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Spectrum of the Lamé operator and application, I: Deformation along $\mathrm{Re}\,\tau = \frac{1}{2}$

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ABSTRACT

In this paper, we study the spectrum of the Lamé operator

$$L = \frac{d^2}{dx^2} - 6\wp(x + z_0; \tau) \quad \text{in} \ \ L^2(\mathbb{R}, \mathbb{C}),$$

where $\wp(z;\tau)$ is the Weierstrass elliptic function with periods 1 and τ, and $z_0 \in \mathbb{C}$ is chosen such that L has no singularities on $\mathbb{R}$.
(i) We completely determine the explicit location of intersection points of spectral arcs.
(ii) We give a complete picture of the deformation of the spectrum as $\tau = \frac{1}{2} + ib$ and $b > 0$ varies. In particular, we show that the spectrum has exactly 9 different types of graphs for different b's, and we also give the explicit range of b for each type of graphs. This solves open problems raised in [17].
(iii) As an application of the spectrum and the deep connection of the Lamé equation with the mean field equation from [4], we prove the existence of $\tau = \frac{1}{2} + ib$ such that the mean field equation $\triangle u + e^u = 16\pi\delta_0$ on the rhombus torus $E_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ has no even axisymmetric solutions but

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does have 2 even not-axisymmetric solutions. This gives the first positive answer to a long-standing open problem.

1. Introduction

Let $\tau \in \mathbb{H} = \{\tau | \operatorname{Im} \tau > 0\}$ and $E_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ be a flat torus. Recall that $\wp(z) = \wp(z;\tau)$ is the Weierstrass elliptic function with basic periods $\omega_1 = 1$ and $\omega_2 = \tau$. Denote also $\omega_3 = 1+\tau$ and $e_k = e_k(\tau) := \wp(\frac{\omega_k}{2};\tau)$ for $k \in \{1,2,3\}$. It is well known that

$$\wp'(z;\tau)^2 = 4\prod_{k=1}^{3}(\wp(z;\tau) - e_k(\tau)) = 4\wp(z;\tau)^3 - g_2(\tau)\wp(z;\tau) - g_3(\tau).$$

Let $\zeta(z) = \zeta(z;\tau) := -\int^z \wp(\xi;\tau)d\xi$ be the Weierstrass zeta function with two quasi-periods $\eta_j = \eta_j(\tau)$, $j = 1,2$:

$$\eta_j(\tau) := 2\zeta(\tfrac{\omega_j}{2};\tau) = \zeta(z+\omega_j;\tau) - \zeta(z;\tau), \quad j = 1,2, \tag{1.1}$$

and $\sigma(z) = \sigma(z;\tau) := \exp\int^z \zeta(\xi;\tau)d\xi$ be the Weierstrass sigma function. Notice that $\zeta(z)$ is an odd meromorphic function with simple poles at $\mathbb{Z}+\mathbb{Z}\tau$, $\sigma(z)$ is an odd entire function with simple zeros at $\mathbb{Z}+\mathbb{Z}\tau$ and η_j satisfies the Legendre relation $\tau\eta_1 - \eta_2 = 2\pi i$. We will use these classical special functions freely.

In this paper, we study the spectrum $\sigma(L_n)$ of the Lamé operator [19]

$$L_n := \frac{d^2}{dx^2} - n(n+1)\wp(x+z_0;\tau), \quad x \in \mathbb{R} \tag{1.2}$$

in $L^2(\mathbb{R},\mathbb{C})$, where $n \in \mathbb{N}$ and $z_0 \in \mathbb{C}$ is chosen such that $\wp(x+z_0;\tau)$ has no singularities on $\mathbb{R}$. Remark that $\sigma(L_n)$ does not depend on the choice of z_0 due to the fact that the Lamé potential $-n(n+1)\wp(z;\tau)$ is a Picard potential in the sense of Gesztesy and Weikard [14] (i.e. all solutions of the Lamé equation

$$y''(z) = [n(n+1)\wp(z;\tau) + E]y(z), \quad z \in \mathbb{C} \tag{1.3}$$

are meromorphic in $\mathbb{C}$). Indeed, we can also consider $z_0 = 0$ where the Lamé potential has singularities on $\mathbb{R}$; see [28] where the spectral theory for Picard potentials with singularities on $\mathbb{R}$ was studied. In particular, the Hill's discriminant $\Delta(E)$ is still well-defined (i.e. $\Delta(E)$ is the trace of the monodromy matrix of (1.3) with respect to $z \to z+1$; see Section 2 for a brief overview of this entire function). The spectral theory of the Schrödinger operator L with complex periodic smooth potentials has attracted significant attention and has been studied widely in the literature; see e.g. [1,2,14,16,17,25] and references therein. In this theory, it is known [25] that the spectrum $\sigma(L)$ satisfies

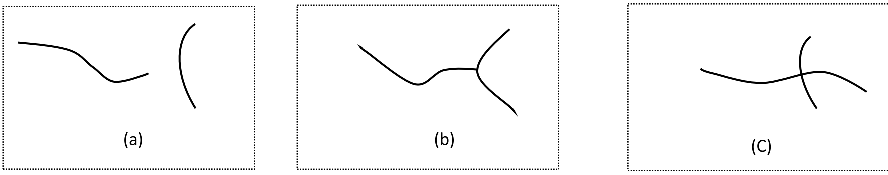

Fig. 1. Rough graphs of three types for the spectrum of the $n = 1$ Lamé operator for general τ.

$$\sigma(L) = \Delta^{-1}([-2, 2]) = \{E \in \mathbb{C} \mid -2 \le \Delta(E) \le 2\}.$$

Furthermore, it was proved in [14] that $\sigma(L)$ consists of finitely many analytic arcs if the potential of L is a Picard potential. In this paper, as in [14] we call the arcs of $\sigma(L) = \Delta^{-1}([-2, 2])$ as *spectral arcs* of the operator L.

Generally, the spectrum is very complicated except for the case when the periodic potential is *real-valued and smooth* on $\mathbb{R}$. If a Picard potential is real-valued but with singularities on $\mathbb{R}$, the spectrum might not be contained in $\mathbb{R}$ in general; see e.g. [16,17,28]. Thus the theory is still far from complete, and more explicit examples with nontrivial spectra are needed to understand the geometry of spectrum arcs.

In this series of papers, we want to study explicitly the spectrum of the classical Lamé operator (1.2) and explore its applications. For the simplest case $\tau \in i\mathbb{R}_{>0}$, since the Lamé potential $-n(n+1)\wp(x + \frac{\omega_k}{2}; \tau)$ with $k \in \{2, 3\}$ is real-valued and smooth on $\mathbb{R}$, Ince [18] discovered a remarkable fact: For $\tau \in i\mathbb{R}_{>0}$,

$$\sigma(L_n) = (-\infty, E_{2n}] \cup [E_{2n-1}, E_{2n-2}] \cup \cdots \cup [E_1, E_0], \tag{1.4}$$

where $E_{2n} < E_{2n-1} < \cdots < E_1 < E_0$ are precisely all roots of the well-known *spectral polynomial* $Q_n(E; \tau)$ (also called the *Lamé polynomial* in the literature) associated to the Lamé potential (see e.g. [3,15]).

However, the spectrum $\sigma(L_n)$ is no longer of the form (1.4) for general τ's and becomes very complicated; see e.g. [1,7,15–17] and references therein. Batchenko and Gesztesy [1] and Haese-Hill et al. [17] concentrated on the $n = 1$ case, for which the spectrum $\sigma(L_1)$ consists of two regular analytic arcs and so there are totally three different types of topological graphs for different τ's as shown in Fig. 1 (see also [15,16]). Note that (b) occurs at those τ's such that $e_k(\tau) + \eta_1(\tau) = 0$ for some k; see [1,17,26]. In particular for $\tau = \frac{1}{2} + ib$ with $b > 0$, [1,17] inferred that $\sigma(L_1) = (-\infty, e_1] \cup \sigma_1$, *where* σ_1 *is a simple arc symmetric with respect to* $\mathbb{R}$ *with endpoints* e_2 *and* $e_3 = \overline{e_2}$, *and* $\{p\} := \sigma_1 \cap \mathbb{R}$ *satisfies* $p = -\eta_1 < e_1$ *for* $b > \tilde{b}$, $p = -\eta_1 = e_1$ *for* $b = \tilde{b}$ *and* $p > e_1$ *for* $0 < b < \tilde{b}$. Here $\tilde{b}$ is the unique zero of $e_1 + \eta_1$ on the line $\tau = \frac{1}{2} + ib$. It was pointed out in [17, Section 5] that the rigorous analysis of $n \ge 2$ cases seems to be difficult since the related explicit formulae quickly become quite complicated as n grows. Indeed the pattern of the spectrum for the case $n = 2$ is still unknown so far.

1.1. Spectrum of the $n = 2$ Lamé operator

Our first subject is to study the spectrum $\sigma(L)$ of the $n = 2$ Lamé operator

$$L = L_2 = \frac{d^2}{dx^2} - 6\wp(x + z_0; \tau), \quad x \in \mathbb{R}. \tag{1.5}$$

It is well known (see e.g. [15,17,21,29]) that the spectral polynomial $Q_2(E; \tau)$ is given by

$$Q_2(E; \tau) = (E^2 - 3g_2(\tau)) \prod_{k=1}^{3} (E + 3e_k(\tau)), \tag{1.6}$$

which has a double zero at $\tau = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. Note that this situation never occurs for $n = 1$. Even along the line $\operatorname{Re} \tau = \frac{1}{2}$, the complexity of spectral arcs of $\sigma(L)$ was briefly commented in [17]. We will show that the spectral arcs have exactly 9 different patterns.

By applying [14, Theorem 4.1] to (1.5), we see that *the spectrum $\sigma(L)$ consists of $\tilde{g} \in \{1, 2\}$ bounded simple analytic arcs σ_k and one semi-infinite simple analytic arc σ_∞ which tends to $-\infty + \langle q \rangle$, with $\langle q \rangle = \int_{x_0}^{x_0+1} q(x)dx$, i.e.*

$$\sigma(L) = \sigma_\infty \cup \cup_{k=1}^{\tilde{g}} \sigma_k, \quad \tilde{g} \in \{1, 2\}, \tag{1.7}$$

where the finite endpoints of such arcs must be zeros of the spectral polynomial $Q_2(E; \tau)$. See Section 2 for a brief overview of this fact. To the best of our knowledge, there seems no more explicit description of $\sigma(L)$ in the literature. To study the geometry of $\sigma(L)$, we have to determine completely the intersection points of different arcs of $\sigma(L)$.

Definition 1.1. Let E be an intersection point of at least two spectral arcs in $\{\sigma_\infty, \sigma_1, \sigma_2\}$.

- We say that E is of type I if E is not an endpoint of these arcs, i.e. $Q_2(E; \tau) \neq 0$ and so E is met by $2k \geq 4$ semi-arcs of the spectrum $\sigma(L)$.
- We say that E is of type II if E is an endpoint of these arcs, i.e. $Q_2(E; \tau) = 0$ or equivalently

$$E \in \left\{ \pm(3g_2(\tau))^{1/2}, -3e_1(\tau), -3e_2(\tau), -3e_3(\tau) \right\}.$$

For example, the intersection point in Fig. 2-(1) is of type I, and the intersection point in Fig. 2-(8) is of type II. Define

$$E_\pm(\tau) := \frac{-3\eta_1(\tau) \pm \sqrt{9\eta_1(\tau)^2 + 6g_2(\tau)}}{2} \tag{1.8}$$

to be zeros of the polynomial $P(E) := E^2 + 3\eta_1(\tau)E - \frac{3}{2}g_2(\tau)$. Our first result completely determines all possible intersection points of $\sigma(L)$.

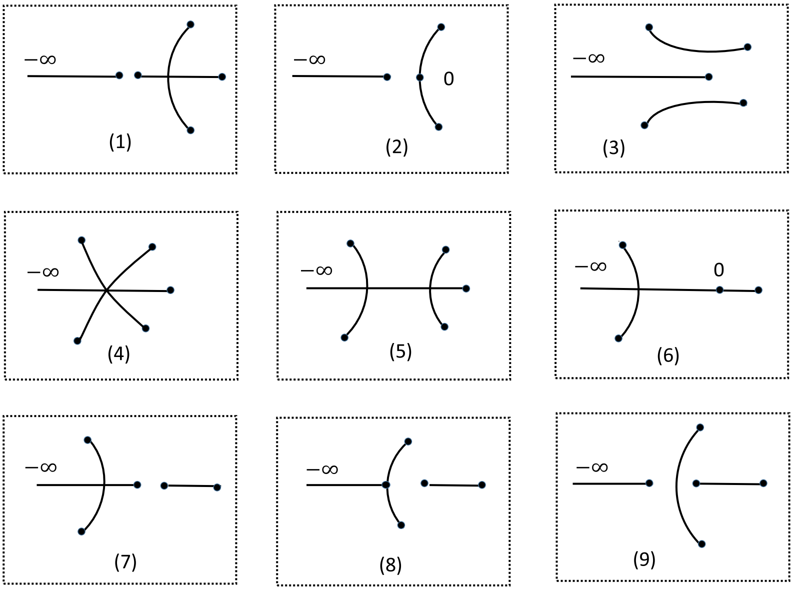

Fig. 2. Nine rough graphs of the spectrum for different choices of b's as stated in Theorem 1.3 and the continuous deformation of the spectrum as b decreases. The dark points denote zeros of $Q_2(E;\tau)$.

Theorem 1.2. *Fix any $\tau \in \mathbb{H}$.*

(1) *E is a type I intersection point if and only if*

$$E \in \{E_\pm(\tau)\} \cap \sigma(L) \quad \textit{and} \quad Q_2(E;\tau) \neq 0.$$

(2) *For $k \in \{1,2,3\}$, $-3e_k(\tau)$ is a type II intersection point if and only if $e_k'(\tau) = 0$.*
(3) *$\pm(3g_2(\tau))^{1/2}$ is a type II intersection point if and only if*

$$6\eta_1(\tau) \pm (3g_2(\tau))^{1/2} = 0.$$

Now we study the deformation of the spectrum $\sigma(L)$ when $\tau = \frac{1}{2} + ib$ with $b > 0$ varying. In the sequel, we use $A \sqcup B$ to denote the disjoint union of A and B, i.e. $A \cap B = \emptyset$. Note that $g_2, e_1, \eta_1 \in \mathbb{R}$ and $e_2 = \overline{e_3} \notin \mathbb{R}$ for $\operatorname{Re}\tau = \frac{1}{2}$. We will prove in Lemma 4.2 that

$$b_1 := \sup\left\{\tilde{b} > \tfrac{1}{2\sqrt{3}} \,|\, (3\eta_1^2 + 2g_2)(\tfrac{1}{2} + ib) > 0 \text{ for } b \in [\tfrac{1}{2\sqrt{3}}, \tilde{b})\right\}, \tag{1.9}$$

is well-defined and $\frac{1}{2\sqrt{3}} < b_1 < \frac{1}{2}$. Clearly $3\eta_1^2 + 2g_2 = 0$ and so $E_+ = E_- = -\frac{3}{2}\eta_1$ at $\frac{1}{2} + ib_1$. On the other hand, it was proved in [9, Corollary 1.5] that

$$\textit{there is a unique } b_0 \in (\tfrac{5}{24}, \tfrac{1}{2\sqrt{3}}) \textit{ such that } \tfrac{d}{db}\eta_1(\tfrac{1}{2} + ib_0) = 0. \tag{1.10}$$

Clearly the explicit values of b_1 and b_0 can be computed numerically, which implies $b_1 \approx 0.3$ and $b_0 \approx 0.24$. Our second main result shows that $\sigma(L)$ has exactly 9 different types of graphs for different b's; see Fig. 2.

Theorem 1.3. *Let $\tau = \frac{1}{2} + ib$, $b > 0$. Then the spectrum $\sigma(L)$ of the $n = 2$ Lamé operator (1.5) is symmetric with respect to the real line $\mathbb{R}$.*

Moreover, the following statements hold, where the notation σ_2 always denotes a simple arc symmetric with respect to $\mathbb{R}$ with endpoints $-3e_2$ and $-3e_3$.

(1) *If $b > \frac{\sqrt{3}}{2}$, then $g_2, e_1 > 0$ and*

$$\sigma(L) = (-\infty, -3e_1] \sqcup [-(3g_2)^{1/2}, (3g_2)^{1/2}] \cup \sigma_2,$$

with

$$\sigma_2 \cap \mathbb{R} = \sigma_2 \cap (-(3g_2)^{1/2}, (3g_2)^{1/2}) = \{E_+(\tfrac{1}{2} + ib)\}.$$

(2) *If $b = \frac{\sqrt{3}}{2}$, then $e_1 > 0$, $g_2 = 0$, i.e. the two endpoints $\pm(3g_2)^{1/2}$ collapse into $E = 0$ and so*

$$\sigma(L) = (-\infty, -3e_1] \sqcup \sigma_2, \quad \text{with} \quad \sigma_2 \cap \mathbb{R} = \{0\}.$$

(3) *Let b_1 be defined in (1.9). Then for $b \in (b_1, \frac{\sqrt{3}}{2})$, we have $g_2 < 0$ and*

$$\sigma(L) = (-\infty, -3e_1] \sqcup \sigma_1 \sqcup \overline{\sigma_1},$$

where σ_1 is a simple arc in $\{E | \operatorname{Im} E > 0\}$ (i.e. $\sigma_1 \cap \mathbb{R} = \emptyset$) with endpoints $-3e_2$ (note $\operatorname{Im} e_2(\frac{1}{2} + ib) < 0$ for all b) and $i|3g_2|^{1/2}$, and $\overline{\sigma_1}$ is the conjugate of σ_1 with endpoints $-3e_3$ and $-i|3g_2|^{1/2}$.

(4) *If $b = b_1$, then $g_2 < 0$ and*

$$\sigma(L) = (-\infty, -3e_1] \cup \sigma_1 \cup \sigma_2,$$

where σ_1 is a simple arc symmetric with respect to $\mathbb{R}$ with endpoints $\pm(3g_2)^{1/2}$ and

$$\sigma_1 \cap \mathbb{R} = \sigma_2 \cap \mathbb{R} = \sigma_1 \cap \sigma_2 = \{-\tfrac{3}{2}\eta_1(\tfrac{1}{2} + ib_1)\} \subset (-\infty, -3e_1).$$

(5) *If $b \in (\frac{1}{2\sqrt{3}}, b_1)$, then $g_2 < 0$ and*

$$\sigma(L) = (-\infty, -3e_1] \cup \sigma_1 \cup \sigma_2,$$

where σ_1 is a simple arc symmetric with respect to $\mathbb{R}$ with endpoints $\pm(3g_2)^{1/2}$ and

$$\sigma_1 \cap \mathbb{R} = \sigma_1 \cap (-\infty, -3e_1) = \{E_+(\tfrac{1}{2} + ib)\},$$

$$\sigma_2 \cap \mathbb{R} = \sigma_2 \cap (-\infty, -3e_1) = \{E_-(\tfrac{1}{2} + ib)\},$$
$$\sigma_1 \cap \sigma_2 = \emptyset, \quad E_+(\tfrac{1}{2} + ib) > E_-(\tfrac{1}{2} + ib).$$

(6) *If* $b = \frac{1}{2\sqrt{3}}$*, then* $g_2 = 0, e_1 < 0$ *and so*

$$\sigma(L) = (-\infty, -3e_1] \cup \sigma_2, \quad \textit{with} \quad \sigma_2 \cap \mathbb{R} = \{-6\sqrt{3}\pi\}.$$

(7) *Recall* $b_0 \in (\frac{5}{24}, \frac{1}{2\sqrt{3}})$ *in (1.10). Then for* $b \in (b_0, \frac{1}{2\sqrt{3}})$*, we have* $g_2 > 0, e_1 < 0$ *and*

$$\sigma(L) = (-\infty, -(3g_2)^{1/2}] \cup \sigma_2 \sqcup [(3g_2)^{1/2}, -3e_1],$$

with

$$\sigma_2 \cap \mathbb{R} = \sigma_2 \cap (-\infty, -(3g_2)^{1/2}) = \{E_-(\tfrac{1}{2} + ib)\}.$$

(8) *If* $b = b_0$*, then* $g_2 > 0, e_1 < 0$ *and*

$$\sigma(L) = (-\infty, -(3g_2)^{1/2}] \cup \sigma_2 \sqcup [(3g_2)^{1/2}, -3e_1],$$

with $\sigma_2 \cap \mathbb{R} = \{-(3g_2)^{1/2}\}$.

(9) *If* $b \in (0, b_0)$*, then* $g_2 > 0, e_1 < 0$ *and*

$$\sigma(L) = (-\infty, -(3g_2)^{1/2}] \sqcup \sigma_2 \sqcup [(3g_2)^{1/2}, -3e_1],$$

with

$$\sigma_2 \cap \mathbb{R} = \sigma_2 \cap (-(3g_2)^{1/2}, (3g_2)^{1/2}) = \textit{one point}.$$

Remark 1.4. Theorem 1.3 gives a complete picture of the spectrum $\sigma(L)$ for the $n = 2$ Lamé operator (1.5) as $\tau = \frac{1}{2} + ib$, and hence completely solve open problems raised in [17, Section 4]. See Fig. 2 for the 9 rough graphs of $\sigma(L)$ for different choices of b's and also the continuous deformation of $\sigma(L)$ as b decreases. It is unexpected to us that Theorem 1.3-(4) happens, i.e. all the arcs $\sigma_\infty, \sigma_1, \sigma_2$ intersect at the same point $E_+ = E_- = -\frac{3}{2}\eta_1$ simultaneously, namely there are 6 semi-arcs meeting at this point $-\frac{3}{2}\eta_1$. By $\sigma(L) = \Delta^{-1}([-2, 2])$ and the local behavior

$$\Delta(E) - \Delta(-\tfrac{3}{2}\eta_1) = c_0(E + \tfrac{3}{2}\eta_1)^k(1 + o(|E + \tfrac{3}{2}\eta_1|)), \quad c_0 \neq 0,$$

we have $k = 3$ if $\Delta(-\frac{3}{2}\eta_1) \in (-2, 2)$ (resp. $k = 6$ if $\Delta(-\frac{3}{2}\eta_1) = \pm 2$), and adjacent semi-arcs meet at $-\frac{3}{2}\eta_1$ with the same angle $\frac{\pi}{3}$.

We emphasize that the explicit expression (1.8) of $E_\pm(\tau)$ plays a crucial role in the proof of Theorem 1.3. For example, in order to obtain Theorem 1.3-(3), we need to rule out the possibility $\sigma_1 \cap \overline{\sigma_1} \neq \emptyset$. By Theorem 1.2-(1), this is equivalent to prove $E_\pm(\tau) \notin \sigma(L)$, which is not trivial at all. Thanks to (1.8), we will overcome this difficulty by applying the pre-modular form theory of the $n = 2$ Lamé equation from [9,21]; see Lemma 4.3.

Remark 1.5. In Part II [10], we will give a complete description of those τ's in $\mathbb{H} = \{\tau | \operatorname{Im}\tau > 0\}$ such that the spectrum $\sigma(L;\tau)$ of the $n = 2$ Lamé operator has a type II intersection point (such as Fig. 2-(8)).

1.2. Applications

Since the Lamé potential is doubly periodic, we can also consider its spectrum along the $\omega_2 = \tau$ direction. Denote the Hill's discriminant and the spectrum by $\Delta_j(E)$ and $\sigma_j(L) := \Delta_j^{-1}([-2,2])$ respectively along the ω_j direction (i.e. $\Delta_j(E)$ is the trace of the monodromy matrix of (1.3) with respect to $z \to z + \omega_j$), $j = 1, 2$. Obviously, the monodromy of the corresponding Lamé equation (1.3) is *unitary* (i.e. the monodromy group is conjugate to a subgroup of $SU(2)$) if and only if

$$E \in \sigma_1(L_n) \cap \sigma_2(L_n) \setminus \{E | Q_n(E;\tau) = 0\}.$$

See e.g. [4]. For example, if (denote $\omega_0 = 0$)

$$L_{\mathbf{n}} := \frac{d^2}{dz^2} - \sum_{k=0}^{3} n_k(n_k+1)\wp(z + \tfrac{\omega_k}{2};\tau), \quad n_k \in \mathbb{Z}_{\geq 0},$$

is the Darboux-Treibich-Verdier operator [12,27] and $\tau \in i\mathbb{R}_{>0}$, then we proved in [8] that

$$\sigma_1(L_{\mathbf{n}}) = (-\infty, E_{2g}] \cup [E_{2g-1}, E_{2g-2}] \cup \cdots \cup [E_1, E_0],$$
$$\sigma_2(L_{\mathbf{n}}) = [E_{2g}, E_{2g-1}] \cup \cdots \cup [E_2, E_1] \cup [E_0, +\infty),$$

provided that (n_0, n_1, n_2, n_3) satisfies

$$\textit{neither} \quad \frac{n_1 + n_2 - n_0 - n_3}{2} \geq 1, \quad n_1 \geq 1, \quad n_2 \geq 1, \tag{1.11}$$

$$\textit{nor} \quad \frac{n_0 + n_3 - n_1 - n_2}{2} \geq 1, \quad n_0 \geq 1, \quad n_3 \geq 1. \tag{1.12}$$

Here $E_{2g} < \cdots < E_1 < E_0$ are all the roots of the associated spectral polynomial $Q_{\mathbf{n}}(E;\tau)$ of $L_{\mathbf{n}}$. Thus

$$\sigma_1(L_{\mathbf{n}}) \cap \sigma_2(L_{\mathbf{n}}) \setminus \{E | Q_{\mathbf{n}}(E;\tau) = 0\} = \emptyset, \tag{1.13}$$

namely the monodromy of $L_{\mathbf{n}} y(z) = E y(z)$ cannot be unitary for any $E \in \mathbb{C}$.

On the other hand, it is known [4,8,13] that the existence of unitary monodromy for $L_{\mathbf{n}}$ is equivalent to the existence of even solutions of the singular Liouville (or mean field) equation

$$\triangle u + e^u = 8\pi \sum_{k=0}^{3} n_k \delta_{\frac{\omega_k}{2}} \quad \text{on} \quad E_\tau, \tag{1.14}$$

where $\triangle = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator and δ_p denotes the Dirac measure at p. Thus (1.13) implies that (1.14) has no even solutions on any rectangular torus provided (1.11)-(1.12) hold. We refer the reader to [8] for more details.

Now we want to apply Theorem 1.3 to prove the existence of solutions to (1.14) on rhombus torus with $(n_0, n_1, n_2, n_3) = (2, 0, 0, 0)$, i.e.

$$\triangle u + e^u = 16\pi\delta_0 \quad \text{on} \quad E_\tau, \quad \operatorname{Re}\tau = \tfrac{1}{2}. \tag{1.15}$$

A solution $u(z)$ (here we use complex variable $z = x + iy$) is called *even* if $u(z) = u(-z)$, and is called *axisymmetric* if $u(z) = u(\bar{z})$. The existence of even axisymmetric solutions have been studied in [6,13], and the following theorem was proved.

Theorem A. *Let $\tau = \frac{1}{2} + ib$ with $b > 0$. Then there exists $\hat{b} > \frac{\sqrt{3}}{2}$ such that*

(1) [13] *If $b \in (0, \frac{1}{4\hat{b}}) \cup (\hat{b}, +\infty)$, then (1.15) has a unique even axisymmetric solution.*
(2) [13] *If $b \in [\frac{1}{4\hat{b}}, \hat{b}]$, then (1.15) has no even axisymmetric solutions.*
(3) [6] *If $b \in \{\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{2\sqrt{3}}\}$, then (1.15) has no solutions.*

Remark that due to the conformal equivalence of E_τ with $E_{\tau'}$, where $\tau' := \frac{\tau-1}{2\tau-1} = \frac{1}{2} + i\frac{1}{4b}$, solving (1.15) on E_τ is equivalent to solving it on $E_{\tau'}$. This simple fact explains the relation of those numbers in Theorem A. The proof of Theorem A is highly non-trivial, and some special techniques were developed in [6,13]. See also [13] for a general result for (1.14). However, both methods in [6,13] cannot be applied to study the long-standing open problem (cf. [8, p.1271]) *whether even but not axisymmetric solutions exist or not.*

Our third main result is to apply the geometry of both spectra $\sigma_1(L)$ and $\sigma_2(L)$ to obtain the following

Theorem 1.6. *Let $\tau = \frac{1}{2} + ib$ with $b > 0$. Then there are $b \in (\frac{1}{2\sqrt{3}}, \frac{\sqrt{3}}{2})$ such that the mean field equation (1.15) has no even axisymmetric solutions, but does have 2 even solutions which are not axisymmetric.*

It is interesting to compare Theorems A and 1.6 with the following result proved in [20].

Theorem B. [20] *Let $\tau = \frac{1}{2} + ib$ with $b > 0$. Then there exists $\frac{1}{2} < \tilde{b} < \frac{\sqrt{3}}{2}$ such that*

(1) *If $b \in (0, \frac{1}{4\tilde{b}}) \cup (\tilde{b}, +\infty)$, then*

$$\triangle u + e^u = 8\pi\delta_0 \quad \text{on} \ \ E_\tau, \quad \operatorname{Re}\tau = \tfrac{1}{2} \tag{1.16}$$

has a unique even solution and hence axisymmetric.

(2) *If $b \in [\frac{1}{4\tilde{b}}, \tilde{b}]$, then (1.16) has no solutions.*

Theorem B indicates that even solutions of (1.16) on rhombus torus must be axisymmetric. To the best of our knowledge, Theorem 1.6 seems to be the first result that confirms the existence of even but not axisymmetric solutions for (1.14) on rhombus torus. Theorem 1.6 indicates that Theorem A might not give the exact number of even solutions, and the solvability of (1.15) with $\tau = \frac{1}{2} + ib$ is still not complete and remains open.

Another application of Theorem 1.3 is that we can obtain $\hat{b} = \frac{1}{4b_0}$, where $\hat{b}$ is the constant in Theorem A and b_0 is given in (1.10). See Remark 5.6.

This paper is organized as follows. In Section 2, first we briefly review the spectral theory of Hill equation from [14] and apply it to the $n = 2$ Lamé potential, then we collect some facts about the monodromy theory of the corresponding Lamé equation from [4,15,21]. In Sections 3-4, we give the proofs of Theorems 1.2 and 1.3 respectively. Finally in Section 5, we apply Theorem 1.3 to the mean field equation and prove Theorem 1.6.

2. Preliminaries

In this section, we collect some preliminary results that are needed in later sections.

2.1. Spectral theory [14]

We briefly review the spectral theory of Hill equation with complex-valued potentials from [14] and apply it to the $n = 2$ Lamé potential. Let $q(x)$ be a complex-valued continuous nonconstant periodic function of period Ω on $\mathbb{R}$. Consider the following Hill equation

$$y''(x) + q(x)y(x) = Ey(x), \quad x \in \mathbb{R}. \tag{2.1}$$

This equation has received an enormous amount of consideration due to its ubiquity in applications as well as its structural richness; see e.g. [14,16] and references therein for historical reviews.

Let $y_1(x)$ and $y_2(x)$ be any two linearly independent solutions of (2.1). Then so do $y_1(x+\Omega)$ and $y_2(x+\Omega)$ and hence there is a monodromy matrix $M(E) \in SL(2,\mathbb{C})$ such that

$$(y_1(x+\Omega), y_2(x+\Omega)) = (y_1(x), y_2(x))M(E). \tag{2.2}$$

Define the *Hill's discriminant* $\Delta(E)$ by

$$\Delta(E) := \mathrm{tr}M(E), \tag{2.3}$$

which is clearly an invariant of (2.1), i.e. does not depend on the choice of linearly independent solutions. This $\Delta(E)$ is an entire function and plays a fundamental role since it encodes all information of the spectrum $\sigma(L)$ of the operator $L = \frac{d^2}{dx^2} + q(x)$; see e.g. [16] and references therein. Indeed, we define

$$\mathcal{S} := \Delta^{-1}([-2,2]) = \{E \in \mathbb{C} \mid -2 \leq \Delta(E) \leq 2\} \tag{2.4}$$

to be the *conditional stability set* of the operator L. Then it was proved by Rofe-Beketov [25] that S coincides with the spectrum $\sigma(L)$:

$$\sigma(L) = \mathcal{S} = \{E \in \mathbb{C} \mid -2 \leq \Delta(E) \leq 2\}. \tag{2.5}$$

Recall that $E \in \mathbb{C}$ is called a periodic (resp. antiperiodic) eigenvalue of L if $Ly = Ey$ has a nonzero solution y satisfying $y(x+\Omega) = y(x)$ (resp. $y(x+\Omega) = -y(x)$). Clearly E is a (anti)periodic eigenvalue if and only if $\Delta(E) = \pm 2$. Define

$$d(E) := \mathrm{ord}_E(\Delta(\cdot)^2 - 4).$$

Then it is well known (cf. [24, Section 2.3]) that $d(E)$ equals to *the algebraic multiplicity of (anti)periodic eigenvalues*. Let $s(E, x, x_0)$ be the special solution of (2.1) satisfying the initial values

$$s(E, x_0, x_0) = 0, \quad s'(E, x_0, x_0) = 1,$$

and define

$$p(E, x_0) := \mathrm{ord}_E s(\cdot, x_0 + \Omega, x_0),$$
$$p_i(E) := \min\{p(E, x_0) : x_0 \in \mathbb{R}\}.$$

It is known that $p(E, x_0)$ is the algebraic multiplicity of a Dirichlet eigenvalue on the interval $[x_0, x_0 + \Omega]$, and $p_i(E)$ denotes the immovable part of $p(E, x_0)$ (cf. [14]). It was proved in [14, Theorem 3.2] that $d(E) - 2p_i(E) \geq 0$.

Now we consider the $n = 2$ Lamé operator L in (1.5), i.e. $q(x) = -6\wp(x + z_0; \tau)$ is smooth on $\mathbb{R}$ with period $\Omega = 1$. Applying the general result [14, Theorem 4.1] to this $q(x)$, we obtain

Theorem 2.A. [14, Theorem 4.1] *For the Lamé potential $q(x) = -6\wp(x + z_0; \tau)$, recall its spectral polynomial $Q_2(E; \tau)$ given in (1.6). Then the spectrum $\sigma(L) = \mathcal{S}$ consists* *of* $\tilde{g} \in \{1, 2\}$ *bounded simple analytic arcs* σ_k, $1 \le k \le \tilde{g}$ *and one semi-infinite simple analytic arc* σ_∞ *which tends to* $-\infty + \langle q \rangle$, *with* $\langle q \rangle = \int_{x_0}^{x_0+1} q(x)dx$, *i.e.*

$$\sigma(L) = \mathcal{S} = \sigma_\infty \cup \cup_{k=1}^{\tilde{g}} \sigma_k, \quad \tilde{g} \in \{1, 2\}.$$

The finite endpoints of such arcs are those E*'s satisfying* $Q_2(E; \tau) = 0$ *with* $d(E) = 2p_i(E) + ord_E Q_2(\cdot; \tau)$ *odd, and there are exactly* $d(E)$*'s semi-arcs of* $\sigma(L)$ *meeting at such* E.

2.2. The $n = 2$ Lamé equation

To study the spectrum $\sigma(L)$ of the $n = 2$ Lamé operator L in (1.5), we need to recall some known results (see e.g. [4,15,21]) for the corresponding $n = 2$ Lamé equation:

$$y''(z) = [6\wp(z; \tau) + E]y(z). \tag{2.6}$$

In the sequel, we omit the notation τ freely for convenience.

(L-1). For any $E \in \mathbb{C}$, there exists a unique pair $\pm\boldsymbol{a} = \pm\{a_1, a_2\} \subset E_\tau \setminus \{0\}$ satisfying $a_1 \ne a_2$ in E_τ and

$$\zeta(a_1 - a_2) - \zeta(a_1) + \zeta(a_2) = 0 \tag{2.7}$$

such that the classical *Hermite-Halphen ansatz*

$$y_{\pm\boldsymbol{a}}(z) := e^{\pm z \sum_{j=1}^2 \zeta(a_j)} \frac{\prod_{j=1}^2 \sigma(z \mp a_j)}{\sigma(z)^2}$$

are solutions of (2.6) with

$$E = 3[\wp(a_1) + \wp(a_2)]. \tag{2.8}$$

The Legendre relation $\tau\eta_1 - \eta_2 = 2\pi i$ implies that there is a unique $(r, s) \in \mathbb{C}^2$ satisfying

$$r + s\tau = a_1 + a_2, \quad r\eta_1 + s\eta_2 = \zeta(a_1) + \zeta(a_2),$$

which is equivalent to

$$\zeta(a_1) + \zeta(a_2) - (a_1 + a_2)\eta_1 = -2\pi is, \tag{2.9}$$

$$\tau(\zeta(a_1) + \zeta(a_2)) - (a_1 + a_2)\eta_2 = 2\pi ir. \tag{2.10}$$

Then the transformation law $\sigma(z + \omega_k) = -e^{(z+\frac{\omega_k}{2})\eta_k}\sigma(z)$ implies

$$y_{\pm\boldsymbol{a}}(z + 1) = e^{\pm(\sum_{j=1}^2 \zeta(a_j) - (a_1+a_2)\eta_1)} y_{\pm\boldsymbol{a}}(z) = e^{\mp 2\pi is} y_{\pm\boldsymbol{a}}(z), \tag{2.11}$$

$$y_{\pm\boldsymbol{a}}(z+\tau)=e^{\pm(\tau\sum_{j=1}^{2}\zeta(a_j)-(a_1+a_2)\eta_2)}y_{\pm\boldsymbol{a}}(z)=e^{\pm 2\pi i r}y_{\pm\boldsymbol{a}}(z), \tag{2.12}$$

namely $y_{\pm\boldsymbol{a}}(z)$ are elliptic of the second kind.

(L-2). Define $w_\pm(x):=y_{\pm\boldsymbol{a}}(x+z_0)$, then $Lw_\pm=Ew_\pm$ and $w_\pm(x+1)=e^{\mp 2\pi i s}w_\pm(x)$, i.e. $e^{\mp 2\pi i s}$ are eigenvalues of the monodromy matrix $M(E)$ in (2.2). Thus (2.3) gives

$$\Delta(E)=e^{-2\pi i s}+e^{2\pi i s}=e^{\sum_{j=1}^{2}\zeta(a_j)-(a_1+a_2)\eta_1}+e^{-(\sum_{j=1}^{2}\zeta(a_j)-(a_1+a_2)\eta_1)}. \tag{2.13}$$

(L-3). Recalling the spectral polynomial $Q_2(E)$ in (1.6), if $Q_2(E)=0$, then $\{a_1,a_2\}=\{-a_1,-a_2\}$, i.e. $y_{\boldsymbol{a}}(z)=y_{-\boldsymbol{a}}(z)$. In this case, $y_{\boldsymbol{a}}(z)$ is known as the *Lamé function* in the literature, and it follows from (2.11)-(2.12) that $2r,2s\in\mathbb{Z}$. Furthermore, the monodromy matrices cannot be diagonalized simultaneously and so the monodromy cannot be unitary.

(L-4). If $Q_2(E)\neq 0$, then $y_{\boldsymbol{a}}(z)$ and $y_{-\boldsymbol{a}}(z)$ are linearly independent, i.e. $\{a_1,a_2\}\cap\{-a_1,-a_2\}=\emptyset$ and so

$$\wp(a_1)\neq\wp(a_2). \tag{2.14}$$

This together with (2.7) and the addition formula

$$\zeta(a_1-a_2)-\zeta(a_1)+\zeta(a_2)=\frac{1}{2}\frac{\wp'(a_1)+\wp'(a_2)}{\wp(a_1)-\wp(a_2)},$$

implies

$$\wp'(a_1)+\wp'(a_2)=0. \tag{2.15}$$

From here and $(\wp')^2=4\wp^3-g_2\wp-g_3$, we easily obtain

$$[\wp(a_1)+\wp(a_2)]^2-\wp(a_1)\wp(a_2)-\tfrac{g_2}{4}=0. \tag{2.16}$$

On the other hand, it follows from (2.11)-(2.12) that the monodromy matrices are given by

$$(y_{\boldsymbol{a}}(z+1),y_{-\boldsymbol{a}}(z+1))=(y_{\boldsymbol{a}}(z),y_{-\boldsymbol{a}}(z))\begin{pmatrix}e^{-2\pi i s} & 0\\ 0 & e^{2\pi i s}\end{pmatrix}, \tag{2.17}$$

$$(y_{\boldsymbol{a}}(z+\tau),y_{-\boldsymbol{a}}(z+\tau))=(y_{\boldsymbol{a}}(z),y_{-\boldsymbol{a}}(z))\begin{pmatrix}e^{2\pi i r} & 0\\ 0 & e^{-2\pi i r}\end{pmatrix}. \tag{2.18}$$

Furthermore, it was proved in [21, Lemma 4.4, Theorem 4.5] that $(r,s)\notin\frac{1}{2}\mathbb{Z}^2$ and satisfies

$$Z_{r,s}^{(2)}(\tau)=0 \quad \textit{as long as} \quad r+s\tau\notin\tfrac{1}{2}\mathbb{Z}+\tfrac{\tau}{2}\mathbb{Z}. \tag{2.19}$$

Clearly *the monodromy is unitary (i.e. the monodromy matrices in (2.17)-(2.18) belong to* $SU(2)$*) if and only if* $(r,s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ and in this case $r + s\tau \notin \frac{1}{2}\mathbb{Z} + \frac{\tau}{2}\mathbb{Z}$ holds automatically. Here $\frac{1}{2}\mathbb{Z}^2 := \{(a,b)|2a, 2b \in \mathbb{Z}\}$ and $Z^{(2)}_{r,s}(\tau)$ is defined by

$$Z^{(2)}_{r,s}(\tau) := Z_{r,s}(\tau)^3 - 3\wp(r+s\tau;\tau)Z_{r,s}(\tau) - \wp'(r+s\tau;\tau),$$

with $Z_{r,s}(\tau) := \zeta(r+s\tau;\tau) - r\eta_1(\tau) - s\eta_2(\tau)$. In other words, [21, Theorem 4.5] proved that for given τ, the monodromy of the Lamé equation (2.6) is given by (2.17)-(2.18) with $r + s\tau \notin \frac{1}{2}\mathbb{Z} + \frac{\tau}{2}\mathbb{Z}$ for some E if and only if $Z^{(2)}_{r,s}(\tau) = 0$. See [21] for the general theory for the general Lamé equation $y'' = [n(n+1)\wp(z;\tau) + E]y$.

Clearly $Z^{(2)}_{r,s}(\tau)$ is holomorphic in τ if $(r,s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$ and

$$Z^{(2)}_{r,s}(\tau) = Z^{(2)}_{r+m,s+n}(\tau) \quad \text{for any} \quad (m,n) \in \mathbb{Z}^2.$$

Moreover, $Z^{(2)}_{r,s}(\tau)$ is a modular form of weight 3 with respect to the principal congruence subgroup $\Gamma(m) := \{\gamma \in SL(2,\mathbb{Z})|\gamma \equiv I_2 \bmod m\}$ if (r,s) is a m-torsion point; see [21]. Due to this property, $Z^{(2)}_{r,s}(\tau)$ is called a *pre-modular form* in [21]. Here we recall the following result concerning the non-vanishing of $Z^{(2)}_{r,s}(\cdot)$, which will play a crucial role in our proof of Theorem 1.3-(3) in Section 4.

Theorem 2.1. [9, Theorem 1.6] *Let* $r \in \mathbb{R} \setminus \frac{1}{2}\mathbb{Z}$*. Then* $Z^{(2)}_{r,0}(\tau) \neq 0$ *for any* $\tau \in F_0 := \{\tau \in \mathbb{H} \mid 0 \leq \operatorname{Re}\tau \leq 1, |\tau - \frac{1}{2}| \geq \frac{1}{2}\}$.

(L-5). Define

$$Y_2 := \left\{ \{a_1, a_2\} \in \operatorname{Sym}^2 E_\tau \;\middle|\; \begin{array}{l} a_i \neq 0,\ a_1 \neq a_2 \text{ in } E_\tau\ , \\ \zeta(a_1 - a_2) - \zeta(a_1) + \zeta(a_2) = 0 \end{array} \right\}.$$

Clearly $-\boldsymbol{a} \in Y_2$ if $\boldsymbol{a} \in Y_2$, and $\boldsymbol{a} \in Y_2$ is a *branch point* of Y_2 if $\boldsymbol{a} = -\boldsymbol{a}$ in E_τ. Then the map $E : Y_2 \to \mathbb{C}$ defined by (2.8) is a ramified covering of degree 2, and there holds (see e.g. [4, Theorem 7.4])

$$Y_2 \cong \{(E,C) \mid C^2 = Q_2(E)\}. \tag{2.20}$$

Therefore, Y_2 is a hyperelliptic curve, known as the *Lamé curve*.

3. Spectrum of the $n = 2$ Lamé operator: general case

In this and next sections, we study the spectrum of the $n = 2$ Lamé operator:

$$L = \frac{d^2}{dx^2} - 6\wp(x + z_0;\tau), \quad x \in \mathbb{R}. \tag{3.1}$$

As mentioned in Section 1,

$$Q_2(E;\tau) = (E^2 - 3g_2(\tau))\prod_{k=1}^{3}(E + 3e_k(\tau)). \tag{3.2}$$

This section is devoted to proving Theorem 1.2. Again we omit the notation τ freely for convenience.

Proof of Theorem 1.2. (1). Consider any E_0 such that $Q_2(E_0) \neq 0$. By (2.20) there is a small neighborhood $O \subset \mathbb{C}$ of E_0 such that $Q_2(E) \neq 0$ for $E \in O$ and $E \in O$ can be a local coordinate for the hyperelliptic curve Y_2, namely $a_1 = a_1(E), a_2 = a_2(E)$ are holomorphic for $E \in O$. Denote $x_j = \wp(a_j(E_0))$ for convenience, then $x_1 \neq x_2$ by (2.14). Since (2.15) holds for $E \in O$, by taking derivative with respect to E, we obtain from $\wp'' = 6\wp^2 - \frac{g_2}{2}$ that

$$(6x_1^2 - \tfrac{g_2}{2})a_1'(E_0) + (6x_2^2 - \tfrac{g_2}{2})a_2'(E_0) = 0. \tag{3.3}$$

Consider the local behavior at E_0:

$$\Delta(E) - \Delta(E_0) = c(E - E_0)^k(1 + O(|E - E_0|)), \quad k \geq 1, \;\; c \neq 0. \tag{3.4}$$

If $\Delta(E_0) \in (-2, 2)$, it follows from (3.4) and $\sigma(L) = \{E| - 2 \leq \Delta(E) \leq 2\}$ that there are precisely $2k$ semi-arcs of $\sigma(L)$ meeting at E_0. If $\Delta(E_0) = \pm 2$, then there are precisely k semi-arcs of $\sigma(L)$ meeting at E_0.

Step 1. We show the necessary part. Suppose that E_0 is a type I intersection point, i.e. E_0 is met by at least 4 semi-arcs of the spectrum $\sigma(L)$ and $Q_2(E_0) \neq 0$. We need to prove $E_0^2 + 3\eta_1 E_0 - \frac{3}{2}g_2 = 0$.

We claim that

$$(x_1 + \eta_1)a_1'(E_0) + (x_2 + \eta_1)a_2'(E_0) = 0. \tag{3.5}$$

Indeed, by (2.13) we have for $E \in O$ that

$$\begin{aligned}\Delta'(E) = &-[e^{\sum_{j=1}^2 \zeta(a_j) - \eta_1(a_1+a_2)} - e^{-(\sum_{j=1}^2 \zeta(a_j) - \eta_1(a_1+a_2))}] \\ &\times[(\wp(a_1) + \eta_1)a_1'(E) + (\wp(a_2) + \eta_1)a_2'(E)],\end{aligned} \tag{3.6}$$

$$\begin{aligned}\Delta''(E) =&\Delta(E)[(\wp(a_1) + \eta_1)a_1'(E) + (\wp(a_2) + \eta_1)a_2'(E)]^2 \\ &- [e^{\sum_{j=1}^2 \zeta(a_j) - \eta_1(a_1+a_2)} - e^{-(\sum_{j=1}^2 \zeta(a_j) - \eta_1(a_1+a_2))}] \\ &\times \tfrac{d}{dE}[(\wp(a_1) + \eta_1)a_1'(E) + (\wp(a_2) + \eta_1)a_2'(E)].\end{aligned} \tag{3.7}$$

If $\Delta(E_0) \neq \pm 2$, i.e. $\Delta(E_0) \in (-2, 2)$, then our assumption implies $2k \geq 4$, i.e. $k \geq 2$ and so $\Delta'(E_0) = 0$. Since $\Delta(E_0) \neq \pm 2$ implies $e^{\sum_{j=1}^2 \zeta(a_j) - \eta_1(a_1+a_2)} \neq \pm 1$ at E_0, we see from (3.6) that (3.5) holds. If $\Delta(E_0) = \pm 2$, then our assumption implies $k \geq 4$, i.e. $\Delta'(E_0) = \Delta''(E_0) = \Delta'''(E_0) = 0$. Since $\Delta(E_0) = \pm 2$ implies $e^{\sum_{j=1}^2 \zeta(a_j) - \eta_1(a_1+a_2)} = \pm 1$ at E_0, again we obtain (3.5) by (3.7).

Noting from (2.8) that $3\wp'(a_1)a_1'(E) + 3\wp'(a_2)a_2'(E) = 1$ and so $(a_1'(E_0), a_2'(E_0)) \neq (0,0)$, we conclude from (3.3) and (3.5) that

$$\det\begin{pmatrix} x_1+\eta_1 & x_2+\eta_1 \\ 6x_1^2-\frac{g_2}{2} & 6x_2^2-\frac{g_2}{2}\end{pmatrix} = 0, \tag{3.8}$$

which gives

$$6x_1x_2 + 6\eta_1(x_1+x_2) + \tfrac{g_2}{2} = 0. \tag{3.9}$$

Since $E_0 = 3(x_1+x_2)$ and (2.16) says $x_1x_2 = (x_1+x_2)^2 - \frac{g_2}{4}$, we cancel the term x_1x_2 and finally obtain $E_0^2 + 3\eta_1 E_0 - \frac{3}{2}g_2 = 0$.

Step 2. Suppose $E_0 \in \{E_\pm(\tau)\} \cap \sigma(L)$ satisfies $Q_2(E_0) \neq 0$. Then $\Delta(E_0) \in [-2,2]$ and $E_0^2 + 3\eta_1 E_0 - \frac{3}{2}g_2 = 0$. This, together (2.16), implies (3.9) and so (3.8). By (3.8) and (3.3), we conclude that (3.5) holds.

If $\Delta(E_0) \in (-2,2)$, then we see from (3.5)-(3.6) that $\Delta'(E_0) = 0$, i.e. $k \geq 2$ in (3.4) and so there are $2k \geq 4$ semi-arcs of $\sigma(L)$ meeting at this E_0. If $\Delta(E_0) = \pm 2$, then $e^{\sum_{j=1}^{2}\zeta(a_j)-\eta_1(a_1+a_2)} = \pm 1$ at E_0. From here and (3.5)-(3.7), we see that $\Delta'(E_0) = \Delta''(E_0) = 0$ and moreover, a direct computation also gives $\Delta'''(E_0) = 0$. This means $k \geq 4$ in (3.4) and so there are $k \geq 4$ semi-arcs of $\sigma(L)$ meeting at this E_0. Therefore, E_0 is a type I intersection point.

(2)-(3). Since $e_1 \neq e_2 \neq e_3 \neq e_1$ and

$$e_1+e_2+e_3 = 0, \quad g_2 = 2(e_1^2+e_2^2+e_3^2), \tag{3.10}$$

it is easy to see

$$\{-3e_1, -3e_2, -3e_3\} \cap \{(3g_2)^{1/2}, -(3g_2)^{1/2}\} = \emptyset,$$

so

$$\begin{gathered} ord_{-3e_k} Q_2(\cdot;\tau) = 1, \quad k = 1,2,3, \\ 1 \leq ord_{\pm(3g_2)^{1/2}} Q_2(\cdot;\tau) \leq 2. \end{gathered} \tag{3.11}$$

Recalling (L-3) in Section 2.2 that $y_{\boldsymbol{a}}(z) = y_{-\boldsymbol{a}}(z)$ for $E \in \{-3e_k, \pm(3g_2)^{1/2}\}$, we proved in [7, Theorem 2.6] that there is a solution $y_2(z)$ linearly independent with $y_{\boldsymbol{a}}(z)$ such that for $E = -3e_k$,

$$y_{\boldsymbol{a}}(z+1) = \varepsilon_k y_{\boldsymbol{a}}(z), \quad y_2(z+1) = \varepsilon_k y_2(z) + \frac{12i\pi\varepsilon_k e_k'(\tau)}{\wp''(\frac{\omega_i}{2})\wp''(\frac{\omega_j}{2})} y_{\boldsymbol{a}}(z), \tag{3.12}$$

where $\{i,j,k\} = \{1,2,3\}$, $\varepsilon_k = 1$ if $k=1$ and $\varepsilon_k = -1$ if $k = 2,3$; and for $E = \pm(3g_2)^{1/2}$,

$$y_{\boldsymbol{a}}(z+1) = y_{\boldsymbol{a}}(z), \quad y_2(z+1) = y_2(z) - \tfrac{1}{3}(6\eta_1 \pm (3g_2)^{1/2}) y_{\boldsymbol{a}}(z). \tag{3.13}$$

From here and [14, Proposition 3.1] which proved that

$$p_i(E) \geq 1 \Leftrightarrow \textit{all solutions of } Ly = Ey \textit{ are (anti)periodic},$$

we immediately obtain

$$p_i(-3e_k(\tau)) \geq 1 \quad \text{if and only if} \quad e_k'(\tau) = 0, \tag{3.14}$$

$$p_i(\pm(3g_2)^{1/2}) \geq 1 \text{ if and only if } 6\eta_1 \pm (3g_2)^{1/2} = 0. \tag{3.15}$$

Therefore, we see from $d(E) = 2p_i(E) + \text{ord}_E Q_2(\cdot)$ in Theorem 2.A that

$$d(-3e_k(\tau)) \geq 3 \quad \text{if and only if} \quad e_k'(\tau) = 0,$$

$$d(\pm(3g_2)^{1/2}) \geq 3 \text{ if and only if } 6\eta_1 \pm (3g_2)^{1/2} = 0.$$

From here and Theorem 2.A, we conclude that $-3e_k(\tau)$ is a type II intersection point (or equivalently, $-3e_k(\tau)$ is met by at least 3 semi-arcs of $\sigma(L)$) if and only if $e_k'(\tau) = 0$, and similar results hold for $\pm(3g_2)^{1/2}$. This proves Theorem 1.2 (2)-(3). □

4. Spectrum of the $n = 2$ Lamé operator: the case $\tau = \frac{1}{2} + ib$

In this section, we always assume $\tau = \frac{1}{2} + ib$ with $b > 0$ and prove Theorem 1.3. First we prove that the spectrum $\sigma(L)$ of the $n = 2$ Lamé operator is symmetric with respect to $\mathbb{R}$, which actually holds for all $n \in \mathbb{N}$.

Lemma 4.1. *The spectrum $\sigma(L_n)$ of the Lamé operator L_n in (1.2) is symmetric with respect to the real line $\mathbb{R}$.*

Proof. Let $\tilde{\tau} = 2ib$ and consider $\tilde{L}_n := \frac{d^2}{dx^2} + q^{(n,0,0,n)}(x + z_0; \tilde{\tau})$, where

$$q^{(n,0,0,n)}(z; \tilde{\tau}) := -n(n+1)(\wp(z; \tilde{\tau}) + \wp(z + \tfrac{1+\tilde{\tau}}{2}; \tilde{\tau}))$$

is the Darboux-Treibich-Verdier potential [12,27]. Since $\tilde{\tau} \in i\mathbb{R}_{>0}$, we proved in [8, Lemma 3.5] that the spectrum $\sigma(\tilde{L}_n)$ is symmetric with respect to $\mathbb{R}$. Since $\frac{1+\tilde{\tau}}{2} = \frac{1}{2} + ib = \tau$, we can rewrite the elliptic function $q^{(n,0,0,n)}(z; \tilde{\tau})$ as

$$q^{(n,0,0,n)}(z; \tilde{\tau}) = -n(n+1)\wp(z; \tau) - n(n+1)e_3(\tilde{\tau}),$$

which implies $\sigma(\tilde{L}_n) = \sigma(L_n) - n(n+1)e_3(\tilde{\tau})$. From here and $e_3(\tilde{\tau}) \in \mathbb{R}$, we conclude that $\sigma(L_n)$ is also symmetric with respect to $\mathbb{R}$. □

Now we continue to study the spectrum of the $n = 2$ Lamé operator

$$L = \frac{d^2}{dx^2} - 6\wp(x + z_0; \tau), \quad x \in \mathbb{R}.$$

Together with Theorem 2.A, we summarize that the following hold:

(P-1) *the spectrum $\sigma(L)$ consists of $\tilde{g} \in \{1,2\}$ bounded simple analytic arcs σ_k and one semi-infinite simple analytic arc σ_∞ which tends to $-\infty$, i.e.*

$$\sigma(L) = \{E| - 2 \le \Delta(E;\tau) \le 2\} = \sigma_\infty \cup \cup_{k=1}^{\tilde{g}} \sigma_k, \quad \tilde{g} \in \{1,2\}, \tag{4.1}$$

where the finite endpoints of such arcs are precisely those

$$E \in \{-3e_1, -3e_2, -3e_3, (3g_2)^{1/2}, -(3g_2)^{1/2}\} \tag{4.2}$$

with $d(E) = 2p_i(E) + ord_E Q_2(\cdot;\tau)$ odd, and there are exactly $d(E)$ semi-arcs of $\sigma(L)$ meeting at such E.

(P-2) *$\sigma(L)$ is symmetric with respect to the real line $\mathbb{R}$. In particular, $\sigma_\infty \subset \mathbb{R}$.*

(P-3) *A classical result (see e.g. [16, Theorem 2.2]) says that $\mathbb{C} \setminus \sigma(L)$ is path-connected.*

Therefore, we need to compute $d(E)$ for E in (4.2) to analyze $\sigma(L)$. Recall $g_2, e_1, \eta_1 \in \mathbb{R}$ and $e_2 = \overline{e_3} \notin \mathbb{R}$ since $\operatorname{Re}\tau = \frac{1}{2}$. It is well known that $g_2(\tau) = 0$ if and only if $\tau \in \left\{\frac{ae^{\pi i/3}+b}{ce^{\pi i/3}+d} \,|\, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})\right\}$, and

$$\left\{ \frac{ae^{\pi i/3}+b}{ce^{\pi i/3}+d} \middle| \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}) \right\} \cap \left(\tfrac{1}{2} + i\mathbb{R}_{>0}\right) = \left\{ e^{\pi i/3}, \frac{e^{\pi i/3}-1}{2e^{\pi i/3}-1} \right\},$$

so

$$g_2(\tfrac{1}{2} + ib) \begin{cases} > 0 & \text{if } b \in (0, \frac{1}{2\sqrt{3}}) \cup (\frac{\sqrt{3}}{2}, +\infty) \\ = 0 & \text{if } b \in \{\frac{1}{2\sqrt{3}}, \frac{\sqrt{3}}{2}\} \\ < 0 & \text{if } b \in (\frac{1}{2\sqrt{3}}, \frac{\sqrt{3}}{2}). \end{cases} \tag{4.3}$$

Together with (3.10)-(3.11), we have

$$\begin{gathered} ord_{-3e_k} Q_2(\cdot;\tau) = 1, \quad k = 1,2,3, \\ ord_{\pm(3g_2)^{1/2}} Q_2(\cdot;\tfrac{1}{2}+ib) = \begin{cases} 1 & \text{if } b \notin \{\frac{1}{2\sqrt{3}}, \frac{\sqrt{3}}{2}\}, \\ 2 & \text{if } b \in \{\frac{1}{2\sqrt{3}}, \frac{\sqrt{3}}{2}\}. \end{cases} \end{gathered} \tag{4.4}$$

Furthermore,

$$g_2 - 3e_1^2 = (e_2 - e_3)^2 = (e_2 - \overline{e_2})^2 < 0, \text{ i.e. } |3e_1| > |(3g_2)^{\frac{1}{2}}| \text{ if } g_2 \ge 0.$$

On the other hand, it is well known that $e_1(\frac{1+i}{2}) = 0$ and it was proved in [20, Theorem 1.7] that $\frac{d}{db} e_1(\frac{1}{2} + ib) > 0$ for $b > 0$, which implies

$$e_1(\tfrac{1}{2}+ib)\begin{cases} >0 & \text{if } b>\frac{1}{2} \\ =0 & \text{if } b=\frac{1}{2} \\ <0 & \text{if } b\in(0,\frac{1}{2}) \end{cases} \tag{4.5}$$

and $e_1'(\tau)\neq 0$ for $\operatorname{Re}\tau=\frac{1}{2}$. This, together with $e_1+e_2+e_3=0$ and $e_3=\overline{e_2}$, implies that $e_k'(\tau)\neq 0$ for $\operatorname{Re}\tau=\frac{1}{2}$, $k=1,2,3$. From here and (3.14) we conclude that $p_i(-3e_k(\tau))=0$ for $\tau=\frac{1}{2}+ib$ and so

$$d(-3e_k(\tau))=1 \quad \text{for } \tau=\tfrac{1}{2}+ib, \quad k=1,2,3. \tag{4.6}$$

To compute $d(\pm(3g_2)^{1/2})$, we use the formula (see e.g. [9, (1.5)])

$$\tfrac{d}{db}\eta_1(\tfrac{1}{2}+ib)=\tfrac{-1}{24\pi}(12\eta_1^2-g_2)=\tfrac{-1}{72\pi}(6\eta_1+(3g_2)^{\frac{1}{2}})(6\eta_1-(3g_2)^{\frac{1}{2}}), \tag{4.7}$$

and we proved in [9, Corollary 1.5] that: *There exists* $b_0\in(\frac{5}{24},\frac{1}{2\sqrt{3}})$ *such that* $\frac{d}{db}\eta_1(\frac{1}{2}+ib)=0$ *if and only if* $b=b_0$, *and*

$$\eta_1(\tfrac{1}{2}+ib_0)=\max_{b>0}\eta_1(\tfrac{1}{2}+ib)>\tfrac{\pi^2}{3}=\lim_{b\to+\infty}\eta_1(\tfrac{1}{2}+ib). \tag{4.8}$$

This, together with (4.7) and (4.3), implies $6\eta_1-(3g_2)^{1/2}=0$ at $\tau=\frac{1}{2}+ib_0$ and

$$(12\eta_1^2-g_2)(\tfrac{1}{2}+ib)\begin{cases} <0 & \text{for } \ b\in(0,b_0) \\ >0 & \text{for } \ b>b_0. \end{cases} \tag{4.9}$$

From here and (3.15) we conclude $p_i((3g_2)^{1/2})=0$ for $\tau=\frac{1}{2}+ib$ with all $b>0$ and

$$p_i(-(3g_2)^{1/2})\begin{cases} \geq 1 & \text{if } b=b_0 \\ =0 & \text{if } b\in(0,b_0)\cup(b_0,+\infty). \end{cases}$$

Together with (4.4), we finally obtain

$$\begin{aligned} &d(\pm(3g_2)^{1/2})=1 \quad \text{for } b\in(0,+\infty)\setminus\{\tfrac{\sqrt{3}}{2},\tfrac{1}{2\sqrt{3}},b_0\}, \\ &d(0)=d(\pm(3g_2)^{1/2})=2 \quad \text{for } \ b\in\{\tfrac{1}{2\sqrt{3}},\tfrac{\sqrt{3}}{2}\}, \\ &d((3g_2)^{\frac{1}{2}})=1,\ d(-(3g_2)^{\frac{1}{2}})=1+2p_i(-(3g_2)^{\frac{1}{2}})\geq 3,\ \text{for } b=b_0. \end{aligned}$$

Finally, it follows from (3.12)-(3.13) that

$$\Delta(-3e_1;\tau)=\Delta(\pm(3g_2)^{\frac{1}{2}};\tau)=2,\ \ \Delta(-3e_2;\tau)=\Delta(-3e_3;\tau)=-2.$$

Recalling the expression of $E_\pm(\tau)$ in (1.8), we need to study the sign of $3\eta_1^2+2g_2$ to see whether $E_\pm(\tau)$ are real or not. Clearly (4.3) and (4.8) imply $(3\eta_1^2+2g_2)(\frac{1}{2}+ib)>0$ and so

$$E_-(\tfrac{1}{2}+ib) < E_+(\tfrac{1}{2}+ib) \text{ for } b \in (0, \tfrac{1}{2\sqrt{3}}] \cup [\tfrac{\sqrt{3}}{2}, +\infty). \tag{4.10}$$

Lemma 4.2. *The quantities*

$$b_1 := \sup\left\{\tilde{b} > \tfrac{1}{2\sqrt{3}} \,|\, (3\eta_1^2 + 2g_2)(\tfrac{1}{2}+ib) > 0 \text{ for } b \in [\tfrac{1}{2\sqrt{3}}, \tilde{b})\right\}, \tag{4.11}$$

$$b_2 := \sup\left\{\tilde{b} > b_1 \,|\, (3\eta_1^2 + 2g_2)(\tfrac{1}{2}+ib) < 0 \text{ for } b \in (b_1, \tilde{b})\right\}, \tag{4.12}$$

are well-defined and satisfy $\frac{1}{2\sqrt{3}} < b_1 < \frac{1}{2} < b_2 < \frac{\sqrt{3}}{2}$*. Furthermore,*

$$(3\eta_1^2 + 2g_2)(\tfrac{1}{2}+ib) \begin{cases} > 0 & \text{if} \quad b \in [\frac{1}{2\sqrt{3}}, b_1) \cup (b_2, \frac{\sqrt{3}}{2}], \\ = 0 & \text{if} \quad b \in \{b_1, b_2\}, \\ < 0 & \text{if} \quad b \in (b_1, b_2). \end{cases}$$

Consequently, $E_- = E_+ = -\frac{3}{2}\eta_1$ *at both* $\frac{1}{2} + ib_1$ *and* $\frac{1}{2} + ib_2$*, and* $E_\pm(\frac{1}{2}+ib) \notin \mathbb{R}$ *for* $b \in (b_1, b_2)$.

Proof. By the well-known Fourier expansion of g_2, numerically $g_2(\frac{1}{2} + i\frac{1}{2}) \approx -76.6\pi^2$. This, together with $\eta_1(\frac{1}{2} + i\frac{1}{2}) = 2\pi$ (see e.g. [9, p32]), implies $(3\eta_1^2 + 2g_2)(\frac{1}{2} + i\frac{1}{2}) < 0$, so b_1 is well-defined and $\frac{1}{2\sqrt{3}} < b_1 < \frac{1}{2}$. Next we need to prove that

$$(3\eta_1^2 + 2g_2)(\tfrac{1}{2}+ib) < 0, \quad \text{for } b \in (b_1, \tfrac{1}{2}]. \tag{4.13}$$

We use $\eta_1(e^{\pi i/3}) = \frac{2\pi}{\sqrt{3}}$ (see e.g. [9, (4.1)]) and the modular property

$$\eta_1\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2\eta_1(\tau) - 2\pi i c(c\tau+d), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}).$$

By these and $\frac{1}{2} + i\frac{1}{2\sqrt{3}} = \frac{e^{\pi i/3}-1}{2e^{\pi i/3}-1}$, we obtain

$$\eta_1(\tfrac{1}{2} + i\tfrac{1}{2\sqrt{3}}) = 2\sqrt{3}\pi. \tag{4.14}$$

Clearly (4.3) implies that $g_2(\frac{1}{2}+ib)$ has a minimum point on $(\frac{1}{2\sqrt{3}}, \frac{\sqrt{3}}{2})$. This fact can be improved as follows (see e.g. [11, Corollary 4.4]): *There exists* $\hat{b} \in (\frac{1}{2\sqrt{3}}, \frac{1}{2})$ *such that* $g_2(\frac{1}{2}+ib)$ *is strictly decreasing for* $b \in (0, \hat{b})$ *and strictly increasing for* $b \in (\hat{b}, +\infty)$. Since (4.8) says that $\eta_1(\frac{1}{2}+ib) > \frac{\pi^2}{3}$ is strictly decreasing for $b \in [\frac{1}{2\sqrt{3}}, +\infty)$, we conclude that $(3\eta_1^2 + 2g_2)(\frac{1}{2}+ib)$ is strictly decreasing for $b \in [\frac{1}{2\sqrt{3}}, \hat{b}]$, so

$$(3\eta_1^2 + 2g_2)(\tfrac{1}{2}+ib) < (3\eta_1^2 + 2g_2)(\tfrac{1}{2}+ib_1) = 0 \quad \text{for } b \in (b_1, \hat{b}].$$

Furthermore,

$$\max_{b\in[\hat{b},\frac{1}{2}]}(3\eta_1^2+2g_2)(\tfrac{1}{2}+ib)<3\eta_1(\tfrac{1}{2}+i\tfrac{1}{2\sqrt{3}})^2+2g_2(\tfrac{1}{2}+i\tfrac{1}{2})<0.$$

This proves (4.13). Together with $(3\eta_1^2+2g_2)(\frac{1}{2}+i\frac{\sqrt{3}}{2})>0$, we conclude that b_2 is well-defined and $b_2\in(\frac{1}{2},\frac{\sqrt{3}}{2})$. Finally, we need to prove

$$(3\eta_1^2+2g_2)(\tfrac{1}{2}+ib)>0\quad\text{for}\ \ b\in(b_2,\tfrac{\sqrt{3}}{2}]. \tag{4.15}$$

Since $(3\eta_1^2+2g_2)(\frac{1}{2}+ib_2)=0<(3\eta_1^2+2g_2)(\frac{1}{2}+i\frac{\sqrt{3}}{2})$, we only to prove that for any $b\in[b_2,\frac{\sqrt{3}}{2})$ satisfying $(3\eta_1^2+2g_2)(\frac{1}{2}+ib)=0$, there holds $\frac{d}{db}(3\eta_1^2+2g_2)(\frac{1}{2}+ib)>0$, which can be easily proved by a direct computation: By (4.7) and (see e.g. [11])

$$g_2'(\tau)=\tfrac{i}{\pi}(2\eta_1g_2-3g_3)(\tau),$$

we have

$$\begin{aligned}\tfrac{d}{db}(3\eta_1^2+2g_2)(\tfrac{1}{2}+ib)=&\tfrac{-1}{4\pi}(12\eta_1^3+15\eta_1g_2-24g_3)\\=&\tfrac{-1}{4\pi}(7\eta_1g_2-24g_3)>0,\end{aligned}$$

where we use $\eta_1>0$, $g_2<0$ and $g_3=4e_1|e_2|^2>0$ for $\tau=\frac{1}{2}+ib$ with $b\in[b_2,\frac{\sqrt{3}}{2})\subset(\frac{1}{2},\frac{\sqrt{3}}{2})$ (note that $g_3(\frac{1}{2}+i\frac{1}{2})=0$). Therefore, (4.15) holds and the proof is complete. □

Lemma 4.3. *Fix any $\tau=\frac{1}{2}+ib$ with $b\in[b_2,\frac{\sqrt{3}}{2})$. Then $Q_2(E_\pm(\tau);\tau)\neq0$, $E_\pm(\tau)>-3e_1(\tau)$ and $\Delta(E_\pm(\tau);\tau)>2$, i.e. $E_\pm(\tau)\notin\sigma(L)$.*

Proof. By $3\eta_1^2+2g_2\geq0$ and $g_2<0$ we have $E_-(\tau)\leq E_+(\tau)<0$. In the following proof, we omit the notation τ freely. Recall

$$E_\pm^2+3\eta_1E_\pm-\tfrac{3}{2}g_2=0. \tag{4.16}$$

If $Q_2(E_\pm)=0$, then $E_\pm=-3e_1$ (because all other roots of $Q_2(\cdot)$ are complex-valued) and so $e_1^2-\eta_1e_1-\frac{1}{6}g_2=0$. However, since $e_1'(\tau)$ is expressed as (see e.g. [7, (2.15)])

$$e_1'(\tau)=\tfrac{-i}{\pi}[e_1^2-\eta_1e_1-\tfrac{1}{6}g_2](\tau),$$

we obtain $e_1'=0$, a contradiction with the aforementioned fact $e_1'\neq0$ for $\operatorname{Re}\tau=\frac{1}{2}$ proved in [20, Theorem 1.7]. Thus $Q_2(E_\pm)\neq0$ and $E_\pm\neq-3e_1$.

By the monodromy theory recalled in Section 2.2, there exists a unique pair $\pm\boldsymbol{a}=\pm\{a_1,a_2\}\subset E_\tau\setminus\{0\}$ satisfying $a_1\neq\pm a_2$ in E_τ such that

$$\begin{aligned}&E_\pm=3(x_1+x_2),\quad\text{where}\quad x_1:=\wp(a_1)\neq x_2=:\wp(a_2),\\&\wp'(a_1)+\wp'(a_2)=0,\quad x_1x_2=(x_1+x_2)^2-\tfrac{g_2}{4},\end{aligned} \tag{4.17}$$

$$\Delta(E_\pm) = e^{\sum_{j=1}^{2}\zeta(a_j)-\eta_1(a_1+a_2)} + e^{-(\sum_{j=1}^{2}\zeta(a_j)-\eta_1(a_1+a_2))}. \tag{4.18}$$

By (4.16)-(4.17), we easily obtain

$$(x_1-x_2)^2 = (x_1+x_2)^2 - 4x_1x_2 = g_2 - \tfrac{E_\pm^2}{3} = \tfrac{1}{2}g_2 + \eta_1 E_\pm < 0.$$

So by renaming a_1 and a_2, we may assume

$$\overline{x_2} = x_1 = \tfrac{E_\pm}{6} + \tfrac{i}{2}\sqrt{-\tfrac{1}{2}g_2 - \eta_1 E_\pm}. \tag{4.19}$$

Consequently,

$$4x_1^2 - g_2 = -\tfrac{2E_\pm^2}{9} + \tfrac{2E_\pm}{3} i\sqrt{-\tfrac{1}{2}g_2 - \eta_1 E_\pm} = -\tfrac{4}{3}E_\pm \overline{x_1},$$

and so

$$\begin{aligned}\wp'(a_1)^2 &= x_1(4x_1^2 - g_2) - g_3 = -\tfrac{E_\pm}{3}\left(\tfrac{E_\pm^2}{9} - \tfrac{1}{2}g_2 - \eta_1 E_\pm\right) - g_3 \\ &= \frac{6\eta_1 g_2 + (g_2 - 12\eta_1^2)E_\pm}{9} - g_3 =: F_\pm(\tau).\end{aligned} \tag{4.20}$$

If $F_\pm = 0$, then $\wp'(a_2) = -\wp'(a_1) = 0$, so $\{a_1, a_2\} = \{\frac{\omega_j}{2}, \frac{\omega_k}{2}\}$ in E_τ and then $E_\pm = 3(e_j + e_k) = -3e_l$, where $\{j,k,l\} = \{1,2,3\}$, which is a contradiction with $Q_2(E_\pm) \neq 0$. Thus $F_\pm \neq 0$. Since

$$\lim_{b\uparrow\frac{\sqrt{3}}{2}} E_+(\tfrac{1}{2}+ib) = 0 > -3e_1(\tfrac{1}{2}+i\tfrac{\sqrt{3}}{2}), \quad \lim_{b\uparrow\frac{\sqrt{3}}{2}} F_+(\tfrac{1}{2}+ib) < 0,$$

so $E_+(\frac{1}{2}+ib) > -3e_1(\frac{1}{2}+ib)$ and $F_+(\frac{1}{2}+ib) < 0$ hold for $\frac{\sqrt{3}}{2} - b > 0$ small and so for all $b \in [b_2, \frac{\sqrt{3}}{2})$ by continuity. Consequently, $E_- = E_+ > -3e_1$ and so $F_- = F_+ < 0$ at $\tau = \frac{1}{2} + ib_2$. Again by continuity we conclude that $E_-(\frac{1}{2}+ib) > -3e_1(\frac{1}{2}+ib)$ and $F_-(\frac{1}{2}+ib) < 0$ for all $b \in [b_2, \frac{\sqrt{3}}{2})$.

On the other hand, since $\tau = \frac{1}{2} + ib$ implies $\bar{\tau} = 1 - \tau$, it follows from the expression of $\wp(z;\tau)$

$$\wp(z;\tau) = \frac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\left(\frac{1}{(z-m-n\tau)^2} - \frac{1}{(m+n\tau)^2}\right)$$

that

$$\overline{\wp(z;\tau)} = \wp(\bar{z};\bar{\tau}) = \wp(\bar{z};1-\tau) = \wp(\bar{z};\tau),$$

and so

$$\overline{\wp'(z)} = \wp'(\bar{z}), \quad \overline{\zeta(z)} = \zeta(\bar{z}).$$

So (4.19) gives $\wp(a_2) = \overline{\wp(a_1)} = \wp(\overline{a_1})$, i.e.

$$\text{either} \quad a_2 = \overline{a_1} \quad \text{or} \quad a_2 = -\overline{a_1} \quad \text{in} \quad E_\tau. \tag{4.21}$$

Since $F_\pm < 0$, it follows from (4.20) that $\wp'(a_1) \in i\mathbb{R} \setminus \{0\}$ and so

$$\wp'(a_2) = -\wp'(a_1) = \overline{\wp'(a_1)} = \wp'(\overline{a_1}).$$

From here and (4.21) we obtain $a_2 = \overline{a_1}$ and so $\zeta(a_2) = \zeta(\overline{a_1}) = \overline{\zeta(a_1)}$, which implies

$$\zeta(a_1) + \zeta(a_2) - \eta_1(a_1 + a_2) \in \mathbb{R}.$$

Now we claim that

$$\zeta(a_1) + \zeta(a_2) - \eta_1(a_1 + a_2) \in \mathbb{R} \setminus \{0\}. \tag{4.22}$$

Indeed, if $\zeta(a_1) + \zeta(a_2) - \eta_1(a_1 + a_2) = 0$, we see from (2.9) that $s = 0$ and so $r = a_1 + a_2 \in \mathbb{R}$. By $Q_2(E_\pm) \neq 0$ and (2.19) we have $r \in \mathbb{R} \setminus \frac{1}{2}\mathbb{Z}$ and $Z^{(2)}_{r,0}(\frac{1}{2} + ib) = 0$. But this is a contradiction with Theorem 2.1 because $b \geq b_2 > \frac{1}{2}$ implies $\frac{1}{2} + ib \in F_0$. Therefore, (4.22) holds and so $\Delta(E_\pm) > 2$ by (4.18). This completes the proof. □

Now we are in a position to prove Theorem 1.3.

Proof of Theorem 1.3. Let $\tau = \frac{1}{2} + ib$ with $b > 0$. Recall $-3e_3 = -3\overline{e_2} \notin \mathbb{R}$ for all $b > 0$. In the following proof, we write $\Delta(E;\tau) = \Delta(E;b)$ for convenience. We divide the proof into several steps.

Step 1. We consider the simple case $b = \frac{\sqrt{3}}{2}$.

Then $-3e_1 < 0 = \pm(3g_2)^{1/2}$, $d(-3e_k) = 1$ for all k and $d(0) = 2$, i.e. 0 is an interior point of $\sigma(L)$ and $-3e_k$ is an endpoint of precisely one semi-arc of $\sigma(L)$. Together with Properties (P-1)-(P-3), we easily conclude that

$$\sigma(L) = (-\infty, -3e_1] \sqcup \sigma_2,$$

where σ_2 is a simple arc symmetric with respect to $\mathbb{R}$ with endpoints $-3e_2$ and $-3e_3$, and $\sigma_2 \cap \mathbb{R} = \{0\}$. This proves Theorem 1.3-(2).

Step 2. We consider the case $b > \frac{\sqrt{3}}{2}$.

Then $-3e_1 < -(3g_2)^{1/2} < (3g_2)^{1/2}$, $d(-3e_k) = d(\pm(3g_2)^{1/2}) = 1$ for all k, i.e. each of $\{-3e_k\}_k \cup \{\pm(3g_2)^{1/2}\}$ is an endpoint of precisely one semi-arc of $\sigma(L)$. Together with (P-1)-(P-3), we easily conclude that

$$\sigma(L) = (-\infty, -3e_1] \sqcup [-(3g_2)^{1/2}, (3g_2)^{1/2}] \cup \sigma_2,$$

where σ_2 is a simple arc symmetric with respect to $\mathbb{R}$ with endpoints $-3e_2$ and $-3e_3$, and $\sigma_2 \cap \mathbb{R} =$ one point $=: \{p_0(b)\}$.

We need to show that $p_0(b) \in (-(3g_2)^{1/2}, (3g_2)^{1/2})$ for all $b > \frac{\sqrt{3}}{2}$. First we prove that this holds for $b - \frac{\sqrt{3}}{2} > 0$ small. If not, there exists $b_n \downarrow \frac{\sqrt{3}}{2}$ such that $p_0(b_n) \notin (-(3g_2)^{1/2}, (3g_2)^{1/2})$ for such b_n, i.e.

$$[-(3g_2)^{1/2}, (3g_2)^{1/2}] \cap \sigma_2 = \emptyset. \tag{4.23}$$

Since $\Delta(E; b_n)$ is holomorphic in E and $\Delta(\pm(3g_2)^{\frac{1}{2}}; b_n) = 2$, there is $\tilde{E}_n \in (-(3g_2)^{1/2}, (3g_2)^{1/2})$ such that

$$\Delta(\tilde{E}_n; b_n) = \min_{E \in [-(3g_2)^{1/2}, (3g_2)^{1/2}]} \Delta(E; b_n) < 2.$$

Then $\frac{d}{dE}\Delta(\tilde{E}_n; b_n) = 0$ and so

$$\Delta(\tilde{E}_n; b_n) = -2. \tag{4.24}$$

Indeed, if $\Delta(\tilde{E}_n; b_n) \in (-2, 2)$, it follows from the Taylor expansion

$$\Delta(E; b_n) - \Delta(\tilde{E}_n; b_n) = a_n(E - \tilde{E}_n)^{k_n}(1 + o(1)),\ a_n \neq 0,\ k_n \geq 2 \tag{4.25}$$

and $\sigma(L) = \{E \in \mathbb{C} \mid -2 \leq \Delta(E; b_n) \leq 2\}$ that there are $2k_n \geq 4$ semi-arcs of $\sigma(L)$ meeting at $\tilde{E}_n$, a contradiction with (4.23). Thus (4.24) holds. Then

$$\lim_{n\to\infty} \tilde{E}_n = \lim_{b_n \to \frac{\sqrt{3}}{2}} \pm(3g_2)^{1/2} = 0, \tag{4.26}$$

and so $-2 = \Delta(\tilde{E}_n; b_n) \to \Delta(0; \frac{\sqrt{3}}{2}) = 2$, a contradiction. So $p_0(b) \in (-(3g_2)^{1/2}, (3g_2)^{1/2})$ for $b - \frac{\sqrt{3}}{2} > 0$ small. Denote

$$\tilde{b} := \sup\{\hat{b} > \tfrac{\sqrt{3}}{2} \mid p_0(b) \in (-(3g_2)^{1/2}, (3g_2)^{1/2}) \text{ for } b \in (\tfrac{\sqrt{3}}{2}, \hat{b})\}.$$

If $\tilde{b} < +\infty$, then $p_0(\tilde{b}) \in \{\pm(3g_2)^{1/2}\}$, say $p_0(\tilde{b}) = (3g_2)^{1/2}$ for example. Then $(3g_2)^{1/2} = [-(3g_2)^{1/2}, (3g_2)^{1/2}] \cap \sigma_2$, i.e. there are 3 semi-arcs of $\sigma(L)$ meeting at $(3g_2)^{1/2}$, a contradiction with $d((3g_2)^{1/2}) = 1$ at $b = \tilde{b}$.

This proves $\tilde{b} = +\infty$, namely $p_0(b) \in (-(3g_2)^{1/2}, (3g_2)^{1/2})$ for all $b > \frac{\sqrt{3}}{2}$. Then Theorem 1.2-(1) says $p_0(b) = E_\pm(\frac{1}{2} + ib)$. Since $\lim_{b\downarrow\frac{\sqrt{3}}{2}} p_0(b) = 0$ and $\lim_{b\downarrow\frac{\sqrt{3}}{2}} E_-(\frac{1}{2} + ib) = -3\eta_1(\frac{1}{2} + i\frac{\sqrt{3}}{2}) < 0$, we conclude that $p_0(b) = E_+(\frac{1}{2} + ib)$ for $b - \frac{\sqrt{3}}{2} > 0$ small and hence for all $b > \frac{\sqrt{3}}{2}$ by continuity and (4.10). This proves Theorem 1.3-(1).

Step 3. We consider $b = b_0 \in (\frac{5}{24}, \frac{1}{2\sqrt{3}})$.

Then $-(3g_2)^{1/2} < (3g_2)^{1/2} < -3e_1$, $d(-3e_k) = d((3g_2)^{1/2}) = 1$ for all k, i.e. each of $\{-3e_k\}_k \cup \{(3g_2)^{1/2}\}$ is an endpoint of precisely one semi-arc of $\sigma(L)$. Furthermore,

$d(-(3g_2)^{\frac{1}{2}}) = 1 + 2p_i(-(3g_2)^{\frac{1}{2}}) \geq 3$ is odd, i.e. there are $d(-(3g_2)^{\frac{1}{2}})$ semi-arcs of $\sigma(L)$ meeting at $-(3g_2)^{1/2}$. Together with (P-1)-(P-3), we easily conclude that

$$\sigma(L) = (-\infty, -(3g_2)^{1/2}] \cup \sigma_2 \sqcup [(3g_2)^{1/2}, -3e_1],$$

where σ_2 is a simple arc symmetric with respect to $\mathbb{R}$ with endpoints $-3e_2$ and $-3e_3$, and $\sigma_2 \cap \mathbb{R} = \{-(3g_2)^{1/2}\}$, i.e. there are 3 semi-arcs meeting at $-(3g_2)^{1/2}$, so $d(-(3g_2)^{\frac{1}{2}}) = 3$. This proves Theorem 1.3-(8).

Step 4. We consider $b \in (0, b_0)$.

Then $-(3g_2)^{1/2} < (3g_2)^{1/2} < -3e_1$, $d(-3e_k) = d(\pm(3g_2)^{1/2}) = 1$ for all k, i.e. each of $\{-3e_k\}_k \cup \{\pm(3g_2)^{1/2}\}$ is an endpoint of precisely one semi-arc of $\sigma(L)$. Together with (P-1)-(P-3), we easily conclude that

$$\sigma(L) = (-\infty, -(3g_2)^{1/2}] \cup \sigma_2 \cup [(3g_2)^{1/2}, -3e_1],$$

where σ_2 is a simple arc symmetric with respect to $\mathbb{R}$ with endpoints $-3e_2$ and $-3e_3$, and

$$\sigma_2 \cap \mathbb{R} = \text{one point} =: \{p_3(b)\}. \tag{4.27}$$

By Step 3 we have $p_3(b) \to -(3g_2)^{1/2}$ as $b \uparrow b_0$, so we obtain either $p_3(b) \in (-\infty, -(3g_2)^{1/2})$ for any $b_0 - b > 0$ small or $p_3(b) \in (-(3g_2)^{1/2}, (3g_2)^{1/2})$ for any $b_0 - b > 0$ small. Then by the same argument as Step 2, we have that either $p_3(b) \in (-\infty, -(3g_2)^{1/2})$ for all $b \in (0, b_0)$ or

$$p_3(b) \in (-(3g_2)^{1/2}, (3g_2)^{1/2}) \quad \text{for all} \quad b \in (0, b_0). \tag{4.28}$$

In particular, $\sigma_2 \cap [(3g_2)^{1/2}, -3e_1] = \emptyset$.

Suppose by contradiction that $p_3(b) \in (-\infty, -(3g_2)^{1/2})$ for all $b \in (0, b_0)$. Then Theorem 1.2-(1) shows $p_3(b) = E_\pm(\frac{1}{2} + ib)$. However, since $b < b_0$ and (4.9) imply $(12\eta_1^2 - g_2)(\frac{1}{2} + ib) < 0$, it is easy to see from the expression of $P(E)$ in (1.8) that $P(-(3g_2)^{1/2}) > 0$, so

$$-(3g_2)^{1/2} < E_- \leq p_3(b) < -(3g_2)^{1/2},$$

a contradiction. This proves (4.28) and so Theorem 1.3-(9) holds.

Step 5. We consider $b \in (b_0, \frac{1}{2\sqrt{3}})$.

Then $-(3g_2)^{1/2} < (3g_2)^{1/2} < -3e_1$, $d(-3e_k) = d(\pm(3g_2)^{1/2}) = 1$ for all k, so the same argument as Step 4 shows

$$\sigma(L) = (-\infty, -(3g_2)^{1/2}] \cup \sigma_2 \sqcup [(3g_2)^{1/2}, -3e_1],$$

where σ_2 is a simple arc symmetric with respect to $\mathbb{R}$ with endpoints $-3e_2$ and $-3e_3$, and

$$\sigma_2 \cap \mathbb{R} = \text{one point} =: \{p_2(b)\},$$

where either $p_2(b) \in (-\infty, -(3g_2)^{1/2})$ for all $b \in (b_0, \frac{1}{2\sqrt{3}})$ or $p_2(b) \in (-(3g_2)^{1/2}, (3g_2)^{1/2})$ for all $b \in (b_0, \frac{1}{2\sqrt{3}})$. We will prove that $p_2(b) \in (-\infty, -(3g_2)^{1/2})$ in the next step. Consequently, $p_2(b) = E_\pm(\frac{1}{2} + ib)$ by Theorem 1.2-(1). Since $P(-(3g_2)^{1/2}) < 0$ and so $E_+(\frac{1}{2} + ib) > -(3g_2)^{1/2} > E_-(\frac{1}{2} + ib)$ for $b \in (b_0, \frac{1}{2\sqrt{3}})$, we obtain $p_2(b) = E_-(\frac{1}{2} + ib)$ and so Theorem 1.3-(7) holds.

Step 6. We consider $b = \frac{1}{2\sqrt{3}}$.

Then $\pm(3g_2)^{1/2} = 0 < -3e_1$, $d(-3e_k) = 1$ for all k and $d(0) = 2$, i.e. $-3e_k$ is an endpoint of precisely one semi-arc of $\sigma(L)$ and 0 is an interior point of $\sigma(L)$, or more precisely there are exactly 2 semi-arcs of $\sigma(L)$ meeting at 0. Together with (P-1)-(P-3), we easily conclude that

$$\sigma(L) = (-\infty, -3e_1] \cup \sigma_2,$$

where σ_2 is a simple arc symmetric with respect to $\mathbb{R}$ with endpoints $-3e_2$ and $-3e_3$, and

$$\sigma_2 \cap \mathbb{R} =: \{p_1\} \subset \mathbb{R} \setminus \{0, -3e_1\}.$$

Recalling $p_2(b)$ in Step 5, we have $p_2(b) \to p_1$ as $b \uparrow \frac{1}{2\sqrt{3}}$. If $p_2(b) \in (-(3g_2)^{1/2}, (3g_2)^{1/2})$ for all $b \in (b_0, \frac{1}{2\sqrt{3}})$, then

$$p_1 = \lim_{b \uparrow \frac{1}{2\sqrt{3}}} p_2(b) = \lim_{b \uparrow \frac{1}{2\sqrt{3}}} \pm(3g_2)^{1/2} = 0,$$

a contradiction. Therefore, $p_2(b) \in (-\infty, -(3g_2)^{1/2})$ for all $b \in (b_0, \frac{1}{2\sqrt{3}})$, i.e. Theorem 1.3-(7) holds. Then

$$p_1 = \lim_{b \uparrow \frac{1}{2\sqrt{3}}} p_2(b) = E_-(\tfrac{1}{2} + i\tfrac{1}{2\sqrt{3}}) = -3\eta_1(\tfrac{1}{2} + i\tfrac{1}{2\sqrt{3}}) = -6\sqrt{3}\pi,$$

where (4.14) is used. This proves Theorem 1.3-(6).

Step 7. We consider $b \in (\frac{1}{2\sqrt{3}}, b_1]$.

Note that for all $b \in (\frac{1}{2\sqrt{3}}, \frac{\sqrt{3}}{2})$, we have $g_2 < 0$, i.e. $\pm(3g_2)^{1/2} \notin \mathbb{R}$, $d(-3e_k) = d(\pm(3g_2)^{1/2}) = 1$ for all k, i.e. each of $\{-3e_k\}_k \cup \{\pm(3g_2)^{1/2}\}$ is an endpoint of precisely one semi-arc of $\sigma(L)$. Together these with (P-1)-(P-3) and the spectrum at $b = \frac{1}{2\sqrt{3}}$ proved in Step 6, we conclude that for $b - \frac{1}{2\sqrt{3}} > 0$ small, we have

$$\sigma(L) = (-\infty, -3e_1] \cup \sigma_1 \cup \sigma_2, \tag{4.29}$$

where σ_2 (resp. σ_1) is a simple arc symmetric with respect to $\mathbb{R}$ with endpoints $-3e_2$ and $-3e_3$ (resp. with endpoints $\pm(3g_2)^{1/2}$), and

$$\sigma_1 \cap \mathbb{R} = \sigma_1 \cap (-\infty, -3e_1) = \text{one point} =: \{p_4(b)\},$$

$$\sigma_2 \cap \mathbb{R} = \sigma_2 \cap (-\infty, -3e_1) = \text{one point} =: \{p_5(b)\}, \tag{4.30}$$

$$p_4(b) > p_5(b), \quad \sigma_1 \cap \sigma_2 = \emptyset, \tag{4.31}$$

because $p_5(b) \to p_1 = -6\sqrt{3}\pi$ and $p_4(b) \to 0$ as $b \downarrow \frac{1}{2\sqrt{3}}$. Indeed by Theorem 1.2-(1), we have

$$p_4(b) = E_+(\tfrac{1}{2} + ib) > p_5(b) = E_-(\tfrac{1}{2} + ib). \tag{4.32}$$

Remark that $\sigma_1 \cap \sigma_2 = \emptyset$ follows from $p_4(b) > p_5(b)$. This fact can be proved by two ways: one is to apply Theorem 1.2-(1) to see that $\sigma_1 \cap \sigma_2 \subset \{E_\pm(\frac{1}{2} + ib)\}$; the other is that if $a \in \sigma_1 \cap \sigma_2$, then $\bar{a} \in \sigma_1 \cap \sigma_2$ and $\bar{a} \neq a$, a contradiction with (P-3).

Define

$$\tilde{b}_1 := \sup\{\tilde{b} > \tfrac{1}{2\sqrt{3}} \mid (4.29)\text{-}(4.31) \text{ hold for any } b \in (\tfrac{1}{2\sqrt{3}}, \tilde{b})\}.$$

By the spectrum at $b = \frac{\sqrt{3}}{2}$ proved in Step 1, we have $\tilde{b}_1 < \frac{\sqrt{3}}{2}$. Then (4.32) holds for any $b \in (\frac{1}{2\sqrt{3}}, \tilde{b}_1)$ and so

$$(3\eta_1^2 + 2g_2)(\tfrac{1}{2} + ib) > 0 \quad \text{for} \quad b \in [\tfrac{1}{2\sqrt{3}}, \tilde{b}_1). \tag{4.33}$$

By the continuity of the spectrum with respect to b and $d(-3e_1) = 1$ for all b, we conclude that (4.29)-(4.30) still hold for $b = \tilde{b}_1$ (i.e. $p_4(\tilde{b}_1) < -3e_1$). Thus the definition of $\tilde{b}_1$ shows that (4.31) does not hold for $\tilde{b}_1$, i.e.

$$E_+(\tfrac{1}{2} + i\tilde{b}_1) = p_4(\tilde{b}_1) = p_5(\tilde{b}_1) = E_-(\tfrac{1}{2} + i\tilde{b}_1),$$

which implies $(3\eta_1^2 + 2g_2)(\frac{1}{2} + i\tilde{b}_1) = 0$. Together with (4.33) and the definition b_1 in (4.11), we conclude $\tilde{b}_1 = b_1$, so Theorem 1.3-(5) holds. Furthermore, at $b = b_1$ we have $\sigma_1 \cap \sigma_2 = \{p_4(b_1)\} = \{-\frac{3}{2}\eta_1(\frac{1}{2} + ib_1)\}$, i.e.

$$\sigma_1 \cap \mathbb{R} = \sigma_2 \cap \mathbb{R} = \sigma_1 \cap \sigma_2 = \{-\tfrac{3}{2}\eta_1(\tfrac{1}{2} + ib_1)\} \subset (-\infty, -3e_1).$$

Thus Theorem 1.3-(4) holds.

Step 8. We consider $b \in (b_1, \frac{\sqrt{3}}{2})$.

Then $\pm(3g_2)^{1/2} \notin \mathbb{R}$, $d(-3e_k) = d(\pm(3g_2)^{1/2}) = 1$ for all k, i.e. each of $\{-3e_k\}_k \cup \{\pm(3g_2)^{1/2}\}$ is an endpoint of precisely one semi-arc of $\sigma(L)$.

Recalling (4.1), if two arcs of $\{\sigma_\infty, \sigma_1, \sigma_2\}$ have an intersection point, by Theorem 1.2 it must be of type I and hence one of $E_\pm(\frac{1}{2}+ib)$, say $E_-(\frac{1}{2}+ib)$ for example, i.e. $E_-(\frac{1}{2}+ib) \in \sigma(L)$. Then Lemma 4.3 implies $b \in (b_1, b_2)$. Consequently, Lemma 4.2 says $E_-(\frac{1}{2}+ib) = \overline{E_+(\frac{1}{2}+ib)} \notin \mathbb{R}$, and then Property (P-2) implies that both $E_-(\frac{1}{2}+ib)$ and $E_+(\frac{1}{2}+ib)$ are type I intersection points of σ_1 and σ_2, which leads to a contradiction with (P-3) which says that $\mathbb{C} \setminus \sigma(L)$ is path-connected.

Therefore, different arcs of $\sigma(L)$ cannot intersect with each other for any $b \in (b_1, \frac{\sqrt{3}}{2})$. Together with (P-1)-(P-3) and the spectrum at $b = b_1$, we conclude from the continuity of the spectrum that

$$\sigma(L) = (-\infty, -3e_1] \sqcup \sigma_1 \sqcup \overline{\sigma_1},$$

where σ_1 is a simple arc with endpoints $-3e_2$ (note $\operatorname{Im} e_2(\frac{1}{2}+ib) < 0$) and $i|3g_2|^{1/2}$, and $\overline{\sigma_1}$ is the conjugate of σ_1 with endpoints $-3e_3$ and $-i|3g_2|^{1/2}$. Clearly $\sigma_1 \cap \mathbb{R} = \emptyset$ (otherwise $\sigma_1 \cap \overline{\sigma_1} \neq \emptyset$, a contradiction) and so Theorem 1.3-(3) holds. The proof is complete. $\square$

5. Application to the mean field equation

The purpose of this section is to apply Theorem 1.3 to the mean field equation (1.15) and prove Theorem 1.6. First we briefly review some basic facts about the mean field equation

$$\triangle u + e^u = 8\pi n \delta_0 \quad \text{on} \quad E_\tau. \tag{5.1}$$

Geometrically, a solution u to (5.1) leads to a metric $\frac{1}{2}e^u|dz|^2$ with constant curvature $+1$ acquiring a conic singularity with angle $2\pi(1+2n)$. Physically, (5.1) appears in statistical physics as the *mean field limit* of the Euler flow, hence the name. It is also related to the self-dual condensates of the Chern-Simons-Higgs model in superconductivity. See [4,5,8,13,20,21,23] and references therein.

The solvability of (5.1) depends on the moduli τ in a sophisticated manner and has been studied in [4,8,13,20,21]. In particular, the connection between (5.1) and the Lamé equation was studied in [4]. Here we recall this relation for the $n = 2$ case for later usage.

Theorem 5.1. [4]

(1) *If there is a solution u to (1.15), then it lies in a scaling family of solutions u_λ through the Liouville formula*

$$u_\lambda(z) = \ln \frac{8e^{2\lambda}|f'(z)|^2}{(1+e^{2\lambda}|f(z)|^2)^2}, \quad \lambda \in \mathbb{R},$$

where $f(z)$ is a meromorphic function on $\mathbb{C}$, known as a developing map and satisfying

$$f(z+\omega_j)=e^{2i\theta_j}f(z),\quad \theta_j\in\mathbb{R},\quad j=1,2.$$

Moreover, there is a unique λ so that u_λ is even, i.e. $u_\lambda(z)=u_\lambda(-z)$.

(2) *Equation (1.15) has a solution if and only if there is $E\in\mathbb{C}$ such that the monodromy of the Lamé equation $y''(z)=[6\wp(z;\tau)+E]y(z)$ is unitary. Furthermore, the number of even solutions equals to the number of those E's such that the monodromy is unitary.*

Now we always assume $\tau=\frac{1}{2}+ib$ with $b>0$, and we write the spectrum $\sigma(L)=\sigma(L;b)$ and the Hill discriminant $\Delta(E;\tau)=\Delta(E;b)$ to emphasize their dependence on b.

5.1. Characterization of even solutions in terms of spectrum

Recall the monodromy theory of the Lamé equation

$$y''(z)=[6\wp(z;\tau)+E]y(z) \tag{5.2}$$

stated in Section 2.2. By (2.11)-(2.13), we have

$$y_{\pm\boldsymbol{a}}(z+2\tau-1)=e^{\pm 2\pi i(2r+s)}y_{\pm\boldsymbol{a}}(z),$$
$$\Delta(E;b)=e^{2\pi is}+e^{-2\pi is}.$$

Define

$$\widetilde{\Delta}(E;b):=e^{2\pi i(2r+s)}+e^{-2\pi i(2r+s)}, \tag{5.3}$$

$$\widetilde{\sigma}(L;b):=\{E\in\mathbb{C}\mid -2\le\widetilde{\Delta}(E;b)\le 2\}. \tag{5.4}$$

This $\widetilde{\sigma}(L;b)$ will play the same role as $\sigma_2(L)$ mentioned in Section 1.2.

For E satisfying $Q_2(E;\tau)=0$, it follows from (L-3) in Section 2.2 that $2r,2s\in\mathbb{Z}$, i.e. $\widetilde{\Delta}(E;b)=\pm 2$. So

$$\{-3e_1,-3e_2,-3e_3,\pm(3g_2)^{1/2}\}\subset\sigma(L;b)\cap\widetilde{\sigma}(L;b).$$

Define

$$\Xi(b):=[\sigma(L;b)\cap\widetilde{\sigma}(L;b)]\setminus\{-3e_1,-3e_2,-3e_3,\pm(3g_2)^{1/2}\}. \tag{5.5}$$

The following result establishes the precise connection between even solutions of the mean field equation and the spectrum.

Lemma 5.2. *The number of even solutions of the mean field equation*

$$\triangle u+e^u=16\pi\delta_0\quad \text{on}\ \ E_\tau \tag{5.6}$$

equals to $\#\Xi(b)$. *Furthermore,*

(1) *The number of even axisymmetric solutions equals to* $\#(\Xi(b)\cap\mathbb{R})$.
(2) *The number of even but not axisymmetric solutions equals to* $\#(\Xi(b)\setminus\mathbb{R})$.

Proof. Recalling (L-3)-(L-4) stated in Section 2.2, we know that the monodromy of (5.2) is unitary if and only if $Q_2(E;\tau)\neq 0$ and the corresponding (r,s) of this E satisfies $(r,s)\in\mathbb{R}^2\setminus\frac{1}{2}\mathbb{Z}^2$, and so if and only if $E\in\Xi(b)$ (note $(r,s)\notin\frac{1}{2}\mathbb{Z}^2$ follows from $Q_2(E;\tau)\neq 0$). Together with Theorem 5.1-(2), we conclude that the number of even solutions of (5.6) equals to $\#\Xi(b)$.

To prove (1)-(2), we need to apply the precise connection between an even solution $u(z)=u(x,y)$ (here we use complex variable $z=x+iy$) and the corresponding $E\in\Xi(b)$ proved in [4]:

$$(u_{zz}-\tfrac{1}{2}u_z^2)(z)=-2[6\wp(z;\tau)+E],$$

and in Theorem 5.1 the developing map $f(z)=y_{\boldsymbol{a}}(z)/y_{-\boldsymbol{a}}(z)$, where $y_{\pm\boldsymbol{a}}(z)$ are solutions of (5.2) stated in (L-1) in Section 2.2.

Clearly $\tilde{u}(z)=\tilde{u}(x,y):=u(x,-y)=u(\bar{z})$ is also an even solution of (5.6) and satisfies (note that $u(z)$ is real-valued as a solution of (5.6))

$$\begin{aligned}(\tilde{u}_{zz}-\tfrac{1}{2}\tilde{u}_z^2)(z)&=\overline{(u_{zz}-\tfrac{1}{2}u_z^2)(\bar{z})}\\&=\overline{-2[6\wp(\bar{z};\tau)+E]}=-2[6\wp(z;\tau)+\overline{E}],\end{aligned}$$

i.e. $\overline{E}\in\Xi(b)$ if $E\in\Xi(b)$. From here and the fact stated in Theorem 5.1-(2) that there is a one-to-one correspondence between $E\in\Xi(b)$ and even solutions of (5.6), we conclude that $E=\overline{E}$ if and only if $u(z)=\tilde{u}(z)$, i.e. $u(z)=u(\bar{z})$ is axisymmetric. Therefore, the assertions (1)-(2) hold. □

5.2. Study of $\Xi(b)$

By Lemma 5.2, we turn to study $\Xi(b)$. Note that

$$\tfrac{\tau-1}{2\tau-1}=\tfrac{1}{2}+i\tfrac{1}{4b}\quad\text{for}\quad\tau=\tfrac{1}{2}+ib.$$

Since $y_{\boldsymbol{a}}(z)$ is a solution of (5.2), then $\tilde{y}(z):=y_{\boldsymbol{a}}((2\tau-1)z)$ satisfies

$$\begin{aligned}\tilde{y}''(z)&=(2\tau-1)^2[6\wp((2\tau-1)z;\tau)+E]\tilde{y}(z)\\&=[6\wp(z;\tfrac{\tau-1}{2\tau-1})+(2\tau-1)^2E]\tilde{y}(z)\\&=[6\wp(z;\tfrac{1}{2}+i\tfrac{1}{4b})-4b^2E]\tilde{y}(z),\end{aligned}$$

and

$$\tilde{y}(z+1) = y_{\boldsymbol{a}}((2\tau-1)z + 2\tau - 1) = e^{2\pi i(2r+s)}\tilde{y}(z).$$

Therefore,

$$\Delta(-4b^2E; \tfrac{1}{4b}) = e^{2\pi i(2r+s)} + e^{-2\pi i(2r+s)} = \widetilde{\Delta}(E; b). \tag{5.7}$$

Consequently, we conclude from (5.4) that

$$\begin{aligned}\widetilde{\sigma}(L; b) &= \{E \mid -2 \le \Delta(-4b^2E; \tfrac{1}{4b}) \le 2\} \\ &= \{E \mid -4b^2E \in \sigma(L; \tfrac{1}{4b})\} = \tfrac{1}{-4b^2}\sigma(L; \tfrac{1}{4b}).\end{aligned} \tag{5.8}$$

Remark that by the modular properties of $e_k(\tau)$ and $g_2(\tau)$, it is easy to see that

$$\begin{aligned}&\tfrac{1}{-4b^2}\{-3e_k(\tfrac{1}{2} + i\tfrac{1}{4b}), \pm(3g_2(\tfrac{1}{2} + i\tfrac{1}{4b}))^{1/2}, k = 1, 2, 3\} \\ =&\{-3e_k(\tfrac{1}{2} + ib), \pm(3g_2(\tfrac{1}{2} + ib))^{1/2}, k = 1, 2, 3\},\end{aligned} \tag{5.9}$$

namely the finite endpoints of arcs of $\widetilde{\sigma}(L; b) = \frac{1}{-4b^2}\sigma(L; \frac{1}{4b})$ also lie in

$$\{-3e_k(\tfrac{1}{2} + ib), \pm(3g_2(\tfrac{1}{2} + ib))^{1/2}, k = 1, 2, 3\},$$

the same as those of $\sigma(L; b)$. Together these with Theorem 1.3, we can give a complete picture of $\widetilde{\sigma}(L; b)$ for all $b > 0$. Here we only list the special case $b \in (\frac{1}{2\sqrt{3}}, \frac{1}{4b_1})$ (i.e. $\frac{1}{4b} \in (b_1, \frac{\sqrt{3}}{2})$) for later usage.

Lemma 5.3. *For $b \in (\frac{1}{2\sqrt{3}}, \frac{1}{4b_1})$,*

$$\widetilde{\sigma}(L; b) = [-3e_1, +\infty) \sqcup \sigma_3 \sqcup \overline{\sigma_3},$$

where σ_3 is a simple arc in $\{E | \operatorname{Im} E > 0\}$ (i.e. $\sigma_3 \cap \mathbb{R} = \emptyset$) with endpoints $-3e_2$ and $i|3g_2|^{1/2}$, and $\overline{\sigma_3}$ is the conjugate of σ_3 with endpoints $-3e_3$ and $-i|3g_2|^{1/2}$.

Now we consider the special case $b = \frac{1}{2}$, where $-e_2 - e_3 = e_1 = 0$ and so

$$e_3 = \overline{e_2} = -e_2 =: iA, \quad \text{i.e.} \;\; A = \operatorname{Im} e_3 > 0. \tag{5.10}$$

Consequently (recall Lemma 4.2 that $g_2(\frac{1}{2} + i\frac{1}{2}) \approx -76.6\pi^2$)

$$-76.6\pi^2 \approx g_2(\tfrac{1}{2} + i\tfrac{1}{2}) = 2(e_1^2 + e_2^2 + e_3^3) = -4A^2, \quad \text{i.e.} \;\; A \approx 4.4\pi,$$

and

$$|3g_2|^{1/2} = 2\sqrt{3}A > 3A = \operatorname{Im}(-3e_2), \tag{5.11}$$

namely the point $i|3g_2|^{1/2}$ is above the point $-3e_2$ on the graph of $\sigma(L;\frac{1}{2})$. Note from (5.8) that

$$\widetilde{\sigma}(L;\tfrac{1}{2}) = -\sigma(L;\tfrac{1}{2}). \tag{5.12}$$

Since $\sigma(L;\frac{1}{2})$ is symmetric with respect to $\mathbb{R}$, we see from (5.12) that $\widetilde{\sigma}(L;\frac{1}{2})$ and $\sigma(L;\frac{1}{2})$ are symmetric with respect to $i\mathbb{R}$. From here we can prove

Lemma 5.4. *For $b = \frac{1}{2}$, recall from Theorem 1.3-(3) that*

$$\sigma(L;\tfrac{1}{2}) = (-\infty, 0] \sqcup \sigma_1 \sqcup \overline{\sigma_1},$$

where σ_1 is a simple arc in $\{E|\operatorname{Im} E > 0\}$ with endpoints $-3e_2 = 3Ai$ and $i|3g_2|^{1/2} = 2\sqrt{3}Ai$, and $\overline{\sigma_1}$ is the conjugate of σ_1. Then

$$\sigma_1 \setminus \{-3e_2, i|3g_2|^{1/2}\} \subset \{E \mid \operatorname{Re} E < 0\}. \tag{5.13}$$

In other words, except the 5 endpoints lying on $i\mathbb{R}$, all other points of $\sigma(L;\frac{1}{2})$ lie in the half plane $\{E \mid \operatorname{Re} E < 0\}$.

Proof. By Theorem A-(3) and Lemma 5.2, we obtain $\Xi(\frac{1}{2}) = \emptyset$. Recall Lemma 5.3 and (5.12) that

$$\widetilde{\sigma}(L;\tfrac{1}{2}) = [0, +\infty) \sqcup \sigma_3 \sqcup \overline{\sigma_3},$$

where $\sigma_3 = -\overline{\sigma_1}$ is symmetric with σ_1 with respect to $i\mathbb{R}$. If there is $E \in (\sigma_1 \setminus \{-3e_2, i|3g_2|^{1/2}\}) \cap i\mathbb{R}$, then $E \in \sigma_3$ and so $E \in \Xi(\frac{1}{2})$, a contradiction. Therefore,

$$(\sigma_1 \setminus \{-3e_2, i|3g_2|^{1/2}\}) \cap i\mathbb{R} = \emptyset, \tag{5.14}$$

$$\sigma_1 \cap \sigma_3 = \{-3e_2, i|3g_2|^{1/2}\}. \tag{5.15}$$

Thanks to (5.14), to prove (5.13) we only to prove $\operatorname{Re} E < 0$ for $E \in \sigma_1$ sufficiently close to $-3e_2$. Recall the hyperelliptic curve

$$Y_2 \cong \{(E, C) \mid C^2 = Q_2(E)\} \tag{5.16}$$

stated in (L-5) in Section 2.2. It is well known that $(a_1, a_2) = (\frac{\omega_1}{2}, \frac{\omega_3}{2})$ at $E = -3e_2$. Note that $C = 0$ at $E = -3e_2$, i.e. $(-3e_2, 0)$ is a branch point of $\{(E, C)|C^2 = Q_2(E)\}$ or equivalently, $(\frac{\omega_1}{2}, \frac{\omega_3}{2})$ is a branch point of the hyperelliptic curve Y_2. Therefore, we can consider C as a local holomorphic coordinate of $(a_1(C), a_2(C)) \in Y_2$ corresponding to (E, C) near the branch point $(\frac{\omega_1}{2}, \frac{\omega_3}{2})$ with

$$(a_1(0), a_2(0)) = (\tfrac{\omega_1}{2}, \tfrac{\omega_3}{2});$$

see [22, Lemma 3.3], where it was also proved that[1]

$$a_1'(0) = \frac{2}{9}\frac{1}{\wp''(\frac{\omega_1}{2})}\frac{1}{e_1-e_3}, \quad a_2'(0) = \frac{2}{9}\frac{1}{\wp''(\frac{\omega_3}{2})}\frac{1}{e_3-e_1}.$$

Inserting $\wp'' = 6\wp^2 - g_2/2$ and (5.10)-(5.11) into the above formula leads to

$$a_1'(0) = \tfrac{i}{9A^3}, \quad a_2'(0) = \tfrac{i}{18A^3}.$$

Therefore for (E, C) close to $(-3e_2, 0)$, the corresponding (a_1, a_2) satisfies

$$\begin{aligned} a_1 = a_1(C) &= \tfrac{\omega_1}{2} + \tfrac{i}{9A^3}C(1+O(C)), \\ a_2 = a_2(C) &= \tfrac{\omega_3}{2} + \tfrac{i}{18A^3}C(1+O(C)). \end{aligned}$$

Inserting these into (2.9), we obtain from

$$\zeta(\tfrac{\omega_1}{2}) + \zeta(\tfrac{\omega_3}{2}) - \eta_1(\tfrac{\omega_1}{2} + \tfrac{\omega_3}{2}) = \tfrac{\eta_2 - \tau\eta_1}{2} = -\pi i$$

(i.e. $s = \frac{1}{2}$ at $E = -3e_2$) and $\zeta' = -\wp$ that

$$\begin{aligned} -2\pi i s &= \zeta(a_1) + \zeta(a_2) - \eta_1(a_1 + a_2) \\ &= -\pi i - \tfrac{i}{18A^3}(e_3 + 3\eta_1)C(1+O(C)) \\ &= -\pi i - \tfrac{i}{18A^3}|e_3 + 3\eta_1||C|e^{i(\theta+\theta_0)}(1+O(C)), \end{aligned} \tag{5.17}$$

where we write $C = |C|e^{i\theta}$, $\theta \in [-\pi, \pi]$, and $e_3 + 3\eta_1 = |e_3 + 3\eta_1|e^{i\theta_0}$. Note that $e_3 = iA$ with $A \approx 4.4\pi$ and $\eta_1(\frac{1}{2} + i\frac{1}{2}) = 2\pi$, we have $\tan\theta_0 = \frac{A}{6\pi} \approx \frac{2.2}{3} \in (\frac{1}{\sqrt{3}}, 1)$, so we can take $\theta_0 \in (\frac{\pi}{6}, \frac{\pi}{4})$.

Now for $E \in \sigma_1$ close to $-3e_2$, i.e. (E, C) close to $(-3e_2, 0)$, we have $\Delta(E) = e^{2\pi i s} + e^{-2\pi i s} \in [-2, 2]$, i.e. $s \in \mathbb{R}$, which implies from (5.17) that $\theta = k\pi - \theta_0 + o(1)$ with $k \in \{0, 1\}$. Consequently, we deduce from $C^2 = Q_2(E)$ and $E = -3e_2 + o(1)$ that

$$E + 3e_2 = \frac{C^2}{(E^2 - 3g_2)(E + 3e_1)(E + 3e_3)} = -\frac{|C|^2}{54A^4}e^{-2i\theta_0}(1 + o(1)),$$

namely $\operatorname{Re} E = \operatorname{Re}(E + 3e_2) < 0$ for $E \in \sigma_1$ close to $-3e_2$. Then by (5.14), we see that (5.13) holds. This completes the proof. □

Remark 5.5. By (5.13) and (5.15) and that σ_3 is symmetric with σ_1 with respect to $i\mathbb{R}$, we see that $\sigma_1 \cup \sigma_3$ is a *simple closed* arc in the half plane $\{E | \operatorname{Im} E > 0\}$. We give an orientation of $\sigma_1 \cup \sigma_3$ by moving from $i|3g_2|^{1/2}$ along σ_1 to $-3e_2$ and then along σ_3 back to $i|3g_2|^{1/2}$. Then the orientation is *counterclockwise* because of (5.11) and (5.13).

[1] In [22, Lemma 3.3] the formula reads $a_1'(0) = \frac{1}{\wp''(\frac{\omega_1}{2})}\frac{1}{e_1-e_3}$ because they used the hyperelliptic curve $C^2 = \frac{4}{81}Q_2(E)$.

Now we are in a position to prove Theorem 1.6.

Proof of Theorem 1.6. Assume by contradiction that

$$\Xi(b) = \emptyset \quad \text{for any} \quad b \in [b_1, \tfrac{1}{2}] \subset (\tfrac{1}{2\sqrt{3}}, \tfrac{1}{4b_1}). \tag{5.18}$$

Step 1. We consider $b = b_1$.

Then $\widetilde{\sigma}(L; b_1) = [-3e_1, +\infty) \sqcup \sigma_3 \sqcup \overline{\sigma_3}$ is given in Lemma 5.3 and

$$\sigma(L; b_1) = (-\infty, -3e_1] \cup \sigma_1 \cup \sigma_2,$$

is given in Theorem 1.3-(4). We rewrite

$$\sigma_1 \cup \sigma_2 = \sigma_1^0 \cup \overline{\sigma_1^0},$$

where $\sigma_1^0 = (\sigma_1 \cup \sigma_2) \cap \{E | \operatorname{Im} E \geq 0\}$, namely σ_1^0 is a *simple* arc connecting $i|3g_2|^{1/2}$ and $-3e_2$ and

$$\sigma_1^0 \setminus \{-\tfrac{3}{2}\eta_1(\tfrac{1}{2} + ib_1)\} \subset \{E | \operatorname{Im} E > 0\}.$$

Since σ_3 is a simple arc in $\{E | \operatorname{Im} E > 0\}$ with endpoints $-3e_2$ and $i|3g_2|^{1/2}$, we conclude from $\Xi(b_1) = \emptyset$ that $\sigma_1^0 \cap \sigma_3 = \{-3e_2, i|3g_2|^{1/2}\}$ and so $\sigma_1^0 \cup \sigma_3$ is a *simple closed* arc which satisfies

$$\sigma_1^0 \cup \sigma_3 \setminus \{-\tfrac{3}{2}\eta_1(\tfrac{1}{2} + ib_1)\} \subset \{E | \operatorname{Im} E > 0\}.$$

Again we give an orientation of $\sigma_1^0 \cup \sigma_3$ by moving from $i|3g_2|^{1/2}$ along σ_1^0 to $-3e_2$ and then along σ_3 back to $i|3g_2|^{1/2}$. Then by letting $b \uparrow b_1$ in Theorem 1.3-(5), we easily see that the orientation of $\sigma_1^0 \cup \sigma_3$ is *clockwise*.

Step 2. We consider $b \in (b_1, \frac{1}{2}]$.

Again $\widetilde{\sigma}(L; b) = [-3e_1, +\infty) \sqcup \sigma_3 \sqcup \overline{\sigma_3}$ is given in Lemma 5.3, where σ_3 is a simple arc in $\{E | \operatorname{Im} E > 0\}$ with endpoints $-3e_2$ and $i|3g_2|^{1/2}$.

On the other hand, Theorem 1.3-(3) says that

$$\sigma(L; b) = (-\infty, -3e_1] \sqcup \sigma_1 \sqcup \overline{\sigma_1},$$

where σ_1 is a simple arc in $\{E | \operatorname{Im} E > 0\}$ with end points $-3e_2$ and $i|3g_2|^{1/2}$. By $\Xi(b) = \emptyset$, we have $\sigma_1 \cap \sigma_3 = \{-3e_2, i|3g_2|^{1/2}\}$ and so $\sigma_1 \cup \sigma_3$ is a *simple closed* arc in $\{E | \operatorname{Im} E > 0\}$. Again we give an orientation of $\sigma_1 \cup \sigma_3$ by moving from $i|3g_2|^{1/2}$ along σ_1 to $-3e_2$ and then along σ_3 back to $i|3g_2|^{1/2}$. Since these simple closed arcs satisfy

$$\sigma_1 \cup \sigma_3 \to \sigma_1^0 \cup \sigma_3 \quad \text{as} \quad b \downarrow b_1,$$

we see that the direction of the orientation is invariant under the continuous deformation. In conclusion, the orientation of $\sigma_1 \cup \sigma_3$ is *clockwise* for any $b \in (b_1, \frac{1}{2}]$, a contradiction with Remark 5.5.

Step 3. We complete the proof.

Steps 1-2 imply that (5.18) is not true, i.e. there exist $b \in [b_1, \frac{1}{2})$ such that $\Xi(b) \neq \emptyset$. Note that $\overline{E} \in \Xi(b)$ if $E \in \Xi(b)$. Besides, it is easy to see from Lemma 5.3 and Theorem 1.3 (3)-(4) that

$$\widetilde{\sigma}(L;b) \cap \sigma(L;b) \cap \mathbb{R} = \{-3e_1\},$$

so $\Xi(b) \cap \mathbb{R} = \emptyset$. Therefore, we conclude from Lemma 5.2 that the mean field equation (5.6) has no even axisymmetric solutions but does have at least 2 even solutions which are not axisymmetric.

This completes the proof. □

Remark 5.6. By (5.8)-(5.9) and Theorem 1.3, we can give a new proof of Theorem A (1)-(2). For example, for $b > \frac{1}{4b_0} > \frac{\sqrt{3}}{2}$ we have

$$\widetilde{\sigma}(L;b) = [-3e_1, -(3g_2)^{1/2}] \sqcup \sigma_3 \sqcup [(3g_2)^{1/2}, +\infty),$$

where σ_3 is a simple arc symmetric with respect to $\mathbb{R}$ with endpoints $-3e_2$ and $-3e_3$, and

$$\sigma_3 \cap \mathbb{R} = \sigma_3 \cap (-(3g_2)^{1/2}, (3g_2)^{1/2}) =: \{E_0(b)\},$$

where $E_0(b) \to (3g_2)^{1/2}$ as $b \downarrow \frac{1}{4b_0}$. Together with Theorem 1.3-(1), we immediately obtain $\Xi(b) \cap \mathbb{R} = \{E_0(b)\}$ for $b > \frac{1}{4b_0}$ and $\Xi(\frac{1}{4b_0}) \cap \mathbb{R} = \emptyset$. From here and Lemma 5.2, we conclude that the number of even axisymmetric solutions of (5.6) is 1 (resp. 0) for $b > \frac{1}{4b_0}$ (resp. for $b = \frac{1}{4b_0}$). This proves Theorem A (1)-(2) for $b \geq \frac{1}{4b_0}$ and $\hat{b} = \frac{1}{4b_0}$. The remaining case $b < \frac{1}{4b_0}$ can be proved similarly and we omit the details here.

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References

[1] V. Batchenko, F. Gesztesy, On the spectrum of Schrödinger operators with quasi-periodic algebro-geometric KdV potentials, J. Anal. Math. 95 (2005) 333–387.
[2] B. Birnir, Complex Hill's equation and the complex periodic Korteweg-de Vries equations, Commun. Pure Appl. Math. 39 (1986) 1–49.

[3] J. Borcea, B. Shapiro, Root asymptotics of spectral polynomials for the Lamé operator, Commun. Math. Phys. 282 (2008) 323–337.
[4] C.L. Chai, C.S. Lin, C.L. Wang, Mean field equations, hyperelliptic curves, and modular forms: I, Camb. J. Math. 3 (2015) 127–274.
[5] C.C. Chen, C.S. Lin, G. Wang, Concentration phenomena of two-vortex solutions in a Chern-Simons model, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) 3 (2004) 367–397.
[6] Z. Chen, T.J. Kuo, C.S. Lin, Nonexistence of solutions for mean field equation on rectangular torus at critical parameter 16π, Commun. Anal. Geom. 27 (2019) 1737–1755.
[7] Z. Chen, C.S. Lin, On algebraic multiplicity of (anti)periodic eigenvalues of Hill's equations, Proc. Am. Math. Soc. 146 (2018) 3039–3047.
[8] Z. Chen, C.S. Lin, Sharp nonexistence results for curvature equations with four singular sources on rectangular tori, Am. J. Math. 142 (2020) 1269–1300.
[9] Z. Chen, C.S. Lin, Critical points of the classical Eisenstein series of weight two, J. Differ. Geom. 113 (2019) 189–226.
[10] Z. Chen, C.S. Lin, Spectrum of the Lamé operator and application, II: when an endpoint is a cusp, Commun. Math. Phys. 378 (2020) 335–368.
[11] Z. Chen, C.S. Lin, Distribution of critical points of the classical Eisenstein series E_4, preprint.
[12] G. Darboux, Sur une équation lineare, C. R. Acad. Sci. Paris XCIV (25) (1882) 1645–1648.
[13] A. Eremenko, A. Gabrielov, Spherical rectangles, Arnold Math. J. 2 (2016) 463–486.
[14] F. Gesztesy, R. Weikard, Picard potentials and Hill's equation on a torus, Acta Math. 176 (1996) 73–107.
[15] F. Gesztesy, R. Weikard, Lamé potentials and the stationary (m)KdV hierarchy, Math. Nachr. 176 (1995) 73–91.
[16] F. Gesztesy, R. Weikard, Floquet theory revisited, in: Differential Equations and Mathematical Physics, Int. Press, Boston, MA, 1995, pp. 67–84.
[17] W. Haese-Hill, M. Hallnäs, A. Veselov, On the spectra of real and complex Lamé operators, SIGMA 13 (2017) 049.
[18] E.L. Ince, Further investigations into the periodic Lamé equations, Proc. R. Soc. Edinb. 60 (1940) 83–99.
[19] G. Lamé, Sur les surfaces isothermes dans les corps homogènes en équilibre de température, J. Math. Pures Appl. 2 (1837) 147–188.
[20] C.S. Lin, C.L. Wang, Elliptic functions, Green functions and the mean field equations on tori, Ann. Math. 172 (2010) 911–954.
[21] C.S. Lin, C.L. Wang, Mean field equations, hyperelliptic curves, and modular forms: II, J. Éc. Polytech. Math. 4 (2017) 557–593.
[22] C.S. Lin, C.L. Wang, Geometric quantities arising from bubbling analysis of mean field equations, Commun. Anal. Geom. 28 (2020) 1289–1313.
[23] C.S. Lin, S. Yan, Existence of bubbling solutions for Chern-Simons model on a torus, Arch. Ration. Mech. Anal. 207 (2013) 353–392.
[24] M.A. Naimark, Linear Differential Operators, Frederick Ungar, New York, 1967.
[25] F. Rofe-Beketov, The spectrum of non-selfadjoint differential operators with periodic coefficients, Sov. Math. Dokl. 4 (1963) 1563–1566.
[26] K. Takemura, Analytic continuation of eigenvalues of the Lamé operator, J. Differ. Equ. 228 (2006) 1–16.
[27] A. Treibich, J.L. Verdier, Revetements exceptionnels et sommes de 4 nombres triangulaires, Duke Math. J. 68 (1992) 217–236.
[28] R. Weikard, On Hill's equation with a singular complex-valued potential, Proc. Lond. Math. Soc. 76 (1998) 603–633.
[29] E.T. Whittaker, G.N. Watson, A Course of Modern Analysis, 4th edition, Cambridge University Press, 1927.