the cone over the convex hull of

$$\bigcup_{i=0}^p (R_i \times \{e_i\})$$

where $\{e_i\}_{i=0}^p$ is the standard basis of \mathbb{Z}^{p+1} . The total space of the family is $\mathcal{X} = X_{\hat{\sigma}}$, and \mathcal{X}_t is defined as fibers of

$$[-t_0, -t_1, \dots, -t_p] = [0, t_0 - t_1, \dots, t_0 - t_p] : \mathcal{X} = X_{\hat{\sigma}} \rightarrow \mathbb{C}^{p+1}/\mathbb{C}\langle(1, \dots, 1)\rangle$$

with $\mathcal{X}_0 = X$. Here t_0, \dots, t_p are functions corresponding to $(0, \check{e}_i) \in \hat{M}$, where \hat{M} is the dual lattice to \hat{N} and $\{\check{e}_i\}_{i=0}^p$ is dual to the standard basis $\{e_i\}_{i=0}^p$ of \mathbb{Z}^{p+1} .

The total space $\mathcal{X} = X_{\hat{\sigma}}$ of the family is a toric variety, but a generic member \mathcal{X}_t is *not* toric. Each member \mathcal{X}_t is equipped with a Kähler structure ω_t induced from \mathcal{X} and a holomorphic volume form $\underline{\Omega}_t := (\iota_{t_1} \dots \iota_{t_p}) \hat{\Omega}|_{\mathcal{X}_t}$, where

$$\hat{\Omega} = d \log z_0 \wedge \dots \wedge d \log z_{n-1} \wedge dt_0 \wedge \dots \wedge dt_p$$

is the volume form on \mathcal{X} .

From now on we assume that \mathcal{X}_t is smooth for generic t . In particular we assume that every summand R_i is a unimodular k_i -simplex for some $k_i = 1, \dots, n-1$.¹ The vertices of R_i are $0, u_1^i, \dots, u_{k_i}^i$ for some lattice points $u_j^i \in \underline{\nu}^{\perp}$.

3. GROSS FIBRATION AND ITS VARIANTS

Let us fix the smoothing parameter $c = [c_0, c_1, \dots, c_p] \in \mathbb{C}^{p+1}/\mathbb{C}\langle(1, \dots, 1)\rangle$ from now on, where $c_0 = 0, c_1, \dots, c_p \in \mathbb{C}$ are taken to be distinct constants, and consider \mathcal{X}_c which is assumed to be smooth. By relabelling c_i 's if necessary, we assume $0 = |c_0| \leq |c_1| \leq \dots \leq |c_p|$. The functions t_i 's restricted to \mathcal{X}_c equals to $t - c_i$, where $t := t_0$.

Gross [19] constructed a proper special Lagrangian fibration on \mathcal{X}_c , which is

$$\pi^K := (\pi_0, |t - K|^2 - K^2) : \mathcal{X}_c \rightarrow B$$

where

$$B = \frac{M_{\mathbb{R}}}{\mathbb{R}\langle\underline{\nu}\rangle} \times \mathbb{R}_{\geq -K^2},$$

$K \in \mathbb{R}_{>0}$, π_0 is the moment map of the action $T^{\perp \underline{\nu}}$ on $(\mathcal{X}_c, \omega_c, \Omega_c)$. Here ω_c is the symplectic form restricted from \mathcal{X} to \mathcal{X}_c , and

$$(3.1) \quad \Omega_c = \iota_{t_1} \dots \iota_{t_p} \hat{\Omega}|_{\mathcal{X}_c} = d \log z_0 \wedge \dots \wedge d \log z_{n-1} \wedge d \log(t - K)$$

and

$$(3.2) \quad \hat{\Omega} = d \log z_0 \wedge \dots \wedge d \log z_{n-1} \wedge d \log(t_0 - K) \wedge dt_1 \wedge \dots \wedge dt_p$$

¹Here a k -simplex with one of its vertex being 0 is said to be unimodular if its vertices generates all the lattice points in the k -plane containing the simplex.

on \mathcal{X} , which are *different* from the ones we described in the end of Section 2. Ω_c is nowhere zero and has a simple pole on the boundary divisor $D_0 = \{t = K\} \subset \mathcal{X}_c$. As K tends to ∞ , π^K tends to the fibration

$$\pi^\infty := (\pi_0, \operatorname{Re} t) : \mathcal{X}_c \rightarrow B$$

which is no longer proper, and we will not use π^∞ in this paper.²

By Proposition 3.3 of [19], away from the boundary ∂B , the discriminant loci have codimension two, and they lie in the hyperplanes

$$H_i = \frac{M_{\mathbb{R}}}{\mathbb{R}\langle \mathcal{L} \rangle} \times \{|c_i - K|^2 - K^2\}.$$

The level sets of $|t - K|^2 - K^2$ gives a fibration of the complex t -plane over $\mathbb{R}_{\geq 0}$ by circles. For a singular fiber of π^K , its image under t is a circle centered at K hitting one of the c_i 's. See Figure 1.

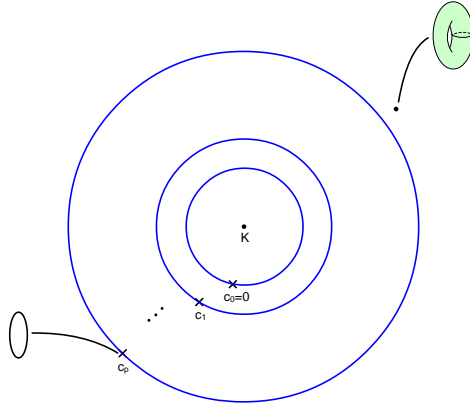


FIGURE 1. The t plane. The image of a fiber of π^K is a circle centered at K , and that of a singular fiber hits one of the c_i 's. t defines a fibration over $\pi_0^{-1}\{a\}$, where the fiber at each point other than c_i 's can be identified as the torus $T^{\perp \nu}$. At c_i 's the $T^{\perp \nu}$ -orbit can degenerate to a smaller torus.

The boundary divisor $\pi^{-1}(\partial B) = \{t = K\}$ is denoted as D_0 , and the discriminant loci supported in H_i are denoted as Γ_i . For a generic choice of K , $|c_i - K|$'s are pairwise distinct for $i = 0, \dots, p$ and so all the hyperplanes H_i 's and ∂B are disjoint. In the rest of this paper we assume such a choice of K .

We can deform the Lagrangian fibration π^K and make all walls H_i collide into one: deform the metric to $d(\cdot, \cdot)$ on the complex t plane from the standard norm $|\cdot|$, so that all c_i 's are of the same distance to K for $i = 0, \dots, p$. Then the Lagrangian fibration $d(\cdot, K)^2 - K^2$ on the t -plane pulls back to a Lagrangian fibration $\underline{\pi}$ on \mathcal{X}_t . Since all c_i 's lie in the same fiber, the discriminant locus away from the boundary ∂B lies in a single hyperplane H . The fibers are no longer special with respect to Ω_c , but this does not hurt since we are only concerned with symplectic geometry. Interesting singular Lagrangian fibers arise for the fibration $\underline{\pi}$. We will

²On the side of toric resolutions, the fibration π^∞ and the related open Gromov-Witten invariants in dimension three has been well-studied by the theory of topological vertex [23].

study the disc potential of a Clifford torus fiber T (see Definition 4.15 and Corollary 4.16) of $\underline{\pi}$ and its behavior under conifold transitions (Theorem 5.2).

For the purpose of SYZ construction along the line of [10], we consider the following compactification of \mathcal{X}_c . Add the ray generated by $-e_0 - \dots - e_p$ to the fan $\hat{\sigma}$ and corresponding cones to produce a complete fan $\bar{\sigma}$. Then consider the corresponding toric variety $\bar{\mathcal{X}} = X_{\bar{\sigma}}$ and fibers $\bar{\mathcal{X}}_c \subset \bar{\mathcal{X}}$ of $(t_0 - t_1, \dots, t_0 - t_p)$ as before, which are preserved under the action of $T^{\perp \underline{\nu}}$. The Gross fibration has the same definition $\bar{\pi}^K = (\pi_0, |t - K|^2 - K^2)$ on $\bar{\mathcal{X}}_c$, which has boundary divisor $D_0 = \{t = K\}$ and $D_\infty = \{t = \infty\}$ (the pole divisor of t). The meromorphic volume form Ω_c defined by the same expression (3.1) has simple poles along D_0 and D_∞ .

4. SYZ OF SMOOTHINGS OF TORIC GORENSTEIN SINGULARITIES

Here we follow the construction of [10] to obtain the SYZ mirror from disc countings. Adapted to this case, the procedures are:

- (1) Define the semi-flat mirror to be

$$\check{X}_0 = \{\text{flat } U(1) \text{ connections } \nabla \text{ on a fiber of } \bar{\pi}^K \text{ over } r \in B_0\}$$

where $B_0 = B - \Gamma$, Γ is the discriminant locus. We have the bundle map $\tilde{\pi} : \check{X}_0 \rightarrow B_0$. We have semi-flat complex coordinates on \check{X}_0 . Each $v \in \underline{\nu}^\perp$ corresponds to a coordinate z^v on \check{X}_0 which is monodromy-invariant.

- (2) Let $H \subset B_0$ be the wall (see Definition 4.3). Define the generating functions of open Gromov-Witten invariants (see Definition 4.4)

$$(4.1) \quad \mathbf{u}(r, \nabla) = \sum_{\beta \cdot D_0 = 1} n_\beta \exp\left(-\int_\beta \omega\right) \text{Hol}_\nabla(\partial\beta)$$

and

$$(4.2) \quad \mathbf{v}(r, \nabla) = \sum_{\beta \cdot D_\infty = 1} n_\beta \exp\left(-\int_\beta \omega\right) \text{Hol}_\nabla(\partial\beta)$$

on the semi-flat mirror $\tilde{\pi}^{-1}(B_0 - H)$ associated to the boundary divisors D_0 and D_∞ . They serve as quantum-corrected complex coordinates.

- (3) Take R to be the ring of functions on $\tilde{\pi}^{-1}(B_0 - H)$ generated by \mathbf{u}, \mathbf{v} and z^v for $v \in \underline{\nu}^\perp$. Then the SYZ mirror is defined to be $\text{Spec}(R)$.

The readers are referred to Section 2 of [10] for more details.

Since π^K is a special Lagrangian fibration with respect to Ω_c which has simple poles along D_0 (and $\bar{\pi}^K$ is special Lagrangian with respect to Ω_c which has simple poles along D_0 and D_∞), by [4] we have the following formula for the Maslov index of a disc class:

Lemma 4.1 (Lemma 3.1 of [4]). *The Maslov index of $\beta \in \pi_2(\mathcal{X}_c, F_r)$, where F_r is a fiber of π^K over $r \in B$, is $\mu(\beta) = 2\beta \cdot D_0$. Similarly the Maslov index of $\beta \in \pi_2(\bar{\mathcal{X}}_c, F_r)$, where F_r is a fiber of $\bar{\pi}^K$, is $\mu(\beta) = 2\beta \cdot (D_0 + D_\infty)$.*

In the following lemma, we see that there is no holomorphic sphere in \mathcal{X}_c (or $\bar{\mathcal{X}}_c$). This simplifies a lot the moduli theory for open Gromov-Witten invariants because no sphere bubbling occurs.

Lemma 4.2. *There is no non-constant holomorphic map from \mathbb{P}^1 to \mathcal{X}_c (or $\bar{\mathcal{X}}_c$).*

Proof. Suppose $u : \mathbb{P}^1 \rightarrow \mathcal{X}_c$ (or $u : \mathbb{P}^1 \rightarrow \bar{\mathcal{X}}_c$) is a holomorphic map. Then $t \circ u$ is a holomorphic function on \mathbb{P}^1 which can only be constant. Thus the image of u lies in a fiber $t^{-1}\{a\}$ for some $a \in \mathbb{C}$. But $t^{-1}\{a\}$ is just a product of $(\mathbb{C}^\times)^k$ with $\{(z_1, \dots, z_{n-k}) \in \mathbb{C}^{n-k} : z_1 \dots z_{n-k} = 0\}$, and so $H_2(t^{-1}\{a\}) = 0$. Hence u can only be a constant. \square

For each i , we may identify H_i with $M_{\mathbb{R}}/\mathbb{R}\langle \underline{\nu} \rangle$. Then the discriminant locus $\Gamma_i \subset H_i$ can be identified as the normal fan of the simplex R_i . Thus $H_i - \Gamma_i$ consists of $k_i + 1$ components which will be referred as chambers in H_i . The chambers are one-to-one corresponding to the vertices of the standard simplex R_i . Each chamber is (up to translation) the dual cone of the cone $(R_i)_v$ obtained by localizing the simplex R_i to its vertex v corresponding to the chamber.

From now on we assume that $|c_{i+1} - K| \geq |c_i - K|$ for all $i = 0, \dots, p-1$, and $K > 0$ is chosen generically such that all the hyperplanes H_i for $i = 1, \dots, p$ are distinct.

To obtain a local trivialization of the torus bundle $(\pi^K)^{-1}(B_0) \rightarrow B_0$, we fix a chamber C_i in H_i for each i . Without loss of generality we fix C_i to be the chamber corresponding to the vertex 0 of the simplex R_i . The vectors normal to the facets of the chamber C_i are $u_1^i, \dots, u_{k_i}^i \in \underline{\nu}^\perp$. Define the open subset

$$U = B_0 - \bigcup_{i=0}^p (H_i - C_i) \subset B_0$$

which is contractible. Thus $(\pi^K)^{-1}(U)$ over U is a trivial torus bundle, and every fiber can be identified as the fiber at a chosen point $p \in U$. We always stick with this local trivialization in the rest of the paper.

Now we consider open Gromov-Witten invariants bounded by a fiber of π^K above points in $U \subset B$. The invariants are well-defined when the fiber has minimal Maslov index two. Recall the following definition of wall from [10]:

Definition 4.3 (Wall). *The wall H of the Lagrangian fibration $\pi^K : \mathcal{X}_t \rightarrow B$ (or $\bar{\pi}^K$) is the set*

$$\{r \in B_0 : \text{the fiber } F_r \text{ bounds non-constant holomorphic discs with Maslov index zero}\}.$$

Away from the wall, the one-pointed genus-zero open Gromov-Witten invariants are well-defined, and we can use them to construct the SYZ mirror.

Definition 4.4 (Open Gromov-Witten invariants). *Let $r \in B_0 - H$ be away from wall H , and $\beta \in \pi_2(\bar{\mathcal{X}}_t, F_r)$ a disc class bounded by the fiber F_r . We have the moduli space $\mathcal{M}_1(\beta)$ of stable discs with one boundary marked point representing β . The open Gromov-Witten invariant associated to β is*

$$n_\beta = \int_{\mathcal{M}_1(\beta)} \text{ev}^*[\text{pt}]$$

where $\text{ev} : \mathcal{M}_1(\beta) \rightarrow F_r$ is the evaluation map at the boundary marked point.

The open Gromov-Witten invariant n_β is non-zero only when the Maslov index $\mu(\beta)$ is two. Moreover, by Lemma 4.2, away from the wall there is no disc bubbling and sphere bubbling

in the moduli. Thus $\mathcal{M}_1(\beta)$ consists of (classes of) holomorphic maps $(\Delta, \partial\Delta) \rightarrow (X, F_r)$. No virtual perturbation theory is needed in this situation.

For the Lagrangian fibration π^K , the wall is a union of disjoint hyperplanes as stated in the following proposition.

Proposition 4.5. *The wall of the Lagrangian fibration $\pi^K : \mathcal{X}_t \rightarrow B$ (or $\bar{\pi}^K$) is the union of hyperplanes H_i for $i = 0, \dots, p$, which are disjoint from each other.*

Proof. Suppose the fiber F_r bounds a non-constant holomorphic disc $u : (\Delta, \partial\Delta) \rightarrow (\mathcal{X}_c, L)$ with Maslov index zero. By the Maslov index formula (Lemma 4.1), the disc does not hit the divisor D_0 (nor D_∞). This means the function $(t - K) \circ u$ is never zero (and never equals to infinity). Since $|t - K| \circ u$ is constant on $\partial\Delta$, by the maximal principle it follows that t is constant on the disc. Thus the disc u lies in the fiber of t . $t^{-1}\{a\}$ for $a \neq 0, c_1, \dots, c_p$ is isomorphic to $(\mathbb{C}^\times)^{n-1}$, and $F_r \cap t^{-1}\{a\}$ is a moment-map fiber of $(\mathbb{C}^\times)^{n-1}$, which cannot bound any non-constant holomorphic disc. Hence we have $t = 0, c_1, \dots, c_p$. This implies r lies in one of the hyperplanes H_i 's. \square

We need to identify all the disc classes in order to compute their open Gromov-Witten invariants. The next definition gives a label to every basic disc class.

Definition 4.6 (Disc classes). *Let F_r be a fiber of the Lagrangian fibration $\bar{\pi}^K$ contained in the trivialization $(\bar{\pi}^K)^{-1}(U)$. The disc class of Maslov index two emanated from the boundary divisor D_0 (or D_∞) is denoted as β_0 (or β_∞ respectively). Moreover, each discriminant locus $\Gamma_i \subset H_i$ gives rise to k_i disc classes of Maslov index zero, which are in one-to-one correspondence with the normal vectors $u_1^i, \dots, u_{k_i}^i \in \underline{\nu}^\perp$ to the facets of the chamber, and they are denoted as β_j^i for $j = 1, \dots, k_i$.*

The disc class β_0 gives a local coordinate function z_0 on the semi-flat mirror $\check{\pi}^{-1}(U) \subset \check{X}_0$ (which is not monodromy invariant):

$$z_0 := \exp\left(-\int_\beta \omega\right) \text{Hol}_\nabla(\partial\beta).$$

All stable disc classes bounded by F_r of Maslov index two must be of the form

$$\beta_0 + \sum_{i=0}^p \sum_{j=1}^{k_i} n_j^i \beta_j^i \quad \text{or} \quad \beta_\infty + \sum_{i=0}^p \sum_{j=1}^{k_i} n_j^i \beta_j^i$$

where $n_j^i \in \mathbb{Z}_{\geq 0}$. In the following theorem we classify all the stable disc classes of Maslov index two and compute their open Gromov-Witten invariants. In the toric case, an analogous result is obtained by [15]. Since here the symplectic manifold \mathcal{X}_t (or $\check{\mathcal{X}}_t$) is non-toric and the Lagrangian-fibration π^K (or $\bar{\pi}^K$) has interior discriminant loci, we need a non-trivial argument here.

Theorem 4.7. *Let r be a regular value between the walls H_l and H_{l+1} of the Lagrangian fibration π^K , and let*

$$\beta_0 + \sum_{i=0}^p \sum_{j=1}^{k_i} n_j^i \beta_j^i$$

be a disc class which has Maslov index two bounded by F_r . Then the moduli space $\mathcal{M}_1(\beta)$ is non-empty only when $n_j^i = 0$ for all i, j except for each $0 \leq i \leq l$, there could be at most one $j = 1, \dots, k_i$ with $n_j^i = 1$. In such a case the open Gromov-Witten invariant is $n_\beta = 1$.

For the Lagrangian fibration $\bar{\pi}^K$, we have exactly the same statement for the disc class $\beta_0 + \sum_{i=0}^p \sum_{j=1}^{k_i} n_j^i \beta_j^i$ stated above. For the disc class

$$\beta_\infty + \sum_{i=0}^p \sum_{j=1}^{k_i} n_j^i \beta_j^i$$

emanated from the boundary divisor D_∞ , the moduli space $\mathcal{M}_1(\beta)$ is non-empty only when $n_j^i = 0$ for all i, j except for each $l+1 \leq i \leq p$, there could be at most one $j = 1, \dots, k_i$ with $n_j^i = 1$. In such a case the open Gromov-Witten invariant is $n_\beta = 1$.

Proof. First consider the disc class $\beta_0 + \sum_{i=0}^p \sum_{j=1}^{k_i} n_j^i \beta_j^i$. The analysis for the disc class $\beta_\infty + \sum_{i=0}^p \sum_{j=1}^{k_i} n_j^i \beta_j^i$ is similar.

Since the disc class has zero intersection with D_∞ , It follows from the maximal principle on $(t - K) \circ u$ that a disc u bounded by F_r must have $|t - K|^2 \leq K^2 + r_n$ (where r_n is the last coordinate entry of r), and hence never hits $(\bar{\pi}^K)^{-1}(H_i)$ for $i > l$. Thus the moduli space $\mathcal{M}_1(\beta)$ is non-empty only when $n_j^i = 0$ for all $i > l$.

Since there is no holomorphic sphere in \mathcal{X}_c (Lemma 4.2), no sphere bubbling occurs in the moduli. Also there is no disc bubbling because the fiber under consideration is not lying over the walls so that minimal Maslov index of non-constant holomorphic discs is two. Thus elements in $\mathcal{M}_1(\beta)$ are holomorphic maps $u : (\Delta, \partial\Delta) \rightarrow (\mathcal{X}_t, F_r)$. Using local coordinates $(\zeta_1, \dots, \zeta_{n-1}, t)$ of \mathcal{X}_c (where $\zeta_1, \dots, \zeta_{n-1}$ corresponds to a basis of $M/\mathbb{Z}\langle \underline{\nu} \rangle$), the map u can be written as $(\zeta_1(z), \dots, \zeta_{n-1}(z), t(z))$ for $z \in \Delta$. Moreover F_r is defined by the equations

$$|\zeta_1|^2 = r_1, \dots, |\zeta_{n-1}|^2 = r_{n-1}, |t - K|^2 = K^2 + r_n.$$

Consider the holomorphic function $t \circ u$ on the disc Δ . Since u represents $\beta_0 + \sum_{i=0}^p \sum_{j=1}^{k_i} n_j^i \beta_j^i$, the winding number of $t \circ u$ around K is 1. Since r is lying between the walls H_l and H_{l+1} , the winding numbers of $t \circ u$ around $c_0 = 0$ and c_1, \dots, c_l are also 1. Thus $t \circ u$ attains each value c_0, \dots, c_l exactly once. On the other hand, u hits the critical fibers over Γ_i for $\sum_{j=1}^{k_i} n_j^i$ times (counting with multiplicity), and so $t \circ u$ attains the value c_i for at least $\sum_{j=1}^{k_i} n_j^i$ times. This forces n_j^i is either zero or one, and among $n_1^i, \dots, n_{k_i}^i$'s at most one of them is one for each i .

Suppose this is the case. Then we equate $\mathcal{M}_1(\beta)$ and $\mathcal{M}_1(\beta_0)$ as follows. Let $u : (\Delta, \partial\Delta) \rightarrow (\mathcal{X}_t, F_r)$ represent the class β . For $i = 0$, extend the vertices $u_1^0, \dots, u_{k_0}^0$ of the polytope R_0 to a basis of $\underline{\nu}^\perp$ and let $\zeta_1, \dots, \zeta_{n-1}$ be functions corresponding to the dual basis of $M/\mathbb{Z}\langle \underline{\nu} \rangle$. u can be written as $(\zeta_1(z), \dots, \zeta_{n-1}(z), t(z))$ in terms of these coordinates. If $n_j^0 = 1$ for a certain $j = 1, \dots, k_0$ (otherwise we do nothing), then there exists $z_0 \in \Delta^\circ$ such that $\zeta_j(z_0) = 0$ and $t(z_0) = 0$. Define

$$\tilde{u}(z) = \left(\zeta_1(z), \dots, \left(\frac{z - z_0}{1 + \bar{z}_0 z} \right)^{-1} \zeta_j(z), \dots, \zeta_{n-1}(z), t(z) \right).$$

Then \tilde{u} does not hit any singular fibers at Γ_0 and hence belongs to the moduli space $\mathcal{M}_1(\beta_0 + \sum_{i=1}^p \sum_{j=1}^{k_i} n_j^i \beta_j^i)$. Conversely let $\tilde{u} : (\Delta, \partial\Delta) \rightarrow (\mathcal{X}_t, F_r)$ represent the class $\beta_0 + \sum_{i=1}^p \sum_{j=1}^{k_i} n_j^i \beta_j^i$. By the above analysis on winding number of t , there exists exactly one $z_0 \in \Delta^\circ$ such that $t(z_0) = 0$. Then define

$$u(z) = \left(\zeta_1(z), \dots, \left(\frac{z - z_0}{1 + \bar{z}_0 z} \right) \zeta_j(z), \dots, \zeta_{n-1}(z), t(z) \right)$$

which belongs to $\mathcal{M}_1(\beta)$. This set up an isomorphism

$$\mathcal{M}_1(\beta) \cong \mathcal{M}_1 \left(\beta_0 + \sum_{i=1}^p \sum_{j=1}^{k_i} n_j^i \beta_j^i \right).$$

Doing the same thing for $i = 1, \dots, p$ inductively, we obtain an isomorphism between $\mathcal{M}_1(\beta)$ and $\mathcal{M}_1(\beta_0)$.

Now for the class β_0 , by the maximum principle any holomorphic map in β_0 have coordinates z_1, \dots, z_{n-1} being constants. The moduli space $\mathcal{M}_1(\beta_0)$ does not have disc bubbling even for a Lagrangian fiber at the walls H_i 's, and thus it remains to be the same under wall-crossing. This means $\mathcal{M}_1(\beta_0)$ for F_r between the walls H_l and H_{l+1} is the same as that for a fiber below the wall H_0 , which is the same as a toric fiber in $\mathbb{C} \times (\mathbb{C}^\times)^{n-1}$. it follows that $n_{\beta_0} = 1$. Since $\mathcal{M}_1(\beta) \cong \mathcal{M}_1(\beta_0)$, we also have $n_\beta = 1$.

For $\beta = \beta_\infty + \sum_{i=0}^p \sum_{j=1}^{k_i} n_j^i \beta_j^i$, we use the winding number of t around ∞ instead of around K , and use the same argument to deduce the statement. \square

Now we are ready to compute the SYZ mirror of $\pi : \mathcal{X}_t \rightarrow B$.

Theorem 4.8. *By the construction defined in the beginning of this section, the SYZ mirror of the Lagrangian fibration $\pi : \mathcal{X}_t \rightarrow B$ is*

$$\check{\mathcal{X}} = \{((\mathbf{u}, \mathbf{v}), z_1, \dots, z_{n-1}) \in \mathbb{C}^2 \times (\mathbb{C}^*)^{n-1} : \mathbf{u}\mathbf{v} = g(z)\}$$

where

$$\begin{aligned} g(z) &= \prod_{i=0}^p \left(1 + \sum_{l=1}^{k_i} z^{u_l^i} \right) \\ &= \sum_{v \in P \cap \underline{\nu}^\perp} n_v z^v \end{aligned}$$

for some explicit positive integers n_v attached to each $v \in P \cap \underline{\nu}^\perp$, and $n_v = 1$ for all vertices v of P . Notice that this is independent of the deformation parameter t .

Proof. By Proposition 4.5, the wall H is a union of $(p+1)$ parallel hyperplanes H_k for $k = 0, \dots, p$. Hence $B_0 - H$ consists of $(p+2)$ components denoted by V_k for $k = -1, \dots, p$, where the hyperplane H_k is a boundary of V_{k-1} and also V_k .

Recall that $\mathbf{u}|_{\pi^{-1}(V_k)}$ is the generating function of open Gromov-Witten invariants n_β of disc classes with $\beta \cdot D_0 = 1$ and $\beta \cdot D_\infty = 0$, bounded by a Lagrangian torus fiber at a point

in V_k (see Equation (4.1)). Such disc classes must take the form

$$\beta = \beta_0 + \sum_{i=0}^p \sum_{j=1}^{k_i} n_j^i \beta_j^i$$

which contributes to the monomial

$$z_0 \prod_{i=0}^p \prod_{j=1}^{k_i} z^{n_j^i u_j^i}.$$

By Theorem 4.7, n_β is either 0 or 1, and it equals to 1 exactly when all $n_j^i = 0$ except for each $i \leq k$ there could be at most one j with $n_j^i = 1$. It means only the monomials

$$z_0 \prod_{i=0}^k z^{u_{j_i}^i}$$

are present in the generating function $\mathbf{u}|_{\pi^{-1}(V_k)}$, where $j_i \in \{0, \dots, k_i\}$ and $u_0^i = 0$, and the coefficient of each of these monomials is 1. Hence

$$\mathbf{u}|_{\pi^{-1}(V_k)} = z_0 \prod_{i=0}^k \left(1 + \sum_{l=1}^{k_i} z^{u_l^i} \right).$$

Similarly $\mathbf{v}|_{\pi^{-1}(V_k)}$ is the generating function of open Gromov-Witten invariants n_β of disc classes with $\beta \cdot D_\infty = 1$ and $\beta \cdot D_0 = 0$ (see Equation (4.2)). Such disc classes must take the form

$$\beta = \beta_\infty + \sum_{i=0}^p \sum_{j=1}^{k_i} n_j^i \beta_j^i$$

which contributes to the monomial

$$z_0^{-1} \prod_{i=0}^p \prod_{j=1}^{k_i} z^{n_j^i u_j^i}.$$

By Theorem 4.7, n_β is either 0 or 1, and it equals to 1 exactly when all $n_j^i = 0$ except for each $i > k$ there could be at most one j with $n_j^i = 1$. It means only the monomials

$$z_0^{-1} \prod_{i=k+1}^p z^{u_{j_i}^i}$$

are present in the generating function $\mathbf{v}|_{\pi^{-1}(V_k)}$, where $j_i \in \{0, \dots, k_i\}$ and $u_0^i = 0$, and the coefficient of each of these monomials is 1. Hence

$$\mathbf{v}|_{\pi^{-1}(V_k)} = z_0^{-1} \prod_{i=k+1}^p \left(1 + \sum_{l=1}^{k_i} z^{u_l^i} \right).$$

It follows that we have the relation

$$\mathbf{u}\mathbf{v} = \prod_{i=0}^p \left(1 + \sum_{l=1}^{k_i} z^{u_l^i} \right)$$

for the functions $\mathbf{u}, \mathbf{v}, z_1, \dots, z_{n-1}$ on $B_0 - H$. Thus $\check{\mathcal{X}} = \text{Spec}(R)$, where R is the subring of functions generated by $\mathbf{u}, \mathbf{v}, z_1, \dots, z_{n-1}$, is the subvariety defined by the above equation. \square

Remark 4.9. *The above calculation of open Gromov-Witten invariants matches with the expectation from Gross-Siebert program [20] (which reconstructs the mirror manifold from tropical geometry instead of directly using symplectic geometry): the wall-crossing function attached to each wall H_i is*

$$1 + \sum_{l=1}^{k_i} z^{u_i^l}$$

and the coefficient of z^v for $v \in \underline{v}^\perp$ in $\prod_{i=0}^p \left(1 + \sum_{l=1}^{k_i} z^{u_i^l}\right)$ equals to $\sum_{\beta'} n_{\beta_0 + \beta'}$ where the summation is over all holomorphic disc classes β' of Maslov index zero with $\partial\beta' = v$.

Remark 4.10. *All the complex geometric information of $\check{\mathcal{X}}$ is recorded by the hypersurface $g(z) = 0$ in $(\mathbb{C}^*)^{n-1}$. Components of the hypersurface, which are products of $(k-1)$ -dimensional pair-of-pants with $(\mathbb{C}^*)^{n-k}$, are one-to-one correspond to discriminant loci of π^K in the hyperplanes H_i 's. It matches with the philosophy of T-duality that big torus is dual to small torus and vice versa.*

Remark 4.11. *We may also consider the dual picture of Minkowski decomposition, namely, decomposition of the dual fan of the polytope. By considering the pair-of-pant factor of each component of the hypersurface $g(z) = 0$ and taking the tropical limit, we obtain a fan (which is standard simplicial of possibly lower dimension). The union of all these fans recovers the dual fan of the original polytope. See Figure 11 in Example 11.*

Remark 4.12. *In the above expression of mirror, only the integer coefficients of g depend on the Minkowski decomposition that we start with. Geometrically it means different smoothings of \mathcal{X} correspond to local patches of different limit points of the same stringy Kähler moduli.*

As a consequence, the open Gromov-Witten invariants of F_r for r above all the walls H_i are non-zero only for β_v where v is a lattice point in P . This is a non-trivial consequence because a priori all stable disc classes of Maslov index two

$$\beta_0 + \sum_{i=0}^p \sum_{j=1}^{k_i} n_j^i \beta_j^i$$

may have non-trivial open Gromov-Witten invariants.

Now consider the Lagrangian fibration $\underline{\pi}$ (defined in Section 3) and its disc potential. There is only one wall and open Gromov-Witten invariants are well-defined away from this wall. We follow the terminologies of [4, Definition 5.1] and make the following analogous definition:

Definition 4.13 (Clifford and Chekanov tori). *A regular fiber T of the Lagrangian fibration $\underline{\pi}$ is called to be a Clifford torus if its based point $r = \underline{\pi}(T)$ is above the wall. It is called to be a Chekanov torus if its based point is below the wall.*

[4] studied an analogous Lagrangian fibration for \mathbb{P}^2 , where there is a wall in the base. A fiber above the wall is called to be a Clifford torus, which is Hamiltonian isotopic to a regular moment-map fiber; a fiber below the wall is called to be a Chekanov torus, which is not a moment-map fiber up to Hamiltonian isotopy.

A Chekanov torus bounds only one class of holomorphic discs, namely β_0 . Thus the disc potential (Definition 4.15) for Chekanov tori is simply z_0 , which is not quite interesting. On the other hand, the disc potential for Clifford tori is an interesting object to study.

There exists a Lagrangian isotopy between the fiber F_r of π^K at $r = (b, a) \in B$ for $a \gg 0$ and a product torus fiber T of $\underline{\pi}$, such that each member in the isotopy never bounds holomorphic discs of Maslov index zero. Thus their open Gromov-Witten invariants are equal:

$$n_{\beta}^{F_r} = n_{\beta}^T$$

where $\beta \in \pi_2(X, T)$ is identified as a disc class in $\pi_2(X, F_r)$ by this isotopy. As a consequence,

Corollary 4.14. *Let T be a product torus fiber of $\underline{\pi}$ and $\beta \in \pi_2(\mathcal{X}_c, T)$. Write*

$$\beta = \beta_0 + \sum_{i=0}^p \sum_{j=1}^{k_i} n_j^i \beta_j^i.$$

Then $n_{\beta}^T = 1$ when n_j^i is either zero or one and for every i there is at most one j such that $n_j^i = 1$. Otherwise $n_{\beta}^T = 0$.

We also consider the disc potential W^T for a product torus fiber T . In Section 5 we will see that W^T can be obtained from the toric disc potential of the conifold transition of \mathcal{X}_t .

Definition 4.15 (Disc potential). *Let T be a product torus fiber of $\underline{\pi}$. The disc potential is defined as*

$$(4.3) \quad W^T = \sum_{\beta \in \pi_2(\mathcal{X}_c, T)} n_{\beta} \exp\left(-\int_{\beta} \omega\right) \text{Hol}_{\nabla}(\partial\beta).$$

By Corollary 4.14, we have

Corollary 4.16. *Let T be a product torus fiber of $\underline{\pi}$ and $\beta \in \pi_2(\mathcal{X}_c, T)$. Then the disc potential for T is*

$$W^T = z_0 \prod_{i=0}^p \left(1 + \sum_{l=1}^{k_i} z^{u_l^i}\right) = z_0 \sum_{v \in P \cap N} n_v z^v$$

where n_v are the same integers as in Theorem 4.8.

Remark 4.17. *If we consider the singular toric variety \mathcal{X}_0 , there are m basic disc classes corresponding to vertices of the polytope. This gives the Laurent polynomial*

$$z_0 \sum_{v: \text{vertex of } P} z^v$$

which is NOT equal to W^T in general.

5. LOCAL CONIFOLD TRANSITIONS

Instead of taking Minkowski decompositions of the polytope P to obtain smoothings of toric Calabi-Yau Gorenstein singularities, one can instead consider triangulations of P giving rise to a fan Σ which corresponds to a toric resolution $Y = Y_{\Sigma}$, and construct the mirror of Y via SYZ. This has been done in [10]. The procedure of degenerating \mathcal{X}_t to a toric Gorenstein

canonical singularity \mathcal{X}_0 and taking toric resolution Y is called conifold transition. On both sides the geometry of Lagrangian fibrations has been explained beautifully in [19].

String theorists expect that quantum geometry undergoes a smooth deformation under conifold transition from Y to \mathcal{X}_t , even though there are singularities developed in the procedure of varying from Y to \mathcal{X}_t . (See Figure 2 showing a picture of the Kähler moduli.) Since we have computed all the open Gromov-Witten invariants of the smoothing \mathcal{X}_t , and from [8] we also have good understanding on open Gromov-Witten invariants of the toric resolution Y , we can compare their disc potentials (which capture the quantum geometry relative to a torus fiber) and also their SYZ mirrors. The phenomenon is the same as that described by Ruan’s crepant resolution conjecture [26] and also its counterpart in the open sector [9]. The main difference is that now we consider crepant resolution of an isolated Gorenstein toric singularity which is *not of orbifold type*, and its smoothing is *no longer a toric manifold*.

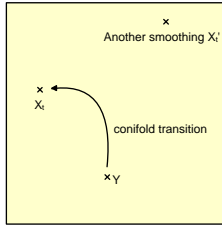


FIGURE 2. The Kähler moduli. For a toric Gorenstein singularity, its resolution Y lies in a neighborhood of a large volume limit. Its smoothing \mathcal{X}_t gives a conifold limit point in the moduli, and different smoothings gives different limit points.

The aim of this section is to deduce Theorem 5.2 which gives a positive response to the string theorists’ expectation. In order to do so, we will use the explicit expression of the disc potential $W^{\mathcal{X}_t}$ given in Corollary 4.16. We also need an explicit expression of the disc potential

$$W^Y(r, \nabla) = \sum_{\beta \in \pi_2(Y, F_r)} n_\beta \exp\left(-\int_\beta \omega\right) \text{Hol}_\nabla(\partial\beta)$$

where F_r is the moment-map fiber at a regular value r and ∇ is any flat $U(1)$ -connection on F_r . W^Y can be expressed in terms of the Kähler parameters q of Y , and it has an explicit expression given by the toric open mirror theorem [8, Theorem 1.5] (which is Conjecture 5.1 of [10]), which states that W_q^Y has a simple combinatorial expression (called the Hori-Vafa mirror) by a change of coordinates $q(\tilde{q})$ called the mirror map. An explicit expression of the mirror map can be found in [11, Definition 3.2]³.

Theorem 5.1 (Toric Open Mirror Theorem [8]). *Let Y be a toric Calabi-Yau manifold arising as a toric resolution of an isolated Gorenstein canonical singularity given by a lattice polytope P in the $(n-1)$ -dimensional affine hyperplane $\{\underline{\nu} = 1\}$. Fix a choice of lattice points $\{v_i\}_{i=1}^n$*

³The paper [11] is about toric semi-Fano toric manifolds rather than toric Calabi-Yau manifolds. Nevertheless the mirror maps have the same combinatorial expressions. The mirror map expressed in [8] is for orbifolds and looks more complicated. Since we will not use the explicit expression of the mirror map $q(\tilde{q})$ in deducing Theorem 5.2, we do not include it here.

in P such that they form a basis of N . For any $v \in (P \cap N) - \{v_i : i = 1, \dots, n\}$, v can be expressed as a linear combination $\sum_{i=1}^n a_i v_i$. Then $\beta_v - \sum_{i=1}^n a_i \beta_i$ gives a curve class in $H_2(Y, \mathbb{Z})$, where β_v and β_i are the basic disc classes corresponding to v and v_i respectively. The corresponding Kähler parameters and mirror complex parameters are denoted as q_v 's and \check{q}_v 's respectively. The mirror map, which expresses q_v 's in terms of \check{q}_v 's, is denoted as $q(\check{q})$.

Then the disc potential W^Y for its regular moment map fibers can be expressed as

$$W^Y(q(\check{q})) = z_0 \left(z^{v_1} + \dots + z^{v_n} + \sum_{v \in (P \cap N) - \{v_i\}_{i=1}^n} \check{q}_v z^v \right)$$

up to an (explicit) affine change of coordinates on $(\mathbb{C}^\times)^n$. Moreover the SYZ mirror \check{Y}_q of Y has the expression

$$\check{Y}_{q(\check{q})} = \left\{ ((u, v), z) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1} : uv = z^{v_1} + \dots + z^{v_n} + \sum_{v \in (P \cap N) - \{v_i\}_{i=1}^n} \check{q}_v z^v \right\}.$$

Now we are ready to deduce that the disc potential undergoes a smooth change under conifold transitions:

Theorem 5.2 (Open Conifold Transition Theorem). *Let Y be a toric resolution of an isolated Gorenstein canonical singularity given by a lattice polytope P , and suppose there exists a Minkowski decomposition of P into standard simplices (of possibly smaller dimensions than P), such that the corresponding smoothing \mathcal{X}_t is smooth. Let $W^Y : (\mathbb{C}^\times)^n \rightarrow \mathbb{C}$ be the disc potential of moment-map fibers of Y , and $W^{\mathcal{X}_t} : (\mathbb{C}^\times)^n \rightarrow \mathbb{C}$ be the disc potential of product torus fibers of $\pi^{\mathcal{X}_t}$ (Definition 4.15). Let q_i 's be the Kähler parameters of Y for $i = 1, \dots, \tilde{m} - n$, where \tilde{m} is the number of lattice points contained in P . Then there exists an invertible change of coordinates $q(\check{q})$ and specialization of parameters $\check{q} = \underline{\check{q}} \in \mathbb{C}^{\tilde{m}-n}$ such that*

- (1) $\check{W}^Y(\check{q}) := W^Y(q(\check{q}))$ can be analytic continued to all $\check{q} \in \mathbb{C}^{\tilde{m}-n}$;
- (2) $\check{W}^Y(\check{q} = \underline{\check{q}})$ equals to $W^{\mathcal{X}_t}$ up to an affine change of coordinates on the domain $(\mathbb{C}^\times)^n$.

The SYZ mirrors $\check{\mathcal{X}}$ and \check{Y}_q also have such a relation, that is, the same change of coordinates $q(\check{q})$ and same specialization of variables $\check{q} = \underline{\check{q}}$ gives

$$\check{\mathcal{X}} = \check{Y}_{q(\check{q})}|_{\check{q}=\underline{\check{q}}}.$$

Proof. By Theorem 5.1, the disc potential W^Y equals to the Hori-Vafa mirror under the mirror map $q(\check{q})$:

$$W^Y(q(\check{q})) = z_0 \left(z^{v_1} + \dots + z^{v_n} + \sum_{v \in (P \cap N) - \{v_i\}_{i=1}^n} \check{q}_v z^v \right).$$

We may choose v_1 to be a vertex of P and v_2, \dots, v_n be lattice points contained in P . Note that $v_2 - v_1, \dots, v_n - v_1$ form a basis of $\underline{\nu}^\perp$. In particular we can write

$$v_1 = a_1(v_2 - v_1) + \dots + a_{n-1}(v_n - v_{n-1})$$

for some integers a_1, \dots, a_{n-1} . Thus $z^{v_1} = (z^{v_2 - v_1})^{a_1} \dots (z^{v_n - v_{n-1}})^{a_{n-1}}$.

The disc potential $W^{\mathcal{X}_t}$ has the expression

$$W^{\mathcal{X}_t} = z_0 \sum_{v \in P \cap N} n_v z^v$$

by Corollary 4.16, where $n_v = 1$ for vertices v of P . (In particular $n_{v_1} = 1$.)

Under the change of coordinates

$$z_0 \mapsto \lambda_0 z_0, \quad z^{v_2 - v_1} \mapsto \lambda_1 z^{v_2 - v_1}, \quad \dots, \quad z^{v_n - v_{n-1}} \mapsto \lambda_{n-1} z^{v_n - v_{n-1}}$$

we have

$$\begin{aligned} z_0 z^{v_1} &\mapsto (\lambda_0 \lambda_1^{a_1} \dots \lambda_{n-1}^{a_{n-1}}) z_0 z^{v_1}, \\ z_0 z^{v_2} &\mapsto (\lambda_0 \lambda_1^{a_1} \dots \lambda_{n-1}^{a_{n-1}}) \lambda_1 z_0 z^{v_2}, \\ &\vdots \\ z_0 z^{v_{n-1}} &\mapsto (\lambda_0 \lambda_1^{a_1} \dots \lambda_{n-1}^{a_{n-1}}) \lambda_{n-1} z_0 z^{v_{n-1}}. \end{aligned}$$

Thus by setting

$$\lambda_1 = n_{v_2}, \lambda_{n-1} = n_{v_n}, \lambda_0 = \frac{1}{n_{v_2}^{a_1} \dots n_{v_n}^{a_{n-1}}},$$

under the change of coordinates $W^Y(q(\check{q}))$ becomes

$$z_0 \left(n_{v_1} z^{v_1} + n_{v_2} z^{v_2} + \dots + n_{v_n} z^{v_n} + \sum_{v \in (P \cap N) - \{v_i\}_{i=1}^n} c_v \check{q}_v z^v \right)$$

for some constants c_v 's. Then the specialization of parameters

$$\check{q}_v = \frac{n_v}{c_v}$$

for $v \in (P \cap N) - \{v_i\}_{i=1}^n$ equates the above expression with $W^{\mathcal{X}_t}$.

The SYZ mirrors have similar expressions as W^Y and $W^{\mathcal{X}_t}$: the SYZ mirror of Y is defined by

$$uv = \left(z^{v_1} + \dots + z^{v_n} + \sum_{v \in (P \cap N) - \{v_i\}_{i=1}^n} \check{q}_v z^v \right)$$

while the SYZ mirror of \mathcal{X}_t is defined by

$$uv = \prod_{i=0}^p \left(1 + \sum_{l=1, \dots, k_i} z^{u_i^l} \right).$$

Thus the mirror map $q(\check{q})$ and the same specialization of parameters give

$$\check{\mathcal{X}} = \check{Y}_{q(\check{q})}|_{\check{q}=\underline{\check{q}}}.$$

□

Remark 5.3. *In general the coefficients of the SYZ mirror \check{Y}_q (or the disc potential W_q^Y) is a series in q , and it may not make sense to specify a value of q (because the series may not converge at that value). A change of variable $q(\check{q})$ is necessary in order to have an analytic continuation of \check{Y}_q from the large complex structure limit to the conifold limit.*

6. EXAMPLES

6.1. **A-type surface singularities.** Consider the A_p singularity $\mathbb{C}^2/\mathbb{Z}_{p+1}$, which is described by the fan whose rays are generated by $(k, 1)$ for $k = 0, \dots, p + 1$. Then the polytope is the line segment $P = [0, p + 1]$ and the fan is obtained by cone over P . By taking the Minkowski decomposition

$$P = R_0 + \dots + R_p$$

where $R_i = [0, 1]$ for all $i = 0, \dots, p$, one obtains a smoothing \mathcal{X}_t of the A_p singularity. See Figure 3.

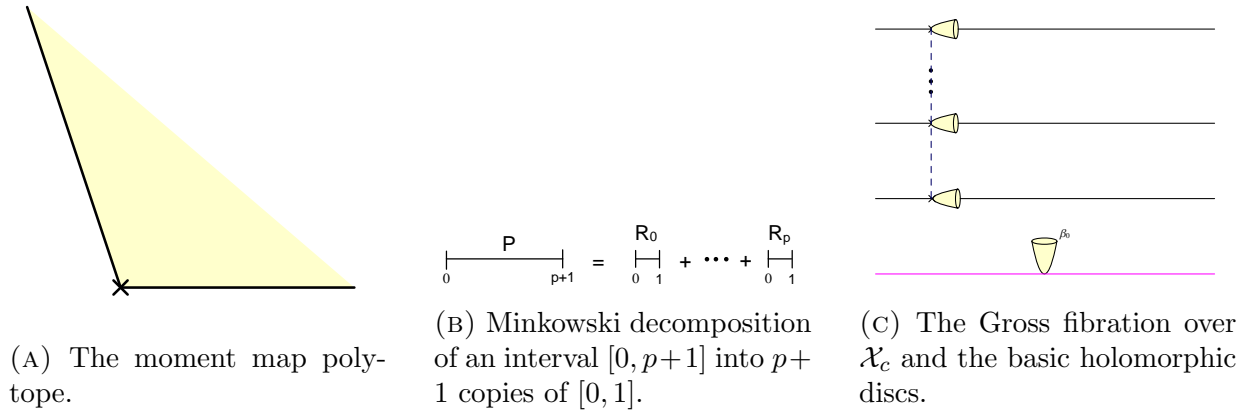


FIGURE 3. Smoothing of A_p singularity $\mathbb{C}^2/\mathbb{Z}_{p+1}$ and the Gross fibration.

The Gross fibration π^K has $p + 1$ parallel walls H_i 's for $i = 0, \dots, p$ in the base, and each of them contains a singular value of the fibration (see Figure 3c). There is an A_p chain of Lagrangian two-spheres hitting the singular fibers, and they do not contribute to computation of open Gromov-Witten invariants (in contrast to the other side of resolution). We can also deform the fibration and make all the walls collapse to one (described in Section 3), and this is denoted as $\underline{\pi}$. Then there is one interior singularity left, and the singular fiber is depicted in Figure 4.

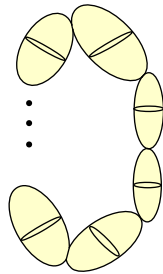


FIGURE 4. The singular fiber of the Lagrangian fibration $\underline{\pi}$ in which all walls collapse to a hyperplane in the base.

From Theorem 4.8, the SYZ mirror is

$$uv = (1 + z)^{p+1} = \sum_{k=0}^{p+1} \binom{p+1}{k} z^k$$

for $(u, v, z) \in \mathbb{C}^2 \times \mathbb{C}^\times$, which is again the A_p singularity. Thus we see that A_p singularity is self-mirror in this sense. Moreover we get a correspondence between the Minkowski decomposition of an interval and the polynomial factorization $(1 + z)^{p+1} = \sum_{k=0}^{p+1} \binom{p+1}{k} z^k$.

On the other side of the transition, we consider toric resolution of A_n singularity. SYZ and relevant open Gromov-Witten invariants have been computed in [22]. The moment map polytope is shown in Figure 5. The SYZ mirror is

$$uv = (1 + z)(1 + q_1 z)(1 + q_1 q_2 z) \dots (1 + q_1 \dots q_p z).$$

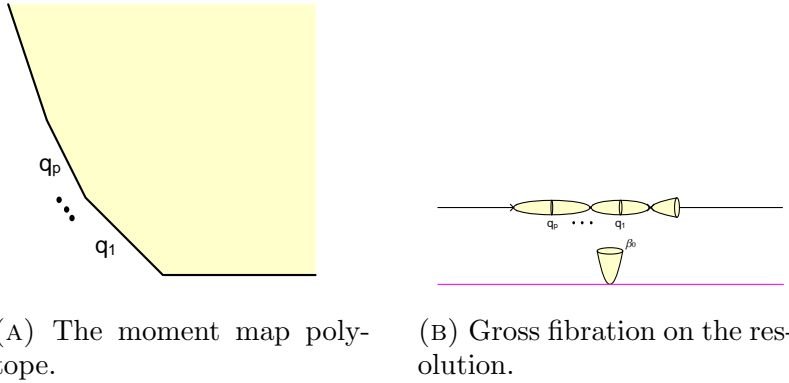


FIGURE 5. The moment map and Gross fibration of resolution of A_p singularity. Each finite line segment on the boundary corresponds to a holomorphic sphere, whose Kähler parameter is denoted as q_i .

It is easy to see that the specialization of Kähler parameters in Theorem 5.2 in this case is

$$q_1 = \dots = q_p = 1$$

and the SYZ mirror of Y reduces to the SYZ mirror of \mathcal{X}_t . (In this case we do not even need analytic continuation.) This gives the conifold limit point which is also an orbifold limit in this case. We can study SYZ via orbifold invariants (defined in [16]) of the orbifold \mathcal{X} instead of its smoothing. This was done in [9] and one obtains a different flat coordinates around the conifold limit.

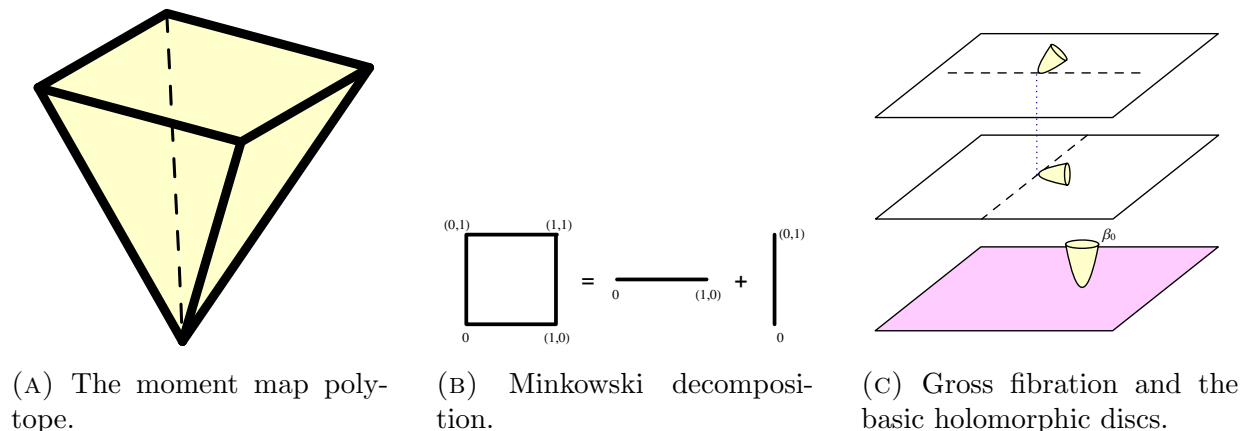
6.2. A three-dimensional conifold. Consider the conifold singularity X defined by $xy = zw$ for $x, y, z, w \in \mathbb{C}$, which is described by the three-dimensional fan whose rays are generated by

$$(0, 0, 1), (1, 0, 1), (0, 1, 1), (1, 1, 1).$$

Then the corresponding polytope is the square $P = [0, 1] \times [0, 1]$ and the fan is obtained by cone over P . By taking the Minkowski decomposition

$$P = R_0 + R_1$$

where $R_0 = [1, 0]$ and $R_1 = [0, 1]$, one obtains a smoothing \mathcal{X}_t of the conifold singularity. See Figure 6. \mathcal{X}_t can be identified with $T^*\mathbb{S}^3$.



(A) The moment map polytope.

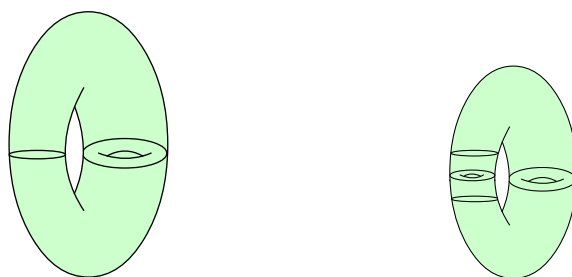
(B) Minkowski decomposition.

(C) Gross fibration and the basic holomorphic discs.

FIGURE 6. The three-dimensional conifold, its smoothing and the Gross fibration. A Lagrangian three-sphere comes up in the smoothing, and its image under the Gross fibration is the vertical dotted line in the rightmost figure.

The Gross fibration π^K has two parallel walls H_0 and H_1 in the base, and the discriminant loci are the boundary of the base, and two lines contained in these two planes respectively (see Figure 6c). There is a Lagrangian three sphere whose image under π^K is a vertical line segment joining the two planes. Under the identification $\mathcal{X}_t \cong T^*\mathbb{S}^3$, the Lagrangian sphere is the zero section of $T^*\mathbb{S}^3 \rightarrow \mathbb{S}^3$. There are no holomorphic two-sphere.

The Lagrangian fibration $\underline{\pi}$ is formed by collapsing the two walls into one. See Figure 7 for the topological type of the singular Lagrangian fibers over the discriminant locus (which is a cross consisting of two lines) in the wall.



(A) A singular fiber over a generic point of the discriminant locus. It is a two-torus fibration over the circle with one singular fiber.

(B) A singular fiber over the central point of the discriminant locus. It is a two-torus fibration over the circle with two singular fibers.

FIGURE 7. Singular fibers of $\underline{\pi}$ which are degenerate tori.

From Theorem 4.8, the SYZ mirror is

$$uv = (1 + z_1)(1 + z_2) = 1 + z_1 + z_2 + z_1z_2.$$

The Minkowski decomposition shown in Figure 6b corresponds to the polynomial factorization $(1 + z_1)(1 + z_2) = 1 + z_1 + z_2 + z_1z_2$.

On the other side of the transition we may consider $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ which is a toric resolution of X (see Figure 8). SYZ and relevant open Gromov-Witten invariants have been computed in Section 5.3.2 of [10]. The SYZ mirror is

$$uv = 1 + z_1 + z_2 + qz_1z_2$$

where q is the Kähler parameter corresponding to the size of the zero section $\mathbb{P}^1 \subset \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

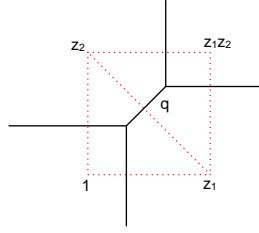


FIGURE 8. $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. The dotted line shows the fan picture (projected to the plane) and the solid line shows the polytope picture. The solid line segment in the middle corresponds to the zero section \mathbb{P}^1 marked by its Kähler parameter q .

Thus the specialization of Kähler parameters in Theorem 5.2 in this case is

$$q = 1.$$

6.3. Cone over del Pezzo surface of degree six. Let $N = \mathbb{Z}^3$, $\underline{\nu} = (0, 0, 1) \in M$. The polytope $P \in \underline{\nu}_{\mathbb{R}}^{\perp}$ has corners $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_3 = (1, 1)$, $v_4 = (-1, 0)$, $v_5 = (0, -1)$ and $v_6 = (-1, -1)$, and $\sigma \subset \mathbb{R}^3$ is the cone over P . This is Example 3.1 of [19]. There are two different Minkowski decompositions as shown in Figure 9 giving rise to two different ways of smoothings of $X = X_{\sigma}$.

The Gross fibrations and holomorphic discs are shown in Figure 10. In the first smoothing, there are Lagrangian three-spheres whose images are vertical line segments between two consecutive walls (shown as a dotted line in Figure 10b). In the second smoothing, there is a Lagrangian cone over the two torus whose image is a vertical line segment between the two walls (shown as a dotted line in Figure 10c).

From Theorem 4.8, the SYZ mirrors are

$$uv = g(z_1, z_2)$$

where the first choice of smoothing gives

$$g = g_1^{\text{con}}(z_1, z_2) = (1 + z_1)(1 + z_2)(1 + z_1^{-1}z_2^{-1}) = 2 + z_1 + z_2 + z_1z_2 + z_1^{-1} + z_2^{-1} + z_1^{-1}z_2^{-1}$$

and the second choice of smoothing gives

$$g = g_2^{\text{con}}(z_1, z_2) = (1 + z_1 + z_2^{-1})(1 + z_1^{-1} + z_2) = 3 + z_1 + z_2 + z_1z_2 + z_1^{-1} + z_2^{-1} + z_1^{-1}z_2^{-1}.$$

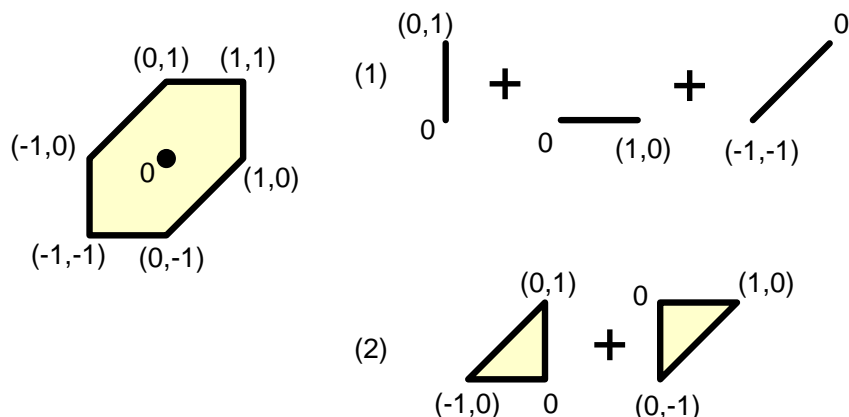


FIGURE 9. Two possible Minkowski decompositions of the fan polytope of dP_6 .

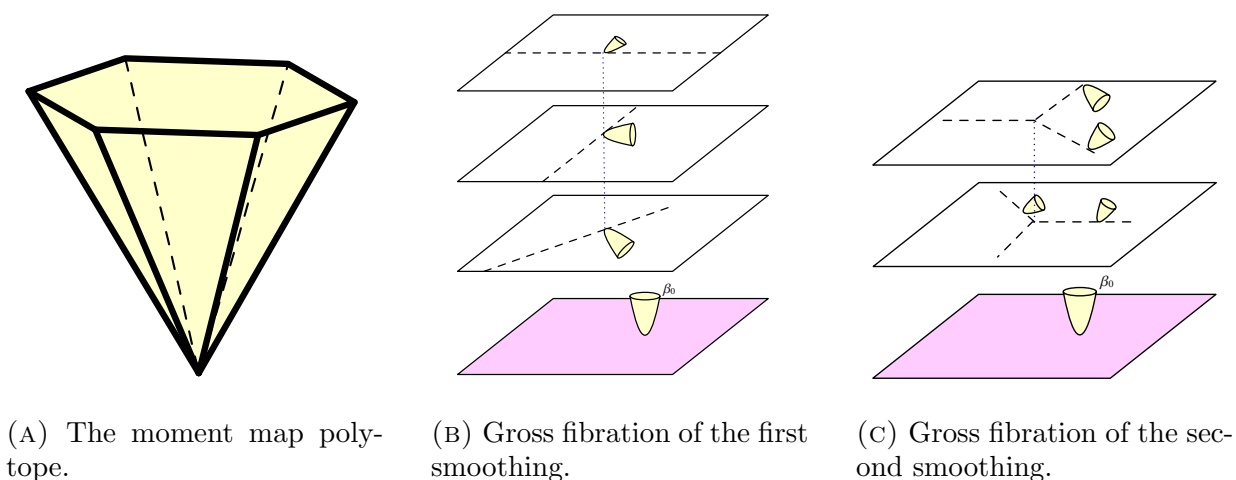


FIGURE 10. Cone over dP_6 and the Gross fibrations over its two different smoothings. The dotted lines are the discriminant loci. The holomorphic disc emanated from the boundary divisor at the bottom is in the class β_0 which has Maslov index two, and the discs emanated from singular fibers have Maslov index zero.

These two equations differ only by their constant terms. We see that Minkowski decompositions of the polytope in this example correspond to (integral) factorizations of polynomials.

Remark 6.1. *The compactification $\bar{\mathcal{X}}_t$ of the first smoothing \mathcal{X}_t (Figure 10b) gives the complete flag manifold $\text{Fl}(1, 2, 3)$ (which is Fano). There are three walls in the base of Lagrangian fibration. If one moves the middle and lowest ones to the bottom boundary, and the highest one to the top boundary, it gives the toric degeneration of $\text{Fl}(1, 2, 3)$ constructed in [24] (see Figure 4 of [24]). By using Theorem 4.7, the disc potential for fibers between the middle and*

highest walls is

$$W = z_0(1+z_1)(1+z_2) + \frac{1}{z_0}(1+z_1^{-1}z_2^{-1}) = z_0 + \frac{1}{z_0} + z_0z_1z_2 + \frac{1}{z_0z_1z_2} + z_0z_1 + z_0z_2$$

(setting all Kähler parameters to be 1 for simplicity). It coincides with the disc potential computed in [24].

For the second choice of smoothing, the tropical diagrams of the components $1+z_1+z_2^{-1}=0$ and $1+z_2+z_1^{-1}=0$ are shown in Figure 11. We see that they give a decomposition of the dual fan of the polytope P .

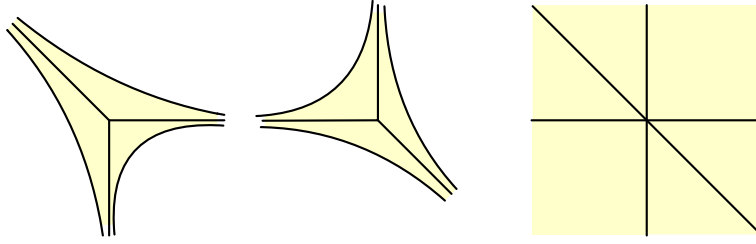


FIGURE 11. The left and middle diagrams show the tropical curves and amoebas associated to $1+z_1+z_2^{-1}=0$ and $1+z_2+z_1^{-1}=0$. The diagram on the right is the dual fan of the polytope P in Figure 9.

On the other side of the transition, we consider toric resolution of the singularity \mathcal{X} . There are three choices Y_A, Y_B, Y_C which differ from each other by flops as shown in Figure 12. In the (complexified) Kähler moduli, they correspond to different chambers around the large volume limit. Their SYZ mirrors are of the same form $uv = g(z_1, z_2)$, where

$$\widetilde{g}_A^{\text{res}}(z_1, z_2) = z_2 + (1 + \delta_A(q^A))z_1z_2 + z_1z_2^2 + q_6^A + q_1^Aq_6^Az_1 + q_1^Aq_2^Az_1^2z_2 + q_5^Az_1^2z_2^2$$

and $q_1^Aq_6^A = q_3^Aq_4^A$, $q_1^Aq_2^A = q_4^Aq_5^A$, $q_2^Aq_3^A = q_5^Aq_6^A$;

$$\widetilde{g}_B^{\text{res}}(z_1, z_2) = z_2 + (1 + \delta_B(q^B))z_1z_2 + z_1z_2^2 + q_6^B + q_1^Bq_6^Bz_1 + q_1^Bq_2^Bz_1^2z_2 + q_4^Bq_1^Bq_2^Bz_1^2z_2^2$$

and $q_1^Bq_6^B = q_3^B$, $q_1^Bq_2^B = q_5^B$, $q_2^Bq_3^B = q_5^Bq_6^B$;

$$\widetilde{g}_C^{\text{res}}(z_1, z_2) = z_2 + (1 + \delta_C(q^C))z_1z_2 + z_1z_2^2 + q_1^Cq_6^C + q_6^Cz_1 + q_2^Cz_1^2z_2 + q_2^Cq_4^Cz_1^2z_2^2$$

and $q_6^C = q_3^C$, $q_2^C = q_5^C$, $q_2^Cq_3^C = q_5^Cq_6^C$. The Kähler parameters q_i 's correspond to the holomorphic spheres as labelled in Figure 12. The Kähler moduli has dimension four.

By the change of coordinates $z_1 \mapsto (1 + \delta(q))^{-1}z_1$, $z_2 \mapsto (1 + \delta(q))z_2$ and $u \mapsto (1 + \delta(q))u$, all the above expressions are brought to the form

$$g_t(z_1, z_2) = z_2 + z_1z_2 + z_1z_2^2 + t_1 + t_2z_1 + t_3z_1^2z_2 + t_4z_1^2z_2^2$$

and so all of them belong to the same mirror family of complex varieties around the same large complex structure limit. The SYZ mirrors after such change of coordinates are denoted as $g_A^{\text{res}}, g_B^{\text{res}}, g_C^{\text{res}}$ respectively.

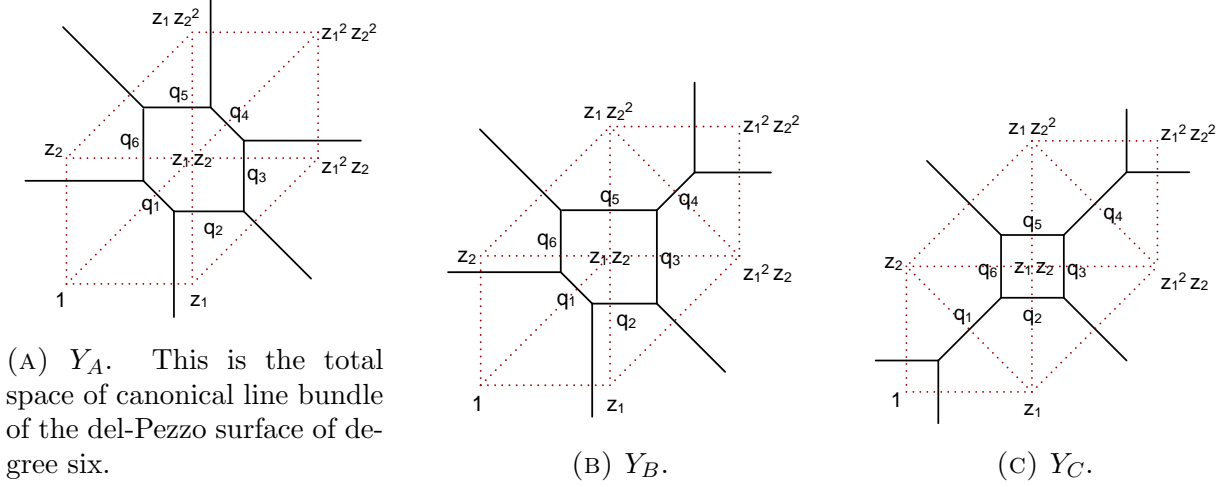


FIGURE 12. The three toric Calabi-Yau resolutions. The dotted lines show the polytope P together with its triangulations. Each vertex corresponds to a monomial of the mirror. The solid lines show the moment map polytope projected to the plane, and each line segment corresponds to a holomorphic sphere which is labelled by a Kähler parameter measuring its area.

By the open mirror theorem for toric Calabi-Yau manifolds [8], g^{res} can be computed from the mirror maps and expressed as the following:

$$\begin{aligned} g_A^{\text{res}}(z_1, z_2) &= z_2 + z_1 z_2 + z_1 z_2^2 + \check{q}_6^A + \check{q}_1^A \check{q}_6^A z_1 + \check{q}_1^A \check{q}_2^A z_1^2 z_2 + \check{q}_5^A z_1^2 z_2^2; \\ g_B^{\text{res}}(z_1, z_2) &= z_2 + z_1 z_2 + z_1 z_2^2 + \check{q}_6^B + \check{q}_1^B \check{q}_6^B z_1 + \check{q}_1^B \check{q}_2^B z_1^2 z_2 + \check{q}_1^B \check{q}_2^B \check{q}_4^B z_1^2 z_2^2; \\ g_C^{\text{res}}(z_1, z_2) &= z_2 + z_1 z_2 + z_1 z_2^2 + \check{q}_1^C \check{q}_6^C + \check{q}_6^C z_1 + \check{q}_2^C z_1^2 z_2 + \check{q}_2^C \check{q}_4^C z_1^2 z_2^2; \end{aligned}$$

where q^A and \check{q}^A (resp. B, C) are related by the mirror maps $q_i^A = q_i^A(\check{q}^A)$ (resp. B, C). \check{q}_i 's satisfy the same relations as q_i 's:

$$\begin{aligned} \check{q}_1^A \check{q}_6^A &= \check{q}_3^A \check{q}_4^A; \check{q}_1^A \check{q}_2^A = \check{q}_4^A \check{q}_5^A; \check{q}_2^A \check{q}_3^A = \check{q}_5^A \check{q}_6^A. \\ \check{q}_1^B \check{q}_6^B &= \check{q}_3^B; \check{q}_1^B \check{q}_2^B = \check{q}_5^B; \check{q}_2^B \check{q}_3^B = \check{q}_5^B \check{q}_6^B. \\ \check{q}_6^C &= \check{q}_3^C; \check{q}_2^C = \check{q}_5^C; \check{q}_2^C \check{q}_3^C = \check{q}_5^C \check{q}_6^C. \end{aligned}$$

We can see that g_A^{res} equals to g_B^{res} by the change of variables

$$(\check{q}_4^A)^{-1} = \check{q}_4^B; \check{q}_1^A = \check{q}_1^B; \check{q}_2^A = \check{q}_2^B; \check{q}_6^A = \check{q}_6^B,$$

and g_B^{res} equals to g_C^{res} by the change of variables

$$(\check{q}_1^B)^{-1} = \check{q}_1^C; \check{q}_3^B = \check{q}_3^C; \check{q}_4^B = \check{q}_4^C; \check{q}_5^B = \check{q}_5^C.$$

Thus the mirror complex variety undergoes no topological change under the flops, which is a well-known prediction by string theorists.

Going back to smoothings of \mathcal{X} , by the change of coordinates $z_1 \mapsto z_1/2$, $z_2 \mapsto 2z_2$ and $u \mapsto 2u$, $uv = g_1^{\text{con}}$ is equivalent to

$$uv = z_2 + z_1 z_2 + z_1 z_2^2 + \frac{1}{2} + \frac{1}{2} z_1 + \frac{1}{2} z_1^2 z_2 + \frac{1}{2} z_1^2 z_2^2.$$

Similarly $uv = g_2^{\text{con}}$ is equivalent to

$$uv = z_2 + z_1 z_2 + z_1 z_2^2 + \frac{1}{3} + \frac{1}{3} z_1 + \frac{1}{3} z_1^2 z_2 + \frac{1}{3} z_1^2 z_2^2.$$

Thus we see that the SYZ mirrors of the smoothings correspond to two conifold limit points of the complex moduli:

$$(t_1, t_2, t_3, t_4) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)$$

and

$$(t_1, t_2, t_3, t_4) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right).$$

Then the specialization of variables in Theorem 5.2 for the conifold transitions are

$$\begin{aligned} \check{q}_1^A &= 1, \check{q}_2^A = \check{q}_5^A = \check{q}_6^A = \frac{1}{2}; \\ \check{q}_1^B &= \check{q}_4^B = 1; \check{q}_2^B = \check{q}_6^B = \frac{1}{2}; \\ \check{q}_1^C &= \check{q}_4^C = 1; \check{q}_2^C = \check{q}_6^C = \frac{1}{2} \end{aligned}$$

for the first smoothing, and

$$\begin{aligned} \check{q}_1^A &= 1, \check{q}_2^A = \check{q}_5^A = \check{q}_6^A = \frac{1}{3}; \\ \check{q}_1^B &= \check{q}_4^B = 1; \check{q}_2^B = \check{q}_6^B = \frac{1}{3}; \\ \check{q}_1^C &= \check{q}_4^C = 1; \check{q}_2^C = \check{q}_6^C = \frac{1}{3} \end{aligned}$$

for the second smoothing.

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