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LAGRANGIAN SUBMANIFOLDS IN HYPERKÄHLER MANIFOLDS, LEGENDRE TRANSFORMATION

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Abstract

We develop the foundation of the *complex symplectic geometry* of Lagrangian subvarieties in a hyperkähler manifold. We establish a characterization, a Chern number inequality, topological and geometrical properties of Lagrangian submanifolds. We discuss a category of Lagrangian subvarieties and its relationship with the theory of Lagrangian intersection.

We also introduce and study extensively a normalized *Legendre transformation* of Lagrangian subvarieties under a birational transformation of projective hyperkähler manifolds. We give a *Plücker type formula* for Lagrangian intersections under this transformation.

1. Introduction

A Riemannian manifold M of real dimension $4n$ is *hyperkähler* if its holonomy group is $\mathrm{Sp}(n)$. Its has three complex structures I, J and K satisfying the Hamilton relation $I^2 = J^2 = K^2 = IJK = -1$. We fix one complex structure J on M and all submanifolds are complex submanifolds with respect to J. When the submanifold is complex, its dimension is always refered to its complex dimension. We denote its Kähler form as ω and its holomorphic two form as Ω which is nondegenerate and defines a (holomorphic) symplectic structure on M.

A submanifold C in M of dimension n is called a *Lagrangian* if the restriction of Ω to it is zero. In the *real* symplectic geometry, Lagrangian submanifolds plays a very key role, for example in geometric quantization, Floer theory, Kontsevich's homological mirror conjecture and the Strominger, Yau and Zaslow geometric mirror conjecture.

The objective of this article is twofold:

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(1) We develop the foundation of the *complex symplectic geometry* of Lagrangian submanifolds in a hyperkähler manifold.[1] This subject was previously studied by Donagi and Markman [11], [12], Hitchin [19] and others.

(2) We introduce a Legendre transformation of Lagrangian subvarieties along $\mathbb{P}^n$ in M and we establish a *Plücker type formula*.

Here we summarize our results in this paper: First, it is not difficult to recognize a Lagrangian:

(i) C is Lagrangian if and only if its Poincaré dual $[C]$ represents a Ω-primitive class in $H^{2n}(M,\mathbb{Z})$. In particular being Lagrangian is invariant under deformation. This also enable us to define singular Lagrangian subvarieties.

(ii) When C is a complete intersection in M, we have

$$\int_C c_2(M)\,\omega^{n-2} \geq 0,$$

moreover equality holds if and only if C is a Lagrangian.

(iii) The rational cobordism class of the manifold C is completely determined by the cohomology class $[C] \in H^{2n}(M,\mathbb{Z})$.

Second, the second fundamental form of C in M defines a cubic vector field $\widetilde{A} \in \Gamma\left(C, \mathrm{Sym}^3 T_C\right)$ and a cubic form,

$$\mathbf{c}_C : \mathrm{Sym}^3 H^0(C, T_C^*) \to \mathbb{C}$$
$$\mathbf{c}_C(\phi,\eta,\zeta) = \int_C \phi_i \eta_j \zeta_k \widetilde{A}^{ijk} \frac{\omega^n}{n!}.$$

$H^0(C, T_C^*)$ can be identified as the tangent space of the moduli space $\mathcal{M}$ of Lagrangian submanifolds. By varying C, the cubic tensor $\mathbf{c}_C$ gives a holomorphic cubic tensor $\mathbf{c} \in H^0\left(\mathcal{M}, \mathrm{Sym}^3 T_{\mathcal{M}}^*\right)$. This cubic form defines a torsion-free flat symplectic connection ∇ on the tangent bundle of $\mathcal{M}$ and it satisfies $\nabla \wedge J_{\mathcal{M}} = 0$. Namely it is a special Kähler structure on $\mathcal{M}$.

[1] Many results presented here can be applied to any holomorphic symplectic manifold.

Third, the category $\mathcal{C}_M$ of Lagrangian subvarieties in M with

$$\operatorname{Hom}_{\mathcal{C}_M}(C_1, C_2) = \Sigma (-1)^q \operatorname{Ext}^q_{O_M}(O_{C_1}, O_{C_2})$$

refines the intersection theory of Lagrangians. For example

$$\dim \operatorname{Hom}_{\mathcal{C}_M}(C_1, C_2) = (-1)^n C_1 \cdot C_2.$$

When C_1 and C_2 intersect cleanly, this equals the Euler characteristic of the intersection up to sign, $C_1 \cdot C_2 = \pm e(C_1 \cap C_2)$. For each individual Ext group, we have:

(i) When C is a complete intersection Lagrangian submanifold in M then

$$\operatorname{Ext}^q_{O_M}(O_C, O_C) = H^q(C, \mathbb{C}), \text{ for all } q.$$

(ii) When C_1 and C_2 intersect transversely then $\operatorname{Ext}^q_{O_M}(O_{C_1}, O_{C_2}) = 0$ for $q < n$ and equals $\mathbb{C}^s$ for $q = n$.

We also study the derived category of Lagrangian coherent sheaves on M, denote $D^b_{\mathrm{Lag}}(M)$. For example the structure sheaf of any Lagrangian subvariety in M defines an object in this category.

Fourth, we have good understanding of coisotropic subvarieties and their corresponding symplectic reductions in hyperkähler manifolds. Coisotropic submanifolds share some of Lagrangian properties. They can be characterized by the Ω-primitivity property of their Poincaré duals. There is also a Chern number inequality in the complete intersection case. Furthermore for a generic complex structure in the twistor family of M there are no isotropic or coisotropic subvarieties.

In *real* symplectic geometry, any well-behaved coisotropic submanifold gives rise to a reduction $\pi_D : D \to B$ with B another symplectic manifold. Lagrangians in M can be reduced to Lagrangians in B or projected to Lagrangians of M inside D. In the *complex* case we can define a reduction functor $\mathbf{R}_D : D^b_{\mathrm{Lag}}(M) \to D^b_{\mathrm{Lag}}(B)$ and a projection functor $\mathbf{P}_D : D^b_{\mathrm{Lag}}(M) \to D^b_{\mathrm{Lag}}(M)$ using Fourier-Mukai type functors.

Suppose the reduction process occurs inside M, namely there is a birational contraction $\pi : M \to Z$ such that D is the exceptional locus and B is the discriminant locus. In this case $\pi_D : D \to B$ is a $\mathbb{P}^k$-bundle and its relative cotangent bundle is the normal bundle of D in M ([33], [27], [20]). Moreover one can replace D by its dual $\mathbb{P}^k$-bundle and produce another holomorphic symplectic manifold M', called the Mukai elementary modification [26] .[2]

Fifth, we can define Legendre transformation on any Mukai elementary modification. For example when $D \cong \mathbb{P}^n$ we associate to each Lagrangian subvariety C in M (not equal to D) another Lagrangian subvariety $C^\vee$ in M'.

We will explain its relationship with the classical Legendre transformation. Roughly speaking it is the hyperkähler quotient by S^1 of the Legendre transformation of defining functions of C on the linear symplectic manifold $T^*\mathbb{C}^{n+1}$. This can also be regarded as a generalization of the dual varieties construction.

We establish the following *Plücker type formula* for the Legendre transformation:

$$C_1 \cdot C_2 + \frac{(C_1 \cdot \mathbb{P}^n)(C_2 \cdot \mathbb{P}^n)}{(-1)^{n+1}(n+1)} = C_1^\vee \cdot C_2^\vee + \frac{(C_1^\vee \cdot \mathbb{P}^{n*})(C_2^\vee \cdot \mathbb{P}^{n*})}{(-1)^{n+1}(n+1)}.$$

When $n = 2$ this formula is essentially equivalent to a classical Plücker formula for plane curves.

Motivated from this formula we define a *normalized Legendre transformation*

$$\mathcal{L}(C) = C^\vee + \frac{(C \cdot \mathbb{P}^n) + (-1)^{n+1}(C^\vee \cdot \mathbb{P}^{n*})}{n+1}\mathbb{P}^{n*} \text{ if } C \neq \mathbb{P}^n$$
$$\mathcal{L}(\mathbb{P}^n) = (-1)^n \mathbb{P}^{n*}.$$

This transformation $\mathcal{L}$ preserves the intersection product:

$$C_1 \cdot C_2 = \mathcal{L}(C_2) \cdot \mathcal{L}(C_2).$$

When $n = 1$ we simply have $M = M'$ and $C = C^\vee$. However the normalized Legendre transformation is interesting; it coincides with the Dehn twist along a (-2)-curve in M.

In the general case, the Plücker type formula and the definition of the normalized Legendre transformation will involve the reduction and the projection of a Lagrangian with respect to the coisotropic exceptional submanifold D.

Next we are going to look at an *explicit* example of a hyperkähler manifold and its flop.

[2] There is also stratified version of this construction by Markman [24].

1.1 The cotangent bundle of $\mathbb{P}^n$

In [9], [10] Calabi showed that the cotangent bundle of the complex projective space is a hyperkähler manifold and its metric can be described explicitly as follows: Let $z^1, \ldots, z^n$ be a local inhomogeneous coordinate system in $\mathbb{P}^n$ and $\zeta_1, \ldots, \zeta_n$ be the corresponding coordinate system on the fibers of $T^*\mathbb{P}^n$, i.e., $\Sigma\zeta_j dz^j$ represents a point in $T^*\mathbb{P}^n$. The symplectic form on $T^*\mathbb{P}^n$ is given by $\Sigma dz^j \wedge d\zeta_j$. The hyperkähler Kähler form on $T^*\mathbb{P}^n$ is given by

$$\omega = \partial\bar{\partial}\left(\log\left(1+|z|^2\right)+f(t)\right),$$

where

$$f(t) = \sqrt{1+4t} - \log\left(1+\sqrt{1+4t}\right) \qquad \text{and}$$
$$t = \left(1+|z|^2\right)\left(|\zeta|^2 + |z\cdot\zeta|^2\right).$$

Calabi's approach is to look for a $\mathrm{U}(n+1)$-invariant hyperkähler metric and reduces the problem to solving an ODE for $f(t)$.

Another approach by Hitchin [18] is to construct the hyperkähler structure on $T^*\mathbb{P}^n$ by the method of hyperkähler quotient, which is analogous to the symplectic quotient (or symplectic reduction) construction. Consider $V = \mathbb{C}^{n+1}$ with the diagonal S^1-action by multiplication. Its induced action on $\mathbb{H}^{n+1} = V \times V^* = T^*V$ is given by

$$S^1 \times T^*V \to T^*V,$$
$$e^{i\theta}\cdot(x,\xi) = \left(e^{i\theta}x, e^{-i\theta}\xi\right).$$

This S^1-action preserves both the Kähler form $dx \wedge d\bar{x} + d\xi \wedge d\bar{\xi}$ and the natural holomorphic symplectic form $dx \wedge d\xi$ on T^*V. Namely it preserves the hyperkähler structure on $\mathbb{H}^{n+1} = T^*V$. The real and complex moment maps are given respectively by

$$\mu_J = i\,|x|^2 - i\,|\xi|^2 \in i\mathbb{R},$$
$$\mu_c = \xi(x) \in \mathbb{C}.$$

We can also combine them to form the hyperkähler moment map:

$$\mu : \mathbb{H}^{n+1} \to i\mathbb{R} + \mathbb{C} = i\mathbb{R}^3.$$
$$\mu = (\mu_J, \mu_c) = (\mu_J, \mu_I, \mu_K).$$

If we take $\lambda = i\,(1,0,0) \in i\mathbb{R}^3$ for the hyperkähler quotient construction, then S^1 acts freely on $\mu^{-1}(\lambda)$ and the quotient is a smooth hyperkähler manifold M:

$$\begin{aligned} M &= T^*V/_{HK}S^1 = \mu^{-1}(\lambda)/S^1 \\ &= \left\{(x,\xi) : \xi(x) = 0, |x|^2 - |\xi|^2 = 1\right\}/S^1. \end{aligned}$$

We can identify M with $T^*\mathbb{P}^n$ explicitly as follows: It is not difficult to see that

$$\begin{gathered} p : M \to \mathbb{P}^n \\ (x,\xi) \to y = x\left(1 + |\xi|^2\right)^{-1/2} \end{gathered}$$

defines a map from M to $\mathbb{P}^n$ with a section given by $\xi = 0$. The fiber of p over the point $y = [1,0,\ldots,0] \in \mathbb{P}^n$ consists of those (x,ξ)'s of the form

$$\begin{aligned} x &= (a,0,\ldots,0) \\ \xi &= (0,b_1,\ldots,b_n), \end{aligned}$$

and satisfying $|a|^2 = 1 + \Sigma\,|b_j|^2$, i.e., $p^{-1}(y)$ is parametrized by $(b_1,\ldots,b_n) \in \mathbb{C}^n$. Note that $p^{-1}(y)$ can be naturally identified with $T_y^*\mathbb{P}^n = \mathrm{Hom}\,(T_y\mathbb{P}^n,\mathbb{C})$ via

$$(0,c_1,\ldots,c_n) \to \Sigma b_j c_j.$$

By using the $\mathrm{U}(n+1)$-symmetry, this gives a natural identification between M and $T^*\mathbb{P}^n$. Moreover $\{\xi = 0\}$ in M corresponds to the zero section in $T^*\mathbb{P}^n$, we simply denote it as $\mathbb{P}^n$.

The only compact Lagrangian submanifold in $T^*\mathbb{P}^n$ is $\mathbb{P}^n$. However there are many noncompact Lagrangian submanifolds. For any submanifold S in $\mathbb{P}^n$, its conormal bundle $N^*_{S/\mathbb{P}^n}$ is a Lagrangian submanifold in $T^*\mathbb{P}^n$, a well-known construction in symplectic geometry.

There is a natural isomorphism $T^*V \cong T^*(V^*)$ because of $(V^*)^* \cong V$. We can therefore carry out the above construction on $T^*(V^*)$ exactly as before and obtain another hyperkähler manifold M', which is of course isomorphic to $T^*\mathbb{P}^{n*}$. In terms of the above coordinate system on V, the only difference is to replace the original S^1-action by its inverse action. This would change the sign of the moment maps and therefore we can identify M' as follows:

$$\begin{aligned} M' &= T^*(V^*)/_{HK}S^1 \\ &= \left\{(\xi, x) \in V^* \times V : \xi(x) = 1, |\xi|^2 - |x|^2 = 1\right\}/S^1. \end{aligned}$$

The two holomorphic symplectic manifolds M and M' are isomorphic outside their zero sections, the birational map $\Phi : M \dashrightarrow M'$ is given explicitly,

$$(x, \xi) \to \left(\left|\frac{x}{\xi}\right|\xi, \left|\frac{\xi}{x}\right|x\right).$$

In fact the zero section $\mathbb{P}^n$ inside $M = T^*\mathbb{P}^n$ can be blown down and we obtain a variety M_0. Both M and M' are two different crepant resolutions [10] of the isolated singularity of M_0 and Φ is usually called a *flop*. This is also the basic structure in the Mukai's elementary modification [26]. Explicitly we have,

$$\pi : M \to M_0 = \{A \in \mathrm{End}\left(\mathbb{C}^{n+1}\right) : \mathrm{Tr}\, A = 0,\ \mathrm{rank}\,(A) \leq 1\}$$

given by $\pi([x, \xi]) \to x \otimes \xi$. It is rather easy to check that π is the blown down morphism of $\mathbb{P}^n$ inside M. The situation for M' is identical.

2. Isotropic and coisotropic submanifolds

The most natural class of submanifolds in a symplectic manifold M consists of those C in M with the property that the restriction of the symplectic form Ω to C is as degenerate as possible. Such a submanifold C is called isotropic, coisotropic or Lagrangian according to $\dim C \leq n$, $\dim C \geq n$ or $\dim C = n$ respectively.

2.1 Definitions and properties

We first look at the linear case, i.e., M and C are a vector space and its linear subspace respectively. The complement of C in M is defined as follows:

$$C^\perp = \{v \in M : \Omega(v, w) = 0 \text{ for all } w \in C\}.$$

C is called *isotropic* (resp. *coisotropic* or *Lagrangian*) if $C \subset C^\perp$ (resp. $C^\perp \subset C$ or $C = C^\perp$). Here is the standard example: $M = \mathbb{C}^{2n}$ with

$\Omega = dz^1 \wedge dz^{n+1} + \cdots + dz^n \wedge dz^{2n}$, then the linear span of $\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^m}$ is an isotropic (resp. coisotropic or Lagrangian) subspace if $m \leq n$ (resp. $m \geq n$ or $m = n$). A useful linear algebra fact is any isotropic or coisotropic subspace of M is equivalent to the one in the standard example up to an automorphism of M which preserves Ω, namely a symplectomorphism. For general symplectic manifolds we have the following standard definition:

Definition 1. If (M, Ω) is a symplectic manifold and C is a submanifold of M, then C is called isotropic (resp. coisotropic or Lagrangian) if $TC \subset TC^{\perp}$ (resp. $TC \supset TC^{\perp}$ or $TC = TC^{\perp}$). Here

$$TC^{\perp} = \{v \in TM|_C : \Omega(v, w) = 0 \text{ for all } w \in TC\}.$$

When $\dim C = 1$ (resp. $2n - 1$), C is automatically isotropic (resp. coisotropic). From the definition it is clear that C being isotropic is equivalent to $\Omega|_C = 0$. This happens if C has no nontrivial holomorphic two form, i.e., $H^{2,0}(C) = 0$. In particular any submanifold C in M with dimension greater than n has $H^{2,0}(C) \neq 0$. More generally if $\dim C = n + k$ then $H^{2j,0}(C) \neq 0$ for $j = 0, \ldots, k$.

Lemma 2. *If M is a hyperkähler manifold and C is a submanifold in M of dimension $n + k$ then the restriction of $(\Omega)^k$ to C is always nowhere vanishing and moreover $(\Omega)^{k+1} = 0$ if and only if C is coisotropic.*

Proof of lemma. In the standard example the above assertion can be verified directly. For the general case the assertion follows from the fact that every coisotropic subspace in a symplectic vector space can be conjugated by a symplectomorphism to the one in the standard example. q.e.d.

With the help of the above lemma we can prove the following useful result for the Lefschetz operator L_Ω, defined by $L_\Omega(c) = c \cup \Omega$, and its adjoint operator Λ_Ω.

Theorem 3. *If M is a compact hyperkähler manifold and C is a submanifold in M then:*

(i) *C is isotropic if and only if $L_\Omega [C] = 0$,*

(ii) *C is coisotropic if and only if $\Lambda_\Omega [C] = 0$.*

Here $[C] \in H^*(M, \mathbb{Z})$ denotes the Poincaré dual of C.

Proof. We need to use the Hard Lefschetz sl_2-action induced by L_Ω and Λ_Ω. For (i) when C is isotropic we have $\Omega|_C = 0$ and therefore $L_\Omega [C] = 0$ by Poincaré duality. For the converse we suppose that $L_\Omega [C] = 0$. By the Hard Lefschetz sl_2-action the dimension of C must be strictly less than n, say m.

If $m = 1$ then C is already isotropic. So we assume $m \geq 2$ and we have

$$0 = \int_M [C] \, \Omega\overline{\Omega}\omega^{m-2},$$

where ω is the Kähler form on M. This is the same as $\int_C \Omega\overline{\Omega}\omega^{n-2} = 0$. Recall that $\omega|_C$ defines a Kähler structure on C. By the Riemann bilinear relation on the Kähler manifold C, the function $\left(\Omega\overline{\Omega}\omega^{n-2}\right)/\omega^n$ is proportional to the norm square of $\Omega|_C$. Therefore the holomorphic form $\Omega|_C$ must vanish, namely C is isotropic.

For (ii) we first suppose that $\Lambda_\Omega [C] = 0$. By the Hard Lefschetz sl_2-action induced by L_Ω and Λ_Ω, we have $\dim C = m = n + k > n$ and

$$[C] \cup (\Omega)^{k+1} = 0 \in H^{2n+2k+2} (M, \mathbb{C}) .$$

Therefore

$$0 = \int_M [C] \, \Omega^{k+1}\overline{\Omega}^{k+1}\omega^{2n-2} = \int_C \Omega^{k+1}\overline{\Omega}^{k+1}\omega^{2n-2}.$$

Using the Riemann bilinear relation as before, we have $(\Omega)^{k+1}|_C = 0$. By the above lemma, C is a coisotropic submanifold. The converse is clear. Hence the result. q.e.d.

The theorem says that a submanifold of M being isotropic (or coisotropic) can be detected cohomologically. An immediate corollary is that such property is invariant under deformation. It also enables us to define isotropic (or coisotropic) subvarieties which might be singular, or even with non-reduced scheme structure.

Definition 4. Suppose M is a compact hyperkähler manifold. A subscheme C of M is called isotropic (resp. coisotropic) if $L_\Omega [C] = 0$ (resp. $\Lambda_\Omega [C] = 0$).

The next proposition gives a good justification of this definition.

Proposition 5. *Suppose M is a compact hyperkähler manifold and C is a subvariety. Then C is isotropic* (*resp. coisotropic*) *if and only if T_xC is an isotropic* (*resp. coisotropic*) *subspace of T_xM for every smooth point $x \in C$.*

Proof. For the isotropic case, the *if* part of the assertion is obvious. For the *only if* part it suffices for us to check T_xC being an isotropic subspace of T_xM for a generic point $x \in C$ because $\Omega|_{T_xC} = 0$ is a closed condition among those x's with constant $\dim T_xC$. If x is a smooth generic point in C at which $\Omega|_{T_xC} \neq 0$, then $\left(\Omega\overline{\Omega}\omega^{k-2}/\omega^k\right)|_x > 0$ with $k = \dim C$, by the Riemann bilinear relation. Therefore

$$\int_C \Omega\overline{\Omega}\omega^{k-2} > 0$$

violating $L_\Omega [C] = 0$ assumption. Hence the claim. The coisotropic case can be treated in a similar way. q.e.d.

Another corollary of the previous theorem is the following proposition which generalizes Fujiki's observation that a generic complex structure in the twistor family of M has no curves or hypersurfaces.

Proposition 6. *Suppose M is a compact hyperkähler manifold. For a general complex structure in its twistor family, M has no nontrivial isotropic or coisotropic submanifold.*

Proof. Suppose $C \subset M$ is an isotropic submanifold of positive dimension k, then $[C] \cup \operatorname{Re}\Omega = [C] \cup \operatorname{Im}\Omega = 0$ but $[C] \cup \omega^k > 0$ in $H^*(M, \mathbb{R})$ because C is a complex submanifold. Other Kähler structures in the same twistor family can be written as $a \operatorname{Re}\Omega + b \operatorname{Im}\Omega + c\omega$ for $(a, b, c) \in \mathbb{R}^3$ satisfying $a^2 + b^2 + c^2 = 1$. This implies that $[C]$ can not be represented by an isotropic complex submanifold in any other Kähler structures in this uncountable family, except possibly $-\omega$ for k even. On the other hand $[C]$ belongs to the integral cohomology of M, which is a countable set. Hence we have our claim for the isotropic case. The coisotropic case can also be argued in a similar way. q.e.d.

We have the following immediate corollary of the above proof:

Corollary 7. *For any given $c \in H^*(M, \mathbb{Z})$ in a compact hyperkähler manifold M, there is at most two complex structures in the twistor family of M such that c can be represented by an isotropic or coisotropic subvariety.*

Note that isotropic or coisotropic submanifolds are plentiful when the whole twistor family of complex structures on M is considered. In the case of a K3 surface or an Abelian surface, they are complex curves and the numbers of such form a beautiful generating function in terms of modular forms, as conjectured by Yau and Zaslow in [35] and proved by Bryan and the author in [5], [6] (also see [7]).

Coisotropic complete intersections

When M is projective, there are many ample hypersurfaces C and they are all coisotropic for trivial reasons. They are varieties of general type and their Chern classes satisfy:[3]

(i) $c_{2k+1}(C) = c_1(C)\, c_{2k}(C)$ for all k;

(ii) $\int_C c_2(C)\, c_1(C)^{2n-3} \le \int_C c_1(C)^{2n-1}$.

The next theorem says that complete intersection coisotropic subvarieties can be characterized by the vanishing of a Chern number. In particular intersecting ample hypersurfaces will not give higher codimension coisotropic subvarieties.

Theorem 8. *Suppose M is a compact hyperkähler manifold and C is a complete intersection subvariety of dimension $n+k$ with $k \le n-2$. Then*

$$\int_C c_2(M)\left(\Omega\overline{\Omega}\right)^k \omega^{n-k-2} \ge 0.$$

Moreover the equality sign holds if and only if C is coisotropic.

Proof. We can express the above quantity in term of the Bogomolov-Beauville quadratic form q by Fujiki's result ([15]) which says for any $D_1, \ldots, D_{2n-2} \in H^2(M, \mathbb{C})$ we have

$$\int_M c_2(M)\, D_1 \ldots D_{2n-2} = c \sum_{\substack{\{i_1,\ldots,i_{2n-2}\}\\ =\{1,\ldots,2n-2\}}} q\left(D_{i_1}, D_{i_2}\right) \ldots q\left(D_{i_{2n-3}}, D_{i_{2n-2}}\right),$$

for some constant c. When D_i's are ample divisors, the left-hand side is strictly positive by the Chern number inequality for Kähler-Einstein manifolds. On the other hand, $q(D, D') > 0$ if D is an ample divisor and D' is an effective divisor. Therefore we have $c > 0$. More generally $q(D, D') \ge 0$ when D and D' are effective divisors and they intersect transversely, moreover the equality sign holds if and only if $(\Omega)^{n-1}|_{D\cap D'} = 0$. This is because

$$q\left(D, D'\right) = c' \int_M [D]\left[D'\right] \Omega^{n-1}\overline{\Omega}^{n-1}$$

[3] These can be proven using the adjunction formula, $c_{2k+1}(M) = 0$ and the Chern number inequality $\int_M c_2(M)\,[C]^{2n-2} \ge 0$ for any Calabi-Yau manifolds [2].

for some explicit positive constant c'.

We recall other basic properties of the Bogomolov-Beauville quadratic form: $q\left(\Omega,\overline{\Omega}\right) > 0$; $q(\Omega,\omega) = q(\Omega,D) = q\left(\overline{\Omega},D\right) = 0$ for any effective divisor D. Now C is a complete intersection subvariety of M, we write

$$C = D_1 \cap \cdots \cap D_{n-k}$$

for some effective divisors D_i's. We have

$$\int_C c_2(M)\left(\Omega\overline{\Omega}\right)^k \omega^{n-k-2} = \int_M c_2(M)\,[D_1]\ldots[D_{n-k}]\,\Omega^k\overline{\Omega}^k\omega^{n-k-2}.$$

We apply Fujiki's result and above properties of q and we get

$$\begin{aligned}
&\int_C c_2(M)\left(\Omega\overline{\Omega}\right)^k \omega^{n-k-2} \\
&= cq\left(\Omega,\overline{\Omega}\right)^k \sum q\left(D_{i_1},D_{i_2}\right) q\left(\omega,D_{i_3}\right)\ldots q\left(\omega,D_{i_{n-k}}\right) \\
&\quad + cq(\omega,\omega)\, q\left(\Omega,\overline{\Omega}\right)^k \sum q\left(D_{i_1},D_{i_2}\right) q\left(D_{i_3},D_{i_4}\right) \\
&\qquad \cdot q\left(\omega,D_{i_3}\right)\ldots q\left(\omega,D_{i_{n-k}}\right) + \ldots.
\end{aligned}$$

Each term on the right-hand side is nonnegative. This implies the Chern number inequality. Moreover it is zero if and only if $q(D_i,D_j) = 0$ for all $i \neq j$. This is equivalent to $\Omega^{n-1}|_{D_i\cap D_j} = 0$ for all $i \neq j$ because $D_i \cap D_j$ is a complete intersection. We will prove in the next lemma that this is equivalent to $\Omega^{k+1}|_{D_1\cap\cdots\cap D_{n-k}} = 0$, i.e., $C = D_1 \cap\cdots\cap D_{n-k}$ is a coisotropic subvariety of M. Hence the result. q.e.d.

The next lemma on linear algebra is needed in the proof of the above theorem and it is also of independent interest.

Lemma 9. *Let $M \cong \mathbb{C}^{2n}$ be a symplectic vector space with its symplectic form Ω and C is a codimension m linear subspace in M. If we write $C = D_1 \cap \cdots \cap D_m$ for some hyperplanes D_i's in M, then C is coisotropic in M if and only if $\Omega^{n-1}|_{D_i\cap D_j} = 0$ for all $i \neq j$.*

Proof. We first prove the *if* part by induction on the dimension of C. The claim is trivial for $m = 1$. When $m = 2$, it says that a codimension two subspace C in M is coisotropic if $\Omega^{n-1}|_C = 0$. This is true and in general we have any codimension m subspace C in M is coisotropic if and only if $\Omega^{n-m+1}|_C = 0$. By induction we assume $C = D_1 \cap \cdots \cap D_{m+1}$ and the claim is true for m, i.e., $D_1 \cap \cdots \cap$

D_m is coisotropic. We can choose coordinates on M such that $\Omega = dz^1 dz^{n+1} + \cdots + dz^n dz^{2n}$ and $D_i = \{z^i = 0\}$ for $i = 1, \ldots, m$. Suppose $\sum_{i=1}^{2n} a_i z^i = 0$ is the defining equation for D_{m+1}. In order for C to be coisotropic, it suffices to show that $a_i = 0$ for $n+1 \leq i \leq n+m$. For instance if $a_{n+1} \neq 0$ then $\Omega^{n-1}|_{D_1 \cap D_{m+1}}$ would be nonzero. It is because $dz^2 dz^{n+2} dz^3 dz^{n+3} \ldots dz^n dz^{2n}|_{D_1 \cap D_{m+1}} \neq 0$ and all other summands of Ω^{n-1} would restrict to zero on $D_1 \cap D_{m+1}$. If any other $a_{n+j} \neq 0$ the same argument applies by replacing D_1 with D_j. This contradicts our assumption $\Omega^{n-1}|_{D_i \cap D_j} = 0$ for all $i \neq j$. Therefore $a_i = 0$ when $n+1 \leq i \leq n+m$ and hence the claim.

For the *only if* part, we need to prove that $\Omega^{n-1}|_{D_1 \cap D_2} \neq 0$ implies $\Omega^{n-l+1}|_{D_1 \cap \cdots \cap D_l} \neq 0$ for any complete intersection $D_1 \cap \cdots \cap D_l$. This is a simple linear algebra exercise. Hence the lemma. q.e.d.

2.2 Contractions and birational transformations

In this subsection we recall known results on the birational geometry of hyperkähler manifolds. In *real* symplectic geometry, coisotropic submanifolds play a role in the *symplectic reductions* which produce symplectic manifolds of smaller dimensions under favorable conditions (see e.g., [1]). The coisotropic condition of a submanifold C in M gives an embedding of $N^*_{C/M}$ in T_C. As a distribution, this is always integrable and called a *characteristic foliation*. When the *leaf space* is a smooth manifold, it carries a natural symplectic form and it is called the *reduction* of the coisotropic submanifold C.

In *complex* symplectic geometry, a coisotropic subvariety arises naturally as the exceptional set C of any *birational contraction*,

$$\pi : M \to Z,$$

that is a projective birational morphism, as proven by Wierzba, Namikawa and Hu-Yau ([33], [27], [20]). Moreover, the restriction $\pi_C : C \to B = \pi(C)$ is its generic characteristic foliation with each smooth fiber being a projective space $\mathbb{P}^k$, provided that $k \geq 2$, and $\dim C \leq 2n - k$. When $\dim C = 2n - k$, Mukai [26] constructs *another* symplectic manifold M' by replacing each fiber $\mathbb{P}^k$ with its dual projective space $(\mathbb{P}^k)^*$. This is called the *Mukai elementary modification.* In dimension four this turns out to be the only possible way to generate birational transformations ([8]).

In Section 4, we will study a Legendre transformation along any $\mathbb{P}^n$ in M, or more generally, any Mukai elementary modification.

3. Lagrangian submanifolds

Lagrangian submanifolds are the smallest coisotropic submanifolds or the biggest isotropic submanifolds. They are the most important objects in symplectic geometry, 'Everything is a Lagrangian manifold' as Weinstein described their roles in real symplectic geometry.

In this section we will first establish some basic topological and geometrical properties of Lagrangian submanifolds in a hyperkähler manifold. We will explain the relationship between the second fundamental form and the special Kähler structure on the moduli space of Lagrangian submanifolds. Then we will study a Lagrangian category and intersection theory of Lagrangian subvarieties, a reduction functor and a projection functor. These structures will be needed to study the Legendre transformation and the Plücker type formula in the next section.

3.1 Properties of Lagrangian submanifolds

When C is a Lagrangian submanifold of M, then $N_{C/M} \cong T_C^*$ and we have the following exact sequence:

$$0 \to T_C \to T_M|_C \to T_C^* \to 0.$$

By the Whitney sum formula we have

$$\begin{aligned} \iota^* c\left(T_M\right) &= c\left(T_C \oplus T_C^*\right) \\ &= c\left(T_C \otimes_{\mathbb{R}} \mathbb{C}\right), \end{aligned}$$

where $\iota : C \to M$ is the inclusion morphism.

This implies that Pontrjagin classes of C are determined by Chern classes of M as follows:

$$p_k\left(C\right) = \left(-1\right)^k \iota^* c_{2k}\left(M\right).$$

In particular, Pontrjagin numbers of C depends only on the cohomology class $[C] \in H^{2n}\left(M, \mathbb{Z}\right)$. By the celebrated theorem of Thom, this determines the rational cobordism type of C and we have proven the following theorem:

Theorem 10. *Given a fixed cohomology class $c \in H^{2n}(M, \mathbb{Z})$, it determines the rational cobordism class of any possible Lagrangian submanifold in M representing c.*

In particular, the signature of the intersection product on $H^*(C, \mathbb{R})$ is also determined by c. Using the Hirzebruch signature formula we have

$$\text{Signature } (C) = \int_M c \cup \sqrt{L_M}.$$

Using $N_{C/M} \cong T_C^*$ we also have the following formula for the Euler characteristic of C:

$$\chi(C) = (-1)^n \int_M c \cup c.$$

Similarly if C is spin, the Â-genus is given by

$$\hat{A}[C] = \int_M c \cup \sqrt{Td_M}.$$

The reason is when restricting to C, using the multiplicative property of the $\hat{A}$-genus, we have

$$Td_M = \hat{A}(T_M) = \hat{A}(T_C + T_C^*) = \hat{A}(T_C)\,\hat{A}(T_C^*) = \hat{A}(T_C)^2.$$

As an immediate corollary of this and the standard Bochner arguments, we can show that if C is an even dimensional spin Lagrangian submanifold of M, then T_C cannot have positive scalar curvature unless $\int_C \sqrt{Td_M} = 0$.

Recall from Section 2.1 that we have following characterizations of a n dimensional submanifold C in M being a Lagrangian. First C is Lagrangian if $H^{2,0}(C) = 0$. Second if C is a complete intersection then

$$\int_C c_2(M)\,\omega^{n-2} \geq 0.$$

Moreover the equality sign holds if and only if C is a Lagrangian. When this happens the tangent bundle of C splits into direct sum of line bundles, this is a strong constraint upon C. For example when $n = 2$ the signature of C would have to be zero. Third C is Lagrangian if and only if its Poincaré dual is a Ω-primitive class, i.e., $\Lambda_\Omega[C] = 0$, or equivalently, $L_\Omega [C] = 0$. This implies that Lagrangian property is invariant under deformations of C and it also allows us to define singular, and possibly non-reduced, Lagrangian subvarieties of M.

Remark. When C is a Lagrangian submanifold, then

$$[C] \in H^{n,n}(M, \mathbb{Z}) \cap \mathrm{Ker}\ (L_\Omega).$$

In fact being a class of type (n,n) follows from the Ω-primitivity. More precisely, we have

$$\mathrm{Ker}\ (L_\Omega) \cap \Omega^{2n}(M, \mathbb{R}) \subset \Omega^{n,n}(M),$$
$$\mathrm{Ker}\ (L_\Omega) \cap H^{2n}(M, \mathbb{R}) \subset H^{n,n}(M).$$

The proof of these inclusions simply uses the hard Lefschetz decomposition of $\Omega^{*,*}$ for the $\mathbf{sl}(2)$-action generated by L_Ω and Λ_Ω. A similar result of Hitchin [19] says that if C is a *real* submanifold of M which is a *real* Lagrangian submanifold with respect to both $\mathrm{Re}\,\Omega$ and $\mathrm{Im}\,\Omega$, then C is a *complex* submanifold and therefore a Lagrangian submanifold of M with respect to Ω.

3.2 Second fundamental form and moduli space

In this subsection we study the differential geometry of a Lagrangian submanifold C in M. The second fundamental form of any Lagrangian submanifold defines a cubic form on $H^0(C, T_C^*)$. By varying C this cubic form will determine a special Kähler structure on the moduli space of Lagrangian submanifolds. Much of these are known to Donagi and Hitchin. We will also give a brief physical explanations of such structure using supersymmetry.

3.2.1 Second fundamental form and a cubic form

Extension class of a Lagrangian subbundle

Suppose E is a symplectic vector bundle over a complex manifold C. Given any Lagrangian subbundle S, we obtain a short exact sequence,

$$0 \to S \to E \to S^* \to 0.$$

Let α be the extenstion class,

$$\alpha \in \mathrm{Ext}\,^1_{O_C}(S^*, S) \cong H^1(C, S \otimes S),$$

associated to the above sequence. Using linear algebra (see Section 2.2 of [31]), one can show that

$$\alpha \in H^1\left(C, \operatorname{Sym}^2 S\right).$$

Given any Hermitian metric h on E there is unique Hermitian connection D_E on E satisfying $(D_E)^{0,1} = \bar{\partial}_E$. Via the orthogonal decomposition of E into $S \oplus S^*$, the above differential form $A \in \Omega^{0,1}(C, S \otimes S)$ is simply the second fundamental form of the subbundle $S \subset E$ in the context of Riemannian geometry. In order for $A \in \Omega^{0,1}\left(C, \operatorname{Sym}^2 S\right)$, we need h to be compatible with Ω in the sense that if we define operators I and K by the following:

$$\Omega(v, w) = h(Iv, w) + ih(Kv, w),$$

then I, J, K defines a fiberwise quaternionic structure on E, i.e., $I^2 = J^2 = K^2 = IJK = -1$.

When we apply this to the tangent bundle $E = T_C$ of a Lagrangian submanifold C in M, we obtain a trilinear symmetric multi-vector field

$$\widetilde{A} \in \Gamma\left(C, \operatorname{Sym}^3 T_C\right),$$

on C given by

$$\begin{aligned} \widetilde{A} &= \sum \widetilde{A}^{ijk} \frac{\partial}{\partial z^i} \otimes \frac{\partial}{\partial z^j} \otimes \frac{\partial}{\partial z^k}, \\ \widetilde{A}^{ijk} &= \sum g^{i\bar{l}} \left(A^{jk}\right)_{\bar{l}}. \end{aligned}$$

This follows from a basic differential geometry fact that the second fundamental form of any submanifold is a symmetric tensor with valued in the normal bundle. We use $\widetilde{A}$ to define a natural cubic form on $H^0(C, T_C^*)$ as follows:

Definition 11. If M is a hyperkähler manifold and C is a compact Lagrangian submanifold of M then we define a cubic form on $H^0(C, T_C^*)$,

$$\begin{aligned} \mathbf{c}_C &: \operatorname{Sym}^3 H^0(C, T_C^*) \to \mathbb{C} \\ \mathbf{c}_C(\phi, \eta, \zeta) &= \int_C \phi_i \eta_j \zeta_k \widetilde{A}^{ijk} \frac{\omega^n}{n!}. \end{aligned}$$

Remark. In fact $\mathbf{c}_C$ depends only on the extension class α since

$$\mathbf{c}_C : H^0\left(C, T_C^*\right)^{\otimes 3} \xrightarrow{\alpha} H^1\left(C, O_C\right) \otimes H^0\left(C, T_C^*\right) \xrightarrow{\otimes} H^1\left(C, T_C^*\right) = H^{1,1}\left(C\right) \xrightarrow{\Lambda} \mathbb{C}.$$

3.2.2 Moduli space of Lagrangian submanifolds

If C is Lagrangian submanifold in M then any deformation of C inside M remains a Lagrangian. Therefore the moduli space of Lagrangian submanifolds is the same as the moduli space of submanifolds representing the same cohomology class in M. We denote it by $\mathcal{M}$. This moduli space is always smooth (McLean [25]). Hitchin [18] shows that it has a natural special Kähler structure. We are going to show that this special Kähler structure on $\mathcal{M}$ is given by the cubic form

$$\mathbf{c} \in \Gamma\left(\mathcal{M}, \operatorname{Sym}^3 T_{\mathcal{M}}^*\right),$$

given by, for any given point $[C] \in \mathcal{M}$,

$$\mathbf{c}\left(C\right) = \mathbf{c}_C : \operatorname{Sym}^3 H^0\left(C, T_C^*\right) \to \mathbb{C}.$$

under the identification $T_{\mathcal{M},[C]} = H^0\left(C, T_C^*\right)$,

Moduli space as a special Kähler manifold

We first give a physical reason for the existence of a natural special Kähler structure on $\mathcal{M}$: σ-model studies maps from Riemann surfaces to a fix target manifold. When the target manifold M is hyperkähler, it is well-known that its σ-model has $N = 4$ supersymmetry (or SUSY). If domain Riemann surfaces have boundary components then we would require their images lie inside a fix Lagrangian submanifold $C \subset M$. In this case only half of the SUSY can be preserved and we have a $N = 2$ SUSY theory. Moduli space of such theories are generally known (physically) to possess special Kähler geometry. In our situation, this is simply the moduli space of Lagrangian submanifolds in M.

Let us recall the definition of a special Kähler manifold (see [13] for details). A special Kähler structure on a Kähler manifold $\mathcal{M}$ is carried by a holomorphic cubic tensor

$$\Xi \in H^0\left(\mathcal{M}, \operatorname{Sym}^3 T_{\mathcal{M}}^*\right).$$

Using the Kähler metric $g_{\mathcal{M}}$ on $\mathcal{M}$ we can identify Ξ with a tensor $\mathbf{A} \in \Omega^{1,0}(\mathcal{M}, \operatorname{End} T_{\mathbb{C}}\mathcal{M})$ as follows:

$$\Xi = -\omega_{\mathcal{M}}\left(\pi^{1,0}, \left[\mathbf{A}, \pi^{1,0}\right]\right)$$

where $\pi^{1,0} \in \Omega^{1,0}(T_{\mathbb{C}}\mathcal{M})$ is constructed from the inclusion homomorphism $T^{1,0} \subset T_{\mathbb{C}}$. In terms of local coordinates we have $(\mathbf{A}_l)_j^{\bar{k}} = i\Xi_{ljm} g_{\mathcal{M}}^{m\bar{k}}$. If we denote the Levi-Civita connection on $\mathcal{M}$ as ∇^{LC}, then the special Kähler condition is $\nabla = \nabla^{LC} + \mathbf{A} + \overline{A}$ defines a torsion-free flat symplectic connection on the tangent bundle and it satisfies $\nabla \wedge J = 0$.

Theorem 12. *Suppose $\mathcal{M}$ is the moduli space of Lagrangian submanifolds in a compact hyperkähler manifold M. Then $\mathbf{c} \in H^0(\mathcal{M}, \operatorname{Sym}^3 T^*_{\mathcal{M}})$ and it determines a special Kähler structure on $\mathcal{M}$.*

Proof. The cubic form that determines the special Kähler structure on $\mathcal{M}$ was described by Donagi and Markman in [11]. First $\mathcal{M}$ can be viewed it as the base space of the moduli space of universal compactified Jacobian $\mathcal{J}$ for Lagrangian submanifolds in M. The space $\mathcal{J}$ has a natural holomorphic symplectic structure and the natural morphism $\mathcal{J} \to \mathcal{M}$ is a Lagrangian fibration. We consider the variation of Hodge structures, i.e., periods, for this family of Abelian varieties over $\mathcal{M}$. Its differential at a point $[C] \in \mathcal{M}$ gives a homomorphism, $T_{\mathcal{M},[C]} \to S^2 V$, where $V = H^1\left(Jac(C), O_{Jac(C)}\right) = H^1(C, O_C)$ and $T_{\mathcal{M},[C]} = H^0(C, T^*_C)$. Using the induced Kähler structure on C, we can identify $H^1(C, O_C)^*$ with $H^0(C, T^*_C)$. We thus obtain a homomorphism

$$\bigotimes^3 T_{\mathcal{M},[C]} \to \mathbb{C}$$

This tensor is a symmetric tensor. The corresponding cubic form $\Gamma(\mathcal{M}, \operatorname{Sym}^3 T^*_{\mathcal{M}})$ is the one that determines the special Kähler structure on $\mathcal{M}$ (see [13]).

The above homomorphism $T_{\mathcal{M},[C]} \to S^2 V$ can be identified as the composition of the natural homomorphism $H^0\left(C, N_{C/M}\right) \to H^1(C, T_C)$ and the natural homomorphism between variation of complex structures on C and on its Jacobian variety, $H^1(C, T_C) \to H^1\left(\operatorname{Jac}(C), T_{\operatorname{Jac}(C)}\right)$. From standard Hodge theory, an Abelian variety is determined by its period, or equivalently its weight one Hodge structure, and its variation is simply given by the usual cup product homomorphism, $H^1(C, T_C) \otimes$

$H^{1,0}(C) \overset{\cup}{\to} H^{0,1}(C)$, or equivalently,

$$H^1(C, T_C) \otimes H^0(C, T_C^*) \overset{\cup}{\to} H^1(C, O_C)$$
$$\left(\phi_{\bar{j}}^i \frac{\partial}{\partial z^i} \otimes d\bar{z}^j\right) \otimes \left(\beta_k dz^k\right) \to \phi_{\bar{j}}^i \beta_i d\bar{z}^j.$$

Recall under the identification $H^0\left(C, N_{C/M}\right) \cong H^0(C, T_C^*)$ we have

$$H^0(C, T_C^*) \to H^1(C, T_C)$$
$$\alpha_i dz^i \to \alpha_i A_{\bar{j}}^{ik} \frac{\partial}{\partial z^k} \otimes d\bar{z}^j,$$

where $A = \Sigma A_{\bar{j}}^{ik} \frac{\partial}{\partial z^i} \otimes \frac{\partial}{\partial z^k} \otimes d\bar{z}^j$ is the second fundamental form of C in M. Therefore the corresponding cubic form $\otimes^3 H^0(C, T_C^*) \to \mathbb{C}$ is given by

$$(\alpha, \beta, \gamma) \to \int_C \alpha_i A_{\bar{j}}^{ik} \beta_k \gamma_l g^{l\bar{j}} \frac{\omega^n}{n!}.$$

This is precisely the cubic form $\mathbf{c}_C$. Hence the result. q.e.d.

Remark. On a special Kähler manifold $\mathcal{M}$ its Riemannian curvature tensor is given by

$$R_{i\bar{j}k\bar{l}} = -g_{\mathcal{M}}^{p\bar{q}} \Xi_{ijp} \bar{\Xi}_{jlq}.$$

This implies that the Ricci curvature of $\mathcal{M}$ is nonnegative and the scalar curvature equals $4|\Xi|^2 \geq 0$. Combining this with our earlier discussions on the second fundamental form, we show that the special Kähler metric on the moduli space of Lagrangian submanifolds is *flat* at $[C] \in \mathcal{M}$ if the natural exact sequence $0 \to T_C \to T_M|_C \to N_{C/M} \to 0$ splits.

3.3 Lagrangian category

On a *real* symplectic manifold (M, ω), Fukaya proposed a category of Lagrangian submanifold using Floer cohomology groups $HF(L_1, L_2)$, which counts holomorphic disks (i.e., instantons). This is an essential ingredient in Kontsevich's homological mirror symmetry conjecture. In this paper we study the holomorphic analog of the Fukaya category. The definition of this category is probably well-known to experts in this subject.

3.3.1 Definition of the category

Non-existence of instanton corrections

We first demonstrate the absent of *instanton corrections* in the hyperkähler setting. Suppose M is a hyperkähler manifold with a preferred complex structure J. When C is a Lagrangian submanifold in M with respect to the symplectic structure $\Omega = \omega_I + i\omega_K$ then it is a *real* Lagrangian submanifold of the *real* symplectic manifold (M, ω_θ) with $\omega_\theta = \cos\theta\omega_I + \sin\theta\omega_K$ for any $\theta \in [0, 2\pi)$. For any fixed θ, *instantons* means J_θ-holomorphic disks bounding C.

Lemma 13. *If C is a Lagrangian submanifold of a compact hyperkähler manifold M, then for all $\theta \in [0, 2\pi)$, with at most one exception, there is no J_θ-holomorphic disk in M bounding C.*

Proof. Suppose D is a J_θ-holomorphic disk in M with $\partial D \subset C$, say $\theta = 0$. For the complex structure $J_0 = I$, $\omega_K + i\omega_J$ is a holomorphic two form on M and therefore restricts to zero of on any I-holomorphic disk. On the other hand ω_I is a Kähler form and hence it is positive on D. Therefore we have

$$\omega_J = \omega_K = 0, \omega_I > 0,$$

on D. We consider the integration of ω_I and ω_K on D, since these two forms restrict to zero on C we have well-defined homomorphisms

$$\int \omega_I : H_2(M, C; \mathbb{Z}) \to \mathbb{R} \text{ and } \int \omega_K : H_2(M, C; \mathbb{Z}) \to \mathbb{R}.$$

From earlier discussions $[D, \partial D]$ represents a class in $H_2(M, C; \mathbb{Z})$ which must lie in the kernel of $\int \omega_K$ and not in the kernel of $\int \omega_I$, in fact $\int_D \omega_I > 0$. Therefore, with a fix class in $H_2(M, C; \mathbb{Z})$, there is at most one θ which can support J_θ-holomorphic disks representing the given class. q.e.d.

Definition of Lagrangian category

Definition 14. Given any hyperkähler manifold M, we define a category $\mathcal{C}_M$ or simply $\mathcal{C}$, called the Lagrangian category of M as follows: An object in $\mathcal{C}$ is a Lagrangian subvariety of M. Given two objects $C_1, C_2 \in \mathcal{C}^{\text{obj}}$ we define the space of morphisms to be the $\mathbb{Z}$-graded Abelian group

$$\operatorname{Hom}_{\mathcal{C}}(C_1, C_2)^{[k]} = \operatorname{Ext}^k_{O_M}(O_{C_1}, O_{C_2}).$$

The composition of morphisms is given by the natural product structure on the *Ext*'s groups. Because of the Serre duality and $K_M = O_M$, the above category carries a natural *duality* property,

$$\operatorname{Ext}^k_{O_M}(O_{C_1}, O_{C_2}) \cong \operatorname{Ext}^{2n-k}_{O_M}(O_{C_2}, O_{C_1})^*,$$

provided that M is compact.

A symplectic 2-category

In our situation an object is a Lagrangian C in a fix hyperkähler manifold M. In *real* symplectic geometry Weinstein ([1] and [32]) defines a symplectic category whose objects are symplectic manifolds and morphisms are immersed Lagrangian submanifolds inside $M_2 \times \overline{M}_1$. Here $\overline{M}$ denote the symplectic manifold M with the symplectic form $-\omega$. When M is a hyperkähler manifold, the complex structure of $\overline{M}$ becomes $-J$. In fact we can combine the two approaches together and define a *symplectic 2-category*: The objects are hyperkähler manifolds, 1-morphisms from M_1 to M_2 are Lagrangian subvarieties in $M_2 \times \overline{M}_1$, 2-morphisms between two Lagrangian subvarieties $C_1, C_2 \subset M_2 \times \overline{M}_1$ are given by $Ext^*_{O_{M_2 \times \overline{M}_1}}(O_{C_1}, O_{C_2})$.

The category of Lagrangian coherent sheaves

The Lagrangian category of M is geometric in nature but it does not have very good functorial properties. Therefore we also need another category.

Definition 15. Let M be a projective hyperkähler manifold. Let $D^b(M)$ be the derived category of coherent sheaves on M. We define the category of Lagrangian coherent sheaves $D^b_{\mathrm{Lag}}(M)$ to be the subcategory of $D^b(M)$ generated by those coherent sheaves $\mathcal{S}$ satisfying $\operatorname{ch}(\mathcal{S}) \cup \Omega \in \oplus_{k>2n+2} H^k(M, \mathbb{C})$.

By the Hard Lefschetz sl_2-action using L_Ω and Λ_Ω, the condition $\operatorname{ch}(\mathcal{S}) \cup \Omega \in \oplus_{k>2n+2} H^k(M, \mathbb{C})$ implies that $\operatorname{ch}_k(\mathcal{S}) = 0$ for $k < n$ and $\operatorname{ch}_n(\mathcal{S})$ is a Ω-primitive cohomology class. In particular the n dimensional support of $\mathcal{S}$ is a Lagrangian in M. For example if C is a Lagrangian subvariety in M then O_C is an object in $D^b_{\mathrm{Lag}}(M)$.

3.3.2 Lagrangian intersection

The Lagrangian category $\mathcal{C}_M$ (and similar for $D^b_{\mathrm{Lag}}(M)$) is closely related to the intersection theory for Lagrangian subvarieties in M.

Theorem 16. *If C_1 and C_2 are two Lagrangian subvarieties of a compact hyperkähler manifold M then*

$$\begin{aligned}\chi\left(\operatorname{Ext}^*_{O_M}(O_{C_1}, O_{C_2})\right) &= \textstyle\sum_k \dim\,(-1)^k \operatorname{Ext}^k_{O_M}(O_{C_1}, O_{C_2})\\ &= (-1)^n\, C_1\cdot C_2.\end{aligned}$$

Proof. We recall the Riemann-Roch formula for the global Ext groups: For any coherent sheaves S_1 and S_2 on M we have,

$$\textstyle\sum_k \dim\,(-1)^k \operatorname{Ext}^k_{O_M}(S_1, S_2) = \displaystyle\int_M \overline{\mathrm{ch}}\,(S_1)\,\mathrm{ch}\,(S_2)\,Td_M$$

where $\overline{\mathrm{ch}}\,(S_1) = \Sigma\,(-1)^k\,\mathrm{ch}_k\,(S_1)$. For $S_i = O_{C_i}$ the structure sheaf of a subvariety C_i of dimension n, we have $\mathrm{ch}_k\,(O_{C_i}) = 0$ for $k < n$ and $\mathrm{ch}_n\,(O_{C_i}) = [C_i]$. Combining these, we have

$$\begin{aligned}&\chi\left(Ext^*_{O_M}(O_{C_1}, O_{C_2})\right)\\ &= \int_M ((-1)^n\,[C_1] + \text{h.o.t.})\,([C_2] + \text{h.o.t.})\,(1 + \text{h.o.t.})\\ &= (-1)^n\, C_1\cdot C_2.\end{aligned}$$

Here h.o.t. refers to *higher order terms* which do not contribute to the outcome of the integral. Hence the result. q.e.d.

If C_1 and C_2 intersect cleanly along D then $C_1\cdot C_2$ equals the Euler characteristic of D up to sign. To prove this we use the following useful lemma whose proof is standard [1]:

Lemma 17. *If C_1 and C_2 are two Lagrangian submanifolds of a hyperkähler manifold M which intersect cleanly along $D = C_1\cap C_2$, then the symplectic form on M induces a nondegenerate pairing*

$$N_{D/C_1}\otimes N_{D/C_2}\to O_D.$$

Using the standard intersection theory, we have

$$\begin{aligned}C_1\cdot C_2 &= e\left(\left(N_{C_1/M}\oplus N_{C_2/M}\right)|_D - N_{D/M}\right) = e\,(T^*_D)\\ &= (-1)^{\dim D}\, e\,(C_1\cap C_2)\,.\end{aligned}$$

Corollary 18. *If C_1 and C_2 are two Lagrangian submanifolds in a compact hyperkähler manifold M which intersect cleanly, then*

$$C_1\cdot C_2 = (-1)^{\dim C_1\cap C_2}\, e\,(C_1\cap C_2)\,.$$

When C_1 and C_2 intersect cleanly we expect that individual $\mathrm{Ext}^k_{O_M}(O_{C_1}, O_{C_2})$ can be computed.

Theorem 19. *If C is a Lagrangian submanifold of a compact hyperkähler manifold M, then there is an isomorphism of vector spaces,*

$$\mathrm{Ext}^k_{O_M}(O_C, O_C) \cong H^k(C, \mathbb{C}),$$

for all k provided that the normal bundle of C can be extended to the whole M. For instance it holds true when C is a complete intersection in M.

Proof. We consider a Koszul resolution of O_C in M,

$$0 \to \Lambda^n E \to \Lambda^{n-1} E \to \cdots \to E \to O_M \to O_C \to 0.$$

The groups $\mathrm{Ext}^*_{O_M}(O_C, O_C)$ can be computed as the hypercohomology of the complex of sheaves $\underline{\mathrm{Hom}}_{O_M}(\Lambda^* E, O_C)$. Note that

$$\underline{\mathrm{Hom}}_{O_M}(\Lambda^q E, O_C) \cong \underline{\mathrm{Hom}}_{O_C}(O_C, \Lambda^q T^*_C) \cong \Omega^q(C).$$

From the definition of the Koszul complex, the restriction of its dual complex $\Lambda^* E^*$ to C has trivial differentials. Therefore

$$\begin{aligned}&\mathrm{Ext}^k_{O_M}(O_C, O_C) \cong \mathbb{H}\left(0 \to O_C \xrightarrow{0} \Omega^1(C) \xrightarrow{0} \dots \xrightarrow{0} \Omega^n(C) \to 0\right)\\ &\cong \textstyle\bigoplus_{p+q=k} H^q(C, \Omega^p(C)) \cong H^k(C, \mathbb{C}).\end{aligned}$$

The last equality uses the Kählerian property of C. Hence the result. q.e.d.

Theorem 20. *If C_1 and C_2 are two Lagrangian submanifolds of a compact hyperkähler manifold M which intersect transversely along $C_1 \cap C_2 = \{p_1, \dots, p_s\}$, then there is an isomorphism of vector spaces,*

$$\mathrm{Ext}^k_{O_M}(O_{C_1}, O_{C_2}) \cong \begin{cases} 0 & \text{if } k \neq n \\ \bigoplus_{p_i} \mathbb{C} \cong \mathbb{C}^s & \text{if } k = n. \end{cases}$$

Proof. We can assume that $C_1 \cap C_2 = \{p\}$ and we take a Koszul resolution of O_{C_1} as before. By transversality, the restriction of this resolution to C_2 gives a Koszul resolution of O_p in C_2,

$$0 \to \Lambda^n F \to \Lambda^{n-1} F \to \cdots \to F \to O_{C_2} \to O_p \to 0,$$

where F is the restriction of E to C_2. And the dual of the Koszul sequence equals the original Koszul sequence, up to tensoring with the top exterior power of the bundle (thanks to the referee).

Now,

$$\begin{aligned}\operatorname{Ext}^k_{O_M}(O_{C_1}, O_{C_2}) &\cong \mathbb{H}^k\left(\operatorname{Hom}_{O_M}(\Lambda^* E, O_{C_2})\right) \\ &\cong \mathbb{H}^k\left(\operatorname{Hom}_{O_{C_2}}(\Lambda^q F, O_{C_2})\right).\end{aligned}$$

Thus the result follows. q.e.d.

Intersection Euler characteristics

From previous discussions the Euler characteristic of any smooth subvariety S in $\mathbb{P}^n$ equals

$$\chi(S) = (-1)^{\dim S} N^*_{S/\mathbb{P}^n} \cdot \mathbb{P}^n.$$

Even though the intersection of the two Lagrangians $N^*_{S/\mathbb{P}^n}$ and $\mathbb{P}^n$ is taken place in a noncompact manifold, namely $T^*\mathbb{P}^n$, their intersection number is well-defined in this case because their intersection occurs inside a compact region and do not intersect even *at infinity*. For singular variety in $\mathbb{P}^n$ we use this as a definition.

Definition 21. For any subvariety S in $\mathbb{P}^n$ we define its intersection Euler characteristic to be the following intersection number inside $T^*\mathbb{P}^n$:

$$\overline{\chi}(S) = (-1)^{\dim S} N^*_{S/\mathbb{P}^n} \cdot \mathbb{P}^n,$$

where $N^*_{S/\mathbb{P}^n}$ is the conormal variety of S.

For example if S is any degree d plane curve with δ doubles points, κ cusps and no other singularities, we have

$$\overline{\chi}(S) = d^2 - 3d + 2\delta + 3\kappa.$$

3.3.3 Reduction functor and projection functor

In this section we assume all subvarities involved are projective. Given any coisotropic submanifold D in M it has an integrable distribution $(T_D)^\perp \subset T_D$. We denote the natural projection to the leave space B as

$$\pi_D : D \to B.$$

When B is smooth, it has a natural holomorphic symplectic structure, called the *reduction* of D, for simplicity we assume B is also hyperkähler.

Reduction and projection of a Lagrangian

Suppose that C is a Lagrangian subvariety of M and we denote its smooth locus as C^{sm}. We construct a subvariety C^{red} in B (resp. C^{proj} in M) called the *reduction* of C (resp. *projection* of C) as follows:

$$C^{\mathrm{red}} = \overline{\pi_D\left(C^{\mathrm{sm}} \cap D\right)} \subset B.$$

and

$$\begin{array}{ccccc} C^{\mathrm{proj}} & \subset & D & \subset & M \\ \downarrow & \square & \downarrow & & \\ C^{\mathrm{red}} & \subset & B. & & \end{array}$$

Both C^{red} in B and C^{proj} in M are Lagrangian subvarieties. This does not yet define functors on Lagrangian categories because it is not so easy to see how to construct the functor on the morphism level. For this purpose the derived category of Lagrangian coherent sheaves $D^b_{\mathrm{Lag}}(\bullet)$ serves a better role.

We define a functor between derived categories called the *reduction functor* as follows: We consider the subvariety $B \times_B D \subset B \times M$ and denote the projection morphism from $B \times M$ to its first and second factor as π_B and π_M respectively then we define

$$\mathbf{R}_D : D^b_{\mathrm{Lag}}(M) \to D^b_{\mathrm{Lag}}(B)$$
$$\mathbf{R}_D(\bullet) = R\pi_{B*}\left(O_{B\times_B D} \overset{L}{\otimes} \pi_M^*(\bullet)\right).$$

We can also define a *projection functor* as follows: We consider a subvariety $D \times_B D \subset M \times M$ and denote the projection morphisms from $M \times M$ to its first and second factors as π_1 and π_2 then we define

$$\mathbf{P}_D : D^b_{\mathrm{Lag}}(M) \to D^b_{\mathrm{Lag}}(M)$$
$$\mathbf{P}_D(\bullet) = R\pi_{1*}\left(O_{D\times_B D} \overset{L}{\otimes} \pi_2^*(\bullet)\right).$$

It can be checked that the image of any Lagrangian coherent sheaf under $\mathbf{P}_D$ or $\mathbf{R}_D$ is indeed a Lagrangian coherent sheaf.[4]

[4]To be precise we should talk about complex of sheaves in the dervied category.

4. Legendre transformation

In this section we will study a Legendre transformation of Lagrangian subvarieties along a coisotropic exceptional subvariety in M and establish a *Plücker type formula* which relates intersection numbers of Lagrangian subvarieties under the Legendre transformation.

4.1 Classical Legendre transformation

Legendre transform on vector spaces

Suppose V is a finite dimensional vector space and $f : V \to \mathbb{C}$ is a function such that its Hessian is nondegenerate at every point. We look at the graph of df in $T^*V = V \times V^*$ then the *Legendre transformation* induced by f is the map,

$$\mathcal{L}_f : V \to V^*,$$
$$\mathcal{L}_f = \pi_{V^*} \circ df,$$

where π_{V^*} is the projection from $V \times V^*$ to its second factor V^*. That is $\mathcal{L}_f(x) = \xi$ if and only if $\xi_i = \frac{\partial f}{\partial x^i}$ for all i in local coordinates.

We also define the Legendre transformation of the function f on V as a function

$$f^\vee : V^* \to \mathbb{C},$$
$$f^\vee(\xi) = \Sigma x^i \xi_i - f(x),$$

with $\xi = \mathcal{L}_f(x)$.

Geometrically this transformation arises from the natural isomorphism,

$$T^*V \cong V \times V^* \cong T^*(V^*).$$

Now the graph C of df in T^*V is a Lagrangian submanifold of T^*V with its canonical symplectic form. Using the above isomorphism we can treat $C \subset T^*(V^*)$ as another Lagrangian submanifold.[5] Under the non-degeneracy assumption C is also a graph of a function on V^*,

[5]Under the natural isomorphism $T^*V \cong T^*(V^*)$, their canonical symplectic forms are identified up to a minus sign. In particular they have identical Lagrangian submanifolds.

this function is precisely the above $f^\vee$. It is obvious that the Legendre transformation is involutive.

A different perspective: conormal bundles

Besides the graph of df, there is another natural Lagrangian submanifold in T^*V associated to f, namely the conormal bundle of the zero set of f, $N^*_{S/V} \subset T^*V$, where $S = \{x \in V : f(x) = 0\}$. When f is *homogeneous* this Lagrangian submanifold, considered inside $T^*(V^*)$, turns out to be the conormal bundle of the zero set of $f^\vee$. This allows us to define the Legendre transformation in much greater generality, at least in the homogeneous case which is just the right setting for the projective geometry.

Theorem 22. *Suppose $f : V \to \mathbb{C}$ is a homogenous polynomial such that its zero set $S = \{x \in V : f(x) = 0\}$ is smooth. Then its conormal bundle $N^*_{S/V} \subset T^*V$, when viewed as a submanifold of $T^*(V^*)$, equals $N^*_{S^\vee/V^*}$ the conormal variety of $S^\vee = \{\xi \in V^* : f^\vee(\xi) = 0\} \subset V^*$.*

Proof. When f is a homogeneous polynomial of degree p, we have $\Sigma x^i \frac{\partial f}{\partial x^i} = pf(x)$. Under the Legendre transformation $\xi_i = \frac{\partial f}{\partial x^i}(x)$ this gives,

$$f^\vee(\xi) = \Sigma x^i \xi_i - f(x) = pf(x) - f(x) = (p-1) f(x).$$

The conormal bundle of S is given by,

$$N^*_{S/V} = \Big\{ (x, \eta) \in V \times V^* : f(x) = 0 \quad \text{and} \quad \eta_i = c \frac{\partial f}{\partial x^i}(x) \text{ for all } i, \text{ for some } c \Big\}.$$

So we need to verify that the same set can be described as

$$\left\{ (x, \eta) \in V \times V^* : f^\vee(\eta) = 0 \text{ and } x^i = b \frac{\partial f^\vee}{\partial \xi_i}(\eta) \text{ for all } i, \text{ for some } b \right\}.$$

Now we suppose $(x, \eta) \in N^*_{S/V}$. Since f is homogenous of degree p, we have

$$\frac{\partial f}{\partial x^i}(ex) = e^{p-1} \frac{\partial f}{\partial x^i}(x),$$

for any number e. Therefore $\eta_i = c\frac{\partial f}{\partial x^i}(x) = \frac{\partial f}{\partial x^i}(c'x)$ for some constant c'. That is $\eta = L_f(c'x)$. Hence

$$f^\vee(\eta) = f^\vee\left(L_f\left(c'x\right)\right) = (p-1)\, f\left(c'x\right) = (p-1)\left(c'\right)^p f(x) = 0.$$

Similarly using the inverse Legendre transformation, we have $\frac{\partial f^\vee}{\partial \xi_i}(\eta) = c'x^i$. Hence the result. q.e.d.

One can also find an indirect proof of this in [16]. Because of this result, we can now define the Legendre transformation for any finite set of homogenous polynomials via the conormal bundle of their common zero set. This approach works particularly well for projective spaces and it is closely related to the dual variety construction.

4.2 Legendre transform in hyperkähler manifolds

4.2.1 Legendre transform in $T^*\mathbb{P}^n$ and dual varieties

Because of Theorem 22 we define the *Legendre transformation* in the projective setting as follows: For any homogenous function $f : V \to \mathbb{C}$ which defines a smooth hypersurface $S = \{f = 0\} \subset \mathbb{P}^n$, the function itself can be recovered from its conormal bundle $N^*_{S/\mathbb{P}^n}$ inside $T^*\mathbb{P}^n$. The Legendre transform $f^\vee : V^* \to \mathbb{C}$ defines the dual hypersurface $S^\vee \subset \mathbb{P}^{n*}$ under the nondegenerate assumption and we have

$$N_{S^\vee/\mathbb{P}^{n*}} = \overline{\Phi\left(N_{S/\mathbb{P}^n} \backslash \mathbb{P}^n\right)},$$

where $\mathbb{P}^n$ denote the zero section in $T^*\mathbb{P}^n$.

An arbitrary Lagrangian subvariety C in $T^*\mathbb{P}^n$ can be regarded as a generalized homogenous function on V unless C is the zero section $\mathbb{P}^n$. Motivated from above discussions, we define the *Legendre transformation* $C^\vee$ of C as follows:

$$C^\vee = \overline{\Phi\left(C \backslash \mathbb{P}^n\right).}$$

It has the following immediate properties:

(i) $C^\vee$ is a Lagrangian subvariety of $T^*\mathbb{P}^{n*}$;

(ii) $C \subset T^*\mathbb{P}^n$ and $C^\vee \subset T^*\mathbb{P}^{n*}$ are isomorphic outside the zero sections;

(iii) the inversion property $\left(C^{\vee}\right)^{\vee} = C$.

Remark. Recall that $T^*\mathbb{P}^n$ is the hyperkähler quotient of T^*V by the natural S^1 action. In fact the Legendre transformation on $T^*\mathbb{P}^n$ can be regarded as a S^1-invariant Legendre transformation on T^*V using the symplectic quotient by S^1: If C is a Lagrangian in $T^*\mathbb{P}^n$ then there is a unique $\mathbb{C}^{\times}$-invariant Lagrangian subvariety of T^*V, denote D such that C is the symplectic quotient of D by S^1. The Legendre transformation of D will be a Lagrangian subvariety $D^{\vee}$ in $T^*(V^*)$ which is again $\mathbb{C}^{\times}$-invariant. The symplectic quotient of $D^{\vee}$ by S^1 would be our transformation $C^{\vee}$ in $T^*\mathbb{P}^{n*}$.

Dual varieties in $\mathbb{P}^n$ and a Plücker formula

For any subvariety S in $\mathbb{P}^n$ we can associated a dual variety $S^{\vee}$ in the dual projective space $\mathbb{P}^{n*}$. The dual variety $S^{\vee}$ is the closure of all hyperplanes in $\mathbb{P}^n$ which are tangent to some smooth point in S. From our previous discussions, the conormal variety of $S^{\vee}$ is simply the Legendre transformation of the conormal variety of S. In this sense our Legendre transformation is a generalization of the dual variety construction. The relationship between dual varieties and the Legendre transformation has been briefly addressed before in various places.

For plane curves, there are Plücker formulae which related various geometric quantities between S and $S^{\vee}$: Suppose $S \subset \mathbb{P}^2$ is a plane curve of degree d with δ double points, κ cusps and no other singularities, we denote the corresponding quantities for $S^{\vee}$ as $d^{\vee}, \delta^{\vee}, \kappa^{\vee}$. Then Plücker formulae say

$$\begin{aligned} d^{\vee} &= d(d-1) - 2\delta - 3\kappa \\ \kappa^{\vee} &= 3d^2 - 6d - 6\delta - 8\kappa. \end{aligned}$$

A similar formula in the higher dimensional setting is obtained by Kleiman in [22].

In the next section we will discuss a similar formula for the Legendre transformation in any hyperkähler manifold. In the case of conormal varieties inside $T^*\mathbb{P}^n$ the formula says: For any subvarieties $S_i \subset \mathbb{P}^n$ of dimension s_i we denote their conormal varieties as $C_i \subset T^*\mathbb{P}^n$ then we have

$$C_1 \cdot C_2 + \frac{(C_1 \cdot \mathbb{P}^n)(C_2 \cdot \mathbb{P}^n)}{(-1)^{n+1}(n+1)} = C_1^{\vee} \cdot C_2^{\vee} + \frac{(C_1^{\vee} \cdot \mathbb{P}^{n*})(C_2^{\vee} \cdot \mathbb{P}^{n*})}{(-1)^{n+1}(n+1)}.$$

In the special case of plane curves, the intersection number of their conormal varieties $C_1 \cdot C_2$ is simply the product of the degree of the curves $d_1 d_2$. In fact the proof of the formula can be reduced to the case when S_2 is a point and it reads as follows: For any plane curve S we have

$$3d^{\vee} = -\overline{\chi}(S) - 2\overline{\chi}\left(S^{\vee}\right).$$

When S has only double point and cusp singularities, the above formula can be proven by the Plücker formulae. Conversely the Plücker formula $d^{\vee} = d(d-1) - 2\delta - 3\kappa$ also follows from it and our earlier formula for $\overline{\chi}(S)$.

4.2.2 Legendre transform along $\mathbb{P}^n$ and a Plücker type formula

Now we study the Legendre transformation on a general hyperkähler manifold M of dimension $2n \geq 4$.

Flop along $\mathbb{P}^n$

Recall that every embedded $\mathbb{P}^n$ in M is a Lagrangian submanifold and therefore its normal bundle is the cotangent bundle of $\mathbb{P}^n$. By the Bott's vanishing results of cohomology groups of $\mathbb{P}^n$, this first order identification along $\mathbb{P}^n$ between $T^*\mathbb{P}^n$ and M extends uniquely to analytic neighborhoods of $\mathbb{P}^n$ (this conclusion also follows from the contractibility of $\mathbb{P}^n$ inside M and the rigidity of the resulting singularity). Then we can apply arguements in Weinstein's Darboux theorem to conclude that $\mathbb{P}^n$ in M has a neighborhood U which is symplectomorphic to a neighborhood of the zero section in $T^*\mathbb{P}^n$ and we continue to denote it by U.

Therefore we can *flop* such a $\mathbb{P}^n$ in M to obtain another holomorphic symplectic manifold M'. To see how this surgery work, we look at the birational transformation $\Phi : T^*\mathbb{P}^n \dashrightarrow T^*\mathbb{P}^{n*}$ and let U' be the image of U, i.e., $U' = \overline{\Phi(U)}$. Then $M' = (M \backslash U) \cup U'$.[6] Since M and M' are isomorphic outside a codimension n subspace, M' inherits a holomorphic two form Ω' from M by the Hartog's theorem. Moreover being a section of the canonical line bundle and non-vanishing outside a codimension n subset, $(\Omega')^n$ must be non-vanishing everywhere. That

[6] To be precise with the holomorphic structure on M' we should write $M' = \left(M \backslash \overline{U_0}\right) \cup U'$ for some open set $U_0 \supset \mathbb{P}^n$ satisfying $\overline{U_0} \subset U$.

is M' is a holomorphic symplectic manifold with the symplectic form Ω'. In most cases M' is actually a hyperkähler manifold, nevertheless Namikawa had a simple example [27] showing that this cannot always be the case.

We will denote the natural birational map between M and M' as

$$\Phi_M : M \dashrightarrow M'.$$

Legendre transformation and a Plücker type formula

Suppose M and M' are hyperkähler manifolds which are related by a flop $\Phi_M : M \dashrightarrow M'$ along a $\mathbb{P}^n \subset M$. For any Lagrangian subvariety C in M which does not contain $\mathbb{P}^n$ we define its Legendre transform to be the Lagrangian subvariety $C^\vee$ in M' defined as

$$C^\vee = \overline{\Phi_M \left(C \backslash \mathbb{P}^n\right)}.$$

Since $\left(C_1 \cup C_2\right)^\vee = C_1^\vee \cup C_2^\vee$, we can extend the definition of the Legendre transformation to the free Abelian group generated by all Lagrangian subvarieties of M except $\mathbb{P}^n$.

Clearly the Legendre transformation has the inversion property, namely $\left(C^\vee\right)^\vee = C$. However the Legendre transformation does not preserve the intersection numbers, i.e., $C_1 \cdot C_2 \neq C_1^\vee \cdot C_2^\vee$. Instead they satisfy the following Plücker type formula [23]:

Theorem 23. *Suppose $\Phi_M : M \dashrightarrow M'$ is a flop along $\mathbb{P}^n$ between projective hyperkähler manifolds then*

$$C_1 \cdot C_2 + \frac{\left(C_1 \cdot \mathbb{P}^n\right)\left(C_2 \cdot \mathbb{P}^n\right)}{(-1)^{n+1}(n+1)} = C_1^\vee \cdot C_2^\vee + \frac{\left(C_1^\vee \cdot \mathbb{P}^{n*}\right)\left(C_2^\vee \cdot \mathbb{P}^{n*}\right)}{(-1)^{n+1}(n+1)},$$

for any Lagrangian subvarieties C_1 and C_2 not containing $\mathbb{P}^n$.

In the above Plücker type formula, it is more natural to interpret the LHS (and similar for the RHS) as follows:

$$\begin{aligned} & C_1 \cdot C_2 + \frac{\left(C_1 \cdot \mathbb{P}^n\right)\left(C_2 \cdot \mathbb{P}^n\right)}{(-1)^{n+1}(n+1)} \\ & = \left(C_1 - \frac{\left(C_1 \cdot \mathbb{P}^n\right)}{\left(\mathbb{P}^n \cdot \mathbb{P}^n\right)}\mathbb{P}^n\right) \cdot \left(C_2 - \frac{\left(C_2 \cdot \mathbb{P}^n\right)}{\left(\mathbb{P}^n \cdot \mathbb{P}^n\right)}\mathbb{P}^n\right). \end{aligned}$$

Note that $\left(C - \frac{(C \cdot \mathbb{P}^n)}{(\mathbb{P}^n \cdot \mathbb{P}^n)}\mathbb{P}^n\right) \cdot \mathbb{P}^n = 0$ for any Lagrangian subvariety C in M, including $C = \mathbb{P}^n$. Roughly speaking the LHS (resp. RHS) of the

Plücker type formula is the intersection number of two Lagrangian subvarieties which do not intersect $\mathbb{P}^n$, and therefore the Legendre transformation along such a $\mathbb{P}^n$ should have no effect to their intersection numbers, thus giving the Plücker type formula in a heuristic way.

Normalized Legendre transformation

It is suggested from above discussions that we should modify the transformation so that

$$C - \frac{(C \cdot \mathbb{P}^n)}{(\mathbb{P}^n \cdot \mathbb{P}^n)} \mathbb{P}^n \to C^\vee - \frac{(C^\vee \cdot \mathbb{P}^{n*})}{(\mathbb{P}^{n*} \cdot \mathbb{P}^{n*})} \mathbb{P}^{n*}.$$

We also want transform $\mathbb{P}^n$, the center of the flop. Thus we arrive to the following definition of a *normalized Legendre transformation* $\mathcal{L}$:

$$\mathcal{L}(C) = C^\vee + \frac{(C \cdot \mathbb{P}^n) + (-1)^{n+1} (C^\vee \cdot \mathbb{P}^{n*})}{n+1} \mathbb{P}^{n*} \text{ if } C \neq \mathbb{P}^n$$
$$\mathcal{L}(\mathbb{P}^n) = (-1)^n \mathbb{P}^{n*}.$$

Now our Plücker type formula can be rephrased as the following simple identity:

$$C_1 \cdot C_2 = \mathcal{L}(C_1) \cdot \mathcal{L}(C_2).$$

Unlike our earlier Legendre transformation, this normalized Legendre transformation is rather nontrivial even when $n = 1$. When M is a K3 surface every embedded $\mathbb{P}^1$ is also called an (-2)-curve because $\mathbb{P}^1 \cdot \mathbb{P}^1 = -2$. Flopping $\mathbb{P}^1$ in M is trivial, i.e., $M = M'$. This is because a point in $\mathbb{P}^1$ is also a hyperplane. Therefore the Legendre transformation $C \to C^\vee = C$ is just the identity transformation. However the normalized Legendre transformation is given by

$$\mathcal{L}(C) = C - \left(C \cdot \mathbb{P}^1\right) \mathbb{P}^1,$$

which is well-defined for any cohomology class of the K3 surface M and induces an automorphism of $H^2(M, \mathbb{Z})$, namely the reflection with respect to the class $\left[\mathbb{P}^1\right]$. We can identify $\mathcal{L}(C)$ as the Dehn twist of C along the Lagrangian $\mathbb{P}^1$, or S^2, in M. If we use all the (-2)-curves in the whole twistor family then their corresponding Dehn twists generate $Aut\left(H^2(M, \mathbb{Z})\right)$, which gives all diffeomorphisms of M up to isotopy by a result of Donaldson.

Remark. It would be interesting to compare our normalized Legendre transformation with the Dehn twist as studied by P. Seidel [28] or the one by Seidel and Thomas [29].

Suppose M is a projective hyperkähler manifold of dimension $2n$ and

$$\pi : M \to Z$$

is a projective contraction with normal exceptional locus D in M. We assume that $\pi(D)$ is a single point, i.e., Z has isolated singularity. Then:

(i) $D \cong \mathbb{P}^n$ if $n \geq 2$;

(ii) D is an ADE configuration of $\mathbb{P}^1$ if $n = 1$.

Conversely every such D can be contracted inside M. Moreover the normalized Legendre transformation is defined in all these cases and is very interesting.

A categorical transformation

Now we have a transformation $\mathcal{L}$ which takes Lagrangians in M to Lagrangians in M' and it respects their intersection numbers. It is then natural to wonder if we have a categorical transformation between the Lagrangian categories of M and M', namely we want to have an isomorphism on the cohomology groups,

$$\operatorname{Ext}^q_{O_M}(O_{C_1}, O_{C_2}) \xrightarrow{\cong} \operatorname{Ext}^q_{O_{M'}}\left(O_{\mathcal{L}(C_1)}, O_{\mathcal{L}(C_2)}\right)$$

rather than just their Euler characteristics, i.e., intersection numbers.

Suppose $\Phi : M \dashrightarrow M'$ is a flop along $P = \mathbb{P}^n$ as before. Let $\widetilde{M}$ be the blow up of M along P and we denote its exceptional divisor as $\widetilde{P}$. $\widetilde{P}$ admits two $\mathbb{P}^{n-1}$-fibration over $\mathbb{P}^n$ provided that $n \geq 2$. A point in $\widetilde{P}$ is a pair (p, H) with $H \subset \mathbb{P}^n$ a hyperplane and $p \in H$. The two $\mathbb{P}^{n-1}$-fibrations correspond to sending the above point to p and H respectively. We can blow down $\widetilde{P}$ along the second fibration to obtain M', we write these morphisms as

$$M \xleftarrow{\pi} \widetilde{M} \xrightarrow{\pi'} M'.$$

Now we define a Legendre functor

$$\mathbf{L} : D^b_{\mathrm{Lag}}(M) \to D^b_{\mathrm{Lag}}\left(M'\right)$$
$$\mathbf{L}(\bullet) = \pi'_* \pi^* (\bullet).$$

It is not difficult to show that the image of any Lagrangian coherent sheaf on M under $\mathbf{L}$ is a Lagrangian coherent sheaf on M'. Using the machinery developed by Bondal and Orlov in [4] $\mathbf{L}$ defines a equivalence of categories [23]. When $\mathcal{S} = O_C$ for a Lagrangian subvariety C in M, then $\mathbf{L}(\mathcal{S})$ should be closely related to $O_{\mathcal{L}(C)}$. This transformation will play an important role in the proof of the Plücker type formula [23].

4.2.3 Legendre transform on Mukai elementary modification

Now we consider a general projective contraction on M, as discussed in Section 2.2, with smooth discriminant locus B,

$$\begin{array}{ccc} D & \subset & M \\ \downarrow & & \downarrow \pi \\ B & \subset & Z. \end{array}$$

The exceptional locus D is a $\mathbb{P}^k$-bundle over B, which has a natural symplectic form. Moreover the normal bundle of D in M is the relative cotangent bundle of $D \to B$. The Mukai elementary modification produces another symplectic manifold M' by replacing the $\mathbb{P}^k$-bundle $D \to B$ with the dual $\mathbb{P}^{k*}$-bundle over B.

$$\begin{array}{ccccccc} D & \subset & M & \overset{\Phi_M}{\dashrightarrow} & M' & \supset & D' \\ \mathbb{P}^k \downarrow & & \downarrow \pi & & \pi' \downarrow & & \downarrow \mathbb{P}^{k*} \\ B & \subset & Z & = & Z & \supset & B \end{array}$$

We define the Legendre transformation of a Lagrangian subvariety C in M not lying inside D as follows:

$$C^\vee = \overline{\Phi(C \backslash D)}.$$

In order to write down the Plücker type formula in this general situation, we need to recall from Section 3.3.3 that C determines:

(i) a Lagrangian subvariety C^{proj} in M lying inside D, called the *projection*;

(ii) a Lagrangian subvariety C^{red} in B, called the *reduction.*

The restriction of π to C^{proj} is a $\mathbb{P}^k$-bundle over $C^{\mathrm{red}} \subset B$ and $C^{\mathrm{proj}} = \pi^{-1}\left(C^{\mathrm{red}}\right) \subset M$. It is not difficult to see that we have $(C^\vee)^{\mathrm{proj}} = (\pi')^{-1}\left(C^{\mathrm{red}}\right) \subset M'$ and

$$C^{\mathrm{proj}} \cdot C^{\mathrm{proj}} = C^{\vee\mathrm{proj}} \cdot C^{\vee\mathrm{proj}}.$$

The *Plücker type formula* in this general case reads as follows:

$$\left(C_1-\frac{C_1\cdot C_1^{\mathrm{proj}}}{C_1^{\mathrm{proj}}\cdot C_1^{\mathrm{proj}}}C_1^{\mathrm{proj}}\right)\cdot\left(C_2-\frac{C_2\cdot C_2^{\mathrm{proj}}}{C_2^{\mathrm{proj}}\cdot C_2^{\mathrm{proj}}}C_2^{\mathrm{proj}}\right)$$
$$=\left(C_1^{\vee}-\frac{C_1^{\vee}\cdot C_1^{\vee\mathrm{proj}}}{C_1^{\vee\mathrm{proj}}\cdot C_1^{\vee\mathrm{proj}}}C_1^{\mathrm{proj}}\right)\cdot\left(C_2^{\vee}-\frac{C_2^{\vee}\cdot C_2^{\vee\mathrm{proj}}}{C_2^{\vee\mathrm{proj}}\cdot C_2^{\vee\mathrm{proj}}}C_2^{\vee\mathrm{proj}}\right).$$

It can be proven using the same method as in the previous situation. *The normalized Legendre transformation* $\mathcal{L}$ for this general case is defined as follows:

$$\begin{aligned}\mathcal{L}(C)&=C^{\vee}+\frac{(-1)^k\,C\cdot C^{\mathrm{proj}}-C^{\vee}\cdot C^{\vee\mathrm{proj}}}{C^{\mathrm{proj}}\cdot C^{\mathrm{proj}}}C^{\vee\mathrm{proj}}\ \text{ when } C\nsubseteq D,\\&=(-1)^k\left(\pi'\right)^{-1}\left(C^{\mathrm{red}}\right)\ \text{ when } C\subset D.\end{aligned}$$

It preserves intersection products of Lagrangian subvarieties in M and M',

$$C_1\cdot C_2=\mathcal{L}(C_1)\cdot\mathcal{L}(C_2)$$

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