EULER CHARACTERISTICS OF UNIVERSAL COTANGENT LINE BUNDLES ON $\overline{\mathcal{M}}_{1,n}$

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Abstract. We give an effective algorithm to compute the Euler characteristics $\chi(\overline{\mathcal{M}}_{1,n}, \otimes_{i=1}^n L_i^{d_i})$. This work is a sequel to [8].

In addition, we give a simple proof of Pandharipande's vanishing theorem $H^j(\overline{\mathcal{M}}_{0,n}, \otimes_{i=1}^n L_i^{d_i}) = 0$ for $j \geq 1, d_i \geq 0$.

0. Introduction

Let $\overline{\mathcal{M}}_{1,n}$ be the moduli stack of n-pointed genus 1 stable curves, $\mathcal{O}$ its structure sheaf, $\mathcal{H}$ the Hodge bundle, and L_i the universal cotangent line bundles at the i-th marked point, $1 \leq i \leq n$. The main result of this paper is the following theorem.

Theorem 0.1. *There is an effective algorithm of computing the Euler characteristics*

$$\chi_{d,d_1,\ldots,d_n} := \chi(\overline{\mathcal{M}}_{1,n}, \mathcal{H}^{-d} \otimes \bigotimes_{i=1}^{n} L_i^{d_i}), \qquad d,\ d_i \geq 0.$$

The details of this algorithm are presented in Section 2.

This work is a sequel to [8], where we calculated the Euler characteristics

$$\chi(\overline{\mathcal{M}}_{0,n}, \otimes_i L_i^{d_i})$$

at genus zero. These are our preliminary attempts in search of a K-theoretic version of Witten–Kontsevich's theory of two-dimensional topological gravity. In the Witten-Kontsevich theory, the correlators are the intersection numbers of tautological classes on the moduli spaces of stable curves

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n}. \tag{0.1}$$

The natural K-theoretic version of intersection numbers (i.e. pushforward to a point in cohomology theory) are the Euler characteristics (i.e. pushforward to a point in K-theory). As Witten–Kontsevich theory states that a generating function of (0.1) is the τ-function of the KdV hierarchy, it is reasonable to surmise that a similar generating function in K-theory could be a τ-function of a version of a discrete KdV hierarchy. Note that the phenomenon of replacing differential equations in quantum cohomology by finite difference equations in quantum K-theory were observed in earlier examples [3] and only very recently demonstrated to hold in general [4].

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Furthermore, since Witten–Kontsevich theory is the Gromov–Witten theory for the target space being a point, it is natural to consider these calculations as basic ingredients for the *quantum K-theory* developed in [2] and [9]. In the calculation of quantum K-invariants at genus one via localization, the Hodge bundle will appear naturally. It is therefore reasonable to consider slightly more general correlators, which have the additional benefits of facilitating the induction process of our algorithm.

Our strategy of proving Theorem 0.1 is to apply the orbifold Riemann-Roch theorem to

$$\chi' := \chi\left(\overline{\mathcal{M}}_{1,n}, \mathcal{H}^{-d} \bigotimes_{i=1}^{n} (L_i^{d_i} - \mathcal{O})\right), \qquad d, d_i \geq 0.$$

We were able to determine χ' by carefully examining and performing computations on the twisted sectors of $\overline{\mathcal{M}}_{1,n}$. In doing so, we find the use of *generating functions* essential. These functions can be found in Section 2, starting with Equation (2.1). It is then not difficult to see that one can determine χ on $\overline{\mathcal{M}}_{1,n}$ by χ' and χ on $\overline{\mathcal{M}}_{1,n-1}$. Hence, we can reduce all calculations to $\overline{\mathcal{M}}_{1,1}$, whose generating function is calculated explicitly in Lemma 2.8 and Proposition 2.9. Note that when $n = 1$ the generating function is a rational function, as in the case of genus zero case. We expect the generating function of $\chi_{d,d_1,\ldots,d_n}$ to be rational as well, but are not able to find the correct closed form. We did, however, perform a consistency check: Our algorithm produces $\chi_{d,d_1,\ldots,d_n}$ as *integers*, even though the intermediate steps require rational numbers, which arise as a consequence of the orbifold Riemann–Roch formula and the stack structure of $\overline{\mathcal{M}}_{1,n}$.

Indeed, the $n = 1$ case is closely related to the theory of *modular forms*. Theorem 0.1 can be considered as a generalization of the following well-known fact.

Proposition 0.2 (See Lemma 2.8 and Proposition 2.9)**.**

$$\chi\left(\overline{\mathcal{M}}_{1,1}, \frac{1}{1 - qL_1}\right) = \frac{1}{(1-q^4)(1-q^6)}.$$

Since $\overline{\mathcal{M}}_{1,1}$ is the moduli stack of elliptic curves, and sections of L_1^k are the modular forms of weight $2k$, Proposition 0.2 can be considered as a rephrase of the classical result that the space of the modular forms is generated by a weight four and a weight six modular forms.

Another result included in this paper, in Appendix A, is a new proof of Pandharipande's vanishing theorem [11] at genus zero.

Theorem 0.3 ([11])**.**

$$H^j(\overline{\mathcal{M}}_{0,n}, \otimes_{i=1}^n L_i^{d_i}) = 0$$

for $j \geq 1$ and $d_i \geq 0$.

Our proof is comparably simpler and shorter, and does not use M. Kapranov's results on $\overline{\mathcal{M}}_{0,n}$ [5]. Only basic definitions and elementary manipulation of spectral sequences are used.

This paper is organized as follows. In Section 1 we recall the necessary background. We then formulate a more precise version of the reduction algorithm in

Section 2, and prove Theorem 0.1 there. In Appendix A the (new) proof of Theorem 0.3 is given.

1. Preliminaries

We work over the ground field $\mathbb{C}$.

1.1. **Twisted sectors of $\overline{\mathcal{M}}_{1,n}$.** We summarize the results we need concerning the inertia stack of $\overline{\mathcal{M}}_{1,n}$ in [10, section 3.b].

For a DM stack $\mathcal{X}$, recall a $\operatorname{Spec}\mathbb{C}$ point of its inertia stack $I\mathcal{X}$ is given by a pair (x,g) with $x\in\mathcal{X}(\operatorname{Spec}\mathbb{C})$ and $g\in Aut_{\mathcal{X}}(x)$. $\mathcal{X}$ is naturally viewed as a component of $I\mathcal{X}$ consists of point (x,g) with g trivial, and we denote this component by $(\mathcal{X},\mathrm{Id})$. A twisted sector is a connected component of $I\mathcal{X}$ disjoint from $(\mathcal{X},\mathrm{Id})$.

Theorem 1.1 ([10] Theorem 3.22, 3.24)**.** *The twisted sectors of $I\overline{\mathcal{M}}_{1,n}$ come from either of the following two sources:*

(1) the closure of a twisted sector of $I\mathcal{M}_{1,n}$ in $I\overline{\mathcal{M}}_{1,n}$,
(2) a twisted sector of $I(\overline{\mathcal{M}}_{0,K\cup\bullet}\times\overline{\mathcal{M}}_{1,K^c\cup\bullet})$ via $I\Delta_K$.

Here $\Delta_K:\overline{\mathcal{M}}_{0,K\cup\bullet}\times\overline{\mathcal{M}}_{1,K^c\cup\bullet}\to\overline{\mathcal{M}}_{1,n}$ is the closed immersion gluing the marked points $\bullet$, K is a subset of $[n]$ with $|K|\geq 2$, and K^c its complement. $I\Delta_K$: $I(\overline{\mathcal{M}}_{0,K\cup\bullet}\times\overline{\mathcal{M}}_{1,K^c\cup\bullet})\to I\overline{\mathcal{M}}_{1,n}$ is the induced closed immersion between the corresponding inertia stacks.

As $I(\overline{\mathcal{M}}_{0,K\cup\bullet}\times\overline{\mathcal{M}}_{1,K^c\cup\bullet})\simeq\overline{\mathcal{M}}_{0,K\cup\bullet}\times I\overline{\mathcal{M}}_{1,K^c\cup\bullet}$, type (2) twisted sectors are built up from type (1) twisted sectors.

The analysis of type (1) twisted sectors in [10] starts with the following description of $\overline{\mathcal{M}}_{1,1}$.

Proposition 1.2. *$\overline{\mathcal{M}}_{1,1}$ is equivalent to the weighted projective space $\mathbb{P}(4,6)$.*

We briefly recall the equivalence appeared in [10, Theorem 3.8], as we will need it to do explicit calculations on $\overline{\mathcal{M}}_{1,1}$, and it also serves to motivate the notations used in [10](Notation 3.9, 3.12; Definition 3.13, 3.16) that we follow.

Let $U=\mathbb{A}^2-(0,0)$ with $\mathbb{C}^*$ action: $\lambda\cdot(a,b)=(\lambda^4a,\lambda^6b)$, where $\lambda\in\mathbb{C}^*$, $(a,b)\in U$. The equivalence from $\mathbb{P}(4,6):=[U/\mathbb{C}^*]$ to $\overline{\mathcal{M}}_{1,1}$ is induced from a $\mathbb{C}^*$-equivariant family of 1 pointed genus one stable curves $C\to U$. where

$$C=\{(a,b)\times[x:y:z]\in U\times\mathbb{P}^2|\ y^2z=x^3+axz^2+bz^3\},$$

with the section

$$s:U\to U\times[0,1,0]\subset C,$$

the $\mathbb{C}^*$ action is given by

$$\lambda\cdot((a,b)\times[x:y:z])=(\lambda^4a,\lambda^6b)\times[\lambda^2x:\lambda^3y:z].$$

Denote by

- A_k: the component of $I\mathcal{M}_{1,k}$ consisting of pairs (x,g) with g of order 2, here $1\leq k\leq 4$.
- C_4: the 1-pointed curve $\{[x:y:z]\in\mathbb{P}^2|\ y^2z=x^3+xz^2\}$ with $[0:1:0]$ marked. Its automorphism group is generated by $i=\sqrt{-1}$.
- C_6: 1-pointed curve $\{[x:y:z]\in\mathbb{P}^2|\ y^2z=x^3+z^3\}$ with $[0:1:0]$ marked. Its automorphism group is generated by $\epsilon=\exp(2\pi i/6)$.
- C_4': C_4 with a 2nd marked point $[0:0:1]$.

- C_6': C_6 with a 2nd marked point $[0:1:1]$.
- C_6'': C_6' with a 3rd marked point $[0:-1:1]$.

Theorem 1.3 ([10] Corollary 3.14)**.**

(1) $I\mathcal{M}_{1,5} = (\mathcal{M}_{1,5}, \mathrm{Id}), n \geq 5$.

(2) For $k \leq 4$, the twisted sectors of $I\mathcal{M}_{1,k}$ are of dimension 1 or 0. A_k is the only 1 dimensional twisted sector. Zero dimensional twisted sectors are of the form $B\mu_r$, and they are determined by
- $(C_4, i), (C_4, -i), (C_6, \epsilon), (C_6, \epsilon^2), (C_6, \epsilon^4), (C_6, \epsilon^5)$ *for* $\mathcal{M}_{1,1}$.
- $(C_4', i), (C_4', -i), (C_6', \epsilon^2), (C_6', \epsilon^4)$ *for* $\mathcal{M}_{1,2}$.
- $(C_6'', \epsilon^2), (C_6'', \epsilon^4)$ *for* $\mathcal{M}_{1,3}$.

Remark 1.4. Given $x \in \mathcal{X}(\mathrm{Spec}\,\mathbb{C})$ and an order r element $g \in Aut_{\mathcal{X}}(x)$, the pair (x, g) determines a representable morphism from $B\mu_r$ to $\mathcal{X}$. (see [1] 3.2) As $B\mu_r$ is proper, it is closed in $I\overline{\mathcal{M}}_{1,k}$.

Theorem 1.5 ([10] Corollary 3.11, lemma 3.17)**.**

Let $\overline{A_k}$ be the closure of A_k in $I\overline{\mathcal{M}}_{1,k}$, then

(1)
- $\overline{A_1}$ *is isomorphic to* $\overline{\mathcal{M}}_{1,1}$.
- $\overline{A_2} \subset I\overline{\mathcal{M}}_{1,2}$ *is isomorphic to* $\mathbb{P}(2,4)$.
- $\overline{A_3} \subset I\overline{\mathcal{M}}_{1,3}$ *is isomorphic to* $\mathbb{P}(2,2)$.
- $\overline{A_4} \subset I\overline{\mathcal{M}}_{1,4}$ *is isomorphic to* $\mathbb{P}(2,2)$.

(2) When viewed as a closed substack of $\overline{\mathcal{M}}_{1,k}$, $\overline{A_k}$ does not intersect with the boundary divisors Δ_K for any $K \subset [k]$, here $2 \leq k \leq 4$.

1.2. **Riemann-Roch formula for Stacks.** We recall the Riemann-Roch formula in a version needed for this paper, adopted from Appendix A of [13].

Theorem 1.6 ([6],[12] Corollary 4.13)**.** *Let $\mathcal{X}$ be a smooth, proper Deligne-Mumford stack with quasi-projective coarse moduli space, E a vector bundle on $\mathcal{X}$. Assume $\mathcal{X}$ has the resolution property, i.e. every coherent sheaf is a quotient of a vector bundle, then we have the following formula for the Euler characteristics of E :*

$$\chi(\mathcal{X}, E) = \int_{I\mathcal{X}} \widetilde{Ch}(E)\widetilde{Td}(\mathcal{X}).$$

Here
- $I\mathcal{X}$ is the inertial stack of $\mathcal{X}$, with projection $p_{\mathcal{X}} : I\mathcal{X} \to \mathcal{X}$.
- $\widetilde{Ch}(E) \in H^*(I\mathcal{X})$ is the Chern character of the bundle $\rho(p_{\mathcal{X}}^* E)$.
- $\rho(F) := \sum_\zeta \zeta F^{(\zeta)} \in K^0(I\mathcal{X})$, if $F = \oplus F^{(\zeta)}$ with $F^{(\zeta)}$ being the eigenbundle of F with eigenvalue ζ.
- $\widetilde{Td}(\mathcal{X}) = \frac{Td(I\mathcal{X})}{Ch(\rho \circ \lambda_{-1}(N^\vee_{I\mathcal{X}/\mathcal{X}}))}$, where Td and Ch are the usual Todd class and Chern character. $N_{I\mathcal{X}/\mathcal{X}}$ is the normal bundle for p, and $N^\vee$ is the dual of N.
- $\lambda_{-1}(V) := \sum_{a \geq 0} (-1)^a \Lambda^a V$ is the λ_{-1} operation in K-theory. If $V = \oplus_i V_i$ is direct sum of line bundles V_i, then $\lambda_{-1}(V) = \prod_i (1 - V_i)$.

Remark 1.7. $\overline{\mathcal{M}}_{1,n}$ satisfies the resolution property. See, e.g. [7] Proposition 5.1.

Remark 1.8. For a vector bundle F over $\mathcal{Y}$ and a map $f : \mathcal{X} \to \mathcal{Y}$ inducing $If : I\mathcal{X} \to I\mathcal{Y}$, it is easy to see $\rho(p_{\mathcal{X}}^* f^*(F)) = If^*(\rho(p_{\mathcal{Y}}^* F))$ in $K^0(I\mathcal{X})$.

1.3. **String Equation.**

Proposition 1.9 ([9] Section 4.4). *Let $\pi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$ be the forgetful map forgetting the n-th marked point, then in terms of generating functions with variables $q, q_i, 1 \le i < n$, we have, when $g = 0$,*

$$\pi_*(\prod_{i<n} \frac{1}{1-q_iL_i}) = \prod_{i<n} \frac{1}{1-q_iL_i} \cdot (1 + \sum_{i<n} \frac{q_i}{1-q_i}),$$

and when $g > 0$,

$$\pi_*(\frac{1}{1-q\mathcal{H}^{-1}} \prod_{i<n} \frac{1}{1-q_iL_i}) = \frac{1}{1-q\mathcal{H}^{-1}} \prod_{i<n} \frac{1}{1-q_iL_i} \cdot (1 - \mathcal{H}^{-1} + \sum_{i<n} \frac{q_i}{1-q_i}).$$

*Here π_* is the K-*theoretic *pushforward.*

2. Euler characteristics of universal cotangent line bundles

In this section, we give an algorithm to compute

$$\chi\left(\overline{\mathcal{M}}_{1,n}, \mathcal{H}^{-d} \bigotimes_{i=1}^{n} L_i^{d_i}\right), d, d_i \ge 0.$$

It is more efficient to encode these numbers into a generating function

$$\begin{aligned} X_n :=& \chi\left(\overline{\mathcal{M}}_{1,n}, \frac{1}{1-q\mathcal{H}^{-1}} \prod_{i=1}^{n} \frac{1}{1-q_iL_i}\right) \\ =& \sum_{d,d_i \ge 0} q^d \prod_{i=1}^{n} q_i^{d_i} \chi\left(\overline{\mathcal{M}}_{1,n}, \mathcal{H}^{-d} \bigotimes_{i=1}^{n} L_i^{d_i}\right). \end{aligned} \tag{2.1}$$

We will first show that the calculation of X_n can be reduced to that of X_{n-1} if another generating function Φ_n in (2.2) can be calculated. We then explicitly determine Φ_n and X_1.

2.1. **Reduction from $\overline{\mathcal{M}}_{1,n}$ to $\overline{\mathcal{M}}_{1,n-1}$.** Let Φ_n be the generating function

$$\Phi_n := \chi\left(\overline{\mathcal{M}}_{1,n}, \frac{1}{1-q\mathcal{H}^{-1}} \prod_{i=1}^{n} \left(\frac{1}{1-q_iL_i} - \frac{1}{1-q_i\mathcal{O}}\right)\right) \tag{2.2}$$

Lemma 2.1. *When $n > 1$, X_n is determined by Φ_n and X_{n-1}. More precisely, we have*

$$\begin{aligned} X_n(q, q_1, \cdots, q_n) =& \Phi_n(q, q_1, \cdots, q_n) + \sum_{I \subset [n], I \neq \emptyset} (-1)^{|I|+1} \prod_{i \in I} \frac{1}{1-q_i} \cdot \\ & \Big(X_{n-1}(q, \{q_j, j \notin I\}, \underbrace{0, \cdots, 0}_{|I|-1})(1 - \frac{1}{q} + \sum_{j \notin I} \frac{q_j}{1-q_j}) \\ & + \frac{1}{q} X_{n-1}(0, \{q_j, j \notin I\}, \underbrace{0, \cdots, 0}_{|I|-1})\Big). \end{aligned}$$

For the last line, note that $X_{n-1}(q, q_1, \cdots, q_{n-1})$ is a symmetric function of the variables $q_1, q_2, \cdots, q_{n-1}$, and it is evaluated at $\{q_j, j \notin I\}$ and $|I| - 1$ zeros.

Proof. This follows directly from the definition and the string equation. Expand the product $\prod_{i=1}^n (\frac{1}{1-q_iL_i} - \frac{1}{1-q_i\mathcal{O}})$ in Φ_n we have

$$\Phi_n = X_n + \sum_{I\subset[n], I\neq\emptyset} (-1)^{|I|} \prod_{i\in I} \frac{1}{1-q_i} \cdot \chi\left(\overline{\mathcal{M}}_{1,n}, \frac{1}{1-q\mathcal{H}^{-1}} \prod_{j\notin I} \frac{1}{1-q_jL_j}\right).$$

By the string equation

$$\begin{aligned}&\chi\left(\overline{\mathcal{M}}_{1,n}, \frac{1}{1-q\mathcal{H}^{-1}} \prod_{j\notin I} \frac{1}{1-q_jL_j}\right)\\ &=\chi\left(\overline{\mathcal{M}}_{1,n-1}, \frac{1}{1-q\mathcal{H}^{-1}} \prod_{j\notin I} \frac{1}{1-q_jL_j} \cdot (1-\mathcal{H}^{-1} + \sum_{j\notin I}\frac{q_j}{1-q_j})\right),\end{aligned}$$

which is

$$\begin{aligned}&X_{n-1}(q, \{q_j, j\notin I\}, \underbrace{0,\cdots,0}_{|I|-1})(1-\frac{1}{q} + \sum_{j\notin I}\frac{q_j}{1-q_j})\\ &+\frac{1}{q}X_{n-1}(0, \{q_j, j\notin I\}, \underbrace{0,\cdots,0}_{|I|-1}).\end{aligned}$$

From here it is easy to see the lemma holds. □

2.2. **Calculation of Φ_n.** By the Riemann-Roch formula (Theorem 1.6), we have

$$\Phi_n = \int_{I\overline{\mathcal{M}}_{1,n}} \widetilde{Ch}(\frac{1}{1-q\mathcal{H}^{-1}} \prod_{i=1}^n (\frac{1}{1-q_iL_i} - \frac{1}{1-q_i\mathcal{O}}))\widetilde{Td}(\overline{\mathcal{M}}_{1,n}).$$

As the inertia stack $I\overline{\mathcal{M}}_{1,n}$ is the disjoint union of the distinguished component $(\overline{\mathcal{M}}_{1,n}, \mathrm{Id})$ and its twisted sectors, the integral is the sum of the contributions from these components.

Proposition 2.2. *The contribution to Φ_n from $(\overline{\mathcal{M}}_{1,n}, \mathrm{Id})$ is*

$$\frac{(n-1)!}{24(1-q)} \prod_{i=1}^n \frac{q_i}{(1-q_i)^2}.$$

Proof. On $(\overline{\mathcal{M}}_{1,n}, \mathrm{Id})$, $\widetilde{Ch}$ and $\widetilde{Td}$ are the same as Ch and Td respectively,

$$\begin{aligned}&Ch(\frac{1}{1-q\mathcal{H}^{-1}} \prod_{i=1}^n (\frac{1}{1-q_iL_i} - \frac{1}{1-q_i\mathcal{O}})\\ =&\frac{1}{1-q}\prod_{i=1}^n \frac{q_i}{(1-q_i)^2} \prod_{i=1}^n c_1(L_i) + \text{higher degree terms.}\end{aligned}$$

Applying the dilaton equation

$$\int_{\overline{\mathcal{M}}_{1,n}} c_1(L_1)\cdots c_1(L_n) = (n-1)\int_{\overline{\mathcal{M}}_{1,n-1}} c_1(L_1)\cdots c_1(L_{n-1}),$$

and

$$\int_{\overline{\mathcal{M}}_{1,1}} c_1(L_1) = \frac{1}{24},$$

we find that

$$\int_{\overline{\mathcal{M}}_{1,n}} Ch\left(\frac{1}{1-q\mathcal{H}^{-1}}\prod_{i=1}^{n}(\frac{1}{1-q_iL_i}-\frac{1}{1-q_i\mathcal{O}})\right)Td(\overline{\mathcal{M}}_{1,n})$$
$$=\frac{(n-1)!}{24(1-q)}\prod_{i=1}^{n}\frac{q_i}{(1-q_i)^2}.$$

□

Proposition 2.3. *The contribution to Φ_n from a twisted sector of type (2) in Theorem 1.1, i.e. from $I\Delta_K$, is zero.*

Proof. Recall such a twisted sector is the product of $\overline{\mathcal{M}}_{0,K\cup\bullet}$ and a twisted sector $\mathcal{I}$ of $\overline{\mathcal{M}}_{1,K^c\cup\bullet}$. The natural map $\overline{\mathcal{M}}_{0,K\cup\bullet}\times\mathcal{I}\to\overline{\mathcal{M}}_{1,n}$ factors through

$$\Delta_K : \overline{\mathcal{M}}_{0,K\cup\bullet}\times\overline{\mathcal{M}}_{1,K^c\cup\bullet}\to\overline{\mathcal{M}}_{1,n}.$$

We quote some known results :

- The dual of the normal bundle for Δ_K is $pr_1^*(L_\bullet)\otimes pr_2^*(L_\bullet)$. Here pr_i is the projection of $\overline{\mathcal{M}}_{0,K\cup\bullet}\times\overline{\mathcal{M}}_{1,K^c\cup\bullet}$ onto its i-th factor.
- $\Delta_K^*(L_i)$ is $pr_1^*(L_i)$ for $i\in K$, and is $pr_2^*(L_i)$ for $i\notin K$.
- $\Delta_K^*(\mathcal{H}) = pr_2^*(\mathcal{H})$.

Using these results, it is then straightforward to see that pushing forward the integrand

$$\widetilde{Ch}(\frac{1}{1-q\mathcal{H}^{-1}}\prod_{i=1}^{n}(\frac{1}{1-q_iL_i}-\frac{1}{1-q_i\mathcal{O}}))\widetilde{Td}(\overline{\mathcal{M}}_{1,n})$$

to $\overline{\mathcal{M}}_{0,K\cup\bullet}$ gives us a class which has a factor $\prod_{i\in K}c_1(L_i)$ coming from $\widetilde{Ch}(\prod_{i\in K}(\frac{1}{1-q_iL_i}-\frac{1}{1-q_i\mathcal{O}}))$. As the degree of $\prod_{i\in K}c_1(L_i)$ exceeds the dimension of $\overline{\mathcal{M}}_{0,K\cup\bullet}$, the contribution is zero. □

Proposition 2.4. *For $2\le k\le 4$, the contribution from $\overline{A_k}$ is*

$$(-1)^k\frac{1}{24(1+q)}\prod_{i=1}^{k}\frac{q_i}{1-q_i^2}\cdot(11+\frac{2q}{1+q}-\sum_{i=1}^{n}\frac{2q_i}{1+q_i})\cdot d_k,$$

where d_k is $6,6,3$ for $k=4,3,2$, respectively. The number d_k is the degree of a maps $\overline{A_k}\to\overline{\mathcal{M}}_{1,1}$ forgetting all but one marked point.

Proof. On $\overline{A_k}$, we have

$$
\begin{aligned}
&\widetilde{Ch}\Big(\frac{1}{1-q\mathcal{H}^{-1}}\prod_{i=1}^{k}\big(\frac{1}{1-q_iL_i}-\frac{1}{1-q_i\mathcal{O}}\big)\Big)\\
=&\frac{1}{1+qe^{c_1(\mathcal{H}^{-1})}}\prod_{i=1}^{k}\big(\frac{1}{1+q_ie^{c_1(L_i)}}-\frac{1}{1-q_i}\big)\\
=&(-2)^k\frac{1}{1+q}\prod_{i=1}^{k}\frac{q_i}{1-q_i^2}\Big(1+\frac{q}{1+q}c_1(\mathcal{H})+\sum_{i=1}^{k}\frac{1-q_i}{2(1+q_i)}c_1(L_i)\Big),
\end{aligned}
$$

and

$$
Ch\big(\rho\circ(\Lambda_{-1}(N^{\vee}_{\overline{A_k}/\overline{\mathcal{M}}_{1,k}}))\big)=2^{k-1}\big(1+\frac{1}{2}c_1(N^{\vee}_{\overline{A_k}/\overline{\mathcal{M}}_{1,k}})\big).
$$

For the above equations, note that over $\overline{A_k}$ the eigenvalues involved in $\widetilde{Ch}$ must be -1 as the nontrivial automorphism is of order 2, also there is no higher degree terms as $\overline{A_k}$ is 1 dimensional.

Thus

$$
\begin{aligned}
&\int_{\overline{A_k}}\widetilde{Ch}\Big(\frac{1}{1-q\mathcal{H}^{-1}}\prod_{i=1}^{k}\big(\frac{1}{1-q_iL_i}-\frac{1}{1-q_i\mathcal{O}}\big)\Big)\widetilde{Td}(\overline{\mathcal{M}}_{1,k}))\\
=&(-1)^k\frac{1}{1+q}\prod_{i=1}^{k}\frac{q_i}{1-q_i^2}\cdot\\
&\cdot\Big(\frac{2q}{1+q}\int_{\overline{A_k}}c_1(H)+\sum_{i=1}^{k}\frac{1-q_i}{1+q_i}\int_{\overline{A_k}}c_1(L_i)+\int_{\overline{A_k}}c_1(T\overline{\mathcal{M}}_{1,k})\Big).
\end{aligned}
$$

It is easy to see

$$
\int_{\overline{A_k}}c_1(H)=d_k\int_{\overline{\mathcal{M}}_{1,1}}c_1(H)=\frac{d_k}{24},
$$

by considering a map $\overline{A_k}\subset\overline{\mathcal{M}}_{1,k}\to\overline{\mathcal{M}}_{1,1}$ forgetting all but one marked point,

$$
\int_{\overline{A_k}}c_1(L_i),1\le i\le k,\ \text{and}\ \int_{\overline{A_k}}c_1(T\overline{\mathcal{M}}_{1,k}).
$$

are determined by Corollary 2.6.

□

Lemma 2.5. *Let $\pi:\overline{\mathcal{M}}_{1,n+1}\to\overline{\mathcal{M}}_{1,n}$ be the forgetful map forgetting the $(n+1)$-th marked point, then*

$$
c_1(L_j)=\pi^*c_1(L_j)+\Delta_{\{j,n+1\}},1\le j\le n;
$$

$$
c_1(T\overline{\mathcal{M}}_{1,n+1})=\pi^*c_1(T\overline{\mathcal{M}}_{1,n})-c_1(L_{n+1})+\sum_{1\le j\le n}\Delta_{\{j,n+1\}}.
$$

Proof. Recall for $\pi:\overline{\mathcal{M}}_{1,n+1}\to\overline{\mathcal{M}}_{1,n}$ we have

$$
L_j=\pi^*L_j\big(\Delta_{\{j,n+1\}}\big),1\le j\le n;\qquad L_{n+1}=\omega_\pi\big(\sum_{1\le j\le n}\Delta_{\{j,n+1\}}\big),
$$

where $\omega_\pi = \omega_{\overline{\mathcal{M}}_{1,n+1}} \otimes \pi^* \omega^{-1}_{\overline{\mathcal{M}}_{1,n}}$ is the relative dualizing sheaf for π. Taking the first Chern class of these equations proves the lemma. □

Corollary 2.6.

$$\int_{\overline{A_k}} c_1(L_j) = \frac{d_k}{24}, 1 \leq j \leq k. \quad \int_{\overline{A_k}} c_1(T\overline{\mathcal{M}}_{1,k}) = \frac{(11-k)d_k}{24}.$$

Proof. This can be easily proved by applying the above lemma and Theorem 1.5 (2) to a forgetful map $\overline{A_k} \to \overline{\mathcal{M}}_{1,1}$.

□

Proposition 2.7. *Over the zero dimensional twisted sectors, the contribution to Φ_n are:*

- *the contribution from $(C_4', i) \sqcup (C_4', -i)$ is*
$$\frac{1}{4}\prod_{j=1,2}\frac{q_j}{(1-q_j)} \cdot \frac{1-q+q_1+q_2-q_1q_2+qq_1+qq_2+qq_1q_2}{(1+q^2)(1+q_1^2)(1+q_2^2)};$$
- *the contribution from $(C_6', \epsilon^2) \sqcup (C_6', \epsilon^4)$ is*
$$\frac{1}{3}\prod_{j=1,2}\frac{q_j}{(1-q_j)} \cdot \frac{1-q+(q+2)(q_1+q_2)+(2q+1)q_1q_2}{(1+q+q^2)(1+q_1+q_1^2)(1+q_2+q_2^2)};$$
- *the contribution from $(C_6'', \epsilon^2) \sqcup (C_6'', \epsilon^4)$ is*

$$-\frac{1}{3}\prod_{j=1,2,3}\frac{q_j}{(1-q_j)} \cdot$$
$$\frac{1-q+(q+2)(q_1+q_2+q_3)+(2q+1)(q_1q_2+q_1q_3+q_2q_3)+(q-1)q_1q_2q_3}{(1+q+q^2)(1+q_1+q_1^2)(1+q_2+q_2^2)(1+q_3+q_3^2)}.$$

Proof. To simplify the notation, we will use (C, λ) to denote a twisted sector.

The integrand are determined by the eigenvalues of the bundles involved in the Riemann Roch formula.

For $(C_4, \lambda), (C_6, \lambda)$ of $I\overline{\mathcal{M}}_{1,1}$, explicit calculation will show that the eigenvalues of L_1, H and $\Omega_{\overline{\mathcal{M}}_{1,1}}$ are λ, λ, λ^2 respectively. Consider the forgetful map $\pi : \mathcal{M}_{1,k} \to \mathcal{M}_{1,k-1}$, as $L_i = \pi^* L_i, i < k$ and $H = \pi^* H$, remark 1.8 implies that the eigenvalues of L_i or H is λ for (C, λ). As π is smooth, we have a short exact sequence $0 \to \pi^*\Omega_{\overline{\mathcal{M}}_{1,k-1}} \to \Omega_{\overline{\mathcal{M}}_{1,k}} \to \omega_\pi \to 0$, and ω_π can be identified with L_k. It is then easy to see that the eigenvalues of $\Omega_{\overline{\mathcal{M}}_{1,2}}$ are λ, λ^2 for (C_4', λ) or (C_6', λ), and the eigenvalues of $\Omega_{\overline{\mathcal{M}}_{1,3}}$ are $\lambda, \lambda, \lambda^2$ for (C_6'', λ).

From the analysis above, on the twisted sector (C, λ) of $\overline{\mathcal{M}}_{1,n}$ we have

$$\widetilde{Ch}\left(\frac{1}{1-q\mathcal{H}^{-1}}\prod_{i=1}^{n}\left(\frac{1}{1-q_iL_i}-\frac{1}{1-q_i\mathcal{O}}\right)\right) = \frac{(\lambda-1)^n}{1-q\lambda^{-1}}\prod_{i=1}^{n}\frac{q_i}{(1-q_i)(1-q_i\lambda)},$$

and

$$\widetilde{Td}(\overline{\mathcal{M}}_{1,n}) = \frac{1}{(1-\lambda^2)(1-\lambda)^{n-1}} = \frac{1}{(1+\lambda)(1-\lambda)^n}.$$

The sum of the integral on $(C_4', i) \sqcup (C_4, -i)$ is then

$$\frac{1}{4} \sum_{\lambda=i,-i} \frac{1}{(1-q\lambda^{-1})(1+\lambda)} \prod_{i=1,2} \frac{q_i}{(1-q_i)(1-q_i\lambda)},$$

which equals

$$\frac{1}{4} \prod_{j=1,2} \frac{q_j}{(1-q_j)} \cdot \frac{1-q+q_1+q_2-q_1q_2+qq_1+qq_2+qq_1q_2}{(1+q^2)(1+q_1^2)(1+q_2^2)}.$$

The remaining cases also follow directly from our formula of $\widetilde{Ch}, \widetilde{Td}$. $\square$

2.3. **Calculation for X_1.** Under the isomorphism $\overline{\mathcal{M}}_{1,1} \simeq \mathbb{P}(4,6)$, the line bundles $\mathcal{H}$ and L_1 all correspond to $\mathcal{O}(1)$, hence

$$\chi(\overline{\mathcal{M}}_{1,1}, \mathcal{H}^{-d} \otimes L_1^{d_1}) = \chi(\mathbb{P}(4,6), \mathcal{O}(d_1-d)),$$

and X_1 is determined by $\chi(\mathbb{P}(4,6), \mathcal{O}(k)), k \in \mathbb{Z}$.

Lemma 2.8. *Let $h^0(\mathcal{O}(k)) = \dim_{\mathbb{C}} H^0(\mathbb{P}(4,6), \mathcal{O}(k))$, then*

$$\sum_{k=0}^{\infty} h^0(\mathcal{O}(k)) q^k = \frac{1}{(1-q^4)(1-q^6)},$$

and $h^0(\mathcal{O}(k)) = 0$ if $k < 0$.

Proof. As a section of $\mathcal{O}(k)$ on $\mathbb{P}(4,6)$ corresponds to a polynomial $f(x,y)$ satisfying $f(\lambda^4 x, \lambda^6 y) = \lambda^k f(x,y)$ for any $\lambda \in \mathbb{C}^*$, monomials $x^a y^b$ such that $4a+6b=k$ form a basis of $H^0(\mathbb{P}(4,6), \mathcal{O}(k))$. Therefore, $h^0(\mathcal{O}(k))$ is given by the coefficient of q^k in the power series $\dfrac{1}{(1-q^4)(1-q^6)}$. $\square$

Proposition 2.9.

$$\chi(\overline{\mathcal{M}}_{1,1}, \frac{1}{1-q\mathcal{H}^{-1}} \frac{1}{1-q_1 L_1}) =$$
$$\frac{(1-qq_1)(1-q^4-q^6-q_1^2q^6-q_1^2q^8-q_1^4q^8+q^2q_1^2+q^4q_1^4+q^6q_1^6+q^8q_1^8)}{(1-q^4)(1-q^6)(1-q_1^4)(1-q_1^6)}.$$

Proof. We have $H^1(\mathbb{P}(4,6), \mathcal{O}(k)) \simeq H^0(\mathbb{P}(4,6), \mathcal{O}(-10-k))^\vee$ by Serre duality, so $\chi(\mathbb{P}(4,6), \mathcal{O}(k)) = h^0(\mathcal{O}(k)) - h^0(\mathcal{O}(-k-10))$, and the proposition now follows from the previous lemma.

$\square$

Appendix A. A simple proof of Pandharipande's vanishing theorem

The purpose of this appendix is to give a very simple and *self-contained* proof of Theorem 0.3, first proved in [11]. Recall that the theorem states that at genus zero

$$H^j(\overline{\mathcal{M}}_{0,n}, \otimes_{i=1}^n L_i^{d_i}) = 0 \tag{A.1}$$

for $j \geq 1$ and $d_i \geq 0$. [1]

We will prove (A.1) by induction on n. Note that (A.1) holds for $n=3$ as $\overline{\mathcal{M}}_{0,3}$ is a point.

[1] The method presented in this appendix can also be used to compute $H^0(\overline{\mathcal{M}}_{0,n}, \otimes L_i^{d_i})$. It is also hoped that this method can help to produce an S_n-equivariant version of our genus zero formula [8], which is needed in the quantum K-theory [9] computation of general target spaces.

For $n > 3$, we first treat the special case that one of the d_i is zero, then reduce the case that all $d_i > 0$ to that special case.

If one of the d_i is zero, up to permutation of the marked points we can assume $d_n = 0$. Consider the forgetful map $\pi : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n-1}$, as $R^1\pi_*(\otimes_{i=1}^{n-1} L_i^{d_i}) = 0$ by cohomology and base change (for C rational and degree of $\mathcal{O}_C(D)$ positive, $H^1(C, \mathcal{O}_C(D)) = 0$), we have a degenerated Leray spectral sequence which gives

$$\begin{aligned} H^j(\overline{\mathcal{M}}_{0,n}, \otimes_{i=1}^{n-1} L_i^{d_i}) &= H^j(\overline{\mathcal{M}}_{0,n-1}, R^0\pi_*(\otimes_{i=1}^{n-1} L_i^{d_i})) \\ =&H^j\Big(\overline{\mathcal{M}}_{0,n-1}, (\otimes_{i=1}^{n-1} L_i^{d_i}) \otimes (\mathcal{O} + \sum_{i, d_i \neq 0} \sum_{m=1}^{d_i} L_i^{-m})\Big), \end{aligned}$$

and this is zero by induction. Here we used the string equation (Prop. 1.9) that K-theoretically

$$\pi_*(\otimes_{i=1}^{n-1} L_i^{d_i}) = (\otimes_{i=1}^{n-1} L_i^{d_i}) \otimes (\mathcal{O} + \sum_{i, d_i \neq 0} \sum_{m=1}^{d_i} L_i^{-m}).$$

If all $d_i > 0$, consider $V := \oplus_{i=1}^{n} L_i^{d_i}$, note that $\bigwedge^n V$ is $\otimes_{i=1}^{n} {L_i}^{d_i}$. Choose sections s_i of L_i such that the zero locus of the section $\oplus_{i=1}^{n} s_i^{\otimes d_i}$ of V is empty.(See the remark below.) Then the Koszul complex

$$0 \to \mathcal{O} \xrightarrow{d} V \xrightarrow{d} \bigwedge^2 V \to \cdots \xrightarrow{d} \bigwedge^n V \to 0,$$

is exact, and we can compute $H^*(\overline{\mathcal{M}}_{0,n}, \otimes {L_i}^{d_i})$ from this resolution.

Form the double complex $(C^p(\underline{U}, \mathcal{K}^q); \delta, d)_{\{p \geq 0, 0 \leq q \leq n-1\}}$, where $\underline{U}$ is an affine covering of $\overline{\mathcal{M}}_{0,n}$, $\mathcal{K}^q = \bigwedge^q V$, C^p are the Čech cochain groups, δ is the Čech differential.

$$\begin{tikzcd} q \\ & & \\ & C^0(\underline{U}, \mathcal{O}(V)) \arrow[u, "d"] \arrow[r, "\delta"] & C^1(\underline{U}, \mathcal{O}(V)) \arrow[u, "d"] \arrow[r, "\delta"] & \\ & C^0(\underline{U}, \mathcal{O}) \arrow[u, "d"] \arrow[r, "\delta"] & C^1(\underline{U}, \mathcal{O}) \arrow[u, "d"] \arrow[r, "\delta"] & \\ & & & p \end{tikzcd}$$

Using two canonical filtrations (by p and q respectively), we obtain two spectral sequences ${}'E_r^{p,q}$ and ${}''E_r^{p,q}$ with

$$\begin{aligned} {}'E_1^{p,q} &= H^p(\overline{\mathcal{M}}_{0,n}, \mathcal{K}^q), \\ {}''E_2^{p,q} &= H^p(\overline{\mathcal{M}}_{0,n}, \mathcal{H}_d^q(\mathcal{K}^*)). \end{aligned}$$

These two spectral sequences abut to the same hyper-cohomology $\mathbb{H}^*(\overline{\mathcal{M}}_{0,n}, \mathcal{K}^*)$.

By induction, $'E_1^{p,q} = 0$ if $p \neq 0$, since $\mathcal{K}^q$ is the direct sum of $\otimes L_i^{d'_i}$'s with some $d'_i = 0$. So $'E_r^{p,q}$ degenerates at $r = 1$, and by our construction $\mathbb{H}^q(\overline{\mathcal{M}}_{0,n}, \mathcal{K}^*) = {}'E_1^{0,q} = 0$ when $q \geq n$.

Note that $''E_2^{p,q} = 0$ if $q \neq n-1$, therefore $''E_r^{p,q}$ degenerates at $r = 2$, and we have $H^p(\overline{\mathcal{M}}_{0,n}, \otimes L_i^{d_i}) = {}''E_2^{p,n-1} = \mathbb{H}^{p+n-1}(\overline{\mathcal{M}}_{0,n}, \mathcal{K}^*) = 0$ when $p \geq 1$.

Remark A.1. The zero locus of the section $\oplus_{i=1}^n s_i^{\otimes d_i}$ is contained in the zero locus of the section $\oplus_{i=1}^{n-2} s_i$ of the vector bundle $\bigoplus_{i=1}^{n-2} L_i$. Since having empty zero locus is an open property for sections, it is easy to show that a generic section of $\bigoplus_{i=1}^{n-2} L_i$ on $\overline{\mathcal{M}}_{0,n}$ has empty zero locus inductively using a forgetful map as follows.

The statement holds for $n = 3, 4$ obviously. For $n > 4$, consider the forgetful map $\pi : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n-1}$. Since $L_i = \pi^* L_i(D_i), 1 \leq i \leq n-2$, where D_i is the image of the i-th section of π, a section t_i of L_i on $\overline{\mathcal{M}}_{0,n-1}$ would induce a section s_i of L_i on $\overline{\mathcal{M}}_{0,n}$ with support $\operatorname{Supp} s_i = \pi^{-1}(\operatorname{Supp} t_i) \cup D_i$. It is straightforward to check $\cap_{i=1}^{n-2} \operatorname{Supp} s_i = \emptyset$ iff for all $1 \leq j \leq n-2, \cap_{i=1, i\neq j}^{n-2} \operatorname{Supp} t_i = \emptyset$, and these conditions hold for generic t_i by induction.

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