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A Conservative Linearly-Implicit Compact Difference Scheme for the Quantum Zakharov System

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Abstract
This paper is devoted to developing and analysing a highly accurate conservative method for solving the quantum Zakharov system. The scheme is based on a linearly-implicit compact finite difference discretization and conserve the mass as well as energy in discrete level. Detailed numerical analysis is presented which shows the method is fourth-order accurate in space and second-order accurate in time. Several numerical examples are reported to confirm the conservation properties and high accuracy of the proposed scheme. Finally the compact scheme is applied to study the convergence rate of the quantum Zakharov system to its limiting model in the semi-classical limit.

Keywords Quantum Zakharov system · Conservative properties · Compact finite difference scheme · Convergence

Mathematics Subject Classification 35Q53 · 65M15 · 65M70

1 Introduction

The classical Zakharov system (ZS) was introduced by Zakharov [46] to describe the propagation of Langmuir waves in plasma. It has also been widely applied to various physical problems, such as the theory of molecular chains [9], hydrodynamics [10] and so on. Furthermore, many researchers have continuously improved the classical ZS and proposed various

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modified ZS which are more consistent with physical phenomena and experimental results. We refer the readers to the book [25], and references therein.

Taking quantum effects into account, the quantum Zakharov system (QZS) has been introduced by Garcia et al. [22] and Haas et al. [28] to describe the nonlinear interaction between high-frequency quantum Langmuir waves and low-frequency quantum ionacoustic waves,

$$\begin{cases} iE_t + \Delta E - \varepsilon^2 \Delta^2 E = NE, \\ N_{tt} - \Delta N + \varepsilon^2 \Delta^2 N = \Delta |E|^2, \quad x \in \mathbb{R}^d, \quad t > 0, \\ E(x,0) = E_0(x), \quad N(x,0) = N_0(x), \quad N_t(x,0) = N_1(x), \end{cases} \tag{1.1}$$

where the complex function $E(x,t)$ is the slowly varying envelope of the rapidly oscillatory electric field, the real function $N(x,t)$ is the deviation of the ion density from its equilibrium value. It is known that $\varepsilon = \frac{\hbar\omega_i}{\kappa_B T_e}$ is the ratio of the ion plasma and electron thermal energies, here $\hbar$ is Planck's constant divided by 2π, ω_i is the ion plasma frequency, T_e is the electron fluid temperature, and κ_B is the Boltzmann constant. Here the quantum effect is characterized by a fourth-order perturbation with a quantum parameter $\varepsilon > 0$ non-negligible when either the ion-plasma frequency is high or the temperature of electrons is low. For a more detailed description of the physical background, we refer the readers to [27,36]. When the quantum effect is absent, i.e., $\varepsilon = 0$, the QZS collapses to the classical ZS [17,26].

Similar to the classical ZS, the QZS (1.1) satisfies the mass conservation law

$$\int_{\mathbb{R}^d} |E(t)|^2 dx = \int_{\mathbb{R}^d} |E(0)|^2 dx, \tag{1.2}$$

and the energy conservation law

$$\int_{\mathbb{R}^d} \left(|\nabla E|^2 + \frac{1}{2}\left(|\nabla u|^2 + N^2\right) + \varepsilon^2 |\Delta E|^2 + \frac{\varepsilon^2}{2} |\nabla N|^2 + N|E|^2 \right) dx = \text{constant}, \tag{1.3}$$

where $\Delta u = N_t$.

In recent years, the QZS (1.1) has been investigated extensively [7,18–20,26,28,30,34–36,45]. For the well-posedness of QZS, we refer to [18,20,26] and references therein. Specifically, the QZS is locally well-posed in $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ for $1 \le d \le 8$ and globally well-posed for dimension up to five [18], which is different from the classical ZS, where the well-posedness for $E_0 \in L^2(\mathbb{R}^d)$ is known only for $d = 1, 2$, and the solution can blow up in $d = 2$ [23]. This suggests that including some more physical effects in the equations which results as a more complicated system may make the mathematical understanding much easier. For the stability of the standing waves of QZS, we refer to [19]. For the hyperchaos and temporal dynamics of the QZS, we refer to [34–36]. For the semi-classical limit of the QZS, i.e., the QZS converges to the classical ZS as the quantum parameter ε goes to zero, we refer to [7,17,26]. For the investigation of the QZS on periodic spatial domains, we refer to a recent work [8].

For the numerical part, different numerical methods for the classical or generalized ZS have been proposed and analyzed in the last decades (cf. [1–6,11–14,16,31,38]). However, there are few results concerning the numerical methods for the QZS (1.1). Recently, Xiao et. al. [43] proposed a conservative linearly-implicit difference scheme for the modified Zakharov system with high-order space fractional quantum correction. The method conserves mass and energy, and converges at the second order in space and time. To improve the accuracy in space, one choice is to apply the compact difference method, whose efficiency has been widely verified in solving a large number of equations, e.g., Schrödinger equation

[29,41,44], Klein–Gordon equation [15,21], Klein–Gordon–Schrödinger equation [39], classical Zakharov system [38], Klein–Gordon–Zakharov equations [42], 2D Rayleigh–Stokes problem [37], and Cahn–Hilliard equations [32,33]. The aim of this paper is to design a high-accuracy conservative scheme, which is based on a linearly-implicit compact difference scheme, for solving the QZS (1.1). This scheme is highly accurate at the second order in time and fourth order in space. It is efficient to solve since one only needs to solve two independent linear systems at each time step. Moreover, the method conserves the mass and energy which is of vital importance for stability and long-time dynamics.

The rest of the paper is organized as follows. In Sect. 2, we present a linearly-implicit compact difference scheme for the QZS (1.1), with accuracy at $O(h^4+\tau^2)$. In Sect. 3, the conservative properties of the scheme are given. A priori bounds and convergence of the scheme are proven in Sect. 4. In Sect. 5, numerical experiments are given to confirm the accuracy of the scheme, to verify the convergence of the QZS in the semi-classical limit, and to investigate the dynamics of the QZS. Finally, a brief conclusion is drawn in Sect. 6. For the sake of simplicity, the symbol C denotes a generic positive constant independent of the mesh size, time step and the quantum parameter ε, which may represent different values in distinct occurrences.

2 A Linearly-Implicit Compact Difference Scheme

Similar to most works for the simulation of the Zakharov system [1–5,31], we will consider the QZS (1.1) in 1D and for numerical implementation we truncate (1.1) into a bounded domain $\Omega=[a,b]$ with periodic boundary conditions:

$$\begin{cases} iE_t+E_{xx}-\varepsilon^2\partial_x^4E-NE=0, \\ N_{tt}-N_{xx}+\varepsilon^2\partial_x^4N-\partial_x^2(|E|^2)=0, \quad x\in(a,b), \quad t>0, \\ E(x,0)=E_0(x), \quad N(x,0)=N_0(x), \quad N_t(x,0)=N_1(x), \quad x\in[a,b], \\ \partial_x^kE(a,t)=\partial_x^kE(b,t), \quad \partial_x^kN(a,t)=\partial_x^kN(b,t), \quad k=0,1,2,3. \end{cases} \tag{2.1}$$

We suppose the periodic Cauchy problem (2.1) possesses a unique solution which is smooth enough and the initial data N_1 satisfies the compatibility condition [24]

$$\int_a^b N_1(x)dx=0, \quad \sum_{k=1}^{M}N_1(a+kh)=0, \quad \text{for } h>0 \text{ with } Mh=b-a, \quad M\in\mathbb{N}. \tag{2.2}$$

To state the difference scheme, let $h=\frac{b-a}{M}$ and $\tau=\frac{T}{J}$ be the spatial and temporal sizes respectively, where M and J are two given integers. Define $\Omega_h=\{x_k\,|\,x_k=a+kh,\ 0\le k\le M\}$, $\Omega_\tau=\{t_n\,|\,t_n=n\tau,\ 0\le n\le J\}$, $\Omega_{h\tau}=\Omega_h\times\Omega_\tau$. Denote the numerical solutions E_k^n and N_k^n as the approximation of the exact solution $E(x,t)$ and $N(x,t)$ at the point (x_k,t_n), respectively. Denote $\mathbb{V}_h:=\left\{v=(v_1,v_2,\ldots,v_M)^T\right\}\subseteq\mathbb{C}^M$ by the periodic grid function space, i.e., $v_j=v_{j+M}$ when involved. For any grid functions $u,v\in\mathbb{V}_h$, define the inner product and norms as

$$\langle u,v\rangle=h\sum_{k=1}^{M}u_k\overline{v_k}, \quad \|v\|_\infty=\max_{1\le k\le M}|v_k|, \quad \|v\|_p=\left(h\sum_{k=1}^{M}|v_k|^p\right)^{1/p} \quad (p\ge 1),$$

where $\overline{v_k}$ represents the complex conjugate of v_k. For simplicity of notation denote $\|u\| = \|u\|_2$. As usual, we use the standard difference operators as

$$\delta_x^+ v_k^n = \frac{v_{k+1}^n - v_k^n}{h}, \quad \delta_x^- v_k^n = \frac{v_k^n - v_{k-1}^n}{h}, \quad \delta_x^2 v_k^n = \frac{v_{k+1}^n - 2v_k^n + v_{k-1}^n}{h^2},$$
$$\delta_t^- v_k^n = \frac{v_k^n - v_k^{n-1}}{\tau}, \quad \delta_t^0 v_k^n = \frac{v_k^{n+1} - v_k^{n-1}}{2\tau}, \quad \delta_t^2 v_k^n = \frac{v_k^{n+1} - 2v_k^n + v_k^{n-1}}{\tau^2}.$$

For simplicity of notation, we denote

$$v_k^{n+\frac{1}{2}} = \frac{v_k^{n+1} + v_k^n}{2}, \qquad v_k^{\overline{n}} = \frac{v_k^{n+1} + v_k^{n-1}}{2}.$$

As we know, the standard central difference operator δ_x^2 approximates the second-order derivative as

$$\partial_x^2 v(x_k, t_n) = \delta_x^2 v_k^n + O(h^2).$$

Through a more detailed expansion, we obtain

$$\delta_x^2 v_k^n = \left(1 + \frac{h^2}{12}\partial_x^2\right) \partial_x^2 v(x_k, t_n) + O(h^4) = \left(1 + \frac{h^2}{12}\delta_x^2\right) \partial_x^2 v(x_k, t_n) + O(h^4),$$

which implies

$$\partial_x^2 v(x_k, t_n) = \mathcal{A}^{-1}\delta_x^2 v_k^n + O(h^4), \tag{2.3}$$

where $\mathcal{A} = I + \frac{h^2}{12}\delta_x^2$ with I being the identity. Based on the periodic condition, the operator $\mathcal{A}$ can also be written by a matrix:

$$A = \frac{1}{12}\begin{pmatrix} 10 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 10 & 1 & \cdots & 0 & 0 \\ & & \ddots & \ddots & \ddots & \\ 0 & 0 & \cdots & 1 & 10 & 1 \\ 1 & 0 & \cdots & 0 & 1 & 10 \end{pmatrix}_{M \times M}.$$

Now, we present the linearly-implicit compact finite difference scheme for the problem (2.1) as follows:

$$i\delta_t^- E_k^n + \mathcal{A}^{-1}\delta_x^2 E_k^{n-\frac{1}{2}} - \varepsilon^2 \mathcal{A}^{-2}\delta_x^4 E_k^{n-\frac{1}{2}} = N_k^{n-\frac{1}{2}} E_k^{n-\frac{1}{2}}, \tag{2.4}$$
$$\delta_t^2 N_k^n - \mathcal{A}^{-1}\delta_x^2 N_k^{\overline{n}} + \varepsilon^2 \mathcal{A}^{-2}\delta_x^4 N_k^{\overline{n}} = \mathcal{A}^{-1}\delta_x^2 \left|E_k^n\right|^2, \quad 1 \le k \le M, \quad 1 \le n < J, \tag{2.5}$$

where when involved, $v_k = v_{k+M}$ for $k \le 0$ and $v_k = v_{k-M}$ for $k > M$ in view of the periodic boundary conditions. Noticing that the real matrix A is symmetric positive definite, there exists a real symmetric positive definite matrix H, such that $H = A^{-1}$. Then (2.4)–(2.5) can be rewritten as the vector form

$$i\delta_t^- E^n + H\delta_x^2 E^{n-\frac{1}{2}} - \varepsilon^2 H^2 \delta_x^4 E^{n-\frac{1}{2}} - N^{n-\frac{1}{2}} E^{n-\frac{1}{2}} = 0, \tag{2.6}$$
$$\delta_t^2 N^n - H\delta_x^2 N^{\overline{n}} + \varepsilon^2 H^2 \delta_x^4 N^{\overline{n}} - H\delta_x^2 \left|E^n\right|^2 = 0, \quad 1 \le n \le J-1. \tag{2.7}$$

To complete the scheme, we need to assign the value for N^1. Suppose (2.7) is also valid for $n = 0$ and the "ghost" vector N^{-1} is given by

$$N_k^1 - N_k^{-1} = 2\tau N_1(x_k),$$

where N_k^{-1} is an approximation of $N(x_k, -\tau)$ which satisfies

$$N(x_k, \tau) - N(x_k, -\tau) = 2\tau \partial_t N(x_k, 0) + O(\tau^3),$$

then N^1 can be calculated by

$$\frac{2}{\tau^2}\left(N^1 - \tau \widetilde{N}_1 - N^0\right) - H\delta_x^2\left(N^1 - \tau\widetilde{N}_1\right) + \varepsilon^2 H^2 \delta_x^4\left(N^1 - \tau\widetilde{N}_1\right) - H\delta_x^2\left|E^0\right|^2 = 0, \tag{2.8}$$

with

$$\begin{aligned} E^0 &= \left(E_0(x_1), \ldots, E_0(x_M)\right)^T, \quad N^0 = \left(N_0(x_1), \ldots, N_0(x_M)\right)^T, \\ \widetilde{N}_1 &= \left(N_1(x_1), \ldots, N_1(x_M)\right)^T. \end{aligned} \tag{2.9}$$

It is worth noticing that the QZS (2.1) is nonlinearly coupled, while the resulting discrete scheme (2.6)–(2.9) is decoupled and solving the large-scale systems of nonlinear algebraic equations is successfully avoided, which makes the computational efficiency greatly improved.

3 Conservative Properties

In this section, the conservation properties are obtained for the difference scheme (2.6)–(2.7), which preserves the conservative properties (1.2)–(1.3) in discrete level. To do this, we need some useful lemmas.

Lemma 3.1 *For $u, v \in \mathbb{V}_h$, we have*

$$\left\langle \delta_x^2 u, v\right\rangle = -\left\langle \delta_x^+ u, \delta_x^+ v\right\rangle, \quad \left\langle \delta_x^4 u, v\right\rangle = \left\langle \delta_x^2 u, \delta_x^2 v\right\rangle.$$

Lemma 3.2 [41] *The operators $\mathcal{A}$ and $\mathcal{A}^{-1}$ (or equivalently the matrices A and H)are commutative with δ_x^+ and δ_x^- , i.e., for any grid function $u \in \mathbb{V}_h$,*

$$\delta_x^+ A u = A\delta_x^+ u, \quad \delta_x^- A u = A\delta_x^- u; \quad \delta_x^+ H u = H\delta_x^+ u, \quad \delta_x^- H u = H\delta_x^- u.$$

Denote $H = R^T R$ by the Cholesky decomposition of H. Then by applying Lemmas 3.1 and 3.2, we get

$$\left\langle H\delta_x^2 u, v\right\rangle = \left\langle \delta_x^2 H u, v\right\rangle = -\left\langle \delta_x^+ H u, \delta_x^+ v\right\rangle = -\left\langle R^T R\delta_x^+ u, \delta_x^+ v\right\rangle = -\left\langle R\delta_x^+ u, R\delta_x^+ v\right\rangle. \tag{3.1}$$

Lemma 3.3 [21] *For any $u \in \mathbb{V}_h$, it holds that*

$$C_0\|u\|^2 \le \langle Hu, u\rangle = \|Ru\|^2 \le C_1\|u\|^2, \quad C_0\|u\|^2 \le \|Hu\|^2 \le C_1\|u\|^2, \tag{3.2}$$

where C_0 and C_1 are two positive constants independent of h.

With the aid of the preceding lemmas, we can now prove the conservation properties of the difference scheme (2.6)–(2.7).

Theorem 3.1 *Suppose the initial function N_1 satisfies (2.2), then the difference scheme (2.6)–(2.7) conserves the discrete mass and energy, i.e.,*

$$\left\|E^n\right\| = \left\|E^0\right\|, \tag{3.3}$$

$$\begin{aligned} \mathcal{E}^n &= 2\left\|R\delta_x^+ E^n\right\|^2 + \left\|R\delta_x^+ U^n\right\|^2 + \frac{1}{2}\left(\left\|N^{n+1}\right\|^2 + \left\|N^n\right\|^2\right) + 2\varepsilon^2\left\|H\delta_x^2 E^n\right\|^2 \\ &\quad + \frac{\varepsilon^2}{2}\left(\left\|R\delta_x^+ N^{n+1}\right\|^2 + \left\|R\delta_x^+ N^n\right\|^2\right) + \left\langle |E^n|^2, 2N^{n+\frac{1}{2}}\right\rangle = \mathcal{E}^0, \end{aligned} \tag{3.4}$$

where $U^n \in \mathbb{V}_h$ is defined by

$$H\delta_x^2 U_k^n = \delta_t^- N_k^{n+1}, \quad 1 \le k \le M, \quad 0 \le n < J, \tag{3.5}$$

with boundary condition $U_M^n = 0$.

Proof Computing the inner product of (2.6) with $E^{n-\frac{1}{2}}$ yields

$$\langle i\delta_t^- E^n, E^{n-\frac{1}{2}}\rangle + \langle H\delta_x^2 E^{n-\frac{1}{2}}, E^{n-\frac{1}{2}}\rangle - \varepsilon^2\langle H^2\delta_x^4 E^{n-\frac{1}{2}}, E^{n-\frac{1}{2}}\rangle = \langle N^{n-\frac{1}{2}} E^{n-\frac{1}{2}}, E^{n-\frac{1}{2}}\rangle. \tag{3.6}$$

Noticing that

$$\langle i\delta_t^- E^n, E^{n-\frac{1}{2}}\rangle = \frac{i}{2\tau}\Big(\|E^n\|^2 - \|E^{n-1}\|^2\Big), \quad \langle H\delta_x^2 E^{n-\frac{1}{2}}, E^{n-\frac{1}{2}}\rangle = -\big\|R\delta_x^+ E^{n-\frac{1}{2}}\big\|^2,$$
$$\langle H^2\delta_x^4 E^{n-\frac{1}{2}}, E^{n-\frac{1}{2}}\rangle = \big\|H\delta_x^2 E^{n-\frac{1}{2}}\big\|^2, \qquad \langle N^{n-\frac{1}{2}} E^{n-\frac{1}{2}}, E^{n-\frac{1}{2}}\rangle = \langle N^{n-\frac{1}{2}}, \big|E^{n-\frac{1}{2}}\big|^2\rangle,$$

thus by taking the imaginary part of (3.6), we obtain

$$\big\|E^n\big\| = \big\|E^{n-1}\big\| = \cdots = \big\|E^0\big\|.$$

Next computing the inner product of (2.6) with $E^n - E^{n-1}$, and taking its real part, we have

$$\begin{aligned}&\mathrm{Re}\langle H\delta_x^2 E^{n-\frac{1}{2}}, E^n - E^{n-1}\rangle - \varepsilon^2\mathrm{Re}\langle H^2\delta_x^4 E^{n-\frac{1}{2}}, E^n - E^{n-1}\rangle\\ &= \frac{1}{4}\langle N^n + N^{n-1}, |E^n|^2 - |E^{n-1}|^2\rangle.\end{aligned}$$

According to (3.1), it holds that

$$\begin{aligned}\mathrm{Re}\left\langle H\delta_x^2 E^{n-\frac{1}{2}}, E^n - E^{n-1}\right\rangle &= -\mathrm{Re}\left\langle R\delta_x^+ E^{n-\frac{1}{2}}, R\delta_x^+(E^n - E^{n-1})\right\rangle\\ &= \frac{1}{2}\left(\|R\delta_x^+ E^{n-1}\|^2 - \|R\delta_x^+ E^n\|^2\right).\end{aligned}$$

Noticing H is symmetric and commutative with δ_x^2, this together with Lemma 3.1 yields that

$$\mathrm{Re}\left\langle H^2\delta_x^4 E^{n-\frac{1}{2}}, E^n - E^{n-1}\right\rangle = \frac{1}{2}\Big(\big\|H\delta_x^2 E^n\big\|^2 - \big\|H\delta_x^2 E^{n-1}\big\|^2\Big).$$

Then, we can obtain

$$\begin{aligned}&2\Big(\big\|R\delta_x^+ E^n\big\|^2 - \big\|R\delta_x^+ E^{n-1}\big\|^2\Big) + 2\varepsilon^2\Big(\big\|H\delta_x^2 E^n\big\|^2 - \big\|H\delta_x^2 E^{n-1}\big\|^2\Big)\\ &= -2\Big\langle N^{n-\frac{1}{2}}, |E^n|^2 - |E^{n-1}|^2\Big\rangle.\end{aligned} \tag{3.7}$$

Taking the inner product of (2.7) with $U^{n-\frac{1}{2}}$ leads to

$$\left\langle \delta_t^2 N^n, U^{n-\frac{1}{2}}\right\rangle - \left\langle H\delta_x^2 N^{\overline{n}}, U^{n-\frac{1}{2}}\right\rangle + \varepsilon^2\left\langle H^2\delta_x^4 N^{\overline{n}}, U^{n-\frac{1}{2}}\right\rangle = \left\langle H\delta_x^2\left|E^n\right|^2, U^{n-\frac{1}{2}}\right\rangle. \tag{3.8}$$

In view of (2.7), (2.8) and the compatibility condition (2.2), we see that

$$\sum_{k=1}^{M} \delta_t^- N_k^{n+1} = 0, \quad n \ge 0, \tag{3.9}$$

this enables (3.5) to be well-defined for $U^n \in \mathbb{V}_h$. For the details, we refer to [24]. It follows from (3.5) that $H\delta_x^2 U^{n-\frac{1}{2}} = \delta_t^- N^{n+\frac{1}{2}} = \delta_t^0 N^n$. Then, we can easily prove that

$$\begin{aligned}
&\langle \delta_t^2 N^n, U^{n-\frac{1}{2}} \rangle = \frac{1}{\tau}\langle H\delta_x^2 U^n - H\delta_x^2 U^{n-1}, U^{n-\frac{1}{2}} \rangle = -\frac{1}{2\tau}\Big(\| R\delta_x^+ U^n \|^2 - \| R\delta_x^+ U^{n-1} \|^2 \Big),\\
&\langle H\delta_x^2 N^{\overline{n}}, U^{n-\frac{1}{2}} \rangle = \langle N^{\overline{n}}, H\delta_x^2 U^{n-\frac{1}{2}} \rangle = \frac{1}{4\tau}\Big(\| N^{n+1} \|^2 - \| N^{n-1} \|^2 \Big),\\
&\langle H^2\delta_x^4 N^{\overline{n}}, U^{n-\frac{1}{2}} \rangle = \langle H\delta_x^2 N^{\overline{n}}, H\delta_x^2 U^{n-\frac{1}{2}} \rangle = \frac{1}{4\tau}\Big(\| R\delta_x^+ N^{n-1} \|^2 - \| R\delta_x^+ N^{n+1} \|^2 \Big),\\
&\langle H\delta_x^2 |E^n|^2, U^{n-\frac{1}{2}} \rangle = \langle |E^n|^2, H\delta_x^2 U^{n-\frac{1}{2}} \rangle = \langle \delta_t^0 N^n, |E^n|^2 \rangle.
\end{aligned} \tag{3.10}$$

Substituting (3.10) into (3.8), we get

$$\begin{aligned}
&\| R\delta_x^+ U^n \|^2 - \| R\delta_x^+ U^{n-1} \|^2 + \frac{1}{2}\Big(\| N^{n+1} \|^2 - \| N^{n-1} \|^2 \Big)\\
&\quad + \frac{\varepsilon^2}{2}\Big(\| R\delta_x^+ N^{n+1} \|^2 - \| R\delta_x^+ N^{n-1} \|^2 \Big)\\
&\quad = -2\tau\langle \delta_t^0 N^n, \big|E^n\big|^2 \rangle.
\end{aligned} \tag{3.11}$$

Summing (3.7) and (3.11), noticing that

$$\Big\langle 2N^{n-\frac{1}{2}}, |E^n|^2 - |E^{n-1}|^2 \Big\rangle + 2\tau \langle \delta_t^0 N^n, |E^n|^2 \rangle = \Big\langle |E^n|^2, 2N^{n+\frac{1}{2}} \Big\rangle - \Big\langle |E^{n-1}|^2, 2N^{n-\frac{1}{2}} \Big\rangle. \tag{3.12}$$

hence (3.4) follows and the proof is completed. □

Remark 3.1 Actually the conservation properties can be established for more generalized QZS with the form as follows:

$$\begin{aligned}
&iE_t + E_{xx} - \varepsilon^2 \partial_x^4 E - Nf(|E|^2)E = 0,\\
&N_{tt} - N_{xx} + \varepsilon^2 \partial_x^4 N - \partial_x^2 (F(|E|^2)) = 0, \quad x \in (a,b), \quad t > 0,
\end{aligned}$$

where f is a smooth, real function, $F(s) = \int_0^s f(\theta)d\theta$. Correspondingly, the compact finite difference scheme is given by

$$\begin{aligned}
&i\delta_t^- E_k^n + \mathcal{A}^{-1}\delta_x^2 E_k^{n-\frac{1}{2}} - \varepsilon^2 \mathcal{A}^{-2}\delta_x^4 E_k^{n-\frac{1}{2}} = N_k^{n-\frac{1}{2}} E_k^{n-\frac{1}{2}} \frac{F(|E_k^n|^2) - F(|E_k^{n-1}|^2)}{|E_k^n|^2 - |E_k^{n-1}|^2},\\
&\delta_t^2 N_k^n - \mathcal{A}^{-1}\delta_x^2 N_k^{\overline{n}} + \varepsilon^2 \mathcal{A}^{-2}\delta_x^4 N_k^{\overline{n}} = \mathcal{A}^{-1}\delta_x^2 F(|E_k^n|^2), \quad 1 \le k \le M, \quad 1 \le n < J,
\end{aligned}$$

and the conserved energy is of the form

$$\begin{aligned}
\mathcal{E}^n = {}& 2\| R\delta_x^+ E^n \|^2 + \| R\delta_x^+ U^n \|^2 + \frac{1}{2}\Big(\| N^{n+1} \|^2 + \| N^n \|^2 \Big) + 2\varepsilon^2 \| H\delta_x^2 E^n \|^2\\
&+ \frac{\varepsilon^2}{2}\Big(\| R\delta_x^+ N^{n+1} \|^2 + \| R\delta_x^+ N^n \|^2 \Big) + \langle F(|E^n|^2), 2N^{n+\frac{1}{2}} \rangle.
\end{aligned}$$

4 Convergence

In this section, we first present an a priori bound result, and obtain the corresponding convergence result of the scheme (2.6)–(2.9) for solving the QZS. We first give the following Lemma.

Lemma 4.1 [47] (Discrete Sobolev inequality) *Suppose that $v \in \mathbb{V}_h$. There exists a constant $C_2 > 0$ independent of v and h such that*

$$\|v\|_\infty \le C_2 \sqrt{\|v\|}\sqrt{\|\delta_x^+ v\| + \|v\|}.$$

Now, we will present an a priori bound of the difference solution for the scheme (2.6)–(2.9).

Theorem 4.1 *Suppose the initial data N_1 satisfies the compatibility condition* (2.2). *Then the difference solution of* (2.6)–(2.9) *satisfies the following estimates*

$$\left\|E^n\right\| + \|\delta_x^+ E^n\| + \left\|E^n\right\|_\infty \le C; \quad \left\|N^n\right\| + \varepsilon\|\delta_x^+ N^n\| \le C; \quad \left\|\delta_x^+ U^n\right\| + \left\|U^n\right\|_\infty \le C. \tag{4.1}$$

Proof It follows from (3.3) that $\|E^n\| \le C$. Using the discrete Cauchy–Schwarz inequality and the Young's inequality, we arrive at

$$\left|\left\langle \left|E^n\right|^2, N^{n+1} + N^n \right\rangle\right| \le \frac{1}{4}\left(\|N^{n+1}\|^2 + \|N^n\|^2\right) + 2\|E^n\|_4^4. \tag{4.2}$$

According to Lemma 4.1, the Young's inequality and Lemma 3.3, we have

$$\begin{aligned} \|E^n\|_4^4 &\le \|E^n\|^2\|E^n\|_\infty^2 \le C\|E^n\|_\infty^2 \le C\|E^n\|\left(\|E^n\| + \|\delta_x^+ E^n\|\right) \\ &\le \frac{C_0}{2}\|\delta_x^+ E^n\|^2 + C\|E^n\|^2 \le \frac{\|R\delta_x^+ E^n\|^2}{2} + C\|E^0\|^2. \end{aligned} \tag{4.3}$$

This together with (3.4) and (4.2) gives

$$\begin{aligned} &\left\|R\delta_x^+ E^n\right\|^2 + \left\|R\delta_x^+ U^n\right\|^2 + \frac{1}{4}\left(\left\|N^n\right\|^2 + \left\|N^{n+1}\right\|^2\right) \\ &+ 2\varepsilon^2\left\|H\delta_x^2 E^n\right\|^2 + \frac{\varepsilon^2}{2}\left(\left\|R\delta_x^+ N^{n+1}\right\|^2 + \left\|R\delta_x^+ N^n\right\|^2\right) \le \mathcal{E}^0 + C\|E^0\|^2, \end{aligned} \tag{4.4}$$

which directly yields

$$\left\|N^n\right\| \le C, \quad \left\|R\delta_x^+ E^n\right\| \le C, \quad \left\|R\delta_x^+ U^n\right\| \le C, \quad \varepsilon\left\|R\delta_x^+ N^n\right\| \le C.$$

Hence by Lemma 3.3, we arrive at

$$\|\delta_x^+ E^n\| \le C, \quad \varepsilon\|\delta_x^+ N^n\| \le C, \quad \|\delta_x^+ U^n\| \le C.$$

Applying the discrete Sobolev inequality (cf. Lemma 4.1) to $E^n \in \mathbb{V}_h$, we get

$$\|E^n\|_\infty \le C_2\sqrt{\|E^n\|}\sqrt{\|E^n\| + \|\delta_x^+ E^n\|} \le C.$$

Recalling $U_M^n = 0$, using the discrete Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} |U_k^n| &= \left|\sum_{s=k}^{M-1}(U_s^n - U_{s+1}^n) + U_M^n\right| \\ &= h\left|\sum_{s=k}^{M-1}\delta_x^+ U_s^n\right| \le (b-a)^{1/2}\|\delta_x^+ U^n\| \le C, \quad 1 \le k < M, \end{aligned} \tag{4.5}$$

which completes the proof. □

Next, we turn to prove the convergence of the difference solution of the linearly-implicit compact difference scheme (2.6)–(2.9). The pointwise errors are defined by

$$e_k^n = E(x_k, t_n) - E_k^n, \qquad \eta_k^n = N(x_k, t_n) - N_k^n.$$

Consider the exact solution of the system (2.1) on the grid points, then the error equations are obtained as

$$i\delta_t^- e^n + H\delta_x^2 e^{n-\frac{1}{2}} - \varepsilon^2 H^2 \delta_x^4 e^{n-\frac{1}{2}} = \widetilde{w}^n + \xi^n, \tag{4.6}$$
$$\delta_t^2 \eta^n - H\delta_x^2 \eta^{\overline{n}} + \varepsilon^2 H^2 \delta_x^4 \eta^{\overline{n}} = H\delta_x^2 \widehat{w}^n + \sigma^n, \quad n \geq 1, \tag{4.7}$$

where $\widetilde{w}^n$, $\widehat{w}^n$ represent the errors for the nonlinearity

$$\begin{aligned}
\widetilde{w}_k^n &= \frac{1}{4}(N(x_k, t_n) + N(x_k, t_{n-1}))(E(x_k, t_n) + E(x_k, t_{n-1})) - N_k^{n-\frac{1}{2}} E_k^{n-\frac{1}{2}} \\
&= \frac{1}{2}(N(x_k, t_n) + N(x_k, t_{n-1}))e_k^{n-\frac{1}{2}} + \eta_k^{n-\frac{1}{2}} E_k^{n-\frac{1}{2}}, \\
\widehat{w}_k^n &= |E(x_k, t_n)|^2 - \left|E_k^n\right|^2,
\end{aligned}$$

and $\xi^n, \sigma^n \in \mathbb{V}_h$ are the truncation errors

$$\begin{aligned}
\xi_k^n = {} & i\delta_t^- E(x_k, t_n) + \frac{1}{2}H\delta_x^2(E(x_k, t_n) + E(x_k, t_{n-1})) - \frac{\varepsilon^2}{2}H^2\delta_x^4(E(x_k, t_n) + E(x_k, t_{n-1})) \\
& - \frac{1}{4}(N(x_k, t_n) + N(x_k, t_{n-1}))(E(x_k, t_n) + E(x_k, t_{n-1})),
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
\sigma_k^n = {} & \delta_t^2 N(x_k, t_n) - \frac{1}{2}H\delta_x^2(N(x_k, t_{n+1}) + N(x_k, t_{n-1})) \\
& + \frac{\varepsilon^2}{2}H^2\delta_x^4(N(x_k, t_{n+1}) + N(x_k, t_{n-1})) - H\delta_x^2 |E(x_k, t_n)|^2.
\end{aligned} \tag{4.9}$$

Applying Taylor's expansion, one easily finds when the solutions are smooth enough, the location truncation errors satisfy

$$\begin{aligned}
\xi_k^n &= O(\tau^2 + h^4), \quad \delta_x^+ \xi_k^n = O(\tau^2 + h^4), \\
\delta_x^2 \xi_k^n &= O(\tau^2 + h^4), \quad \sigma_k^n = O(\tau^2 + h^4), \quad n \geq 1.
\end{aligned}$$

Theorem 4.2 *Suppose $\varepsilon \leq 1$ and the solutions $E(x,t)$, $N(x,t)$ are sufficiently smooth with initial conditions satisfying the compatibility condition (2.2). Then the solution (E_k^n, N_k^n) of the scheme (2.6)–(2.9) converges to the solution $(E(x_k, t_n), N(x_k, t_n))$ of the QZS (2.1) with order $O(\tau^2 + h^4)$:*

$$\|e^n\| + \|\delta_x^+ e^n\| + \varepsilon\left\|\delta_x^2 e^n\right\| \leq C\big(\tau^2 + h^4\big), \quad \|\eta^n\| + \varepsilon\|\delta_x^+ \eta^n\| \leq C\big(\tau^2 + h^4\big), \tag{4.10}$$

where C is dependent of h, τ and ε.

Proof First of all, computing the inner product of (4.6) with $e^n + e^{n-1}$, and taking the imaginary part yields

$$\frac{1}{\tau}\Big(\|e^n\|^2 - \|e^{n-1}\|^2\Big) = \mathrm{Im}\big\langle\widetilde{w}^n, e^n + e^{n-1}\big\rangle + \mathrm{Im}\big\langle\xi^n, e^n + e^{n-1}\big\rangle. \tag{4.11}$$

In view of Theorem 4.1, one easily gets

$$\left|\widetilde{w}_k^n\right| \leq C\left(|e_k^n| + |e_k^{n-1}| + |\eta_k^n| + |\eta_k^{n-1}|\right).$$

Applying the discrete Cauchy–Schwarz inequality, we easily obtain the following estimates

$$\left|\mathrm{Im}\langle \widetilde{w}^n, e^n + e^{n-1}\rangle\right| \le C\Big(\|\eta^n\|^2 + \|\eta^{n-1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2\Big), \tag{4.12}$$

$$\left|\mathrm{Im}\langle \xi^n, e^n + e^{n-1}\rangle\right| \le C\big(\tau^2 + h^4\big)^2 + C\Big(\|e^n\|^2 + \|e^{n-1}\|^2\Big). \tag{4.13}$$

Thus, substituting (4.12) and (4.13) into (4.11), we get

$$\|e^n\|^2 - \|e^{n-1}\|^2 \le C\tau\Big(\|e^n\|^2 + \|e^{n-1}\|^2 + \|\eta^n\|^2 + \|\eta^{n-1}\|^2\Big) + C\tau\big(\tau^2 + h^4\big)^2. \tag{4.14}$$

Secondly, computing the inner product of (4.6) with $\delta_t^- e^n$, and taking the real part, we get

$$\begin{aligned} &-\frac{1}{2\tau}\Big(\|R\delta_x^+ e^n\|^2 - \|R\delta_x^+ e^{n-1}\|^2\Big) - \frac{\varepsilon^2}{2\tau}\Big(\big\|H\delta_x^2 e^n\big\|^2 - \big\|H\delta_x^2 e^{n-1}\big\|^2\Big) \\ &= \mathrm{Re}\langle \widetilde{w}^n, \delta_t^- e^n\rangle + \mathrm{Re}\langle \xi^n, \delta_t^- e^n\rangle. \end{aligned} \tag{4.15}$$

Based on (4.6), applying Lemma 3.3, we derive

$$\begin{aligned} \left|\mathrm{Re}\langle \xi^n, \delta_t^- e^n\rangle\right| &= \left|\mathrm{Re}\Big\langle \xi^n, iH\delta_x^2 e^{n-\frac{1}{2}} - i\varepsilon^2 H^2\delta_x^4 e^{n-\frac{1}{2}} - i\widetilde{w}^n - i\xi^n\Big\rangle\right| \\ &\le \left|\Big\langle R\delta_x^+\xi^n, R\delta_x^+ e^{n-\frac{1}{2}}\Big\rangle\right| + \varepsilon^2\left|\langle H\delta_x^2\xi^n, H\delta_x^2 e^{n-\frac{1}{2}}\rangle\right| + \left|\langle \xi^n, \widetilde{w}^n\rangle\right| \\ &\le C\,\Big(\big\|R\delta_x^+ e^n\big\|^2 + \big\|R\delta_x^+ e^{n-1}\big\|^2 + \big\|R\delta_x^+\xi^n\big\|^2 + \varepsilon^2\big\|H\delta_x^2\xi^n\big\|^2 \\ &\quad +\varepsilon^2\big\|H\delta_x^2 e^n\big\|^2 + \varepsilon^2\big\|H\delta_x^2 e^{n-1}\big\|^2 + \big\|\xi^n\big\|^2 + \big\|\widetilde{w}^n\big\|^2\Big) \\ &\le C(\tau^2 + h^4)^2 + C\,\Big(\big\|e^n\big\|^2 + \big\|e^{n-1}\big\|^2 + \big\|\eta^n\big\|^2 + \big\|\eta^{n-1}\big\|^2 \\ &\quad +\big\|R\delta_x^+ e^n\big\|^2 + \big\|R\delta_x^+ e^{n-1}\big\|^2 + \varepsilon^2\big\|H\delta_x^2 e^n\big\|^2 + \varepsilon^2\big\|H\delta_x^2 e^{n-1}\big\|^2\Big). \end{aligned} \tag{4.16}$$

To estimate $\mathrm{Re}\langle \widetilde{w}^n, \delta_t^- e^n\rangle$, we denote $\widetilde{w}_k^n = \widetilde{w}_k^{1,n} + \widetilde{w}_k^{2,n}$ with

$$\widetilde{w}_k^{1,n} = \frac{1}{2}\left(N(x_k, t_n) + N(x_k, t_{n-1})\right) e_k^{n-\frac{1}{2}}, \quad \widetilde{w}_k^{2,n} = \eta_k^{n-\frac{1}{2}} E_k^{n-\frac{1}{2}}.$$

Suppose N is smooth enough, then we have

$$\begin{aligned} \left|\widetilde{w}_k^{1,n}\right| &\le C\left(|e_k^n| + \left|e_k^{n-1}\right|\right), \\ \left|\delta_x^+\widetilde{w}_k^{1,n}\right| &= \frac{1}{2}\left|(N(x_k, t_n) + N(x_k, t_{n-1}))\,\delta_x^+ e_k^{n-\frac{1}{2}} + e_{k+1}^{n-\frac{1}{2}}\left(\delta_x^+ N(x_k, t_n) + \delta_x^+ N(x_k, t_{n-1})\right)\right| \\ &\le \|N\|_{L^\infty}\left|\delta_x^+ e_k^{n-\frac{1}{2}}\right| + \|N_x\|_{L^\infty}\left|e_{k+1}^{n-\frac{1}{2}}\right| \\ &\le C\left(\left|e_{k+1}^n\right| + \left|e_{k+1}^{n-1}\right| + \left|\delta_x^+ e_k^n\right| + \left|\delta_x^+ e_k^{n-1}\right|\right), \\ \left|\delta_x^2\widetilde{w}_k^{1,n}\right| &= \frac{1}{2}\Big|\,(N(x_{k+1}, t_n) + N(x_{k+1}, t_{n-1}))\,\delta_x^2 e_k^{n-\frac{1}{2}} + 2\delta_x^- e_k^{n-\frac{1}{2}}\delta_x^+ N(x_k, t_n) \\ &\quad + 2\delta_x^- e_k^{n-\frac{1}{2}}\delta_x^+ N(x_k, t_{n-1}) + e_{k-1}^{n-\frac{1}{2}}\left(\delta_x^2 N(x_k, t_n) + \delta_x^2 N(x_k, t_{n-1})\right)\Big| \\ &\le \|N\|_{L^\infty}\left|\delta_x^2 e_k^{n-\frac{1}{2}}\right| + 2\|N_x\|_{L^\infty}\left|\delta_x^- e_k^{n-\frac{1}{2}}\right| + \|N_{xx}\|_{L^\infty}\left|e_{k-1}^{n-\frac{1}{2}}\right| \\ &\le C\left(\left|e_{k-1}^n\right| + \left|e_{k-1}^{n-1}\right| + \left|\delta_x^+ e_{k-1}^n\right| + \left|\delta_x^+ e_{k-1}^{n-1}\right| + \left|\delta_x^2 e_k^n\right| + \left|\delta_x^2 e_k^{n-1}\right|\right). \end{aligned}$$

Hence

$$
\begin{aligned}
\left|\mathrm{Re}\left\langle\widetilde{w}^{1,n},\delta_t^- e^n\right\rangle\right| &= \left|\mathrm{Re}\left\langle\widetilde{w}^{1,n}, iH\delta_x^2 e^{n-\frac{1}{2}} - i\varepsilon^2 H^2\delta_x^4 e^{n-\frac{1}{2}} - i\widetilde{w}^n - i\xi^n\right\rangle\right|\\
&\le \left|\left\langle R\delta_x^+\widetilde{w}^{1,n}, R\delta_x^+ e^{n-\frac{1}{2}}\right\rangle\right| + \varepsilon^2\left|\left\langle H\delta_x^2\widetilde{w}^{1,n}, H\delta_x^2 e^{n-\frac{1}{2}}\right\rangle\right|\\
&\quad + \left|\left\langle\widetilde{w}^{1,n},\widetilde{w}^n\right\rangle\right| + \left|\left\langle\widetilde{w}^{1,n},\xi^n\right\rangle\right|\\
&\le C\left(\left\|R\delta_x^+ e^n\right\|^2 + \left\|R\delta_x^+ e^{n-1}\right\|^2 + \left\|\delta_x^+\widetilde{w}^{1,n}\right\|^2 + \varepsilon^2\left\|H\delta_x^2 e^n\right\|^2\right.\\
&\quad \left. +\varepsilon^2\left\|H\delta_x^2 e^{n-1}\right\|^2 + \varepsilon^2\left\|\delta_x^2\widetilde{w}^{1,n}\right\|^2 + \left\|\xi^n\right\|^2 + \left\|\widetilde{w}^n\right\|^2 + \left\|\widetilde{w}^{1,n}\right\|^2\right)\\
&\le C(h^4+\tau^2)^2 + C\left(\left\|e^n\right\|^2 + \left\|e^{n-1}\right\|^2 + \left\|\eta^n\right\|^2 + \left\|\eta^{n-1}\right\|^2\right.\\
&\quad \left. +\left\|R\delta_x^+ e^n\right\|^2 + \left\|R\delta_x^+ e^{n-1}\right\|^2 + \varepsilon^2\left\|H\delta_x^2 e^n\right\|^2 + \varepsilon^2\left\|H\delta_x^2 e^{n-1}\right\|^2\right).
\end{aligned}
\tag{4.17}
$$

On the other hand,

$$
\begin{aligned}
\mathrm{Re}\left\langle\widetilde{w}^{2,n},\delta_t^- e^n\right\rangle &= \frac{1}{\tau}\mathrm{Re}\left\langle\eta^{n-\frac{1}{2}}E^{n-\frac{1}{2}}, e^n - e^{n-1}\right\rangle\\
&= \frac{1}{\tau}\mathrm{Re}\left\langle\eta^{n-\frac{1}{2}}E^{n-\frac{1}{2}}, E(\cdot,t_n) - E(\cdot,t_{n-1})\right\rangle - \frac{1}{\tau}\mathrm{Re}\left\langle\eta^{n-\frac{1}{2}}E^{n-\frac{1}{2}}, E^n - E^{n-1}\right\rangle\\
&= \frac{1}{\tau}\mathrm{Re}\left\langle\eta^{n-\frac{1}{2}}\left(\frac{E(\cdot,t_n)+E(\cdot,t_{n-1})}{2} - e^{n-\frac{1}{2}}\right), E(\cdot,t_n) - E(\cdot,t_{n-1})\right\rangle\\
&\quad - \frac{1}{2\tau}\left\langle\eta^{n-\frac{1}{2}}, |E^n|^2 - |E^{n-1}|^2\right\rangle\\
&= \frac{1}{2\tau}\left\langle\eta^{n-\frac{1}{2}}, |E(\cdot,t_n)|^2 - |E(\cdot,t_{n-1})|^2 - |E^n|^2 + |E^{n-1}|^2\right\rangle\\
&\quad - \frac{1}{\tau}\mathrm{Re}\left\langle\eta^{n-\frac{1}{2}}e^{n-\frac{1}{2}}, E(\cdot,t_n) - E(\cdot,t_{n-1})\right\rangle\\
&= \frac{1}{2\tau}\left\langle\eta^{n-\frac{1}{2}}, \widehat{w}^n - \widehat{w}^{n-1}\right\rangle - \frac{1}{\tau}\mathrm{Re}\left\langle\eta^{n-\frac{1}{2}}e^{n-\frac{1}{2}}, E(\cdot,t_n) - E(\cdot,t_{n-1})\right\rangle,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\left|\mathrm{Re}\left\langle\widetilde{w}^{2,n},\delta_t^- e^n\right\rangle - \frac{1}{2\tau}\left\langle\eta^{n-\frac{1}{2}},\widehat{w}^n - \widehat{w}^{n-1}\right\rangle\right| \le \frac{1}{2}\|E_t\|_{L^\infty}\left(\left\|\eta^{n-\frac{1}{2}}\right\|^2 + \left\|e^{n-\frac{1}{2}}\right\|^2\right)\\
&\quad \le C\left(\left\|\eta^n\right\|^2 + \left\|\eta^{n-1}\right\|^2 + \left\|e^n\right\|^2 + \left\|e^{n-1}\right\|^2\right).
\end{aligned}
\tag{4.18}
$$

Combining (4.15)–(4.18), we arrive at

$$
\begin{aligned}
&\left\|R\delta_x^+ e^n\right\|^2 - \left\|R\delta_x^+ e^{n-1}\right\|^2 + \varepsilon^2\left\|H\delta_x^2 e^n\right\|^2 - \varepsilon^2\left\|H\delta_x^2 e^{n-1}\right\|^2 + \left\langle\eta^{n-\frac{1}{2}},\widehat{w}^n - \widehat{w}^{n-1}\right\rangle\\
&\quad \le C\tau(h^4+\tau^2)^2 + C\tau\left(\left\|e^n\right\|^2 + \left\|e^{n-1}\right\|^2 + \left\|\eta^n\right\|^2 + \left\|\eta^{n-1}\right\|^2\right.\\
&\qquad \left. +\left\|R\delta_x^+ e^n\right\|^2 + \left\|R\delta_x^+ e^{n-1}\right\|^2 + \varepsilon^2\left\|H\delta_x^2 e^n\right\|^2 + \varepsilon^2\left\|H\delta_x^2 e^{n-1}\right\|^2\right).
\end{aligned}
\tag{4.19}
$$

In view of the compatibility condition (2.2), the second equation in (2.1) and the periodic boundary conditions, one can derive that (cf. [24])

$$
\sum_{k=1}^{M}\left(N(x_k,t_{n+1}) - N(x_k,t_n)\right) = 0, \quad n \ge 0,
$$

which together with (3.9) yields $\sum_{k=1}^{M} \delta_t^- \eta_k^{n+1} = 0$ for $n \geq 0$. Hence following (3.5), we are able to define $\theta^n \in \mathbb{V}_h$ for $0 \leq n < J$ by

$$H\delta_x^2 \theta_k^n = \delta_t^- \eta_k^{n+1}, \quad k = 1, 2, \dots, M; \quad \theta_M^n = 0. \tag{4.20}$$

Taking the inner product of (4.7) with $\theta^{n-\frac{1}{2}}$ gives

$$\langle \delta_t^2 \eta^n, \theta^{n-\frac{1}{2}} \rangle - \langle H\delta_x^2 \eta^{\overline{n}}, \theta^{n-\frac{1}{2}} \rangle + \varepsilon^2 \langle H^2 \delta_x^4 \eta^{\overline{n}}, \theta^{n-\frac{1}{2}} \rangle = \langle H\delta_x^2 \widehat{w}^n, \theta^{n-\frac{1}{2}} \rangle + \langle \sigma^n, \theta^{n-\frac{1}{2}} \rangle. \tag{4.21}$$

Applying Lemma 3.1, (3.1), one easily obtains

$$\begin{aligned} \langle \delta_t^2 \eta^n, \theta^{n-\frac{1}{2}} \rangle &= -\frac{1}{2\tau}\Big(\| R\delta_x^+ \theta^n \|^2 - \| R\delta_x^+ \theta^{n-1} \|^2 \Big), \\ \langle H\delta_x^2 \eta^{\overline{n}}, \theta^{n-\frac{1}{2}} \rangle &= \frac{1}{4\tau}\Big(\| \eta^{n+1} \|^2 - \| \eta^{n-1} \|^2 \Big), \\ \langle H^2 \delta_x^4 \eta^{\overline{n}}, \theta^{n-\frac{1}{2}} \rangle &= -\frac{1}{4\tau}\Big(\| R\delta_x^+ \eta^{n+1} \|^2 - \| R\delta_x^+ \eta^{n-1} \|^2 \Big). \end{aligned} \tag{4.22}$$

It follows from Lemmas 3.1 and 3.2 that

$$\langle H\delta_x^2 \widehat{w}^n, \theta^{n-\frac{1}{2}} \rangle = \langle \widehat{w}^n, H\delta_x^2 \theta^{n-\frac{1}{2}} \rangle = \langle \widehat{w}^n, \delta_t^0 \eta^n \rangle. \tag{4.23}$$

In view of the homogeneous boundary condition of θ^n, similar to (4.5), it can be seen that $\|\theta^n\| \leq C\|\delta_x^+ \theta^n\|$, this yields

$$\begin{aligned} \left| \langle \sigma^n, \theta^{n-\frac{1}{2}} \rangle \right| &\leq C\left(\| \theta^n \|^2 + \| \theta^{n-1} \|^2 \right) + C\| \sigma^n \|^2 \\ &\leq C\big(\tau^2 + h^4\big)^2 + C\Big(\| R\delta_x^+ \theta^n \|^2 + \| R\delta_x^+ \theta^{n-1} \|^2 \Big). \end{aligned} \tag{4.24}$$

Thus, plugging (4.22), (4.23) and (4.24) into (4.21), we obtain

$$\begin{aligned} &\| R\delta_x^+ \theta^n \|^2 - \| R\delta_x^+ \theta^{n-1} \|^2 + \frac{1}{2}\left(\| \eta^{n+1} \|^2 - \| \eta^{n-1} \|^2 \right) \\ &+ \frac{\varepsilon^2}{2}\left(\| R\delta_x^+ \eta^{n+1} \|^2 - \| R\delta_x^+ \eta^{n-1} \|^2 \right) + \langle \widehat{w}^n, 2\tau\delta_t^0 \eta^n \rangle \\ &\leq C\tau\Big(\| R\delta_x^+ \theta^n \|^2 + \| R\delta_x^+ \theta^{n-1} \|^2 \Big) + C\tau\big(\tau^2 + h^4\big)^2. \end{aligned} \tag{4.25}$$

Based on Theorem 4.1, $\|E^n\|_\infty$ is bounded. Suppose

$$\|E^n\|_\infty \leq C_E, \quad 0 \leq n \leq J,$$

where C_E is independent of h, τ and ε. Denote

$$M_E := C_E + \|E\|_{L^\infty([0,T];L^\infty(\Omega))}.$$

Noticing that

$$\langle 2\eta^{n-\frac{1}{2}}, \widehat{w}^n - \widehat{w}^{n-1} \rangle + \langle \widehat{w}^n, 2\tau\delta_t^0 \eta^n \rangle = \langle \widehat{w}^n, 2\eta^{n+\frac{1}{2}} \rangle - \langle \widehat{w}^{n-1}, 2\eta^{n-\frac{1}{2}} \rangle,$$

combining $4M_E^2\cdot$(4.14)+2$\cdot$(4.19)+(4.25), we are led to

$$
\begin{aligned}
&4M_E^2\left(\|e^n\|^2-\|e^{n-1}\|^2\right)+\frac{1}{2}\left(\|\eta^{n+1}\|^2-\|\eta^{n-1}\|^2\right)+2\left(\|R\delta_x^+e^n\|^2-\|R\delta_x^+e^{n-1}\|^2\right)\\
&\quad+\|R\delta_x^+\theta^n\|^2-\|R\delta_x^+\theta^{n-1}\|^2+\frac{\varepsilon^2}{2}\left(\|R\delta_x^+\eta^{n+1}\|^2-\|R\delta_x^+\eta^{n-1}\|^2\right)\\
&\quad+2\varepsilon^2\left(\|H\delta_x^2e^n\|^2-\|H\delta_x^2e^{n-1}\|^2\right)+\langle\widehat{w}^n,2\eta^{n+\frac{1}{2}}\rangle-\langle\widehat{w}^{n-1},2\eta^{n-\frac{1}{2}}\rangle\\
&\le C\tau(h^4+\tau^2)^2+C\tau\left(\|e^n\|^2+\|e^{n-1}\|^2+\|\eta^n\|^2+\|\eta^{n-1}\|^2+\|R\delta_x^+e^n\|^2+\|R\delta_x^+e^{n-1}\|^2\right.\\
&\quad\left.+\|R\delta_x^+\theta^n\|^2+\|R\delta_x^+\theta^{n-1}\|^2+\varepsilon^2\|H\delta_x^2e^n\|^2+\varepsilon^2\|H\delta_x^2e^{n-1}\|^2\right).
\end{aligned}\tag{4.26}
$$

Denote

$$
\begin{aligned}
\mathcal{F}^n&=4M_E^2\|e^n\|^2+\frac{1}{2}\left(\|\eta^n\|^2+\|\eta^{n+1}\|^2\right)+2\|R\delta_x^+e^n\|^2+\|R\delta_x^+\theta^n\|^2\\
&\quad+\frac{\varepsilon^2}{2}\left(\|R\delta_x^+\eta^n\|^2+\|R\delta_x^+\eta^{n+1}\|^2\right)+2\varepsilon^2\|H\delta_x^2e^n\|^2,\quad n\ge 0.
\end{aligned}
$$

Then (4.26) can be rewritten as

$$
\mathcal{F}^n-\mathcal{F}^{n-1}+\langle\widehat{w}^n,2\eta^{n+\frac{1}{2}}\rangle-\langle\widehat{w}^{n-1},2\eta^{n-\frac{1}{2}}\rangle\le C\tau(h^4+\tau^2)^2+C\tau\big(\mathcal{F}^n+\mathcal{F}^{n-1}\big).
$$

Summing the above equation for $n=1,2,\ldots,m\le J-1$, we get

$$
\mathcal{F}^m-\mathcal{F}^0+\langle\widehat{w}^m,2\eta^{m+\frac{1}{2}}\rangle-\langle\widehat{w}^0,2\eta^{\frac{1}{2}}\rangle\le C(\tau^2+h^4)^2+C\tau\sum_{n=0}^{m}\mathcal{F}^n.
$$

Noticing that

$$
\begin{aligned}
\left|\langle\widehat{w}^m,2\eta^{m+\frac{1}{2}}\rangle\right|&\le 2\|\widehat{w}^m\|^2+\frac{1}{8}\|\eta^m+\eta^{m+1}\|^2\\
&\le 2M_E^2\|e^m\|^2+\frac{1}{4}\left(\|\eta^m\|^2+\|\eta^{m+1}\|^2\right)\le\frac{1}{2}\mathcal{F}^m,
\end{aligned}
$$

which implies

$$
\mathcal{F}^m\le 2\mathcal{F}^0+C(\tau^2+h^4)^2+C\tau\sum_{n=0}^{m}\mathcal{F}^n,
$$

by recalling that $\widehat{w}^0=0$. Noticing that $e^0=\eta^0=0$, it remains to estimate η^1 and $\delta_x^+\theta^0$. Denote

$$
\begin{aligned}
\sigma_k^0&=\delta_t^2N(x_k,0)-\frac{1}{2}H\delta_x^2(N(x_k,\tau)+N(x_k,-\tau))+\frac{\varepsilon^2}{2}H^2\delta_x^4(N(x_k,\tau)\\
&\quad+N(x_k,-\tau))-H\delta_x^2\,|E(x_k,0)|^2,
\end{aligned}
$$

which represents the truncation error at $t=0$. It can be clearly seen that

$$
|\sigma_k^0|=O(\tau^2+h^4),
$$

when the solutions are smooth enough. Denote $s_k=\frac{N(x_k,\tau)+N(x_k,-\tau)}{2}-(N_k^1-\tau N_1(x_k))$, thus by definition (2.8), we have

$$
\frac{2}{\tau^2}s-H\delta_x^2s+\varepsilon^2H^2\delta_x^4s=\sigma^0.
$$

Taking the inner product of this equation with s yields

$$\begin{aligned}\|s\|^2 + \frac{\tau^2}{2}\left(\|R\delta_x^+ s\|^2 + \varepsilon^2\|H\delta_x^2 s\|^2\right) &= \frac{\tau^2}{2}\langle\sigma^0, s\rangle \\ &\leq \tau^4\|\sigma^0\|^2 + \frac{1}{2}\|s\|^2 \leq C\tau^4(\tau^2+h^4)^2 + \frac{1}{2}\|s\|^2,\end{aligned}$$

which implies

$$\|s\| \leq C\tau^2(\tau^2+h^4), \quad \|R\delta_x^+ s\| \leq C\tau(\tau^2+h^4).$$

Noticing that $N(x_k, -\tau) = N(x_k, \tau) - 2\tau N_1(x_k) + O(\tau^3)$, this means

$$s_k = \eta_k^1 + O(\tau^3), \quad \delta_x^+ s_k = \delta_x^+ \eta_k^1 + O(\tau^3),$$

thus we get

$$\|\eta^1\| \leq C\tau\left(\tau^2+h^4\right), \quad \|R\delta_x^+\eta^1\| \leq C\tau(\tau^2+h^4).$$

Applying definition (4.20), the Young's inequality and $\|\theta^0\| \leq C\|\delta_x^+\theta^0\|$, one easily finds

$$\|R\delta_x^+\theta^0\|^2 = -\left\langle H\delta_x^2\theta^0, \theta^0\right\rangle = -\frac{1}{\tau}\left\langle\eta^1, \theta^0\right\rangle \leq \frac{1}{2}\|R\delta_x^+\theta^0\|^2 + \frac{C}{\tau^2}\|\eta^1\|^2,$$

which immediately yields

$$\|R\delta_x^+\theta^0\| \leq \frac{C}{\tau}\|\eta^1\| \leq C(\tau^2+h^4).$$

Now we can conclude that $\mathcal{F}^0 \leq C(\tau^2+h^4)^2$. Applying the Gronwall's inequality, there exists $\tau_0 > 0$ such that

$$\mathcal{F}^m \leq C(\tau^2+h^4)^2, \quad m \geq 1,$$

when $\tau < \tau_0$. According to the definition of $\mathcal{F}^n$, it follows that

$$\left\|e^n\right\| + \left\|R\delta_x^+ e^n\right\| + \varepsilon\left\|H\delta_x^2 e^n\right\| \leq C\left(\tau^2+h^4\right), \quad \left\|\eta^n\right\| + \varepsilon\left\|R\delta_x^+\eta^n\right\| \leq C\left(\tau^2+h^4\right).$$

This together with Lemma 3.3 yields (4.10) and the proof is completed. □

Remark 4.1 It is easy to verify that the compact finite difference scheme and the corresponding convergence can be established for the QZS with homogeneous Dirichlet boundary condition

$$\begin{aligned}&E(a,t) = E(b,t) = N(a,t) = N(b,t) = 0, \\ &\partial_x^2 E(a,t) = \partial_x^2 E(b,t) = \partial_x^2 N(a,t) = \partial_x^2 N(b,t) = 0.\end{aligned}$$

Remark 4.2 Noticing the proof of Theorem 4.2 relies strongly on the a prior bound $\|E^n\|_\infty \leq C$ with C independent of ε. For higher dimensions, e.g., $d = 2$, applying the Discrete Sobolev inequality [47]

$$\|E^n\|_\infty \leq C\sqrt{\|E^n\|}\sqrt{\|E^n\| + \|\delta_x^2 E^n\|},$$

one gets

$$\|E^n\|_4^4 \leq \|E^n\|_\infty^2\|E^n\|^2 \leq C\|E^n\|^3(\|E^n\| + \|\delta_x^2 E^n\|) \leq \varepsilon^2\|H\delta_x^2 E^n\|^2 + C/\varepsilon^2.$$

This together with the conservative energy implies an a prior bound $\|E^n\|_\infty \leq C/\varepsilon$. Thus repeating the similar argument will lead to the same convergence with the constant depending on ε, which suggests that the convergence is not uniform for $\varepsilon \in (0, 1]$. To overcome this problem and get a uniform convergence, another idea is to use the cut-off technique [1,5] by truncating the nonlinearity to a global Lipschitz function with a compact support and the ε-independent bound of the difference solution can be derived by the triangle inequality

$$\|E^n\|_\infty \le \|e^n\|_\infty + \|E\|_{L^\infty([0,T];L^\infty(\Omega))},$$

and the inverse inequality [5,40]

$$\|e^n\|_\infty \le \frac{C}{C_d(h)}\left(\|e^n\| + \|\delta_x^+ e^n\|\right), \quad C_d(h) = \begin{cases} 1/|\ln(h)|, & d=2, \\ h^{1/2}, & d=3, \end{cases}$$

under the constraint that $\tau \le C\sqrt{C_d(h)}$. Finally the uniform ε-independent error bound of $\|e^n\| + \|\delta_x^+ e^n\| + \|\eta^n\|$ can be established under the constraint condition $\tau \le C\sqrt{C_d(h)}$ for higher dimension problem.

5 Numerical Results

In this section, we report some numerical results to demonstrate the accuracy of the proposed compact difference scheme (2.6)–(2.9) for solving the QZS (2.1). Furthermore, we apply the method to numerically study the convergence of the QZS to ZS in the semiclassical limit.

5.1 Accuracy Test

Several examples are presented to verify the spatial and temporal accuracy of the scheme. For 1D problem, we solve the QZS by the scheme (2.6)–(2.9) on a bounded interval $\Omega = [-32, 32]$ with periodic boundary conditions. In order to quantify the numerical errors, we introduce the following error functions:

$$e^\varepsilon(t_n) = \|e^{\varepsilon,n}\| + \|\delta_x^+ e^{\varepsilon,n}\| + \varepsilon\|\delta_x^2 e^{\varepsilon,n}\|, \quad \eta^\varepsilon(t_n) = \|\eta^{\varepsilon,n}\| + \varepsilon\|\delta_x^+ \eta^{\varepsilon,n}\|,$$

where

$$e_j^{\varepsilon,n} = E^\varepsilon(x_j, t_n) - E_j^{\varepsilon,n}, \quad \eta_j^{\varepsilon,n} = N^\varepsilon(x_j, t_n) - N_j^{\varepsilon,n}$$

are the numerical errors with $(E^\varepsilon(x_j, t_n), N^\varepsilon(x_j, t_n))$ and $(E_j^{\varepsilon,n}, N_j^{\varepsilon,n})$ being the exact and numerical solutions of the QZS with quantum parameter ε, respectively. Furthermore, to testify the conservation properties, we define the discrete mass and energy errors as

$$Error_{\|E^n\|} = \left|\|E^n\| - \|E^0\|\right|, \quad Error_{\mathcal{E}^n} = |\mathcal{E}^n - \mathcal{E}^0|.$$

Example 5.1 Taking $\varepsilon = 0$ in (2.1), this corresponds to the classical ZS

$$\begin{cases} iE_t + E_{xx} = NE, \\ N_{tt} - N_{xx} = \partial_x^2\left(|E|^2\right). \end{cases} \tag{5.1}$$

The Langmuir solitary solution of the classical ZS is given by (cf. e.g., [24])

$$\begin{aligned} &E(x,t) = i\sqrt{2B^2\left(1-v^2\right)}\,\mathrm{sech}\left(B\left(x-x_0-vt\right)\right)e^{i\left(v(x-x_0)/2-\left(v^2/4-B^2\right)t\right)}, \\ &N(x,t) = -2B^2\,\mathrm{sech}^2\left(B\left(x-x_0-vt\right)\right), \end{aligned} \tag{5.2}$$

Table 1 Temporal accuracy of the scheme at $T = 1$ for (5.1), $h = 1/2^5$

τ	$e^0(1)$	Order	$\eta^0(1)$	Order
1/20	4.9718e−04	–	1.7612e−03	–
1/40	1.2445e−04	1.9982	4.4115e−04	1.9972
1/80	3.1128e−05	1.9993	1.1044e−04	1.9981
1/160	7.7872e−06	1.9990	2.7714e−05	1.9945
1/320	1.9521e−06	1.9961	7.0309e−06	1.9788

Table 2 Spatial accuracy of the scheme at $T = 1$ for (5.1), $\tau = 1/10{,}000$

h	$e^0(1)$	Order	$\eta^0(1)$	Order
1/2	4.1929e−03	–	1.1281e−02	–
1/4	2.4118e−04	4.1198	6.5459e−04	4.1072
1/8	1.4814e−05	4.0250	3.9861e−05	4.0375
1/16	9.2259e−07	4.0052	2.4818e−06	4.0056
1/32	5.8620e−08	3.9762	1.6128e−07	3.9438

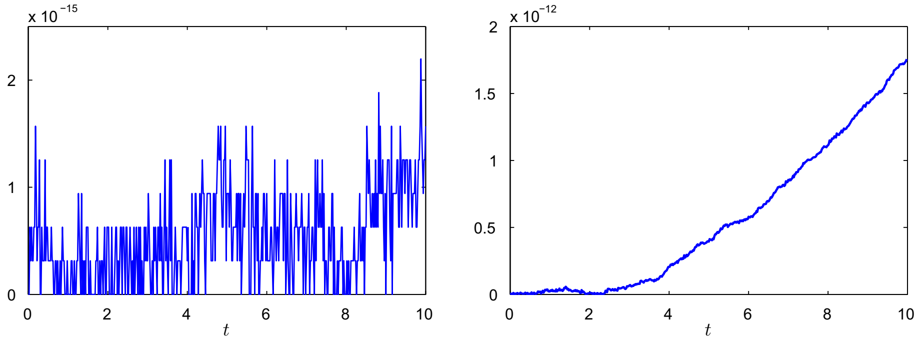

Fig. 1 Conservation of mass (left) and energy (right) of the compact finite difference scheme for the classical ZS (5.1)

where B is a constant, x_0 represents the initial displacement, and v represents the propagation velocity of the soliton. Set $B = 1, v = \frac{1}{2}, x_0 = 0$. The initial and boundary conditions $E_0(x)$, $N_0(x)$ and $N_1(x)$ are obtained from (5.2) by setting $t = 0$, i.e.,

$$E_0(x) = i\sqrt{1.5}\,\mathrm{sech}(x)e^{ix/4}, \quad N_0(x) = -2\,\mathrm{sech}^2(x), \quad N_1(x) = -2\,\mathrm{sech}^2(x)\tanh(x). \tag{5.3}$$

Tables 1 and 2 list the temporal and spatial errors of the compact finite difference method (2.6)–(2.9) for the classical ZS at $T = 1$ under different choices of τ and h. To testify the spatial accuracy, we take a tiny time step $\tau = 1/10{,}000$ such that the temporal error is negligible; for temporal error analysis, we set the mesh size $h = 1/32$ such that the spatial error can be ignorable. It can be clearly observed that the scheme converges quartically and quadratically in space and time, respectively, which agrees with the theoretical estimate in Theorem 4.2. Figures 1 and 2 display the conservation and the numerical error with respect to time, respectively, where the computation was performed with $h = 1/8$, $\tau = 1/50$ until $T = 10$. We observe that the numerical scheme preserves the discrete mass and energy of the ZS very well. Furthermore, Fig. 2 suggests that the scheme keeps stable well, too.

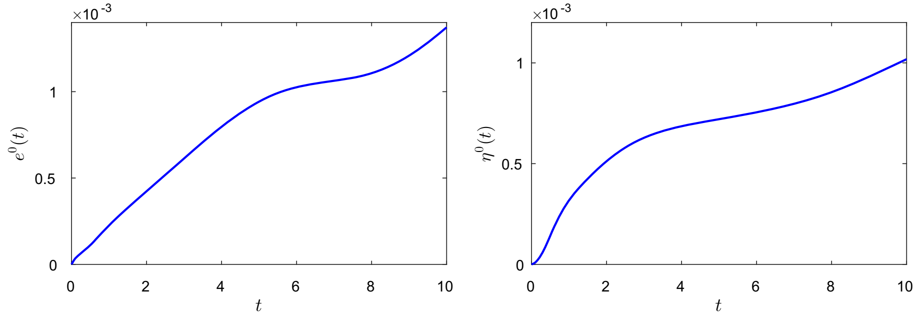

Fig. 2 Errors $e^0(t)$ (left) and $\eta^0(t)$ (right) of the compact finite difference scheme for the classical ZS (5.1)

Table 3 Comparison of maximum norm error at $T = 1$ for (5.1), $\tau = 1/1000$

	h	1/2	1/4	1/8	1/16
CFDM (2.6)–(2.9)	$\|e^{0,n}\|_\infty$	3.6589e−03	2.0592e−04	1.2646e−05	8.5976e−07
	$\|\eta^{0,n}\|_\infty$	1.0453e−02	6.5363e−04	4.4865e−05	3.5011e−06
LIFDM in [43]	$\|e^{0,n}\|_\infty$	4.6583e−02	1.0849e−02	2.6985e−03	6.7317e−04
	$\|\eta^{0,n}\|_\infty$	8.5400e−02	2.5568e−02	6.1770e−03	1.5321e−03

To verify the accuracy and efficiency, we compare the compact finite difference scheme (2.6)–(2.9) (CFDM) with another conservative linearly-implicit finite difference method (LIFDM) proposed by [43] in Table 3, where we employ the maximum norm error. It should be pointed out that the LIFDM [43] converges at the second order in space while (2.6)–(2.9) converges at the fourth order in space.

Example 5.2 Consider the QZS (2.1) with initial conditions (5.3) and different $\varepsilon > 0$. Since the exact solution is not known, we take the numerical solution obtained by the proposed compact difference scheme with $h = 1/2^6$, $\tau = 1/1600$ as the reference solution.

The errors at $T = 1$ of the numerical solutions under various h and τ are listed in Tables 4 and 5. It is observed that the CFDM (2.6)–(2.9) gives second- and fourth-order accuracies in time and space, respectively, which confirms the error estimate in Theorem 4.2. The time evolution, conservation properties, the errors $e^{\varepsilon}(t)$ and $\eta^{\varepsilon}(t)$ of the solution for the QZS with $\varepsilon = 0.01$ are shown in Figs. 3, 4 and 5, respectively, where the computation was performed with $h = 1/8$, $\tau = 1/50$. It can be observed that the difference scheme preserves the conservation laws very well and the dynamics of the soliton for the QZS coincides with that for the classical ZS (5.2) when ε is small. Comparison between the CFDM (2.6)–(2.9) and the LIFDM [43] is listed in Table 6, which shows the superiority in spatial accuracy of the CFDM.

Example 5.3 We test the accuracy of the scheme for two-dimensional problem. The initial conditions are selected as

$$E_0(x, y) = \sin(\pi x + \pi y), \quad N_0(x, y) = 0, \quad N_1(x, y) = 0, \tag{5.4}$$

We confine this problem on a periodical cell $\Omega = [-4, 4] \times [-4, 4]$. Since the exact solution is not known, we take the numerical solution obtained by the compact difference

Table 4 Temporal errors of the scheme at $T = 1$ for QZS (2.1) with $h = \frac{1}{128}$ and different ε

ε	τ	$e^{\varepsilon}(1)$	Order	$\eta^{\varepsilon}(1)$	Order
$\varepsilon = \frac{1}{2^4}$	1/20	7.6010e−04	–	1.8798e−03	–
	1/40	2.4811e−04	1.6152	4.7066e−04	1.9978
	1/80	7.3944e−05	1.7464	1.1750e−04	2.0020
	1/160	1.9867e−05	1.8960	2.9158e−05	2.0107
	1/320	4.9326e−06	2.0100	7.0689e−06	2.0443
$\varepsilon = \frac{1}{2^6}$	1/20	4.9894e−04	–	1.7676e−03	–
	1/40	1.2511e−04	1.9957	4.4245e−04	1.9982
	1/80	3.1280e−05	1.9998	1.1045e−04	2.0021
	1/160	7.7669e−06	2.0099	2.7408e−05	2.0107
	1/320	1.8831e−06	2.0442	6.6445e−06	2.0444
$\varepsilon = \frac{1}{2^8}$	1/20	4.9714e−04	–	1.7612e−03	–
	1/40	1.2438e−04	1.9989	4.4084e−04	1.9983
	1/80	3.1047e−05	2.0023	1.1005e−04	2.0021
	1/160	7.7040e−06	2.0108	2.7309e−05	2.0107
	1/320	1.8677e−06	2.0444	6.6204e−06	2.0444

Table 5 Spatial errors of the scheme at $T = 1$ for QZS (2.1) with $\tau = 1/1600$ and different ε

ε	h	$e^{\varepsilon}(1)$	Order	$\eta^{\varepsilon}(1)$	Order
$\varepsilon = \frac{1}{2^4}$	1/2	5.1113e−03	–	1.2114e−02	–
	1/4	3.3262e−04	3.9417	6.6569e−04	4.1857
	1/8	2.2289e−05	3.8995	4.0535e−05	4.0376
	1/16	1.5114e−06	3.8824	2.5169e−06	4.0094
	1/32	1.0576e−07	3.8370	1.5648e−07	4.0076
$\varepsilon = \frac{1}{2^6}$	1/2	4.4290e−03	–	1.1345e−02	–
	1/4	2.6348e−04	4.0712	6.5521e−04	4.1139
	1/8	1.6524e−05	3.9951	3.9892e−05	4.0378
	1/16	1.0346e−06	3.9974	2.4771e−06	4.0094
	1/32	6.4518e−08	4.0033	1.5404e−07	4.0073
$\varepsilon = \frac{1}{2^8}$	1/2	4.2529e−03	–	1.1285e−02	–
	1/4	2.4683e−04	4.1069	6.5462e−04	4.1076
	1/8	1.5244e−05	4.0172	3.9856e−05	4.0378
	1/16	9.5014e−07	4.0040	2.4749e−06	4.0094
	1/32	5.9164e−08	4.0054	1.5390e−07	4.0073

scheme with $h_1 = h_2 = 1/2^6$, $\tau = 1/3200$ as the reference solution. The errors at $T = 0.5$ of the numerical solutions under various h and τ are listed on Tables 7 and 8. It can be clearly seen that the CFDM (2.6)–(2.9) gives second- and fourth-order accuracies in time and space, respectively, for 2D problem. Similarly, comparison between the CFDM (2.6)–(2.9) and the LIFDM [43] is displayed in Table 9, which confirms the superior accuracy of the CFDM.

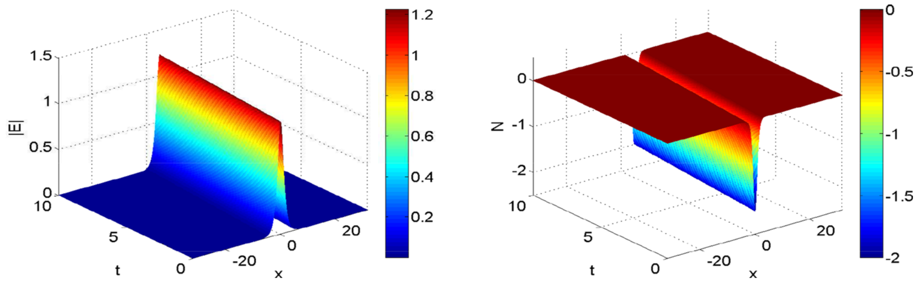

Fig. 3 Time evolution of the solitary wave for the QZS (2.1) with $\varepsilon = 0.01$: $|E|$ (left) and N (right)

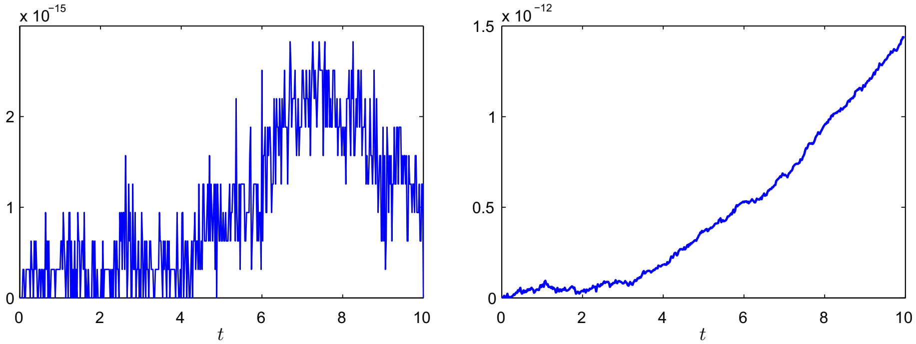

Fig. 4 Conservation of mass (left) and energy (right) of the compact finite difference scheme for QZS (2.1) with $\varepsilon = 0.01$

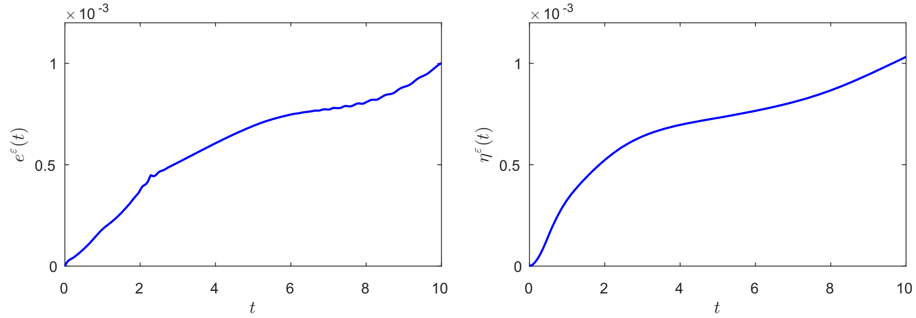

Fig. 5 Errors $e^{\varepsilon}(t)$ (left) and $\eta^{\varepsilon}(t)$ (right) of the compact finite difference scheme for the QZS (2.1) with $\varepsilon = 0.01$

5.2 Convergence of the QZS to the Classical ZS in the Semi-classical Limit ($\varepsilon \to 0$)

Here we apply the proposed CFDM (2.6)–(2.9) to study the convergence of the QZS (2.1) to its limiting model, i.e., the classical ZS (5.1). In order to do so, we choose the same initial conditions as (5.3), and define the error functions as

$$\eta_E(t) := \|E^{\varepsilon}(\cdot, t) - E(\cdot, t)\|_{H^2}, \quad \eta_N(t) := \|N^{\varepsilon}(\cdot, t) - N(\cdot, t)\|_{H^1},$$

where $(E^{\varepsilon}, N^{\varepsilon})$ and (E, N) are the solution of the QZS (2.1) and the classical ZS (5.1), respectively, and $(E^{\varepsilon}, N^{\varepsilon})$ are obtained by the proposed scheme (2.6)–(2.9) with $h = 1/16$,

Table 6 Comparison of maximum norm error at $T = 1$ for QZS (2.1) with $\varepsilon = 1/64$, $\tau = 1/1600$

	h	1/2	1/4	1/8	1/16
CFDM (2.6)–(2.9)	$\|e^{\varepsilon,n}\|_\infty$	3.6639e−03	2.0543e−04	1.2528e−05	7.9014e−07
	$\|\eta^{\varepsilon,n}\|_\infty$	1.0515e−02	6.4443e−04	4.3821e−05	2.7082e−06
LIFDM in [43]	$\|e^{\varepsilon,n}\|_\infty$	4.6561e−02	1.0844e−02	2.6952e−03	6.7252e−04
	$\|\eta^{\varepsilon,n}\|_\infty$	8.5063e−02	2.5505e−02	6.1594e−03	1.5276e−03

Table 7 Temporal errors of the scheme at $T = 0.5$ for 2D QZS (5.4) with $h = 1/64$

ε	τ	$e^\varepsilon(0.5)$	Order	$\eta^\varepsilon(0.5)$	Order
$\varepsilon = \frac{1}{2^4}$	1/160	6.7516e−01	–	1.5365e−02	–
	1/320	1.6785e−01	2.0081	3.8138e−03	2.0104
	1/640	4.0712e−02	2.0436	9.2463e−04	2.0443
	1/1280	8.9071e−03	2.1924	2.0226e−04	2.1926
$\varepsilon = \frac{1}{2^6}$	1/160	5.0774e−01	–	9.4749e−03	–
	1/320	1.2620e−01	2.0084	2.3533e−03	2.0094
	1/640	3.0608e−02	2.0437	5.7060e−04	2.0441
	1/1280	6.6962e−03	2.1925	1.2483e−04	2.1926
$\varepsilon = \frac{1}{2^8}$	1/160	4.9249e−01	–	8.1508e−03	–
	1/320	1.2240e−01	2.0085	2.0244e−03	2.0094
	1/640	2.9686e−02	2.0438	4.9094e−04	2.0439
	1/1280	6.4945e−03	2.1925	1.0740e−04	2.1925

$\tau = 1/400$. Figure 6 displays the errors between the solutions of the QZS (2.1) and the classical ZS (5.1), which indicates that the QZS converges to the ZS quadratically in terms of ε. This is consistent with the analytical result in [17]:

$$\left\| E^\varepsilon(\cdot,t) - E(\cdot,t)\right\|_{H^k} + \left\| N^\varepsilon(\cdot,t) - N(\cdot,t)\right\|_{H^{k-1}} \le C\varepsilon^2, \quad k \ge 2, \quad 0 \le t \le T,$$

where C is a positive constant which is independent of $\varepsilon \in (0, 1]$.

5.3 Collision Between Two Solitons

In this section, we apply the finite difference scheme to investigate the interaction between two solitary waves traveling in the opposite directions.

Example 5.4 The initial data is chosen as (cf. [43])

$$\begin{aligned} E_0(x) &= i\sum_{k=1}^{2}\sqrt{2\left(1-v_k^2\right)}\,\mathrm{sech}\,(x-x_k)\,e^{iv_k(x-x_k)/2}, \\ N_0(x) &= -2\sum_{k=1}^{2}\mathrm{sech}^2\,(x-x_k)\,, \quad N_1(x) = -4\sum_{k=1}^{2} v_k\,\mathrm{sech}^2\,(x-x_k)\tanh\,(x-x_k)\,, \end{aligned} \tag{5.5}$$

which represents two solitary waves located initially at the positions $x = x_1$ and $x = x_2$, respectively, moving to the right or left depending on the sign of the velocity v_k. Computations

Table 8 Spatial errors of the scheme at $T = 0.5$ for 2D QZS (5.4) with $\tau = 1/3200$

ε	h	$e^{\varepsilon}(0.5)$	Order	$\eta^{\varepsilon}(0.5)$	Order
$\varepsilon = \frac{1}{2^4}$	1/4	8.4035e−01	–	3.2288e−01	–
	1/8	5.2705e−02	3.9950	1.9339e−02	4.0614
	1/16	3.2818e−03	4.0054	1.1920e−03	4.0201
	1/32	1.9303e−04	4.0876	6.9942e−05	4.0910
$\varepsilon = \frac{1}{2^6}$	1/4	6.6577e−01	–	2.0825e−01	–
	1/8	4.1687e−02	3.9973	1.2402e−02	4.0697
	1/16	2.5956e−03	4.0054	7.6107e−04	4.0264
	1/32	1.5267e−04	4.0876	4.4598e−05	4.0930
$\varepsilon = \frac{1}{2^8}$	1/4	6.5012e−01	–	1.8005e−01	–
	1/8	4.0642e−02	3.9997	1.0623e−02	4.0831
	1/16	2.5294e−03	4.0061	6.5044e−04	4.0297
	1/32	1.4877e−04	4.0876	3.8093e−05	4.0938

Table 9 Comparison of maximum norm error at $T = 0.5$ for 2D QZS (5.4) with $\varepsilon = 1/64$, $\tau = 1/1600$

	h	1/4	1/8	1/16	1/32
CFDM (2.6)–(2.9)	$\|e^{\varepsilon,n}\|_\infty$	1.6344e−02	1.0073e−03	6.2363e−05	3.6638e−06
	$\|\eta^{\varepsilon,n}\|_\infty$	3.3086e−02	1.8682e−03	1.1402e−04	6.6750e−06
LIFDM in [43]	$\|e^{\varepsilon,n}\|_\infty$	4.9596e−01	1.2758e−01	3.2077e−02	8.0287e−03
	$\|\eta^{\varepsilon,n}\|_\infty$	2.1163e−01	5.6904e−02	1.4405e−02	3.6113e−03

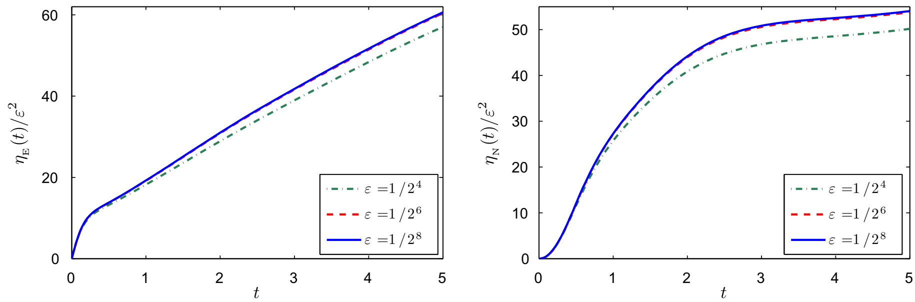

Fig. 6 Convergence between the QZS and classical ZS under initial data (5.3)

are done with $h = 1/16$ and $\tau = 1/500$ on the interval $\Omega = [-400, 400]$ which is large enough that the truncation error to the original whole space problem (1.1) can be negligible due to the periodic boundary condition. We consider the following cases:

(i) $x_2 = -x_1 = 30$, $v_1 = -v_2 = 1/2$;
(ii) $x_2 = -x_1 = 30$, $v_1 = 3/4$, $v_2 = -1/2$;
(iii) $x_2 = -x_1 = 5$, $v_1 = 3/4$, $v_2 = -1/2$.

Figures 7 and 8 display the interaction of two solitary waves for the ZS ($\varepsilon = 0$) and QZS ($\varepsilon = 1/2^4$) under cases (i)–(iii), respectively. It can be clearly seen that the interactions are not elastic under all cases. For the dynamics of the ZS, when two initially well-separated

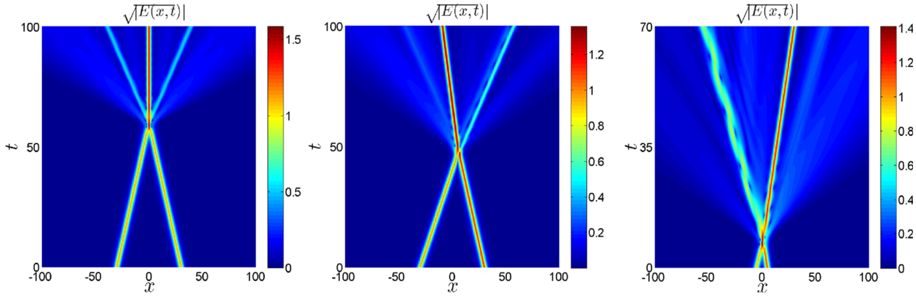

Fig. 7 Inelastic collision between two solitons for the ZS (5.1) under cases (i)–(iii) (from left to right)

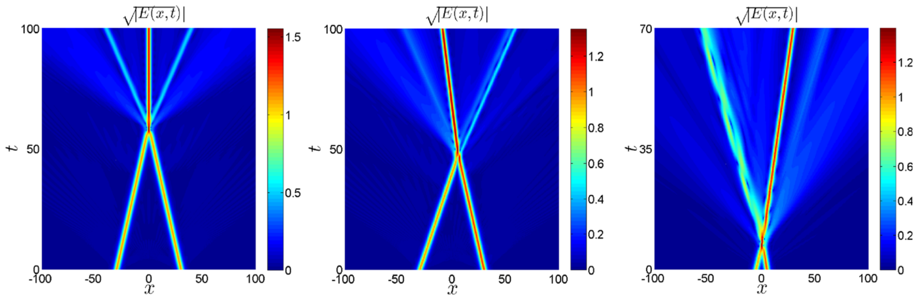

Fig. 8 Inelastic collision between two solitons for the QZS (2.1) with $\varepsilon = 1/2^4$ under cases (i)–(iii) (from left to right)

solitons move towards each other at the same velocity (cf. case (i) in Fig. 7), they collide and split after collision as two amplitude-weakened solitons with velocities changed. At the same time, a static pulse with strengthened amplitude and some small radiation are generated during the collision. When the velocities are different (cf. case (ii) in Fig. 7), the static soliton is replaced by a moving solitary wave. The dynamics is much more complicated when the two solitons are not initially well separated (cf. case (iii) in Fig. 7). Unlike the former cases where the soliton remains as a soliton with constant density and velocity after collision, here there is a periodic perturbation on the position of some localized pulse. From Fig. 8, we find that the collision between two solitons for the QZS differs from the ZS in two aspects: (1) there are small outgoing waves emitting before collision; (2) the quantum effect makes the chaos more obvious after collision.

6 Conclusion

We proposed a conservative linearly-implicit compact finite difference scheme to solve the quantum Zakharov system (QZS). The method is decoupled and only two linear systems are solved at each time step, which is very efficient in implementation. The convergence of the scheme was established at $O(\tau^2 + h^4)$, which was confirmed by the reported numerical examples. By adopting our numerical method, we observed numerically that the QZS converges to the classical Zakharov system (ZS) quadratically in the semi-classical limit. Finally some examples are provided to show the distinction between the dynamics of the QZS and the classical ZS.

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Declarations

Conflict of interest Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

1. Bao, W., Su, C.: Uniform error bounds of a finite difference method for the Zakharov system in the subsonic limit regime via an asymptotic consistent formulation. Multiscale Model. Simul. **15**, 977–1002 (2017)
2. Bao, W., Su, C.: A uniformly and optically accurate method for the Zakharov system in the subsonic limit regime. SIAM J. Sci. Comput. **40**, A929–A953 (2018)
3. Bao, W., Sun, F.: Efficient and stable numerical methods for the generalized and vector Zakharov system. SIAM J. Sci. Comput. **26**(3), 1057–1088 (2005)
4. Bao, W., Sun, F., Wei, G.W.: Numerical methods for the generalized Zakharov system. J. Comput. Phys. **190**, 201–228 (2003)
5. Cai, Y., Yuan, Y.: Uniform error estimates of the conservative finite difference method for the Zakharov system in the subsonic limit regime. Math. Comput. **87**, 1191–1225 (2018)
6. Chang, Q., Guo, B., Jiang, H.: Finite difference method for generalized Zakharov equations. Math. Comput. **64**(210), 537–553 (1995)
7. Choi, B.J.: Global well-posedness of the adiabatic limit of quantum Zakharov system in 1D (2019). arXiv:1906.10807v2
8. Choi, B.J.: Multilinear weighted estimates and quantum Zakharov system (2020). arXiv: 2004.08952v1
9. Davydov, A.S.: Solitons in molecular systems. Phys. Scr. **20**, 387–394 (1979)
10. Degtyarev, L.M., Nakhankov, V.G., Rudakov, L.I.: Dynamics of the formation and interaction of Langmuir solitons and strong turbulence. Sov. Phys. JETP **40**, 264–268 (1974)
11. Dehghan, M., Abbaszadeh, M.: Numerical investigation based on direct meshless local Petrov Galerkin (direct MLPG) method for solving generalized Zakharov system in one and two dimensions and generalized Gross–Pitaevskii equation. Eng. Comput. **33**(4), 983–996 (2017)
12. Dehghan, M., Abbaszadeh, M.: Solution of multi-dimensional Klein–Gordon–Zakharov and Schrödinger/Gross–Pitaevskii equations via local radial basis functions-differential quadrature (RBF-DQ) technique on non-rectangular computational domains. Eng. Anal. Bound. Elem. **92**, 156–170 (2018)
13. Dehghan, M., Mohammadi, V.: Two numerical meshless techniques based on radial basis functions (RBFs) and the method of generalized moving least squares (GMLS) for simulation of coupled Klein–Gordon–Schrödinger (KGS) equations. Comput. Math. Appl. **71**(4), 892–921 (2016)
14. Dehghan, M., Mohammadi, V.: A numerical scheme based on radial basis function finite difference (RBF-FD) technique for solving the high-dimensional nonlinear Schrödinger equations using an explicit time discretization: Runge–Kutta method. Comput. Phys. Commun. **217**, 23–34 (2017)
15. Dehghan, M., Mohebbi, A., Asgari, Z.: Fourth-order compact solution of the nonlinear Klein–Gordon equation. Numer. Algorithm **52**, 523–540 (2009)
16. Dehghan, M., Nikpour, A.: The solitary wave solution of coupled Klein–Gordon–Zakharov equations via two different numerical methods. Comp. Phys. Commun. **184**(9), 2145–2158 (2013)
17. Fang, Y., Kuo, H., Shih, H., Wang, K.: Semi-classical limit for the quantum Zakharov system. Taiwan. J. Math. **23**(4), 925–949 (2019)
18. Fang, Y., Nakanishi, K.: Global well-posedness and scattering for the quantum Zakharov system in L^2. Proc. Am. Math. Soc. **6**, 21–32 (2019)
19. Fang, Y., Segata, J., Wu, T.: On the standing waves of quantum Zakharov system. J. Math. Anal. Appl. **458**, 1427–1448 (2018)
20. Fang, Y., Shih, H., Wang, K.: Local well-posedness for the quantum Zakharov system in one spatial dimension. J. Hyperbolic Differ. Equ. **14**(01), 157–192 (2017)
21. Feng, Y.: Long time error analysis of the fourth-order compact finite difference methods for the nonlinear Klein–Gordon equation with weak nonlinearity. Numer. Methods Partial Differ. Equ. **37**(1), 897–914 (2021)

22. Garcia, L.G., Haas, F., de Oliveira, L.P.L., Goedert, J.: Modified Zakharov equations for plasmas with a quantum correction. Phys. Plasmas **12**(1), 012302 (2005)
23. Glangetas, L., Merle, F.: Existence of self-similar blow-up solutions for Zakharov equation in dimension two I. Commun. Math. Phys. **160**, 173–215 (1994)
24. Glassey, R.T.: Convergence of energy-preserving scheme for the Zakharov equations in one space dimension. Math. Comput. **58**, 83–102 (1992)
25. Guo, B., Gan, Z., Kong, L., Zhang, J.: The Zakharov System and Its Soliton Solutions. Science Press, Beijing (2016)
26. Guo, Y., Zhang, J., Guo, B.: Global well-posedness and the classical limit of the solution for the quantum Zakharov system. Z. Angew. Math. Phys. **64**(1), 53–68 (2013)
27. Haas F.: Quantum Plasmas. Springer Series on Atomics, Optical and Plasma Physics, vol. 65 (2011)
28. Haas, F., Shukla, P.K.: Quantum and classical dynamics of Langmuir wave packets. Phys. Rev. E **79**(6), 066402 (2009)
29. Hu, X., Zhang, L.: Conservative compact difference schemes for the coupled nonlinear Schrödinger system. Numer. Methods Partial Differ. Equ. **30**, 749–772 (2014)
30. Jiang, J.C., Lin, C.K., Shao, S.: On one dimensional quantum Zakharov system. Discrete Contin. Dyn. Syst. **36**(10), 5445–5475 (2016)
31. Jin, S., Markowich, P.A., Zheng, C.: Numerical simulation of a generalized Zakharov system. J. Comput. Phys. **201**, 376–395 (2004)
32. Lee, S., Shin, J.: Energy stable compact scheme for Cahn–Hilliard equation with periodic boundary condition. Comput. Math. Appl. **77**(1), 189–198 (2019)
33. Li, J., Sun, Z., Zhao, X.: A three level linearized compact difference scheme for the Cahn–Hilliard equation. Sci. China Math. **55**(04), 805–826 (2012)
34. Misra, A.P., Banerjee, S., Haas, F., Shukla, P.K., Assis, L.P.G.: Temporal dynamics in the one-dimensional quantum Zakharov equations for plasmas. Phys. Plasmas **17**(3), 032307 (2010)
35. Misra, A.P., Ghosh, D., Chowdhury, A.R.: A novel hyperchaos in the quantum Zakharov system for plasmas. Phys. Lett. A **372**(9), 1469–1476 (2008)
36. Misra, A.P., Shukla, P.K.: Pattern dynamics and spatiotemporal chaos in the quantum Zakharov equations. Phys. Rev. E **79**(5), 056401 (2009)
37. Mohebbi, A., Abbaszadeh, M., Dehghan, M.: Compact finite difference scheme and RBF meshless approach for solving 2D Rayleigh–Stokes problem for a heated generalized second grade fluid with fractional derivatives. Comput. Methods Appl. Mech. Eng. **264**, 163–177 (2013)
38. Pan, X., Zhang, L.: On the convergence of a high-accuracy conservative scheme for the Zakharov equations. Appl. Math. Comput. **297**, 79–91 (2017)
39. Sun, Q., Zhang, L., Wang, S., Hu, X.: A conservative compact difference scheme for the coupled Klein–Gordon–Schrödinger equation. Numer. Methods Partial Differ. Equ. **29**, 1657–1674 (2013)
40. Thomée, V.: Galerkin Finite Element Methods for Parabolic Problems. Springer, Berlin (1997)
41. Wang, T., Guo, B., Xu, Q.: Fourth-order compact and energy conservative difference scheme for the nonlinear Schrödinger equation in two dimensions. J. Comput. Phys. **243**, 382–399 (2013)
42. Wang, T., Zhang, L., Jiang, Y.: Convergence of an efficient and compact finite difference scheme for the Klein–Gordon–Zakharov equation. Appl. Math. Comput. **221**, 433–443 (2013)
43. Xiao, A., Wang, C., Wang, J.: Conservative linearly-implicit difference scheme for a class of modified Zakharov systems with high-order space fractional quantum correction. Appl. Numer. Math. **146**, 379–399 (2019)
44. Xie, S., Li, G., Yi, S.: Compact finite difference schemes with high accuracy for one-dimensional nonlinear Schrödinger equation. Comput. Methods Appl. Mech. Eng. **198**, 1052–1060 (2009)
45. Yao, S., Sun, J., Wu, T.: Stationary quantum Zakharov systems involving a higher competing perturbation. Electron. J. Differ. Equ. **2020**(6), 1–18 (2020)
46. Zakharov, V.E.: Collapse of Langmuir waves. Sov. Phys. JETP **35**(5), 908–914 (1972)
47. Zhou, Y.: Application of Discrete Functional Analysis to the Finite Difference Method. Inter Academy Publishers, Beijing (1990)