

## Notes on Axiomatic Gromov–Witten theory and applications

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### 0. Introduction

The purpose of these notes is to give their readers some idea of Givental’s *axiomatic Gromov–Witten theory*, and a few applications. Due to the scope of these notes, some statements are not precisely formulated and almost all proofs are omitted. However, we try to point out some subtleties, and to give references for further reading whenever desirable. The readers are assumed to be familiar with the rudiments of geometric Gromov–Witten theory.

We start with a very brief review of geometric Gromov–Witten theory in Section 1, mainly to fix the necessary notations. In Section 2 the *genus zero axiomatic Gromov–Witten theory* is introduced. Among the important properties of the axiomatic theory is the following theorem: The “moduli space” of the genus zero axiomatic theories of a fixed rank is acted upon by the twisted loop group. Furthermore, the subspace of semisimple theories is a homogeneous space of the twisted loop group. Then, in Section 3, the semisimple genus zero theories are *quantized* to obtain the higher genus theories. Here, the above theorem of genus zero theories plays an important role. The implications of the axiomatic formulation to *Virasoro constraints* are discussed in Section 4. Finally, the notion of *invariance of tautological equations* and its applications are briefly discussed in Section 5.

*Warning:* This article does not aim to give a historical account of Gromov–Witten theory, which is preferably left to other experts. It rather emphasizes upon some highlights centered at the axiomatic theory which captures our imagination. Therefore, the choice of the topics is entirely personal, and some important progress is completely left out when its intersection with axiomatic theory is minimal.

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### 1. Review of geometric Gromov–Witten theory

Gromov–Witten theory studies the tautological intersection theory on  $\overline{M}_{g,n}(X, \beta)$ , the moduli stacks of stable maps from curves  $C$  of genus  $g$  with  $n$  marked points to a smooth projective variety  $X$ . The intersection numbers, or *Gromov–Witten*

*invariants*, are integrals of tautological classes over the virtual fundamental classes of  $\overline{M}_{g,n}(X, \beta)$

$$\int_{[\overline{M}_{g,n}(X, \beta)]^{\text{vir}}} \prod_{i=1}^n \text{ev}_i^*(\gamma_i) \psi_i^{k_i}.$$

Here  $\gamma_i \in H^*(X)$  and  $\psi_i$  are the *cotangent classes* (gravitational descendents).

For the sake of the later reference, let us fix some notations.

- (i)  $H := H^*(X, \mathbb{Q})$  is a  $\mathbb{Q}$ -vector space, assumed of rank  $N$ . Let  $\{\phi_\mu\}_{\mu=1}^N$  be a basis of  $H$ .
- (ii)  $H$  carries a symmetric bilinear form, Poincaré pairing,

$$\langle \cdot, \cdot \rangle : H \otimes H \rightarrow \mathbb{Q}.$$

Define

$$g_{\mu\nu} := \langle \phi_\mu, \phi_\nu \rangle$$

and  $g^{\mu\nu}$  to be the inverse matrix.

- (iii) Let  $\mathcal{H}_t := \bigoplus_{k=0}^{\infty} H$  be the infinite dimensional complex vector space with basis  $\{\phi_\mu \psi^k\}$ .  $\mathcal{H}_t$  has a natural  $\mathbb{Q}$ -algebra structure:

$$\phi_\mu \psi^{k_1} \otimes \phi_\nu \psi^{k_2} \mapsto (\phi_\mu \cdot \phi_\nu) \psi^{k_1+k_2},$$

where  $\phi_\mu \cdot \phi_\nu$  is the cup product in  $H$ .

- (iv) Let  $\{t_k^\mu\}$ ,  $\mu = 1, \dots, N$ ,  $k = 0, \dots, \infty$ , be the dual coordinates of the basis  $\{\phi_\mu \psi^k\}$ .

We note that at each marked point, the insertion is  $\mathcal{H}_t$ -valued. Let

$$t := \sum_{k, \mu} t_k^\mu \phi_\mu \psi^k$$

denote a general element in the vector space  $\mathcal{H}_t$ .

- (v) Define

$$\langle \partial_{k_1}^{\mu_1} \dots \partial_{k_n}^{\mu_n} \rangle_{g,n,\beta} := \int_{[\overline{M}_{g,n}(X, \beta)]^{\text{vir}}} \prod_{i=1}^n \text{ev}_i^*(\phi_{\mu_i}) \psi_i^{k_i}$$

and define

$$\langle t^n \rangle_{g,n,\beta} = \langle t \dots t \rangle_{g,n,\beta}$$

by multi-linearity.

- (vi) Let

$$F_g^X(t) := \sum_{n,\beta} \frac{1}{n!} \langle t^n \rangle_{g,n,\beta}$$

be the generating function of all genus  $g$  Gromov–Witten invariants. <sup>1</sup>

The  $\tau$ -function of  $X$  is the formal expression

$$(1) \quad \tau_{GW}^X := e^{\sum_{g=0}^{\infty} \hbar^{g-1} F_g^X}.$$

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<sup>1</sup>In Gromov–Witten theory, one usually has to deal with the coefficients in the *Novikov ring*, due to some convergence issues. We shall not touch upon this subtleties here but refer the readers to [33].

## 2. Genus zero axiomatic Gromov–Witten theory

Let  $H$  be a  $\mathbb{Q}$ -vector space of dimension  $N$  with a distinguished element  $\mathbf{1}$ . Let  $\{\phi_\mu\}$  be a basis of  $H$  and  $\phi_1 = \mathbf{1}$ . Assume that  $H$  is endowed with a nondegenerate symmetric  $\mathbb{Q}$ -bilinear form, or metric,  $\langle \cdot, \cdot \rangle$ . Let  $\mathcal{H}$  denote the infinite dimensional vector space  $H[z, z^{-1}]$  consisting of Laurent polynomials with coefficients in  $H$ .<sup>2</sup> Introduce a symplectic form  $\Omega$  on  $\mathcal{H}$ :

$$\Omega(f(z), g(z)) := \text{Res}_{z=0}(f(-z), g(z)),$$

where the symbol  $\text{Res}_{z=0}$  means to take the residue at  $z = 0$ .

There is a natural polarization  $\mathcal{H} = \mathcal{H}_q \oplus \mathcal{H}_p$  by the Lagrangian subspaces  $\mathcal{H}_q := H[z]$  and  $\mathcal{H}_p := z^{-1}H[z^{-1}]$  which provides a symplectic identification of  $(\mathcal{H}, \Omega)$  with the cotangent bundle  $T^*\mathcal{H}_q$  with the natural symplectic structure.  $\mathcal{H}_q$  has a basis

$$\{\phi_\mu z^k\}, \quad 1 \leq \mu \leq N, \quad 0 \leq k$$

with dual coordinates  $\{q_\mu^k\}$ . The corresponding basis for  $\mathcal{H}_p$  is

$$\{\phi_\mu z^{-k-1}\}, \quad 1 \leq \mu \leq N, \quad 0 \leq k$$

with dual coordinates  $\{p_\mu^k\}$ .

For example, let  $\{\phi_i\}$  be an *orthonormal* basis of  $H$ . An  $H$ -valued Laurent formal series can be written in this basis as

$$\begin{aligned} \dots + (p_1^1, \dots, p_1^N) \frac{1}{(-z)^2} + (p_0^1, \dots, p_0^N) \frac{1}{(-z)} \\ + (q_0^1, \dots, q_0^N) + (q_1^1, \dots, q_1^N)z + \dots \end{aligned}$$

In fact,  $\{p_k^i, q_k^i\}$  for  $k = 0, 1, 2, \dots$  and  $i = 1, \dots, N$  are the Darboux coordinates compatible with this polarization in the sense that

$$\Omega = \sum_{i,k} dp_k^i \wedge dq_k^i.$$

The parallel between  $\mathcal{H}_q$  and  $\mathcal{H}_t$  is evident, and is in fact given by the following affine coordinate transformation, called the *dilaton shift*,

$$t_k^\mu = q_k^\mu + \delta^{\mu 1} \delta_{k1}.$$

**Definition 1.** Let  $G_0(t)$  be a (formal) function on  $\mathcal{H}_t$ . The pair  $T := (\mathcal{H}, G_0)$  is called a  $g = 0$  *axiomatic theory* if  $G_0$  satisfies three sets of genus zero tautological equations: the *Dilaton Equation* (2), the *String Equation* (3) and the *Topological Recursion Relations* (TRR) (4).

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<sup>2</sup>Different completions of  $\mathcal{H}$  are used in different places. Although there is not a single completion which works for all theorems quoted in this context, the final results nonetheless make sense as a coherent theory. This subtlety will not be discussed in the present article. See [33] for the details.

$$(2) \quad \frac{\partial G_0(t)}{\partial t_1^1}(t) = \sum_{k=0}^{\infty} \sum_{\mu} t_k^{\mu} \frac{\partial G_0(t)}{\partial t_k^{\mu}} - 2G_0(t),$$

$$(3) \quad \frac{\partial G_0(t)}{\partial t_0^1} = \frac{1}{2} \langle t_0, t_0 \rangle + \sum_{k=0}^{\infty} \sum_{\nu} t_{k+1}^{\nu} \frac{\partial G_0(t)}{\partial t_k^{\nu}},$$

$$(4) \quad \frac{\partial^3 G_0(t)}{\partial t_{k+1}^{\alpha} \partial t_l^{\beta} \partial t_m^{\gamma}} = \sum_{\mu\nu} \frac{\partial^2 G_0(t)}{\partial t_k^{\alpha} \partial t_0^{\mu}} g^{\mu\nu} \frac{\partial^3 G_0(t)}{\partial t_0^{\nu} \partial t_l^{\beta} \partial t_m^{\gamma}}, \quad \forall \alpha, \beta, \gamma, k, l, m.$$

To simplify the notations,  $p_k$  will stand for the vector  $(p_k^1, \dots, p_k^N)$  and  $p^{\mu}$  for  $(p_0^{\mu}, p_1^{\mu}, \dots)$ . Similarly for  $q, t$ .

In the case of geometric theory,  $G_0 = F_0^X$ . It is well known that  $F_0^X$  satisfies the above three sets of equations (2) (3) (4). The main advantage of viewing the genus zero theory through this formulation, seems to us, is to replace  $\mathcal{H}_t$  by  $\mathcal{H}$  where a symplectic structure is available. Therefore many properties can be reformulated in terms of the symplectic structure  $\Omega$  and hence independent of the choice of the polarization. This suggests that the space of genus zero axiomatic Gromov–Witten theories, i.e. the space of functions  $G_0$  satisfying the string equation, dilaton equation and TRRs, has a huge symmetry group.

**Definition 2.** Let  $L^{(2)}GL(H)$  denote the *twisted loop group* which consists of  $\text{End}(H)$ -valued formal Laurent series  $M(z)$  in the indeterminate  $z^{-1}$  satisfying  $M^*(-z)M(z) = \mathbf{I}$ . Here  $*$  denotes the adjoint with respect to  $(\cdot, \cdot)$ .

The condition  $M^*(-z)M(z) = \mathbf{I}$  means that  $M(z)$  is a symplectic transformation on  $\mathcal{H}$ .

**Theorem 1.** [21] *The twisted loop group acts on the space of axiomatic genus zero theories. Furthermore, the action is transitive on the semisimple theories of a fixed rank  $N$ .*

**Remarks.** (i) In the geometric theory,  $F_0^X(t)$  is usually a formal function in  $t$ . Therefore, the corresponding function in  $q$  would be formal at  $q = -\mathbf{1}z$ . Furthermore, the Novikov rings are usually needed to ensure the well-definedness of  $F_0^X(t)$ . (cf. Footnote 1.)

(ii) It can be shown that the axiomatic genus zero theory over complex numbers is equivalent to the definition of abstract (formal) Frobenius manifolds, not necessarily conformal. The coordinates on the corresponding Frobenius manifold is given by the following map [9]

$$(5) \quad s^{\mu} := \frac{\partial}{\partial t_0^{\mu}} \frac{\partial}{\partial t_1^1} G_0(t).$$

*From now on, the term “genus zero axiomatic theory” is identified with “Frobenius manifold”.*

(iii) The above formulation (or the Frobenius manifold formulation) does not include the divisor axiom, which is true for any geometric theory.

(iv) Coates–Givental [8] and Givental [21] gave a beautiful geometric reformation of the genus axiomatic theory in terms of *Lagrangian cones* in  $\mathcal{H}$ . When viewed in the Lagrangian cone formulation, Theorem 1 becomes transparent and a proof is almost unnecessary.

Roughly, the descendent Lagrangian cones are constructed in the following way. Denote by  $\mathcal{L}$  the graph of the differential  $dG_0$ :

$$\mathcal{L} = \{(p, q) \in T^*\mathcal{H}_q : p_k^\mu = \frac{\partial}{\partial q_k^\mu} G_0\}.$$

It is considered as a formal germ at  $q = -z$  (i.e.  $t = 0$ ) of a Lagrangian section of the cotangent bundle  $T^*\mathcal{H}_q = \mathcal{H}$ , due to the convergence issues of  $G_0$ .  $\mathcal{L}$  is therefore considered as a formal germ of a Lagrangian submanifold in the space  $(\mathcal{H}, \Omega)$ .

**Theorem.** *( $\mathcal{H}, G_0$ ) defines an axiomatic theory if the corresponding Lagrangian cone  $\mathcal{L} \subset \mathcal{H}$  satisfies the following properties:  $\mathcal{L}$  is a Lagrangian cone with the vertex at the origin of  $q$  such that its tangent spaces  $L$  are tangent to  $\mathcal{L}$  exactly along  $zL$ .*

A Lagrangian cone with the above property is also called *over-ruled* (descendent) Lagrangian cones.

### 3. Quantization and higher genus axiomatic theory

**3.1. Preliminaries on quantization.** To quantize an infinitesimal symplectic transformation, or its corresponding quadratic hamiltonians, we recall the standard Weyl quantization. A polarization  $\mathcal{H} = T^*\mathcal{H}_q$  on the symplectic vector space  $\mathcal{H}$  (the phase space) defines a configuration space  $\mathcal{H}_q$ . The quantum ‘‘Fock space’’ will be a certain class of functions  $f(\hbar, q)$  on  $\mathcal{H}_q$  (containing at least polynomial functions), with additional formal variable  $\hbar$  (‘‘Planck’s constant’’). The classical observables are certain functions of  $p, q$ . The quantization process is to find for the classical mechanical system on  $\mathcal{H}$  a ‘‘quantum mechanical’’ system on the Fock space such that the classical observables, like the hamiltonians  $h(q, p)$  on  $\mathcal{H}$ , are quantized to become operators  $\widehat{h}(q, \frac{\partial}{\partial q})$  on the Fock space.

Let  $A(z)$  be an  $\text{End}(H)$ -valued Laurent formal series in  $z$  satisfying

$$(A(-z)f(-z), g(z)) + (f(-z), A(z)g(z)) = 0,$$

then  $A(z)$  defines an infinitesimal symplectic transformation

$$\Omega(Af, g) + \Omega(f, Ag) = 0.$$

An infinitesimal symplectic transformation  $A$  of  $\mathcal{H}$  corresponds to a quadratic polynomial <sup>3</sup>  $P(A)$  in  $p, q$

$$P(A)(f) := \frac{1}{2}\Omega(Af, f).$$

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<sup>3</sup>Due to the nature of the infinite dimensional vector spaces involved, the ‘‘polynomials’’ here might have infinite many terms, but the degrees remain finite.

Choose a Darboux coordinate system  $\{q_k^i, p_k^i\}$ . The quantization  $P \mapsto \widehat{P}$  assigns

$$(6) \quad \begin{aligned} \widehat{1} &= 1, \quad \widehat{p}_k^i = \sqrt{\hbar} \frac{\partial}{\partial q_k^i}, \quad \widehat{q}_k^i = q_k^i / \sqrt{\hbar}, \\ \widehat{p}_k^i p_l^j &= \widehat{p}_k^i \widehat{p}_l^j = \hbar \frac{\partial}{\partial q_k^i} \frac{\partial}{\partial q_l^j}, \\ \widehat{p}_k^i q_l^j &= q_l^j \frac{\partial}{\partial q_k^i}, \\ \widehat{q}_k^i q_l^j &= q_k^i q_l^j / \hbar, \end{aligned}$$

In summary, the quantization is the process

$$\begin{array}{ccccc} A & \mapsto & P(A) & \mapsto & \widehat{P(A)} \\ \text{inf. sympl. transf.} & \mapsto & \text{quadr. hamilt.} & \mapsto & \text{operator on Fock sp..} \end{array}$$

It can be readily checked that the first map is a Lie algebra isomorphism: The Lie bracket on the left is defined by  $[A_1, A_2] = A_1 A_2 - A_2 A_1$  and the Lie bracket in the middle is defined by Poisson bracket

$$\{P_1(p, q), P_2(p, q)\} = \sum_{k,i} \frac{\partial P_1}{\partial p_k^i} \frac{\partial P_2}{\partial q_k^i} - \frac{\partial P_2}{\partial p_k^i} \frac{\partial P_1}{\partial q_k^i}.$$

The second map is not a Lie algebra homomorphism, but is very close to being one.

**Lemma 1.**

$$[\widehat{P}_1, \widehat{P}_2] = \{\widehat{P}_1, \widehat{P}_2\} + \mathcal{C}(P_1, P_2),$$

where the cocycle  $\mathcal{C}$ , in orthonormal coordinates, vanishes except

$$\mathcal{C}(p_k^i p_l^j, q_k^i q_l^j) = -\mathcal{C}(q_k^i q_l^j, p_k^i p_l^j) = 1 + \delta^{ij} \delta_{kl}.$$

**Example.** Let  $\dim H = 1$  and  $A(z)$  be multiplication by  $z^{-1}$ . It is easy to see that  $A(z)$  is infinitesimally symplectic.

$$(7) \quad \begin{aligned} P(z^{-1}) &= -\frac{q_0^2}{2} - \sum_{m=0}^{\infty} q_{m+1} p_m \\ \widehat{P}(z^{-1}) &= -\frac{q_0^2}{2} - \sum_{m=0}^{\infty} q_{m+1} \frac{\partial}{\partial q_m}. \end{aligned}$$

Note that one often has to quantize the symplectic instead of the infinitesimal symplectic transformations. Following the common practice in physics, define

$$(8) \quad \widehat{e^{A(z)}} := e^{\widehat{A(z)}},$$

for  $e^{A(z)}$  an element in the twisted loop group.

**3.2.  $\tau$ -function for the axiomatic theory.** Let  $X$  be the space of  $N$  points and  $H^{Npt} := H^*(X)$ . Let  $\phi_i$  be the delta-function at the  $i$ -th point. Then  $\{\phi_i\}_{i=1}^N$  form an orthonormal basis and are the idempotents of the quantum product

$$\phi_i * \phi_j = \delta_{ij} \phi_i.$$

The genus zero potential for  $N$  points is nothing but a sum of genus zero potentials of a point

$$F_0^{Npt}(t^1, \dots, t^N) = F_0^{pt}(t^1) + \dots + F_0^{pt}(t^N).$$

In particular, the genus zero theory of  $N$  points is semisimple.

By Theorem 1, any *semisimple* genus zero axiomatic theory  $T$  of rank  $N$  can be obtained from  $H^{Npt}$  by action of an element  $O^T$  in the twisted loop group. By Birkhoff factorization,  $O^T = S^T(z^{-1})R^T(z)$ , where  $S(z^{-1})$  (resp.  $R(z)$ ) is a matrix-valued function in  $z^{-1}$  (resp.  $z$ ).

In order to define the axiomatic higher genus potentials  $G_g^T$  for the semisimple theory  $T$ , one first introduces the “ $\tau$ -function of  $T$ ”.

**Definition 3.** [19] Define the *axiomatic  $\tau$ -function* as

$$(9) \quad \tau_G^T := \widehat{S^T}(\widehat{R^T} \tau_{GW}^{Npt}),$$

where  $\tau_{GW}^{Npt}$  is defined in (1). Define the *axiomatic genus  $g$  potential*  $G_g^T$  via the formula (cf. (1))

$$(10) \quad \tau_G^T =: e^{\sum_{g=0}^{\infty} \hbar^{g-1} G_g^T}.$$

**Remarks.** (i) It is not obvious that the above definitions make sense. The function  $\widehat{S^T}(\widehat{R^T} \tau_{GW}^{Npt})$  is well-defined, due to some finiteness properties of  $\tau^{pt}$ , called the  $(3g-2)$ -jet properties [19][17]. The fact that  $\log \tau_G^T$  can be written as  $\sum_{g=0}^{\infty} \hbar^{g-1}$  (formal function in  $t$ ) is also nontrivial. The interested readers are referred to the original article [19] or [33] for details.

(ii) What makes Givental’s axiomatic theory especially attractive are the facts that

- (a) It works for any semisimple Frobenius manifolds, not necessarily coming from geometry.
- (b) It enjoys properties often complementary to the geometric theory.

These will be put into use in the following sections.

## 4. Virasoro constraints

**4.1. Virasoro operators for points via quantization.** Let  $H^{Npt}$  be the genus zero theory for  $X$  being  $N$  points. Define the differential operators  $D$  on the corresponding  $\mathcal{H} = H^{Npt}((z^{-1}))$

$$D := z\left(\frac{d}{dz}\right)z = z^2 \frac{d}{dz} + z.$$

Define the operators  $\{L_m\}$  for  $m = -1, 0, 1, 2, \dots$

$$L_m := -z^{-1/2} D^{m+1} z^{-1/2}.$$

The operators  $L_m$  have only integer exponents of  $z$ .

**Lemma 2.** (i)

$$(11) \quad [L_m, L_n] = (m-n)L_{m+n}.$$

(ii)  $L_m$  are infinitesimal symplectic transformations.

PROOF. Part (i) can be proved in the following way. First perform a change of variables  $w = 1/z$ . Then

$$z^{1/2} L_m z^{-1/2} = (-1)^m w \frac{d^{m+1}}{dw^{m+1}}.$$

The RHS has a Fourier transform to the standard vector fields on the disk  $(-1)^m w^{m+1} \frac{d}{dw}$ , which obviously satisfies the Virasoro relations of (i).

Part (i) implies that the Lie algebra spanned by  $L_m$  is generated by  $L_2$  and  $L_{-1}$ . These two operators can be verified to satisfy (ii).  $\square$

Due to Lemma 2 (ii),  $L_m$  can be quantized to  $\widehat{L}_m$ . The Lie algebra generated by  $\{\widehat{L}_m\}_{m \geq -1}$  satisfies the Virasoro relations due to Lemma 1. The Virasoro operators  $\{\widehat{L}_m\}_{m \geq -1}$  constructed above are the same as  $N$  copies of those used in Witten's conjecture in relation to KdV hierarchies [39].

**4.2. Virasoro operators for semisimple axiomatic theories.** Now for any axiomatic theory of rank  $N$ , one may define the Virasoro operators. The notations in Definition 3 will be followed.

**Definition 4.** [19]

$$(12) \quad \widehat{L}_m^T := \widehat{S}^T (\widehat{R}^T \widehat{L}_m^{H^{Npt}} \widehat{R}^T)^{-1} \widehat{S}^T{}^{-1}.$$

**Lemma 3.** (i)  $\{\widehat{L}_m^T\}_{m \geq -1}$  satisfy the Virasoro relations (11).

$$(ii) \quad \widehat{L}_m^T \tau_G^T = 0.$$

PROOF. (i) is obviously true as the conjugation does not change the commutation relations. (ii) follows from the fact that  $\widehat{L}_m^{H^{Npt}} \tau^{H^{Npt}} = 0$ , which is  $N$  copies of Witten's conjecture.  $\square$

### 4.3. Virasoro constraints.

**Virasoro Conjecture.** [12] *For any projective manifold  $X$ , there exist "Virasoro operators"  $\{\widehat{L}_m^X\}_{m \geq -1}$ , satisfying the relations (11), such that*

$$\widehat{L}_m^X \tau_{GW}^X = 0, \quad \forall m \geq -1.$$

A good reference for a precise statement can be found in [16].

With Lemma 3, a clear path to prove Virasoro conjecture would be to show, when  $T^X$ , the genus zero Gromov–Witten theory of  $X$ , is semisimple, the following two statements.

- (a) Definition 4 of Virasoro operators coincide with the definition of Eguchi–Hori–Xiong [12] in the semisimple case.
- (b)  $\tau_{GW}^X = \tau_G^{T^X}$ .

**Remarks.** (i) (a) can be proved with some efforts. See [19] and [33]. In fact, the operators defined by (12) are also equivalent to those defined by Dubrovin–Zhang [10].

(ii) Givental uses the axiomatic framework and a clever observation to give a one-line proof of genus zero Virasoro Conjecture [18].

(b) will be called *Givental's Conjecture*. The proof of (b) is more complicated.

Assume that  $X$  has a torus action. Then the equivariant Gromov–Witten invariants are defined and hence the  $\tau$ -function  $\tau_{e_{GW}}^X$  for the equivariant Gromov–Witten theory of  $X$ . Suppose furthermore that the torus action has isolated fixed points and one-dimensional orbits. It is not hard to see that the genus zero equivariant Gromov–Witten invariants define a semisimple axiomatic theory  $T_e^X$  [33]. Therefore the above formulation works in this case and

**Theorem 2.** [18, 19] *Givental's Conjecture holds in the above (equivariant) case.*



Givental’s proof of this theorem is a tour de force and involves deep theory of Frobenius manifolds and localization. See [33] for details.

If the non-equivariant genus zero Gromov–Witten theory  $T^X$  is semisimple, then it is reasonable to expect that non-equivariant limit of  $T_e^X$  exists and equals  $T^X$ . This will prove (non-equivariant) Virasoro Conjecture in a great deal of examples. Unfortunately, this proves harder than one expects. Givental provided a key reduction to a much simpler statement of the existence of the non-equivariant limit of a certain  $R_e^{T^X}$  restricted to the “small quantum cohomology”, which we will not state but refer the interested readers to [19]. This last statement has been proved for the toric Fano manifolds [19], for general toric manifolds [24], and for some classical flag varieties [26] [3] [4].

Another approach to Givental’s conjecture is through the invariance of the tautological equations. See Section 5 for an explanation of the following result.

**Theorem 3.** [22, 30, 31] *Givental’s Conjecture holds for genus one and two.*

In 2005, C. Teleman announces a very strong classification theorem of all semisimple Frobenius manifolds. This is a very exciting progress as Givental’s Conjecture follows as a corollary.

**Theorem 4.** [36] *Givental’s Conjecture is true. Therefore, Virasoro constraints also holds for semisimple Gromov–Witten theory. Furthermore, the equality holds at the level of cycles.*

## 5. Invariance of tautological relations

**5.1. Tautological rings.** A basic reference for tautological rings is [38], where the history of the subject is explained.

The tautological rings  $R^*(\overline{M}_{g,n})$  are subrings of  $A^*(\overline{M}_{g,n})_{\mathbb{Q}}$ , or subrings of  $H^{2*}(\overline{M}_{g,n})$  via cycle maps, generated by some “geometric classes” which will be described below.

There are two types of natural morphisms between moduli stacks of curves. The *forgetful morphisms*

$$(13) \quad \text{ft}_i : \overline{M}_{g,n+1} \rightarrow \overline{M}_{g,n}$$

forget one of the  $n + 1$  marked points. The *gluing morphisms*

$$(14) \quad \overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \rightarrow \overline{M}_{g_1+g_2,n_1+n_2}, \quad \overline{M}_{g-1,n+2} \rightarrow \overline{M}_{g,n},$$

glue two marked points to form a curve with a new node. Note that the boundary strata are the images (of the repeated applications) of the gluing morphisms, up to factors in  $\mathbb{Q}$  due to automorphisms.

**Definition 5.** The system of tautological rings  $\{R^*(\overline{M}_{g,n})\}_{g,n}$  is the smallest system of  $\mathbb{Q}$ -unital subalgebra closed under the forgetful and gluing morphisms.

As it contains the units, the fundamental classes of the boundary strata are contained in  $R^*(\overline{M}_{g,n})$ . From some elementary manipulations, one can also produce Chern classes of certain tautological vector bundles:  $\psi$ -classes,  $\lambda$ -classes and  $\kappa$ -classes.

**5.2. Invariance constraints.** Due to the existence of natural *stabilization morphisms*

$$(15) \quad \text{st} : \overline{M}_{g,n}(X, \beta) \rightarrow \overline{M}_{g,n}$$

defined by forgetting morphisms to  $X$  and stabilizing the domain curves if necessary, any relations in  $H^*(\overline{M}_{g,n})_{\mathbb{Q}}$  can be pull-backed to  $H^*(\overline{M}_{g,n}(X, \beta))_{\mathbb{Q}}$ . Therefore, the induced equation will hold for any geometric Gromov–Witten theory. It is natural to expect that the same is true for any axiomatic theory.

For any axiomatic theory  $T$ ,  $\tau$ -function can be obtained as  $\tau_G^T = \widehat{S}^T(\widehat{R}^T \tau_{GW}^{Npt})$ . The  $\tau$ -function of  $N$  points certainly satisfies any induced equations. Therefore, in order to show that the induced tautological relations hold for  $T$ , it is enough to show that these relations are *invariant* under the action of  $\widehat{R}^T$  and  $\widehat{S}^T$ .

**Theorem 5.** [30, 31, 14] *The tautological relations are invariant under the action of twisted loop groups.*

Since the twisted loop groups can be (Birkhoff) factorized into  $\widehat{S}$  and  $\widehat{R}$ , one can prove Theorem 5 by proving invariance individually. The  $\widehat{S}$  invariance was shown in [30, 31] to be a consequence of geometry on moduli of stable maps. The proof of  $\widehat{R}$  invariance was originally given in [30], using Theorem 4.<sup>4</sup> It was later independently discovered by Faber–Shadrin–Zvonkine, and R. Pandharipande and the author, that the  $\widehat{R}$  invariance is a simple consequence of geometry on moduli of curves. See Section 3 of [14] for details.

**Remark.** The invariance under the action of  $\widehat{S}$  imposes little restriction. It is a consequence of the fact that there is an accountable difference [27] [19] between  $\psi$ -classes defined on  $\overline{M}_{g,n}(X, \beta)$  and the pull-backs of  $\psi$ -classes from  $\overline{M}_{g,n}$  via the stabilization morphism (15). On the other hand, the invariance under the action of  $\widehat{R}$  imposes very strong constraints on the structure of tautological relations, and hence can be used to derive many tautological relations.

**5.3. Application I: Virasoro constraints.** As alluded in Theorem 3, the invariance conjecture can be used to prove Givental’s Conjecture for  $g \leq 2$ . The idea is to first show some uniqueness theorem: If two genus  $g$  potentials  $F_g$  and  $G_g$  satisfy enough tautological relations, then they are identical up to some initial conditions. For  $g = 1$  and 2, this has been done in [10] and [34]. Therefore, Givental’s conjecture would follow from the statement that  $G_g$  satisfy those tautological relations. As explained above, that in turns follows from the invariance of the tautological relations under the action of  $\widehat{R}$ . That is how Theorem 3 is proved. One immediate corollary of Theorem 3 is Virasoro conjecture in  $g \leq 2$  in the semisimple cases.

**5.4. Application II: Witten’s (generalized) conjecture.** Another consequence, which seems less obvious but more interesting to us, is the following conjecture of Witten.

Witten has proposed a conjectural relation between invariants on the moduli space of higher spin curves and the Gelfand–Dickey hierarchies [40], generalizing his

<sup>4</sup>In fact, only the “cycle-form” of Theorem 4 for  $X = \mathbb{P}^1$  was used. The  $X = \mathbb{P}^1$  case can be proved via localization and clever resummation trick of Givental [19], *without* Teleman’s result. However, a localization proof does not yet exist in the literature. (It is part of project [33].)

previous conjecture [39]. It turns out that the genus zero case defines a semisimple theory, which is isomorphic to the Frobenius structure of the miniversal deformation of  $A_r$ -singularities. The axiomatic theory built on that satisfies the Gelfand–Dickey hierarchies [20]. Although the “geometric” theory defined by the invariants on the moduli space of higher spin curves is, strictly speaking, not a geometric Gromov–Witten theory, it also fit into the framework of the axiomatic theory. Therefore the uniqueness theorems apply and one has the following corollary.

**Theorem 6.** [14] *Witten’s generalized conjecture is true.*

This would be a simple consequence of Theorem 4, combined with an earlier result by Givental [20]. However, the proof in [14] is independent of Teleman’s results. It uses the ingredients outlined in [28, 31], while expertly combining with ingredients from well-known results in tautological rings, which we failed to see four years ago.

**5.5. Application III: Finding tautological relations.** Another application is going the opposite direction: a feedback from Gromov–Witten theory to tautological rings on moduli of curves. Using Theorem 5, all the known tautological relations are obtained by requiring invariance under  $\widehat{R}$ . It seems plausible that  $\widehat{R}$  invariance can be used to derive many tautological relations. In fact, combining Theorem 5 with some known results (e.g. Betti number calculation of Getzler in [15]), all known tautological relations can be proved uniformly within this framework. See [29] [1] [2] [22] for discussions and computations.

## 6. Other applications

**6.1. Twisted Gromov–Witten invariants.** In the axiomatic theory, Coates and Givental [8] was able to find the right framework to express the twisted Gromov–Witten invariants in terms of the untwisted Gromov–Witten invariants.

Given a vector bundle  $E$  on  $X$ , one can define a element  $\mathcal{R}$  in the  $K$ -theory of  $\overline{M} := \overline{M}_{g,n}(X, \beta)$  in the following way. Let  $\pi : \mathcal{C} \rightarrow \overline{M}$  be the universal curve and  $f : \mathcal{C} \rightarrow X$  be the universal morphism.  $\mathcal{R} := R^0\pi_* f^* E \rightarrow R^1\pi_* f^* E$  is then an element in the  $K$ -theory of  $\overline{M}$ . The twisted Gromov–Witten invariants are the integration, over the virtual fundamental classes  $[\overline{M}]^{\text{vir}}$  of the usual insertions  $H^*(X)$  and descendents, but “twisted” with certain multiplicative characteristic classes of the virtual vector bundle  $\mathcal{R}$ .

It is known in [13] that these twisted classes can be expressed in terms of untwisted classes in the case  $E = \mathcal{O}$ . The underlying geometry was the Grothendieck–Riemann–Roch calculation in [35]. The case  $E$  being any virtual bundle proceed similarly without difficulty. However, there was no “closed form” to express this relation. The axiomatic framework made this possible. Of course, it is a truly tour de force to carry out this program even within the axiomatic framework.

**6.2. Crepant resolution conjecture.** In another direction, this framework is applied to formulate a precise conjecture regarding the relations between the Gromov–Witten invariants of an orbifold and those of its crepant resolutions. A

recent preprint [7] gives an excellent account of this conjecture, so we will be very brief.<sup>5</sup>

A morphism  $f : Y \rightarrow \mathcal{X}$  is called a *crepant resolution*, if  $Y$  is smooth and  $\mathcal{X}$  is  $\mathbb{Q}$ -Gorenstein (e.g. an orbifold) such that  $f^*(K_{\mathcal{X}}) = K_Y$ . In the case  $\mathcal{X}$  is an orbifold, there is a well-defined orbifold Gromov–Witten theory due to Chen–Ruan [5]. The *crepant resolution conjecture* asserts a close relation between the Gromov–Witten theory of  $Y$  and that of  $\mathcal{X}$ . Under suitable conditions (e.g. hard Lefschetz), the conjecture asserts that the Gromov–Witten theory of  $Y$  and  $\mathcal{X}$  are identified, up to analytic continuation on the Novikov variables. When the orbifold  $\mathcal{X}$  does not satisfy the hard Lefschetz condition, it is speculated that a very weak relation still hold. Coates–Corti–Iritani–Tseng and Ruan believe that this weak relation could and should only be formulated in terms of a symplectic transformation of the special type. The interested readers might consult [7, 6] and references therein, as well as the next subsection, for more information.

**6.3. Invariance of Gromov–Witten theory under simple flops.** Crepant resolution is a special case of  $K$ -equivalence.  $X$  and  $X'$  are called  $K$ -equivalent if there is a common resolution  $Y$  such that  $K_X$  and  $K_{X'}$  are equal after pulling back to  $Y$ . In [32], the case when  $X$  and  $X'$  are smooth and related by a simple flop is studied. It was shown that the big quantum rings are isomorphic *after* an analytic continuation in the quantum variables.

In [25], axiomatic framework is used to generalize this result to higher genus. It is first shown, by degeneration to the normal cone, that the statement can be reduced to the a statement about toric varieties and toric flops. Then the quantization formulation reduces the higher genus statement to a genus zero statement. An explicit calculation was carried out in genus zero to show the invariance of *ancestor* invariants, after analytic continuation. In fact, results in [25] suggest that the ancestor formulation is the right framework to study crepant resolution conjecture.

## 7. Final remarks

**7.1. Orbits of twisted loop group action on the moduli spaces of Frobenius manifolds.** We learned from Theorem 1 that the twisted loop groups acts on the space of all Frobenius manifolds of a given rank. Furthermore, the semisimple theories lie in a single orbit. It is obvious that the group action has to preserve the “degree of diagonalizability” (or degree of semisimplicity) of the theories. However, it is not known (to me) how many orbits one is to have with a fixed degree of diagonalizability. It would be very interesting to investigate the orbit structure of the twisted loop group action.

**7.2. Integrable hierarchies.** In the proof of the Virasoro conjecture, a basic trick is to move  $\widehat{L}_m^{Npt}$  to the left of  $\widehat{S}^T \widehat{R}^T$  and to identify it as the  $\widehat{L}_m^T$ . In the same spirit, since the  $\tau^{Npt}$  satisfies the Hirota equations of KdV hierarchies, one may try to commute Hirota operators of KdV hierarchies with  $\widehat{S}^T \widehat{R}^T$ . The result could be some yet unknown integrable hierarchies. This would realize Dubrovin’s program of finding a correspondence between certain classes of integrable hierarchies and

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<sup>5</sup>The only *difference* of our points of view lies on the our choices of the (over-ruled) Lagrangian cones. While the authors of [7] insists in using descendent cones, we think the ancestor cones would make the statement a lot cleaner.

the Gromov–Witten theory. However, the task of commuting operators are rather difficult, due to some convergence issues. Successful examples include [20] and [23].

**7.3. Axiomatic relative/orbifold/open-string theory.** Givental’s theory has so far been developed upon the “original” Gromov–Witten theory. It is reasonable to ask whether this theory can be generalized to cover some ramifications of the original Gromov–Witten theory, such as *relative Gromov–Witten theory* or *orbifold Gromov–Witten theory*. My personal guess is that orbifold theory is probably easier. Indeed, it is not very difficult to see that Givental’s theory of quantization should work for orbifold theory as well. At least in the equivariant context, the original localization scheme seems to work. See Tseng’s work [37] for some progress along this direction.

In a proper sense that the orbifold compactification is a “minimal” one inside the relative compactification, which involves bubbling off the target spaces and is more complicated.<sup>6</sup> Therefore, one expects that the axiomatization of relative theory will be harder.

Furthermore, Gromov–Witten theory is considered as a topological field theory associated to closed strings. There is an open string analogue. It is also reasonable to ponder the possibility of an open-string axiomatic theory, whose geometric theory has not been successfully constructed. This might further our understanding of the open-string GW theory.

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<sup>6</sup>This was discovered in a discussion with Y. Ruan more than three years ago. One example fully worked out is the comparison of the TQFT developed by Bryan–Pandharipande in arXiv:math/0411037 (relative compactification) and one developed by R. Cavalieri in arXiv:math/0411500 and arXiv:math/0512225 (orbifold compactification). More recently, the relationship between the relative invariants and orbifold invariants have been extensively studied by D. Abramovich, C. Cadman, B. Fantechi, and J. Wise.

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